QUALITATIVE BEHAVIOR OF SOLUTIONS TO CROSS-DIFFUSION SYSTEMS FROM POPULATION DYNAMICS

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ABSTRACT. A general class of cross-diffusion systems for two population species in a bounded domain with no-flux boundary conditions and Lotka-Volterra-type source terms is analyzed. Although the diffusion coefficients are assumed to depend linearly on the population densities, the equations are strongly coupled. Generally, the diffusion matrix is neither symmetric nor positive definite. Three main results are proved: the existence of global uniformly bounded weak solutions, their convergence to the constant steady state in the weak competition case, and the uniqueness of weak solutions. The results hold under appropriate conditions on the diffusion parameters which are made explicit and which contain simplified Shigesada-Kawasaki-Teramoto population models as a special case. The proofs are based on entropy methods, which rely on convexity properties of suitable Lyapunov functionals.

1. INTRODUCTION

Many multi-species systems in biology, chemistry, and physics can be described by reaction-diffusion systems with cross-diffusion effects. The analysis of such problems is challenging since generally neither maximum principles nor regularity theory can be applied. Moreover, many systems have diffusion matrices that are neither symmetric nor positive definite such that even the local-in-time existence of solutions is a nontrivial task. In this paper, we apply and extend the boundedness-by-entropy method of [14] to a class of cross-diffusion systems for two species, which are motivated from population dynamics. Compared to our previous work [14], we are here interested in the qualitative behavior of weak solutions, namely their uniform boundedness, positivity, large-time asymptotics, and uniqueness.

1.1. **Setting.** We consider reaction-diffusion systems of the form

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \quad \text{in } \Omega, \; t > 0,$$
(1)

subject to the homogeneous Neumann boundary and initial conditions

$$(A(u)\nabla u) \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad u(0) = u^0 \quad \text{in } \Omega,$$
(2)

where $u = (u_1, u_2)^\top$ represents the vector of the densities of the species, $A(u) = (A_{ij}(u)) \in \mathbb{R}^{2\times 2}$ is the diffusion matrix, and the birth-death processes are modeled by the function $f = (f_1, f_2)$. Furthermore, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with Lipschitz boundary and ν is the exterior unit normal vector to $\partial\Omega$. Our main assumption is that the diffusivities depend linearly on the densities,

$$(3) \qquad A_{ij}(u) = \alpha_{ij} + \beta_{ij}u_1 + \gamma_{ij}u_2. \quad i,j = 1,2,$$

where α_{ij}, β_{ij}, γ_{ij} are real numbers.

Such models can be formally derived from a master equation for a random walk on a lattice in the diffusion limit with transition rates which depend linearly on the species' densities [14, Appendix B]. They can be also deduced as the limit equations of an interacting particle system modeled by stochastic differential equations with interaction forces which depend linearly on the corresponding stochastic processes [12, 19].

The most prominent example of (3) is probably the population model of Shigesada, Kawasaki, and Teramoto [20] (abbreviated SKT model):

$$(4) \qquad A(u) = \begin{pmatrix} a_{10} + 2a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + 2a_{22}u_2 \end{pmatrix},$$

where the coefficients a_{ij} are nonnegative, and the source terms in (1) are given by

$$(5) \qquad f_i(u) = (b_{i0} - b_{i1}u_1 - b_{i2}u_2)u_i, \quad i = 1,2,$$

and the coefficients b_{ij} are nonnegative. The existence of global weak solutions without any restriction on the diffusivities (except positivity) was achieved in [11] in one space dimension and in [6, 7] in several space dimensions. Global classical solutions for constant A_{ij} were shown to exist in [17]. Galiano [10] proved the uniqueness of bounded weak solutions to the SKT model with either diagonal diffusion matrix or the regularity assumption $\nabla u_i \in L^\infty$. Uniqueness of strong solutions was shown by Amann [1] in the triangular case ($a_{21} = 0$ in (4)).

There are much less results in the literature concerning L^∞ bounds and large-time asymptotics. In one space dimension and with coefficients $a_{10} = a_{20}$, Shim [21] proved uniform upper bounds. Moreover, if cross-diffusion is weaker than self-diffusion (i.e. $a_{12} < a_{22}$, $a_{21} < a_{11}$), weak solutions are bounded and Hölder continuous [16]. The existence of global bounded solutions in the triangular case (i.e. $a_{21} = 0$) was shown in [8]. In the triangular case, Le [15] proved the existence of a global attractor. With vanishing birth-death terms, it was shown in [7] that the solution to the SKT model converges exponentially fast to the constant steady state.

It cannot be expected that such results hold for any choice of the parameters appearing in (3) and (5). For instance, system (1) with

$$A(u) = \begin{pmatrix} 1 & -u_1 \\ 0 & 1 \end{pmatrix}, \quad f(u) = \begin{pmatrix} 0 \\ u_1 - u_2 \end{pmatrix}$$

corresponds to the parabolic-parabolic Keller-Segel model which exhibits the phenomenon of cell aggregation. If the cell density is sufficiently large initially, finite-time L^∞ blow-up

of solutions in two and three space dimensions occurs (see, e.g., [13]), and bounded weak solutions cannot be generally expected.

We wish to determine conditions on the parameters in (3) for which the weak solutions to (1)-(2) are uniformly bounded, positive, converge to the steady state, and are unique. The key idea is to apply and refine entropy methods. Here, an entropy is a convex Lyapunov functional which provides additional gradient estimates. Special entropies may also allow for uniform L^∞ bounds, see below. The advantage of these methods is a separation of the analytical and algebraic properties of the parabolic system. Often, it is sufficient to analyze the algebraic structure of the diffusion matrix, which simplifies the proofs, while achieving new results.

1.2. **Main results.** We introduce the triangle

$$(6) \qquad D = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > 0, \ u_2 > 0, \ u_1 + u_2 < 1\}.$$

First, we prove the existence of global bounded weak solutions to (1)-(3) for diffusion matrices of the form

$$A(u) = \begin{pmatrix} \alpha_{11} + \beta_{11}u_1 + \gamma_{11}u_2 & \beta_{12}u_1 \\ \gamma_{21}u_2 & \alpha_{22} + \beta_{22}u_1 + \gamma_{22}u_2 \end{pmatrix}.$$

Theorem 1 (Bounded weak solutions to (1)). *Let* $u^0 = (u_1^0, u_2^0) \in L^1(\Omega; \mathbb{R}^2)$ *be such that* $u^0(x) \in D$ *for* $x \in \Omega$, *let* $A(u)$ *be given by* (3) *with coefficients satisfying*

$$(7) \quad \alpha_{12} = \alpha_{21} = \beta_{21} = \gamma_{12} = 0,$$

$$(8) \quad \beta_{22} = \beta_{11} - \gamma_{21}, \quad \gamma_{11} = \gamma_{22} - \beta_{12}, \quad \gamma_{21} = \alpha_{22} - \alpha_{11} + \beta_{12},$$

$$(9) \quad \alpha_{11} > 0, \quad \alpha_{22} > 0, \quad \beta_{12} < \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\}, \quad \alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0,$$

and let $f_i(u) = u_i g_i(u)$, *where* $g_i(u)$ *is continuous in* $\overline{D}$ *and nonpositive in* $\{1 - \varepsilon < u_1 + u_2 < 1\}$ *for some* $\varepsilon > 0$ ($i = 1, 2$). *Then there exists a bounded nonnegative weak solution* $u = (u_1, u_2)$ *to* (1)-(2) *satisfying* $u(x, t) \in \overline{D}$ *for* $x \in \Omega$, $t > 0$,

$$(10) \qquad u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)'),$$

and the initial datum is satisfied in the sense of L^2.

