GLOBAL MARTINGALE SOLUTIONS FOR A STOCHASTIC POPULATION CROSS-DIFFUSION SYSTEM

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ABSTRACT. The existence of global nonnegative martingale solutions to a stochastic cross-diffusion system for an arbitrary but finite number of interacting population species is shown. The random influence of the environment is modeled by a multiplicative noise term. The diffusion matrix is generally neither symmetric nor positive definite, but it possesses a quadratic entropy structure. This structure allows us to work in a Hilbert space framework and to apply a stochastic Galerkin method. The existence proof is based on energy-type estimates, the tightness criterion of Brzeźniak and co-workers, and Jakubowski's generalization of the Skorokhod theorem. The nonnegativity is proved by an extension of Stampacchia's truncation method due to Chekroun, Park, and Temam.

1. Introduction

The dynamics of interacting population species can be described macroscopically by cross-diffusion equations. A well-known model example is the deterministic Shigesada-Kawasaki-Teramoto population system [36]. It can be derived formally from a random-walk model on lattices for transition rates which depend linearly on the population densities [38, Appendix A]. Generalized population cross-diffusion models are obtained when the dependence of the transition rates on the densities is nonlinear. The existence of global weak solutions to these deterministic models was proved for an arbitrary number of species in [13]. In this paper, we allow for a random influence of the environment and prove the existence of global nonnegative martingale solutions to the corresponding stochastic cross-diffusion system.

More precisely, we consider the cross-diffusion equations

$$du_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right)dt = \sum_{j=1}^n \sigma_{ij}(u)dW_j(t) \quad \text{in } \mathcal{O},\ t>0,\ i=1,\ldots,n, \tag{1}$$

with no-flux boundary and initial conditions

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j\cdot\nu = 0 \quad \text{on } \partial\mathcal{O},\ t>0, \quad u_i(0)=u_i^0 \quad \text{in } \mathcal{O},\ i=1,\ldots,n, \tag{2}$$

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where $\mathcal{O} \subset \mathbb{R}^d$ with $d = 2, 3$ is a bounded domain with Lipschitz boundary, ν is the exterior unit normal vector to $\partial\mathcal{O}$, and u_i^0 is a possibly random initial datum. The solution $u = (u_1, \ldots, u_n) : \mathcal{O} \times [0,T] \times \Omega \to \mathbb{R}^n$ models the density of the $i^{\rm th}$ population species, where $x \in \mathcal{O}$ represents the spatial variable, $t \in (0,T)$ the time, and $\omega \in \Omega$ the stochastic variable. The matrix $A(u) = (A_{ij}(u))$ is the diffusion matrix, $\sigma_{ij}(u)$ is a multiplicative noise term, and $W = (W_1, \ldots, W_n)$ is an n-dimensional cylindrical Wiener process. Details on the stochastic framework will be given in section 1.3.

The diffusion coefficients are given by

$$A_{ij}(u) = \delta_{ij}\bigg(a_{i0} + \sum_{k=1}^n a_{ik}u_k^2\bigg) + 2a_{ij}u_iu_j, \quad i,j = 1, \ldots, n, \tag{3}$$

where $a_{i0} > 0$ and $a_{ij} > 0$. This model is derived from an on-lattice model with transition rates $p_i(u)$, which depend quadratically on the densities, i.e. $p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k^2$ for $i = 1, \ldots, n$ [38]. This quadratic structure is essential for our analysis. To understand this, we need to explain the entropy structure of equations (1).

1.1. **Entropy structure.** Generally, the diffusion matrix in (1), originating from general transition rates in the lattice model, is neither symmetric nor positive definite which significantly complicates the analysis. However, the equations possess a formal gradient-flow or entropy structure under certain conditions. For the sake of simplicity, we sketch this structure in the deterministic context only and refer to [23, Chapter 4] for details. By entropy structure, we mean that there exists a so-called entropy density $h : \mathbb{R}_+^n \to \mathbb{R}$ such that, still in the deterministic context, system (1) in the entropy variables $w_i := \partial h/\partial u_i$, $i = 1, \ldots, n$, has a positive semi-definite diffusion matrix $B = (B_{ij})$,