Note that the L^∞ bound on u is uniform in time. We show in the appendix that (7) (and two further conditions) are necessary to apply the entropy method. Thus, in the framework of such techniques, conditions (7) cannot be improved. The theorem also holds true if $\alpha_{11} = \alpha_{22} = 0$ but $\beta_{11} > 0$ and $\gamma_{22} > 0$; see Remark 7. The condition $u_1^0 + u_2^0 < 1$ can be satisfied after a suitable scaling of the positive function $u^0 \in L^\infty(\Omega; \mathbb{R}^2)$ and is therefore not a restriction. The assumption on $f(u)$ guarantees that the triangle D is an invariant region under the action of the reaction terms. Theorem 1 generalizes the global existence result in [12], where the positive definiteness of A was needed. To the best of our knowledge, this is the first general existence result for uniformly bounded weak solutions to cross-diffusion systems with linear diffusivities.

The proof is based on the boundedness-by-entropy method, first used in [3] for a ion-transport model and later extended in [14]. The key idea is to formulate conditions under which the functional

$$\mathcal{H}[u] = \int_\Omega h(u)dx, \quad \text{where } h(u) = \sum_{i=1}^{3} u_i(\log u_i - 1), \ u_3 := 1 - u_1 - u_2,$$

is an entropy for (1). More precisely, assume that the derivative of the entropy density $h : D \to \mathbb{R}$ is invertible and the matrix $h''(u)A(u)$ is positive semidefinite, where $h''(u)$ is the Hessian of $h(u)$. We introduce the entropy variable $w = h'(u)$. Then (1) is equivalent to

$$\partial_t u - \text{div}(B(w)\nabla w) = f(u(w)),$$

where $B(w) = A(u)h''(u)^{-1}$ and $u(w) = (h')^{-1}(w)$. Now, if $f(u) \cdot w \leq 0$,

$$\frac{d}{dt}\mathcal{H}[u] \leq -\int_\Omega \nabla w : B(w)\nabla w dx = -\int_\Omega \nabla u : h''(u)A(u)\nabla u dx \leq 0,$$

where ":" denotes summation over both matrix indices. This shows that $\mathcal{H}[u]$ is a Lyapunov functional for (1). There is a second consequence: Since the triangle D in (6) is bounded, the original variable $u = (h')^{-1}(w)$ maps into D which is bounded. Therefore, $u(x,t) \in D$ and the solutions to (1) are bounded. This result holds without the use of a maximum principle.

Theorem 1 can be applied to the SKT model (4) to determine conditions under which this model possesses bounded weak solutions; see Section 2.3. The novelty is not the global existence (which has been proven in [6]) but the uniform boundedness of weak solutions.

The second main result is concerned with the large-time behavior of the solutions to (1). The steady state of (1)-(2) is defined as the only constant solution $U = (U_1, U_2)$ to (1)-(2). (There may be also non-constant steady states [18] but we are interested only in constant solutions.) The steady state is a solution to the algebraic equation $f(U) = 0$. If f is given by (5) and $(b_{ij})_{i,j=1,2}$ is positive definite, equation $f(U) = 0$ admits the unique solution

$$(11) \qquad U_1 = \frac{b_{10}b_{22} - b_{20}b_{12}}{b_{11}b_{22} - b_{12}b_{21}}, \quad U_2 = \frac{b_{20}b_{11} - b_{10}b_{21}}{b_{11}b_{22} - b_{12}b_{21}}.$$

Theorem 2 (Convergence to the steady state). *Let the hypotheses of Theorem 1 hold and let $f(u)$ be given by (5). Let the matrix $(b_{ij})_{i,j=1,2}$ be positive definite and assume that*

$$(12) \qquad b_{10} = b_{12} < b_{11}, \quad b_{20} = b_{21} < b_{22},$$

as well as

$$(13) \qquad \begin{aligned} (\alpha_{11} + \beta_{11})(\alpha_{11} + \beta_{11} - \beta_{12}) - 4\gamma_{21}^2 \frac{U_2}{U_1} > 0, \\ (\alpha_{22} + \gamma_{22})(\alpha_{22} + \gamma_{22} - \gamma_{21}) - 4\beta_{12}^2 \frac{U_1}{U_2} > 0. \end{aligned}$$

Then the solution to (1)-(3) *constructed in Theorem 1 satisfies* $u_i(x,t) > 0$ *a.e. in* $\Omega \times (0,\infty)$, $u_i - U_i$, $\nabla \log u_i \in L^2(\Omega \times (0,\infty))$, *and*

$$u_i(t) \to U_i \quad \text{strongly in } L^2(\Omega) \text{ as } t \to \infty, \quad i = 1, 2.$$

Assumption (12) is a special case of the weak competition case,

$$(14) \qquad \frac{b_{11}}{b_{21}} > \frac{b_{10}}{b_{20}} > \frac{b_{12}}{b_{22}},$$

which allows for coexistence of species in the Lotka-Volterra differential equations [2]. This condition guarantees that $U \in D$, i.e., $U_1, U_2 > 0$ and $U_1 + U_2 < 1$. The idea of the proof is to show that under the stated conditions on the parameters, the functional

$$\Phi(u|U) = \sum_{i=1}^{2} \int_{\Omega} \left(u_i - U_i + U_i \log \frac{u_i}{U_i} \right) dx$$

is a Lyapunov functional and satisfies

$$(15) \qquad \frac{d}{dt}\Phi(u(t)|U) + c_b \int_0^t \|u(s) - U\|_{L^2(\Omega)}^2 ds + c \sum_{i=1}^{2} \int_0^t \int_{\Omega} |\nabla \log u_i|^2 dx ds \leq 0,$$

where $c_b > 0$ is the smallest (positive) eigenvalue of $(b_{ij})_{i,j=1,2}$ and $c > 0$ is another constant. For this property, we need condition (13). Clearly, (15) is only formal as u_i may vanish, and we need to regularize to make this inequality rigorous (see Section 3). Inequality (15) is the key step to deduce the properties mentioned in the theorem.

Our final result is the uniqueness of weak solutions to (1).

Theorem 3 (Uniqueness of weak solutions). *Let the assumptions of Theorem 1 hold. Furthermore, let $f = 0$ and*

$$(16) \qquad \alpha_{22} = \alpha_{11}, \quad \gamma_{21} = \beta_{12}, \quad \gamma_{22} = \beta_{11}.$$

Then the weak solution to (1)-(3) *is unique.*

Summarizing the assumptions on the parameters, the uniqueness result holds for diffusion matrices of the form

$$A(u) = \begin{pmatrix} \alpha_{11} + \beta_{11}u_1 + (\beta_{11} - \beta_{12})u_2 & \beta_{12}u_1 \\ \beta_{12}u_2 & \alpha_{11} + (\beta_{11} - \beta_{12})u_1 + \beta_{11}u_2 \end{pmatrix}.$$

For the proof of Theorem 3, we first observe that under the conditions imposed on the parameters in (3), the sum $\rho := u_1 + u_2$ satisfies the diffusion equation $\partial_t \rho = \Delta F(\rho)$ for a certain nondecreasing function F. By the H^{-1} method, this equation is uniquely solvable. Furthermore, the difference $\sigma := u_1 - u_2$ solves the drift-diffusion equation $\partial_t \sigma = \text{div}(d(\rho)\nabla\sigma + \sigma\nabla V(\rho))$ for certain functions $d(\rho) > 0$ and $V(\rho)$. To prove the uniqueness of *weak* solutions to this equation, we employ the method of Gajewski [9]. We stress the fact that we require only the regularity $V(\rho) \in L^2(0, T; H^1(\Omega))$, which excludes many uniqueness techniques. The idea of Gajewski is to differentiate the semimetric

$$\Xi[\sigma_1, \sigma_2] = S[\rho_1] + S[\rho_2] - 2S\left[\frac{\sigma_1 + \sigma_2}{2}\right], \quad \text{where } S[\sigma] = \int_{\Omega} \sigma \log \sigma \, dx,$$

for two solutions σ_1 and σ_2 with respect to time and to show that $\partial_t \Xi[\sigma_1(t), \sigma_2(t)] \leq 0$ for $t > 0$. Since $\Xi[\sigma_1(0), \sigma_2(0)] = 0$, we infer from the nonnegativity of Ξ that $\Xi[\sigma_1(t), \sigma_2(t)] = 0$ for all $t \geq 0$, and the convexity of $\sigma \log \sigma$ shows that $\sigma_1(t) = \sigma_2(t) = 0$ for $t \geq 0$.

This paper is organized as follows. Theorems 1, 2, 3 are proved in, respectively, Sections 2, 3, 4. In the Appendix, we derive some necessary conditions on the parameters in (3) to apply the entropy method.

2. Proof of Theorem 1

We apply the following theorem from [14, Theorem 2], here in a formulation which is adapted to our situation.

Theorem 4 ([14]). *Let $D \subset (0,1)^2$ be a bounded domain, $u^0 \in L^1(\Omega; \mathbb{R}^2)$ with $u^0(x) \in D$ for $x \in \Omega$ and assume that*

   H1: *There exists a convex function $h \in C^2(D; [0, \infty))$ such that its derivative h' : $D \to \mathbb{R}^n$ is invertible.*
   H2: *Let $\alpha^* > 0$, $0 \leq m_i \leq 1$ $(i = 1, 2)$ be such that for all $z = (z_1, z_2)^\top \in \mathbb{R}^2$ and $u = (u_1, u_2)^\top \in D$,*

$$z^\top h''(u) A(u) z \geq \alpha^* \sum_{i=1}^{2} u_i^{2(m_i - 1)} z_i^2.$$

   H3: *It holds $A \in C^0(D; \mathbb{R}^{2 \times 2})$, $f \in C^0(D; \mathbb{R}^2)$, and there exists $c_f > 0$ such that for all $u \in D$, $f(u) \cdot h'(u) \leq c_f(1 + h(u))$.*

Then there exists a weak solution u to (1)-(2) satisfying $u(x, t) \in \overline{D}$ for $x \in \Omega$, $t > 0$ and

$$u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)), \quad \partial_t u \in L^2_{\mathrm{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^2)').$$

The initial datum is satisfied in the sense of L^2. Moreover, if $h \in C^0(\overline{D})$ and $f(u) \cdot h'(u) \leq 0$ for all $u \in D$, the entropy $\mathcal{H}[u(\cdot, t)] = \int_\Omega h(u(x,t)) dx$ is nonincreasing in time.

The last statement is a consequence of the proof of the theorem in [14].
Now, choose the entropy density

$$(17) \qquad h(u | \overline{u}) = \sum_{i=1}^{3} \overline{u}_i \left(\frac{u_i}{\overline{u}_i} \log \frac{u_i}{\overline{u}_i} - \frac{u_i}{\overline{u}_i} + 1 \right), \quad u_3 = 1 - u_1 - u_2, \quad \overline{u}_3 = 1 - \overline{u}_1 - \overline{u}_2,$$

defined on D (see (6)). This function fulfills Hypothesis H1. It remains to verify Hypotheses H2 and H3.

2.1. Verification of Hypothesis H2.
Let $H(u) = h''(u)$. We require that the matrix $H(u)A(u)$ is symmetric. This leads to conditions (7)-(8), and we are left with the five parameters α_{11}, α_{22}, β_{11}, β_{12}, and γ_{22}. We prove that $H(u)A(u)$ is positive definite under additional assumptions.

Lemma 5. *Let conditions* (7)-(9) *hold. Then there exists $\varepsilon > 0$ such that for all $z \in \mathbb{R}^2$ and all $u \in D$,*

$$(18) \qquad z^\top H(u)A(u)z \geq \varepsilon \left(\frac{z_1^2}{u_1} + \frac{z_2^2}{u_2} \right).$$

The lemma shows that Hypothesis H2 is fulfilled with $m_i = \frac{1}{2}$. First, we verify the following result.

Lemma 6. *The matrix $H(u)A(u)$ is positive semidefinite for all $u \in D$ if and only if*

$$(19) \quad \alpha_{11} \geq 0, \quad \alpha_{22} \geq 0, \quad \beta_{12} \leq \alpha_{11} + \min\{\beta_{11}, \gamma_{22}\}, \quad \alpha_{11} + \beta_{11} \geq 0, \quad \alpha_{22} + \gamma_{22} \geq 0.$$

Proof. Step 1: equations (19) *are necessary.* We first prove that the positive semidefiniteness of $H(u)A(u)$ implies (19) by studying $H(u)A(u)$ close to the vertices of D. To this end, we define the matrix-valued functions

$$F_1(s) = sH(s, s)A(s, s), \quad F_2(s) = sH(1 - 2s, s)A(1 - 2s, s),$$
$$F_3(s) = sH(s, 1 - 2s)A(s, 1 - 2s) \quad \text{for } s \in (0, \tfrac{1}{2}).$$

A straightforward computation shows that

$$\lim_{s \to 0+} F_1(s) = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{22} \end{pmatrix}, \quad \lim_{s \to 0+} F_2(s) = \begin{pmatrix} \alpha_{11} + \beta_{11} & \alpha_{11} + \beta_{11} \\ \alpha_{11} + \beta_{11} & 2(\alpha_{11} + \beta_{11}) - \beta_{12} \end{pmatrix},$$
$$\lim_{s \to 0+} F_3(s) = \begin{pmatrix} \alpha_{11} + \alpha_{22} + 2\gamma_{22} - \beta_{12} & \alpha_{22} + \gamma_{22} \\ \alpha_{22} + \gamma_{22} & \alpha_{22} + \gamma_{22} \end{pmatrix}.$$

Since $H(u)A(u)$ is assumed to be positive semidefinite on D, also $\lim_{s \to 0+} F_i(s)$ must be positive semidefinite for $i = 1, 2, 3$. Sylvester's criterion applied to these matrices yields (19) since

$$\det \left(\lim_{s \to 0+} F_2(s) \right) = (\alpha_{11} + \beta_{11})(\alpha_{11} + \beta_{11} - \beta_{12}) \geq 0,$$
$$\det \left(\lim_{s \to 0+} F_3(s) \right) = (\alpha_{22} + \gamma_{22})(\alpha_{11} + \gamma_{22} - \beta_{12}) \geq 0.$$