$$\partial_t u_i(w) - \operatorname{div}\bigg(\sum_{j=1}^n B_{ij}\nabla w_j\bigg) = 0, \tag{4}$$

where $B = A(u)h''(u)^{-1}$ is the product of $A(u)$ and the inverse of the Hessian of $h(u)$, and $u(w) = (h')^{-1}(w)$ is the back transformation. When the transition rates are given by $p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k^s$ for some $s \geq 1$, the entropy density can be chosen as $h(u) = \sum_{i=1}^n \pi_i h_s(u_i)ds$, where $\pi_i > 0$ are some numbers and

$$h_s(z) = \begin{cases} z(\log z - 1) + 1 & \text{for } s = 1, \\ z^s/s & \text{for } s \neq 1. \end{cases}$$

It was shown in [12] that $B = (B_{ij})$ in (4) is positive semi-definite in the two-species case $n = 2$ with $\pi_1 = \pi_2 = 1$. This property generally does not hold for the n-species system. It turns out [13] that B is symmetric, positive semi-definite if the numbers π_i are chosen such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i,j = 1, \ldots, n.$$

This condition is recognized as the detailed-balance condition for the Markov chain associated to (a_{ij}) and $(\pi_1, \ldots, \pi_n)$ is the reversible measure. The detailed-balance condition is sufficient but not necessary for the positive semi-definiteness of B; in fact, when self-diffusion dominates cross-diffusion (see (8) for the precise statement) then B is still positive semi-definite.

The entropy structure also yields a priori estimates. Indeed, let $H(u) = \int_{\mathcal{O}} h(u)dx$ be the so-called entropy. A computation shows that, still in the absence of the stochastic term,

$$\frac{dH}{dt} + \int_{\mathcal{O}} \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial u_i \partial u_j}(u) A_{ij}(u) \nabla u_i \cdot \nabla u_j dx = 0.$$

Since $B = A(u)h''(u)^{-1}$ is positive semi-definite, this holds true for $h''(u)A(u)$. Thus, taking into account the special structure of $A(u)$, this yields gradient estimates (see Lemma 3 below).

The gradient-flow structure is the key of the analysis of the deterministic analog to (1), but there are severe difficulties in the stochastic context. Indeed, neither semigroup techniques [15, 26] nor monotonicity arguments [30] can be applied because of the properties of the differential operator in (1). Stochastic Galerkin methods usually work in Hilbert spaces, and generally they cannot be used since the transformation to entropy variables is nonlinear. In order to overcome these difficulties, we consider quadratic transition rates with $s = 2$ which makes the transformation to entropy variable linear,

$$w_i = \frac{\partial h}{\partial u_i} = \pi_i h_2'(u_i) = \pi_i u_i.$$

Still, the diffusion matrix $A(u)$ is not positive definite, but the new diffusion matrix $B = A(u)\operatorname{diag}(1/\pi_1, \ldots, 1/\pi_n)$ is positive semi-definite; see Lemma 3. This allows us to combine entropy methods for diffusive equations and stochastic techniques.

1.2. **State of the art.** Before stating our main existence result, let us review the literature. Fundamental results on stochastic partial differential equations of monotone type were obtained already in the 1970s by Pardoux [34]. More recently, abstract stochastic evolution equations with locally monotone nonlinearities [30] or maximal monotone operators [4] were analyzed. The existence of (mild or pathwise strong) solutions to quasilinear stochastic evolution equations was proved in, e.g., [17, 21]. For these solutions, the driving noise is given in advance. A weaker concept is given by martingale solutions, where the stochastic basis is unknown a priori and is given as part of the solution. Existence proofs of such solutions to nonlinear stochastic evolution equations can be found in [6, 14].

Stochastic reaction-diffusion equations are a special class of evolution equations, and they are investigated in many papers starting from the 1980s [19, 20]. There are less results on *systems* of stochastic reaction-diffusion equations. In [10], the existence and uniqueness of mild solutions with Lipschitz continuous multiplicative noise was shown. The result was generalized in [29] to Hölder continuous multiplicative noise. The existence of maximal pathwise solutions to stochastic reaction-diffusion systems with polynomial reaction terms was proved in [33]. More general quasilinear systems were investigated recently in [28], proving the existence of local pathwise mild solutions, including the Shigesada-Kawasaki-Teramoto cross-diffusion system. The local-in-time results are not surprising since even in the deterministic case, certain reaction terms may lead to finite-time blow-up of solutions. The work [31] also analyzes population systems and provides the existence of pathwise unique solutions, but only for two species and for Lipschitz continuous nonlinearities.