Step 2: sign of the diagonal elements of HA. Let conditions (19) hold. We claim that either $HA := H(u)A(u)$ is positive semidefinite or one of the two coefficients $(HA)_{11}$ or $(HA)_{22}$ is positive in D. For this, we introduce the functions

$$f_1(u_2, u_3) = (1 - u_2 - u_3)u_3(HA)_{11}(1 - u_2 - u_3, u_2), \quad (u_2, u_3) \in D,$$
$$f_2(u_1, u_3) = (1 - u_1 - u_3)u_3(HA)_{22}(u_1, 1 - u_1 - u_3), \quad (u_1, u_3) \in D.$$

We wish to apply the strong maximum principle to f_1 and f_2. In fact, f_1 and f_2 are nonnegative on ∂D since (19) implies that

$$(20) \qquad f_1|_{u_3 = 1 - u_2} = (1 - u_2)\big(\alpha_{11} + (\gamma_{22} - \beta_{12})u_2\big) \geq \alpha_{11}(1 - u_2)^2 \geq 0,$$
$$(21) \qquad f_1|_{u_2 = 0} = \alpha_{11} + \beta_{11}(1 - u_3) \geq \alpha_{11}u_3 \geq 0,$$
$$(22) \qquad f_1|_{u_3 = 0} = (1 - u_2)\big((\alpha_{11} + \beta_{11})(1 - u_2) + \alpha_{22} + \gamma_{22}\big) \geq 0,$$
$$(23) \qquad f_2|_{u_1 = 0} = \alpha_{22} + \gamma_{22}(1 - u_2) \geq \alpha_{22}u_2 \geq 0,$$

(24) $f_2|_{u_3=1-u_1} = (1-u_1)\big(\alpha_{22}(1-u_1) + (\alpha_{11} + \beta_{11} - \beta_{12})u_1\big) \geq \alpha_{22}(1-u_1)^2 \geq 0,$

(25) $f_2|_{u_3=0} = (1-u_1)\big((\alpha_{22} + \gamma_{22})(1-u_1) + (\alpha_{11} + \beta_{11})u_1\big) \geq 0.$

Furthermore, a straightforward computation gives

$$\Delta_{(u_2,u_3)}f_1 = -\Delta_{(u_1,u_3)}f_2 = 2(\alpha_{11} - \alpha_{22} + \beta_{11} - \gamma_{22}) \quad \text{in } D.$$

Consequently, either $\Delta_{(u_2,u_3)}f_1 \leq 0$ or $\Delta_{(u_1,u_3)}f_2 \leq 0$ in D. By the strong maximum principle, there exists $i \in \{1,2\}$ such that $f_i > 0$ in D unless $f_i \equiv 0$ in D. This means that $(HA)_{ii} > 0$ in D unless $(HA)_{ii} \equiv 0$ in D.

To complete the claim, we show that if one of the coefficients $(HA)_{11}$ or $(HA)_{22}$ is identically zero in D, then HA is positive semidefinite in D. Consider first the case $(HA)_{11} \equiv 0$ in D, i.e. $f_1 \equiv 0$ in D. Then also $f_1 \equiv 0$ on ∂D. We deduce from (20)-(22) the relations $\alpha_{11} = \beta_{11} = 0$, $\alpha_{22} = -\gamma_{22}$, and $\gamma_{22} = \beta_{12}$ and so,

$$HA = \alpha_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1/u_2 \end{pmatrix}.$$

Since $\alpha_{22} \geq 0$, HA is positive semidefinite. In the remaining case $(HA)_{22} \equiv 0$ in D, (23)-(25) lead to

$$HA = \alpha_{11} \begin{pmatrix} 1/u_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and because of $\alpha_{11} \geq 0$, this matrix is positive semidefinite. This shows the claim.

Step 3: sign of the determinant of HA. By Step 2, we can assume that one of the two coefficients $(HA)_{11}$ or $(HA)_{22}$ is positive in D. We show that $\det A \geq 0$ in D. Then $\det(HA) = \det H \det A \geq 0$ in D, and by Sylvester's criterion, these properties give the positive semidefiniteness of HA. This proves that conditions (19) are sufficient for the positive semidefiniteness of HA.

We consider $\det A$ on ∂D. Taking into account conditions (19), we find that

$$\det A(0, u_2) = (\alpha_{22} + \gamma_{22}u_2)\big(\alpha_{11} + (\gamma_{22} - \beta_{12})u_2\big) \geq \alpha_{22}(1-u_2)\alpha_{11}(1-u_2) \geq 0,$$

$$\det A(u_1, 0) = (\alpha_{11} + \beta_{11}u_1)\big(\alpha_{22}(1-u_1) + (\alpha_{11} + \beta_{11} - \beta_{12})u_1\big)$$
$$\geq \alpha_{22}(1-u_1)\alpha_{11}(1-u_1) \geq 0,$$

$$\det A(u_1, 1-u_1) = \big((\alpha_{22} + \gamma_{22})(1-u_1) + \alpha_{11} + \beta_{11}\big)$$
$$\times \big(\alpha_{11} - \beta_{12} + \gamma_{22} + (\beta_{11} - \gamma_{22})u_1\big)$$
$$\geq (\alpha_{11} + \beta_{11})\big(-\min\{\beta_{11} - \gamma_{22}, 0\} + (\beta_{11} - \gamma_{22})u_1\big) \geq 0.$$

We conclude that $\det A \geq 0$ on ∂D.

Next, we consider the Hessian $C := (\det A)''(u)$ with respect to u. Since $\det A$ is a (multivariate) quadratic polynomial in u, C is a symmetric constant matrix satisfying

$$\det C = -\big(\beta_{11}\beta_{12} + \gamma_{22}(\alpha_{11} - \alpha_{22} - \beta_{12})\big)^2 \leq 0.$$

Thus, one of the two eigenvalues of C is nonpositive, say $\lambda \leq 0$. Let $v \in \mathbb{R}^2 \backslash \{0\}$ be a corresponding eigenvector, i.e. $Cv = \lambda v$. Furthermore, let $u \in D$ be arbitrary and let $I_u \subset \mathbb{R}$ be the (unique) bounded open interval containing zero with the property that

the segment $u + I_u v$ is contained in D and its extreme points belong to ∂D. Define $\phi(r) = \det A(u + rv)$ for $r \in I_u$. Then $\phi''(r) = v^\top C v = \lambda|v|^2 \leq 0$ for all $r \in I_u$. We infer that ϕ is concave and attains its minimum at the border of I_u. Since $\det A \geq 0$ on ∂D, this implies that $\det A(u + rv) \geq 0$ for all $r \in I_u$. By choosing $r = 0 \in I_u$, we conclude that $\det A(u) \geq 0$. As $u \in D$ was arbitrary, this finishes the proof. $\square$

Proof of Lemma 5. The claim (18) is equivalent to the positive semidefiniteness of the matrix $HA - \varepsilon\Lambda$ for a suitable $\varepsilon > 0$, where

$$\Lambda = \begin{pmatrix} 1/u_1 & 0 \\ 0 & 1/u_2 \end{pmatrix}.$$

Since $\Lambda = HP$, where

$$P = \begin{pmatrix} 1 - u_1 & -u_1 \\ -u_2 & 1 - u_2 \end{pmatrix},$$

we can write $HA - \varepsilon\Lambda = HA^\varepsilon$ with $A^\varepsilon = A - \varepsilon P$. We observe that A^ε has the same structure as A with the parameters

$$\alpha_{11}^\varepsilon = \alpha_{11} - \varepsilon, \quad \alpha_{22}^\varepsilon = \alpha_{22} - \varepsilon, \quad \beta_{11}^\varepsilon = \beta_{11} + \varepsilon, \quad \beta_{12}^\varepsilon = \beta_{12} + \varepsilon, \quad \gamma_{22}^\varepsilon = \gamma_{22} + \varepsilon.$$

From Lemma 5 we conclude that HA^ε is positive semidefinite if and only if (19) holds for the parameters $(\alpha_{11}^\varepsilon, \alpha_{22}^\varepsilon, \beta_{11}^\varepsilon, \beta_{12}^\varepsilon, \gamma_{22}^\varepsilon)$ instead of $(\alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{12}, \gamma_{22})$. This means that $HA - \varepsilon\Lambda$ is positive semidefinite for a suitable $\varepsilon > 0$ if and only if (9) holds. $\square$

Remark 7. Let $\alpha_{11} = \alpha_{22} = 0$ but $\beta_{11} > 0$ and $\gamma_{22} > 0$. We claim that there exists $\varepsilon > 0$ such that for all $z \in \mathbb{R}^2$ and $u \in D$,

$$z^\top H(u)A(u)z \geq \varepsilon|z|^2$$

holds, i.e., Hypothesis H2 is satisfied for $m_i = 1$, and the conclusion of Theorem 1 holds. We show that $HA - \varepsilon\mathbb{I}$ is positive semidefinite, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{2\times2}$. The matrix can be written as

$$HA - \varepsilon\mathbb{I} = (HA)^\varepsilon + \frac{\varepsilon}{1 - u_1 - u_2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

where $(HA)^\varepsilon$ has the same structure as HA but with β_{11}, β_{12}, γ_{22} replaced by $\beta_{11}^\varepsilon = \beta_{11} - \varepsilon$, $\beta_{12}^\varepsilon = \beta_{12} - \varepsilon$, $\gamma_{22}^\varepsilon = \gamma_{22} - \varepsilon$. Choosing $0 < \varepsilon \leq \min\{\beta_{11}, \gamma_{22}\}$, conditions (19) are satisfied for these parameters. Thus, Lemma 5 shows that $(HA)^\varepsilon$ is positive semidefinite and we conclude that also $HA - \varepsilon\mathbb{I}$ is positive semidefinite, proving the claim. $\square$

2.2. Verification of H3.

By definition of f_i, we write

$$f_i(u)\partial_{u_i}h(u) = u_i g_i(u)\log u_i - u_i g_i(u)\log(1 - u_1 - u_2) - u_i g_i(u)\log(\overline{u}_i/\overline{u}_3).$$

Since $g_i(u)$ and $u_i\log u_i$ are bounded in $\overline{D}$, the first term on the right-hand side is bounded. The second term is bounded in $\{0 < u_1 + u_2 \leq 1 - \varepsilon\}$ by a constant which depends on ε. Moreover, we have $g_i(u) \leq 0$ in $\{1 - \varepsilon < u_1 + u_2 < 1\}$ by assumption, which implies that $-u_i g_i(u)\log(1 - u_1 - u_2) \leq 0$ in $\{1 - \varepsilon < u_1 + u_2 < 1\}$. Finally, the third term is trivially bounded. Thus, $f_i(u)\partial_{u_i}(u) \leq c$ for a suitable constant $c > 0$.

2.3. **Bounded weak solutions to the SKT model.** Applying Theorem 1 to (1)-(2) with diffusion matrix (4), we infer the following corollary.

Corollary 8 (Bounded weak solutions to (4))**.** *Let the assumptions of Theorem 1 hold except that the coefficients of A, defined in (4), are nonnegative and satisfy $a_{10} > 0$, $a_{20} > 0$ as well as*

$$(26) \qquad a_{21} = a_{11}, \quad a_{22} = a_{12}, \quad a_{20} - a_{10} = a_{11} - a_{22} \geq 0.$$

Furthermore, let $f(u)$ be given by the Lotka-Volterra terms (5) satisfying

$$(27) \qquad b_{10} \leq \min\{b_{11}, b_{12}\}, \quad b_{20} \leq \min\{b_{21}, b_{22}\}.$$

Then there exists a bounded weak solution $u = (u_1, u_2)$ to (1)-(2) satisfying u_1, $u_2 \geq 0$, $u_1 + u_2 \leq 1$ in $\Omega \times (0, \infty)$, and (10).

Proof. The corollary follows from Theorem 4 and Theorem 1 by specifying the diffusivities according to (4). The requirement of the symmetry of $H(u)A(u)$ leads to the conditions $a_{11} = a_{21}$, $a_{22} = a_{12}$, and $a_{20} - a_{10} = a_{11} - a_{22}$, whereas (9) becomes $a_{10} > 0$, $a_{20} > 0$, and $-a_{12} < a_{10} + 2\min\{a_{20} - a_{10}, 0\}$. Taking into account that $a_{10} \leq a_{20}$, the last condition is equivalent to $-a_{12} < a_{10}$, and this inequality holds since a_{10} is positive. Finally, Hypothesis H3 follows from the inequality $g_i(u) = b_{i0} - b_{i1}u_1 - b_{i2}u_2 \leq b_{i0} - \min\{b_{i1}, b_{i2}\}(u_1 + u_2) \leq 0$ for $1 - \varepsilon < u_1 + u_2 < 1$, where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_i = 1 - b_{i0}/\min\{b_{i1}, b_{i2}\} \in (0, 1)$. $\square$

3. Proof of Theorem 2

First, we observe that condition (12) is a special case of the weak competition condition (14) which implies that $U_1 > 0$ and $U_2 > 0$. It holds that $U_1 + U_2 < 1$ since otherwise, the assumption $U_1 + U_2 \geq 1$ leads in view of condition (12) to

$$0 = f_1(U) = (b_{10} - b_{11}U_1 - b_{12}U_2)U_1 < (b_{10} - b_{12}U_1 - b_{12}U_2)U_1 \leq b_{10} - b_{12} = 0,$$

which is a contradiction. Thus, $U \in D$. Furthermore, the identity $b_{i0} = b_{i1}U_1 + b_{i2}U_2$ allows us to we rewrite $f_i(u)$ as

$$(28) \qquad f_i(u) = -u_i \sum_{j=1}^{2} b_{ij}(u_j - U_j), \quad i = 1, 2,$$

and the additional condition (12) leads to

$$f_i(u) = -b_{i0}u_iU_3\left(\frac{u_i}{U_i} - \frac{u_3}{U_3}\right), \quad \text{where } U_3 := 1 - U_1 - U_2.$$

For later use, we observe that the entropy density (17) satisfies

$$(29) \qquad f(u) \cdot h'(u|U) = -\sum_{i=1}^{2} b_{i0}u_iU_3\left(\frac{u_i}{U_i} - \frac{u_3}{U_3}\right)\left(\log\frac{u_i}{U_i} - \log\frac{u_3}{U_3}\right) \leq 0$$

for all $u \in D$, and we conclude from Theorem 4 that $t \mapsto \mathcal{H}[u(t)|U] := \int_\Omega h(u(x,t)|U)dx$ is nonincreasing.