Up to our knowledge, the population model (1) with coefficients (3) was not studied in the literature. In this paper, we prove the existence of global martingale solutions using the techniques of [8, 9]. We show that the solutions are nonnegative under a natural condition on the operators $\sigma_{ij}(u)$ using the stochastic maximum principle of [11]. Since even the uniqueness of weak solutions to the deterministic analog of (1)-(3) is not known (see the partial result in [24]), we cannot expect to obtain pathwise unique strong solutions.

1.3. **Stochastic framework and main results.** Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space endowed with a complete right continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ and let H be a Hilbert space. The space $L^2(\mathcal{O})$ is the vector space of all square integrable functions $u:\mathcal{O}\to\mathbb{R}$ with the inner product $(\cdot,\cdot)_{L^2(\mathcal{O})}$. We fix a Hilbert basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{O})$. The space $L^2(\Omega;H)$ consists of all H-valued random variables u with

$$\mathbb{E}\|u\|_H^2 := \int_\Omega \|u(\omega)\|_H^2\mathbb{P}(d\omega) < \infty.$$

Furthermore, the space $H^1(\mathcal{O})$ contains all functions $u\in L^2(\mathcal{O})$ such that the distributional derivatives $\partial u/\partial x_1,\ldots,\partial u/\partial x_d$ belong to $L^2(\mathcal{O})$. Let Y be any separable Hilbert space with orthonormal basis $(\eta_k)_{k\in\mathbb{N}}$. We denote by

$$\mathcal{L}_2(Y;L^2(\mathcal{O})) = \bigg\{L:Y\to L^2(\mathcal{O}) \text{ linear continuous: } \sum_{k=1}^\infty \|L\eta_k\|_{L^2(\mathcal{O})}^2 < \infty\bigg\}$$

the space of Hilbert-Schmidt operators from Y to $L^2(\mathcal{O})$ endowed with the norm

$$\|L\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2 := \sum_{k=1}^\infty \|L\eta_k\|_{L^2(\mathcal{O})}^2.$$

Let $(\beta_{jk})_{j=1,\cdots,n,\,k\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions and for $j=1,\ldots,n$, let $W_j(x,t,\omega)=\sum_{k\in\mathbb{N}}\eta_k(x)\beta_{jk}(t,\omega)$ be a cylindrical Brownian motion. If $Y_0\supset Y$ is a second auxiliary Hilbert space such that the map $Y\ni u\mapsto u\in Y_0$ is Hilbert-Schmidt, the series $W_j=\sum_{k\in\mathbb{N}}\eta_k\beta_{jk}$ converges in $\mathcal{L}_2(\Omega;Y_0)$.

The multiplicative noise terms $\sigma:=\sigma_{ij}(u,t,\omega):L^2(\mathcal{O})\times[0,T]\times\Omega\to\mathcal{L}_2(Y;L^2(\mathcal{O}))$ are assumed to be $\mathcal{B}(L^2(\mathcal{O})\otimes[0,T]\otimes\mathcal{F};\mathcal{B}(\mathcal{L}_2(Y;L^2(\mathcal{O}))))$-measurable and $\mathbb{F}$-adapted with the property that there exists one constant $C_\sigma>0$ such that for all u, $v\in L^2(\mathcal{O})$ and $i,j=1,\ldots,n$,

$$\begin{aligned} \|\sigma_{ij}(u)\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2 &\leq C_\sigma\big(1+\|u\|_{L^2(\mathcal{O})}^2\big), \\ \|\sigma_{ij}(u)-\sigma_{ij}(v)\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2 &\leq C_\sigma\|u-v\|_{L^2(\mathcal{O})}^2. \end{aligned} \tag{5}$$

Here, the $L^2(\mathcal{O})$ norm of the function $u=(u_1,\ldots,u_n)$ is understood as $\|u\|_{L^2(\mathcal{O})}^2 = \sum_{i=1}^n\|u_i\|_{L^2(\mathcal{O})}^2$, and we use this notation also for other vector-valued or tensor-valued functions. The expression $\sigma_{ij}(u)dW_j(t)$ formally means that

$$\sigma_{ij}(u)dW_j(t) = \sum_{k,\ell\in\mathbb{N}} \sigma_{ij}^{k\ell}(u)e_\ell d\beta_{jk}(t), \quad \text{where } \sigma_{ij}^{k\ell}(u) := \big(\sigma_{ij}(u)\eta_k, e_\ell\big)_{L^2(\mathcal{O})}. \tag{6}$$

Next, we define our concept of solution.