For the positivity and large-time behavior, we need another functional. Define

$$\Phi_\varepsilon(u|U) = \int_\Omega \phi_\varepsilon(u|U)dx, \quad \text{where}$$

$$\phi_\varepsilon(u|U) = \sum_{i=1}^2 \left(u_i - U_i - (U_i + \varepsilon) \log \frac{u_i + \varepsilon}{U_i + \varepsilon} \right), \quad u \in D.$$

We will show that $\Phi_\varepsilon(u|U)$ is an entropy for (1)-(2). For this, let $K = \phi''_\varepsilon(u|U)$ be the Hessian of ϕ_ε with respect to u. Because of the ε-regularization, $\phi'_\varepsilon(u|U)$ is an admissible test function for (1):

$$(30) \quad \Phi_\varepsilon(u(t)|U) + \int_0^t \int_\Omega \nabla u : KA(u)\nabla u\,dxds = \Phi_\varepsilon(u^0|U) + \int_0^t \int_\Omega f(u) \cdot \phi'_\varepsilon(u|U)dxds.$$

First, we estimate the last term on the right-hand side. We infer from (28) and $\partial_{u_i} \phi_\varepsilon(u|U) = (u_i - U_i)/(u_i + \varepsilon)$ that

$$\int_0^t \int_\Omega f(u) \cdot \phi'_\varepsilon(u|U)dxds = -\sum_{i,j=1}^2 \int_0^t \int_\Omega b_{ij}(u_i - U_i)(u_j - U_j)dxds$$

$$+ \varepsilon \sum_{i,j=1}^2 \int_0^t \int_\Omega \frac{b_{ij}}{u_i + \varepsilon}(u_i - U_i)(u_j - U_j)dxds.$$

Since (b_{ij}) is positive definite and u_i is bounded, there are constants $c_b > 0$ and $C > 0$ such that

$$(31) \quad \int_0^t \int_\Omega f(u) \cdot \phi'_\varepsilon(u|U)dxds \le -c_b \int_0^t \|u - U\|_{L^2(\Omega)}^2 ds + \varepsilon C \sum_{i=1}^2 \int_0^t \int_\Omega \frac{dxds}{u_i + \varepsilon}.$$

Next, the second term on the left-hand side of (30) is estimated with the help of the following lemma.

Lemma 9. *There exists $\varepsilon_0 > 0$ and $c_{KA} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $u \in D$, $z = (z_1, z_2) \in \mathbb{R}^2$,*

$$z^\top KA(u)z \ge c_{KA} \sum_{i=1}^2 \frac{z_i^2}{(u_i + \varepsilon)^2}.$$

Proof. The matrix coefficients of K are explicitly given by $K_{ij} = (U_i + \varepsilon)\delta_{ij}/(u_i + \varepsilon)^2$. In order to estimate the product $z^\top KA(u)z$, we rewrite the coefficients of the diffusion matrix as $A_{ij}(u) = \sum_{k=1}^3 a_{ij}^{(k)} u_k$, where

$$a_{ij}^{(1)} = \alpha_{ij} + \beta_{ij}, \quad a_{ij}^{(2)} = \alpha_{ij} + \gamma_{ij}, \quad a_{ij}^{(3)} = \alpha_{ij}, \quad i, j = 1, 2.$$

Then we need to treat the quadratic form

$$z^\top KA(u)z = \sum_{k=1}^3 u_k \sum_{i,j=1}^2 \frac{U_i + \varepsilon}{(u_i + \varepsilon)^2} a_{ij}^{(k)} z_i z_j$$

$$= \sum_{k=1}^{3} u_k \left(a_{11}^{(k)} w_1^2 + \left(a_{12}^{(k)} \sqrt{\frac{U_1 + \varepsilon}{U_2 + \varepsilon}} \frac{u_2 + \varepsilon}{u_1 + \varepsilon} + a_{21}^{(k)} \sqrt{\frac{U_2 + \varepsilon}{U_1 + \varepsilon}} \frac{u_1 + \varepsilon}{u_2 + \varepsilon} \right) w_1 w_2 + a_{22}^{(k)} w_2^2 \right),$$

where $w_i = z_i \sqrt{U_i + \varepsilon}/(u_i + \varepsilon)$, $i = 1, 2$. Because of condition (7), $a_{12}^{(2)} = a_{12}^{(3)} = a_{21}^{(1)} = a_{21}^{(3)} = 0$, and so, the quadratic form simplifies to

$$z^\top K A(u) z = \sum_{k=1}^{3} u_k \left(a_{11}^{(k)} w_1^2 + a_{22}^{(k)} w_2^2 \right)$$

$$+ \left(a_{12}^{(1)} \sqrt{\frac{U_1 + \varepsilon}{U_2 + \varepsilon}} \frac{u_1(u_2 + \varepsilon)}{u_1 + \varepsilon} + a_{21}^{(2)} \sqrt{\frac{U_2 + \varepsilon}{U_1 + \varepsilon}} \frac{u_2(u_1 + \varepsilon)}{u_2 + \varepsilon} \right) w_1 w_2$$

$$\geq \sum_{k=1}^{3} u_k \left(a_{11}^{(k)} w_1^2 + a_{22}^{(k)} w_2^2 \right)$$

$$- \left(|a_{12}^{(1)}| \sqrt{\frac{U_1 + \varepsilon}{U_2 + \varepsilon}} (u_2 + \varepsilon) + |a_{21}^{(2)}| \sqrt{\frac{U_2 + \varepsilon}{U_1 + \varepsilon}} (u_1 + \varepsilon) \right) |w_1||w_2|$$

$$(32) \qquad = \sum_{k=1}^{3} u_k I_k - \varepsilon \left(|a_{12}^{(1)}| \sqrt{\frac{U_1 + \varepsilon}{U_2 + \varepsilon}} + |a_{21}^{(2)}| \sqrt{\frac{U_2 + \varepsilon}{U_1 + \varepsilon}} \right) |w_1||w_2|,$$

where

$$I_1 = a_{11}^{(1)} w_1^2 + a_{22}^{(1)} w_2^2 - |a_{21}^{(2)}| \sqrt{\frac{U_2 + \varepsilon}{U_1 + \varepsilon}} |w_1||w_2|,$$

$$I_2 = a_{11}^{(2)} w_1^2 + a_{22}^{(2)} w_2^2 - |a_{12}^{(1)}| \sqrt{\frac{U_1 + \varepsilon}{U_2 + \varepsilon}} |w_1||w_2|,$$

$$I_2 = a_{11}^{(3)} w_1^2 + a_{22}^{(3)} w_2^2.$$

Condition (9) shows that $a_{ii}^{(3)} > 0$, $a_{ii}^{(i)} \geq 0$ for $i = 1, 2$, and conditions (7) and (8) lead to

$$a_{11}^{(1)} a_{22}^{(1)} - 4\frac{U_2}{U_1} |a_{21}^{(2)}|^2 = (\alpha_{11} + \beta_{11})(\alpha_{22} + \beta_{22}) - 4\frac{U_2}{U_1}(\alpha_{21} + \gamma_{21})^2$$

$$= (\alpha_{11} + \beta_{11})(\alpha_{11} + \beta_{11} - \beta_{12}) - 4\frac{U_2}{U_1}\gamma_{21}^2 > 0,$$

$$a_{11}^{(2)} a_{22}^{(2)} - 4\frac{U_1}{U_2} |a_{12}^{(1)}|^2 = (\alpha_{11} + \gamma_{11})(\alpha_{22} + \gamma_{22}) - 4\frac{U_1}{U_2}(\alpha_{12} + \beta_{12})^2$$

$$= (\alpha_{22} + \gamma_{22} - \gamma_{21})(\alpha_{22} + \gamma_{22}) - 4\frac{U_1}{U_2}\beta_{12}^2 > 0,$$

and the positivity of the discriminants follows from assumption (13). As $\sqrt{(U_i + \varepsilon/(U_j + \varepsilon)}$ is an ε-perturbation of $\sqrt{U_i/U_j}$, there exist $\delta > 0$ and $C > 0$ such that, for sufficiently small $\varepsilon > 0$,

$$I_k \geq 2\delta(w_1^2 + w_2^2) - \varepsilon C |w_1||w_2| \geq \delta(w_1^2 + w_2^2).$$

Therefore, still for sufficiently small $\varepsilon > 0$, (32) yields

$$z^\top KA(u)z \geq \frac{\delta}{2}(w_1^2 + w_2^2) = \frac{\delta}{2}\left(\frac{U_1 + \varepsilon}{(u_1 + \varepsilon)^2}z_1^2 + \frac{U_2 + \varepsilon}{(u_2 + \varepsilon)^2}z_2^2\right).$$

Since $U_1 > 0$, $U_2 > 0$, the conclusion follows with $c_{KA} = \delta \min\{U_1, U_2\}/2$. □

We proceed with the proof of Theorem 2. Employing Lemma 9 and estimate (31) in the entropy inequality (30), it follows that

$$\Phi_\varepsilon(u(t)|U) + c_b \int_0^t \|u(s) - U\|_{L^2(\Omega)}ds + c_{KA}\sum_{i=1}^2 \int_0^t \int_\Omega \frac{|\nabla u_i|^2}{(u_i + \varepsilon)^2}dxds$$

$$(33) \qquad \leq \Phi_\varepsilon(u^0|U) + \varepsilon C \sum_{i=1}^2 \int_0^t \int_\Omega \frac{dxds}{u_i + \varepsilon}.$$

We wish to pass to the limit $\varepsilon \to 0$. First, we focus on the integral on the right-hand side of (33):

$$\varepsilon \sum_{i=1}^2 \int_0^t \int_\Omega \frac{dxds}{u_i + \varepsilon} = \sum_{i=1}^2 \int_0^t \int_{\{u_i>0\}} \frac{\varepsilon}{u_i + \varepsilon}dxds + \sum_{i=1}^2 \int_0^t \text{meas}(\{x : u_i(x,s) = 0\})ds.$$

By dominated convergence, we have

$$\lim_{\varepsilon \to 0} \sum_{i=1}^2 \int_0^t \int_{\{u_i>0\}} \frac{\varepsilon}{u_i + \varepsilon}dxds = 0.$$

Thus, performing the limit inferior $\varepsilon \to 0$ in (33) and applying Fatou's lemma, we obtain

$$\Phi(u(t)|U) + c_b \int_0^t \|u(s) - U\|_{L^2(\Omega)}ds + c_{KA}\sum_{i=1}^2 \int_0^t \int_\Omega \frac{|\nabla u_i|^2}{u_i^2}dxds$$

$$(34) \qquad \leq \Phi(u^0|U) + C\sum_{i=1}^2 \int_0^t \text{meas}(\{x : u_i(x,s) = 0\})ds,$$

where

$$\Phi(u|U) = \lim_{\varepsilon \to 0} \Phi_\varepsilon(u|U) = \sum_{i=1}^2 \int_\Omega U_i\left(\frac{u_i}{U_i} - 1 - \log\frac{u_i}{U_i}\right)dx.$$

If $\text{meas}(\{x : u_i(x,t) = 0\}) > 0$ for some $t > 0$ and some $i \in \{1,2\}$ then $\Phi(u(t)|U) = +\infty$, which contradicts (34). Thus, $\text{meas}(\{x : u_i(x,t) = 0\}) = 0$ for all $t > 0$ and $i = 1,2$. This means that $u_i(x,t) > 0$ for a.e. $x \in \Omega$, $t > 0$, which shows the first property stated in the theorem. It follows from (34) that $u_i - U_i$, $\nabla \log u_i \in L^2(0,\infty; L^2(\Omega))$. In particular, (34) implies that

$$\int_0^\infty \|u(s) - U\|_{L^2(\Omega)}^2 ds < \infty.$$

Hence, there exists a sequence $t_n \to \infty$ such that $u(t_n) \to U$ strongly in $L^2(\Omega)$ as $n \to \infty$. In view of (29) and Theorem 4, the mapping $t \mapsto \mathcal{H}[u(t)|U]$ is nonincreasing. Since

$h(u(t_n)|U) \to 0$ as $n \to \infty$, the dominated convergence theorem and the continuity of h in $\overline{D}$ (see (17)), we infer that $\mathcal{H}[u(t_n)|U] \to 0$ as $n \to \infty$. Then the monotonicity of $t \mapsto \mathcal{H}[u(t)|U]$ implies that this convergence holds for any sequence and $\mathcal{H}[u(t)|U] \to 0$ as $t \to \infty$. This finishes the proof of Theorem 2.

4. Proof of Theorem 3

Set $\rho = u_1 + u_2$ and $\sigma = u_1 - u_2$. A straightforward computation shows that, thanks to assumptions (7)-(8) and (16), ρ and σ solve

(35) $\quad \partial_t \rho = \Delta F(\rho), \ t > 0, \quad \nabla\rho \cdot \nu = 0$ on $\partial\Omega, \quad \rho(0) = u_1^0 + u_2^0$ in Ω,

(36) $\quad \partial_t \sigma = \operatorname{div}\big(d(\rho)\nabla\sigma + \sigma\nabla V(\rho)\big), \ t > 0, \quad \nabla\sigma \cdot \nu = 0$ on $\partial\Omega, \quad \sigma(0) = u_1^0 - u_2^0$ in Ω,

where

$$F(\rho) = \begin{cases} (\alpha_{11} + \beta_{11}\rho)^2/(2\beta_{11}) & \text{if } \beta_{11} \neq 0, \\ \alpha_{11}\rho & \text{if } \beta_{11} = 0, \end{cases}, \quad d(\rho) = \alpha_{11} + (\beta_{11} - \beta_{12})\rho,$$

and $V(\rho) = \beta_{12}\rho$. Observe that, by assumption (9), $\alpha_{11} + \beta_{11} - \beta_{12} > 0$ and hence, together with $\rho = u_1 + u_2 \leq 1$, it holds that $d(\rho) > 0$. Clearly, the bounded weak solution $u = (u_1, u_2)$ to (1)-(2) is unique if and only if the weak solution (ρ, σ) to (35)-(36) is unique. First, we prove that (35) possesses at most one weak solution. Then the uniqueness result is shown for (36).