Definition 1. *Let $T > 0$ be arbitrary. We say that the system $(\widetilde{U}, \widetilde{W}, \widetilde{u})$ is a* global martingale solution *to (1)-(3) if $\widetilde{U} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}})$ is a stochastic basis with filtration $\widetilde{\mathbb{F}} = (\widetilde{\mathcal{F}}_t)_{t\in[0,T]}$, $\widetilde{W}$ is a cylindrical Wiener process, and $\widetilde{u}(t) = (\widetilde{u}_1(t), \ldots, \widetilde{u}_n(t))$ is an $\widetilde{\mathcal{F}}_t$-adapted stochastic process for all $t \in [0, T]$ such that for all $i = 1, \ldots, n$,*

$$\widetilde{u}_i \in L^2(\widetilde{\Omega}; C^0([0, T]; L^2_w(\mathcal{O}))) \cap L^2(\widetilde{\Omega}; L^2(0, T; H^1(\mathcal{O}))),$$

the law of $\widetilde{u}_i(0)$ is the same as for u_i^0, and $\widetilde{u}$ satisfies for all $\phi \in H^1(\mathcal{O})$ and all $i = 1, \ldots, n$,

$$(\widetilde{u}_i(t), \phi)_{L^2(\mathcal{O})} = (\widetilde{u}_i(0), \phi)_{L^2(\mathcal{O})} - \sum_{j=1}^n \int_0^t \big\langle \operatorname{div} \big(A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s)\big), \phi\big\rangle ds$$
$$+ \bigg(\sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{u}(s))d\widetilde{W}_j(s), \phi\bigg)_{L^2(\mathcal{O})}.$$

The brackets $\langle\cdot, \cdot\rangle$ signify the duality pairing between $H^1(\mathcal{O})'$ and $H^1(\mathcal{O})$, i.e.

$$\big\langle \operatorname{div} \big(A_{ij}(\widetilde{u})\nabla\widetilde{u}_j\big), \phi\big\rangle = -\int_{\mathcal{O}} A_{ij}(\widetilde{u})\nabla\widetilde{u}_j \cdot \nabla\phi dx.$$

As mentioned before, the new diffusion matrix B in (4) is positive definite only under an additional assumption, namely either

$$\pi_i a_{ij} = \pi_j a_{ji} \text{ for } i \neq j \quad \text{and} \quad \alpha_1 := \min_{i=1,\ldots,n} \bigg(a_{ii} - \frac{1}{3}\sum_{j=1,\, j\neq i}^n a_{ij}\bigg) > 0, \quad \text{or} \tag{7}$$

$$\alpha_2 := \min_{i=1,\ldots,n} \bigg(a_{ii} - \frac{1}{3}\sum_{j=1,\, j\neq i}^n \big((a_{ij} + a_{ji}) - 2\sqrt{a_{ij}a_{ji}}\big)\bigg) > 0. \tag{8}$$

Our main result is as follows.

Theorem 1 (Existence of global martingale solution)**.** *Let $T > 0$ be arbitrary, $d \le 3$, and $u_0 \in L^2(\mathcal{O})$. Let $\sigma = (\sigma_{ij})_{i,j=1}^n$ with $\sigma_{ij} : L^2(\mathcal{O}) \times [0, T] \times \Omega \to \mathcal{L}_2(Y; L^2(\mathcal{O}))$ satisfy (5), $a_{i0} > 0$, $a_{ij} > 0$ for $i, j = 1, \ldots, n$, and let either (7) or (8) hold. Then there exists a global martingale solution to (1)-(3). If additionally, $u_i^0 \ge 0$ a.e. in $\mathcal{O}$, $\mathbb{P}$-a.s. for $i = 1, \ldots, n$ and*

$$\sum_{j=1}^n \|\sigma_{ij}(u)\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))} \le C\|u_i\|_{L^2(\mathcal{O})}, \tag{9}$$

then the population densities are nonnegative $\mathbb{P}$-a.s.

Remark 2 (Discussion of the assumptions)**.** (i) We can also choose random initial data, see Remark 18. We need additionally that $\mathbb{E}\|u^0\|^p_{L^2(\mathcal{O})} < \infty$ for $p = 24/(4-d)$. This condition is needed to derive a higher-order estimate for u_i. It can be weakened to smaller values of p by refining the Gagliardo-Nirenberg argument in the proof of Lemma 7.

(ii) Assumption (5) on σ_{ij} seems to be quite natural. In [29], the multiplicative noise was assumed to be only Hölder continuous, but the matrix $(\sigma_{ij}(u))$ is needed to be diagonal, which we do not assume. Condition (9) implies that $\sum_{j=1}^n \sigma_{ij}(u) = 0$ if $u_i = 0$, which is a natural condition to obtain the nonnegativity of u_i.

(iii) The existence of solutions to the deterministic version of (1)-(3) can be shown also for vanishing coefficients $a_{i0} = 0$ [13]. This seems to be not possible in the stochastic framework, since the condition $a_{i0} > 0$ is needed to derive estimates for ∇u_i in $L^2(\mathcal{O})$ $\mathbb{P}$-a.s., and these estimates are necessary to work in the Hilbert space $H^1(\mathcal{O})$.

(iv) Conditions (7) and (8) on the matrix coefficients are probably not optimal. For local-in-time existence of solutions to the determinstic analog of (1), only the positivity of the real parts of the eigenvalues of $A(u)$ is needed [1]. This condition is generally not sufficient to ensure global solvability. A sufficient condition for the global existence for general quasilinear evolution equations is provided by uniform $W^{1,p}(\mathcal{O})$ bounds with $p > d$ [2, Theorem 15.3], but it is difficult to prove this regularity for solutions to cross-diffusion systems. Conditions (7) and (8) are currently the best available assumptions to guarantee the existence of global solutions, even in the deterministic framework. □

1.4. **Ideas of the proof of Theorem 1.** We sketch the main steps of the proof. The full proof is given in section 2. First, we show the existence of a pathwise unique strong solution $u^{(N)}$ to a stochastic Galerkin approximation of (1)-(3), where $N \in \mathbb{N}$ is the Galerkin dimension. Estimates uniform in N are derived from a stochastic version of the entropy inequality (which is made rigorous using Itô's formula in section 2.3)

$$\begin{aligned}&\mathbb{E}H(u^{(N)}(t)) - \mathbb{E}H(u^{(N)}(0)) + \sum_{i,j=1}^n \mathbb{E}\int_0^t\int_{\mathcal{O}} \pi_i A_{ij}(u^{(N)})\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}dxds \\ &\quad\le \frac12\mathbb{E}\int_0^t \|P^{1/2}\Pi_N\sigma(u^{(N)})\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds + \sum_{i,j=1}^n \mathbb{E}\int_0^t\int_{\mathcal{O}} \pi_i u_i^{(N)}\sigma_{ij}(u^{(N)})dW_j(s)dx,\end{aligned}$$

where Π_N is the projection on the finite-dimensional Galerkin space,

$$H(u) = \sum_{i=1}^n \int_{\mathcal{O}} \pi_i h_2(u_i)dx = \sum_{i=1}^n \frac{\pi_i}{2}\int_{\mathcal{O}} u_i^2 dx = \frac12\|P^{1/2}u\|^2_{L^2(\mathcal{O})}$$

is the quadratic entropy, and $P = \operatorname{diag}(\pi_1,\ldots,\pi_n)$, $P^{1/2} = \operatorname{diag}(\pi_1^{1/2},\ldots,\pi_n^{1/2})$. Since $PA(u^{(N)})$ is positive definite, the last term on the left-hand side yields uniform gradient estimates. The first integral on the right-hand side is bounded from above by the entropy H (up to some additive constant), using assumption (5), and the second integral is estimated using the Burkholder-Davis-Gundy inequality (see Proposition 21 in the appendix).