The function F is nondecreasing since $\beta_{11} > 0$. Thus, by the H^{-1} method, the solution to (35) is unique. Indeed, if ρ_1, ρ_2 are two weak solutions to (35), their difference satisfies

(37) $$\partial_t(\rho_1 - \rho_2) = \Delta(F(\rho_1) - F(\rho_2)) \quad \text{in } \Omega.$$

Let $w(t)$ be the weak solution to the dual problem

$$-\Delta w(t) = \rho_1(t) - \rho_2(t) \quad \text{in } \Omega, \quad \nabla w(t) \cdot \nu = 0 \quad \text{on } \partial\Omega, \ t > 0.$$

Then $w \in L^2(0, T; H^1(\Omega))$ and using this function as a test function in the weak formulation of (37):

$$0 = \langle \partial_t(\rho_1 - \rho_2), w \rangle + \int_\Omega \nabla(F(\rho_1) - F(\rho_2)) \cdot \nabla w \, dx$$

$$= -\langle \partial_t \Delta w, w \rangle - \int_\Omega (F(\rho_1) - F(\rho_2))\Delta w \, dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \int_\Omega (F(\rho_1) - F(\rho_2))(\rho_1 - \rho_2) dx.$$

By the monotonicity of F, the last integral is nonnegative, so $\int_\Omega |\nabla w(t)|^2 dx$ is nonincreasing in time. But $\int_\Omega |\nabla w(0)|^2 dx = 0$, and therefore $w(t) = 0$ which implies that $\rho_1(t) = \rho_2(t)$ for $t > 0$.

Next, we consider (36) with ρ being a given function. Let σ_1, σ_2 be two weak solutions to (36). As in [9], we introduce the semimetric

$$\Xi[\sigma_1, \sigma_2] = S[\sigma_1] + S[\sigma_2] - 2S\left[\frac{\sigma_1 + \sigma_2}{2}\right], \quad S[\sigma] = \int_\Omega \sigma \log \sigma \, dx.$$

Because of the strict convexity of $\sigma \mapsto \sigma \log \sigma$, it holds that $\Xi[\sigma_1, \sigma_2] \geq 0$ and $\Xi[\sigma_1, \sigma_2] = 0$ if and only if $\sigma_1 = \sigma_2$. Computing the time derivative of $\Xi[\sigma_1, \sigma_2]$, we see that the drift terms cancel and we end up with

$$\frac{d}{dt}\Xi[\sigma_1, \sigma_2] = -4 \int_\Omega d(\rho)\big(|\nabla\sqrt{\sigma_1}|^2 + |\nabla\sqrt{\sigma_2}|^2 - |\nabla\sqrt{\sigma_1 + \sigma_2}|^2\big)dx.$$

It was shown in, for instance, [22, Lemma 10] that the integral is nonnegative (since the Fisher information $\int_\Omega d(\rho)|\nabla\sqrt{\sigma_i}|^2 dx$ is subadditive). We infer that $\Xi[\sigma_1(t), \sigma_2(t)] \leq \Xi[\sigma_1(0), \sigma_2(0)]$ for $t > 0$. As σ_1 and σ_2 have the same initial data, $\Xi[\sigma_1(0), \sigma_2(0)] = 0$ and consequently, $\Xi[\sigma_1(t), \sigma_2(t)] = 0$ for $t > 0$. Since Ξ is a semimetric, we infer that $\sigma_1(t) = \sigma_2(t)$ for $t > 0$, finishing the proof.

Appendix A. Necessary conditions for positive semidefiniteness

We show that conditions (7) and a part of conditions (8) are necessary to apply the boundedness-by-entropy method. More precisely, we prove the following result.

Lemma 10 (Necessary conditions). *We define $h(u) = \sum_{k=1}^3 \phi_k(u_k)$, where $u = (u_1, u_2)$, $u_3 = 1 - u_1 - u_2$, and $\phi_k \in C^2(0, 1)$ are convex functions satisfying $\lim_{s \to 0+} \phi_k''(s) = \infty$ for $k = 1, 2, 3$. Let $H = h''(u) \in \mathbb{R}^{2 \times 2}$ be the Hessian of $h(u)$ and let $A(u)$ be given by (3). If $HA(u)$ is positive semidefinite then*

(38) $$\alpha_{12} = \alpha_{21} = \beta_{21} = \gamma_{12} = 0,$$

(39) $$\beta_{12} = \alpha_{11} - \alpha_{22} + \beta_{11} - \beta_{22}, \quad \gamma_{21} = \alpha_{22} - \alpha_{11} + \gamma_{22} - \gamma_{11}.$$

Conditions (38) correspond to (7) needed in Theorem 1. If the coefficients fulfill conditions (8) from Theorem 1 then also (39) holds. Functions which satisfy the assumptions stated above are $\phi(s) = s \log s$, $\phi(s) = s - \log s$, and $\phi(s) = s^b$ with $b < 2$, $b \neq 1$.

Proof. We write $A(u) = \sum_{k=1}^3 u_k A^{(k)}$, where $A^{(k)} = (a_{ij}^{(k)})_{i,j=1,2}$ are constant matrices and

$$a_{ij}^{(1)} = \alpha_{ij} + \beta_{ij}, \quad a_{ij}^{(2)} = \alpha_{ij} + \gamma_{ij}, \quad a_{ij}^{(3)} = \alpha_{ij}.$$

Furthermore, we formulate $H = \sum_{k=1}^3 \phi_k''(u_k)H^{(k)}$, where

$$H^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H^{(3)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$HA(u) = \sum_{k,\ell=1}^3 \phi_k''(u_k)u_\ell H^{(k)}A^{(\ell)}.$$

The idea is to study the behavior of $HA(u)$ at the border of the triangle D. We take $u_1 = (1 - \varepsilon)s$, $u_2 = (1 - \varepsilon)(1 - s)$, and consequently $u_3 = \varepsilon$ for some $\varepsilon, s \in (0, 1)$ in

$$\frac{1}{\phi_3''(u_3)}HA(u) = \sum_{\ell=1}^3 u_\ell H^{(3)}A^{(\ell)} + \frac{1}{\phi_3''(u_3)}\sum_{\ell=1}^3 u_\ell\big(\phi_1''(u_1)H^{(1)} + \phi_2''(u_2)H^{(2)}\big)A^{(\ell)}$$

and pass to the limit $\varepsilon \to 0$. By assumption, the left-hand side is a positive semidefinite matrix. Moreover, since $\phi_3''(u_3) = \phi_3''(\varepsilon) \to \infty$ as $\varepsilon \to 0$, the last sum on the right-hand side vanishes in the limit. We deduce that

$$\lim_{\varepsilon \to 0} \sum_{\ell=1}^{3} u_\ell H^{(3)} A^{(\ell)} = H^{(3)}\big(s A^{(1)} + (1-s) A^{(2)}\big)$$

is positive semidefinite for all $s \in (0,1)$, which implies that $H^{(3)} A^{(1)}$ and $H^{(3)} A^{(2)}$ are positive semidefinite. By exchanging the rule of u_1, u_2, u_3, a similar argument shows that $H^{(i)} A^{(j)}$ is positive semidefinite for all $i = 1,2,3$, $j \neq i$. For any matrix $M = (m_{ij})_{i,j=1,2}$, we have

$$H^{(1)} M = \begin{pmatrix} m_{11} & m_{12} \\ 0 & 0 \end{pmatrix}, \; H^{(2)} M = \begin{pmatrix} 0 & 0 \\ m_{21} & m_{22} \end{pmatrix}, \; H^{(3)} M = \begin{pmatrix} m_{11} + m_{21} & m_{12} + m_{12} \\ m_{11} + m_{21} & m_{12} + m_{22} \end{pmatrix}.$$

We verify that $H^{(i)} A^{(j)}$ is positive semidefinite for all $i = 1,2,3$, $j \neq i$ if and only if (38)-(39) hold. $\qquad\square$

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