Next, the tightness of the laws $\mathcal{L}(u^{(N)})$ in the topological space Z_T, defined in (23) below, is proved by applying a criterion of Brzeźniak, Goldys, and Jegaraj [7]. Because of the low regularity properties of the solutions, Z_T cannot be chosen to be a metric space and we cannot apply the Skorokhod representation theorem, as usually done in the literature (e.g. [16, 33]). This problem is overcome by using Jakubowski's generalization of the Skorokhod theorem, which holds for topological spaces with a separating-points property (Theorem 23). Then there exists a subsequence of $(u^{(N)})$ (not relabeled), another probability space, and random variables $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ having the same law as $(u^{(N)},W)$ and $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ converges to $(\widetilde{u},\widetilde{W})$ in the topology of Z_T. Because of the gradient estimates, we conclude in particular the strong convergence $\widetilde{u}^{(N)}\to\widetilde{u}$ in $L^2(\mathcal{O}\times(0,T))$ $\mathbb{P}$-a.s. This, together with further convergences resulting from the relative compactness in Z_T, allows us to pass to the limit $N\to\infty$ in the Galerkin approximation, showing that $(\widetilde{u},\widetilde{W})$ is a global martingale solution to (1).

From the application viewpoint, we expect that the population densities $u_i(t)$ are nonnegative $\mathbb{P}$-a.s. if this holds initially. The problem is that generally, maximum principle arguments cannot be applied to cross-diffusion systems. System (1), (3), however, possesses a special structure. Indeed, we may write (1) as

$$du_i-\operatorname{div}\left(\left(a_{i0}+\sum_{k=1}^n a_{ik}u_k^2\right)\nabla u_i+u_iF_i[u]\right)=\sum_{j=1}^n\sigma_{ij}(u)dW_j(t),$$

and F_i depends on u_j and ∇u_j for $j\neq i$. The term $u_iF_i[u]$ can be interpreted as a drift term which vanishes if $u_i=0$. If we assume that $\sigma_{ij}(u)=0$ if $u_i=0$ then a maximum principle can be applied.

More precisely, we employ the stochastic Stampacchia-type maximum principle due to Chekroun, Park, and Temam [11]. The idea is to regularize the test function $(\widetilde{u}_i^{(N)})^-=\max\{0,-\widetilde{u}_i^{(N)}\}$ by some smooth function $F_\varepsilon(\widetilde{u}_i^{(N)})$, to apply the Itô formula for $\mathbb{E}\int F_\varepsilon(\widetilde{u}_i^{(N)})dx$, and then to pass to the limits $N\to\infty$ and $\varepsilon\to 0$ leading to the inequality

$$\mathbb{E}\|\widetilde{u}_i(t)^-\|^2_{L^2(\mathcal{O})}\le\mathbb{E}\int_0^t\|\widetilde{u}_i(s)^-\|^2_{L^2(\mathcal{O})}ds.$$

Gronwall's lemma show that $\widetilde{u}_i(t)^-=0$ a.e. in $\mathcal{O}$, which proves the nonnegativity of $\widetilde{u}_i$ $\mathbb{P}$-a.s.

In order to make the manuscript accessible also to non-experts of stochastic partial differential equations, we recall some known results from stochastic analysis used in this paper in Appendix A. As the tightness criterion of [7] is probably less known, we present the details directly in the proof of Theorem 1 in section 2.4.

2. Proof of the existence theorem

2.1. **An algebraic property.** We recall the following result on the positive definiteness of the new diffusion matrix, taken from [13, Lemma 3] by choosing $s=2$.

Lemma 3. *Let $\pi_1,\ldots,\pi_n>0$ and $P=\operatorname{diag}(\pi_1,\ldots,\pi_n)\in\mathbb{R}^{n\times n}$. Let either condition (7) or (8) hold. Then $PA(u)$ is positive definite, i.e., it holds for any $z=(z_1,\ldots,z_n)\in\mathbb{R}^n$*

and $u=(u_1,\ldots,u_n)\in\mathbb{R}^n$,

$$\sum_{i,j=1}^n \pi_i A_{ij}(u)z_iz_j \ge \sum_{i=1}^n \pi_i a_{i0}z_i^2 + 3\alpha\sum_{i=1}^n \pi_i u_i^2 z_i^2,$$

where $\alpha=\alpha_1$ *if* (7) *holds and* $\alpha=\alpha_2$ *if* (8) *is satisfied. In the latter case, we may choose* $\pi_i=1$ *for all* $i=1,\ldots,n$.

2.2. **Stochastic Galerkin approximation.** We fix an orthonormal basis $(e_k)_{k\ge 1}$ of $L^2(\mathcal{O})$ and a number $N\in\mathbb{N}$ and set $H_N=\operatorname{span}\{e_1,\ldots,e_N\}$. We introduce the projection operator $\Pi_N: L^2(\mathcal{O})\to H_N$,

$$\Pi_N(v)=\sum_{i=1}^N (v,e_i)_{L^2(\mathcal{O})}e_i, \quad v\in L^2(\mathcal{O}).$$

The approximate problem is the following system of stochastic differential equations,

$$du_i^{(N)} - \Pi_N \operatorname{div}\bigg(\sum_{j=1}^n A_{ij}(u^{(N)})\nabla u_j^{(N)}\bigg)dt = \Pi_N\bigg(\sum_{j=1}^n \sigma_{ij}(u^{(N)})\bigg)dW_j(t), \tag{10}$$

$$u_i^{(N)}(0)=\Pi_N(u_i^0), \quad i=1,\ldots,n. \tag{11}$$

Lemma 4. *Let Assumptions* (7) *or* (8) *hold. Then there exists a pathwise unique strong solution to* (10)-(11).

Proof. We apply Theorem 22 in Appendix A to

$$\pi\cdot du = a(u)dt + b(u)dW(t), \quad t>0, \quad u(0)=\Pi_N(u^0), \tag{12}$$

where

$$a=(a_1,\ldots,a_n): H_N\to\mathbb{R}^n, \quad a_i(u)=\Pi_N\operatorname{div}\bigg(\sum_{j=1}^n \pi_i A_{ij}(u)\nabla u_j\bigg),$$
$$b_{ij}: H_N\to\mathcal{L}_2(Y;H_N), \quad b_{ij}(u)=\pi_i\Pi_N\sigma_{ij}(u),$$

and the numbers $\pi_1,\ldots,\pi_n>0$ are given by (7). Observe that this problem is equivalent to (10) after componentwise division by π_i. It is sufficient to verify Assumptions (48)-(49). Let $R>0$, $T>0$, and $\omega\in\Omega$ and let u, $v\in H_N$ with $\|u\|_{H_N}$, $\|v\|_{H_N}\le R$. Then, using the positive definiteness of PA, according to Lemma 3, and the equivalence of norms on H_N,

$$\begin{aligned}(a(u)-a(v),u-v)_{H_N} &= -\sum_{i,j=1}^n\int_{\mathcal{O}}\pi_i A_{ij}(u)\nabla(u_i-v_i)\cdot\nabla(u_j-v_j)dx \\ &\quad + \sum_{i,j=1}^n\int_{\mathcal{O}}\pi_i(A_{ij}(u)-A_{ij}(v))\nabla(u_i-v_i)\cdot\nabla v_j dx \\ &\le C\|A(u)-A(v)\|_{L^2(\mathcal{O})}\|\nabla(u-v)\|_{L^2(\mathcal{O})}\|\nabla v\|_{L^\infty(\mathcal{O})} \\ &\le C(N,R)\|u-v\|_{H_N}^2,\end{aligned}$$

where the constant $C(N,R)>0$ depends on N and R. In the last step we have used the fact that $A_{ij}(u)$ is locally Lipschitz continuous. Hence, together with assumption (5) on σ, the local weak monotonicity condition (48) holds. To verify the weak coercivity condition (49), we take $u\in H_N$ with $\|u\|_{H_N}\le R$ and employ again the positive definiteness of PA:

$$\begin{aligned}(a(u),u)_{H_N}+\|b(u)\|^2_{\mathcal{L}_2(Y;H_N)} &= -\sum_{i,j=1}^n\int_{\mathcal{O}}\pi_iA_{ij}(u)\nabla u_i\cdot\nabla u_j dx+\|P^{1/2}\sigma(u)\|^2_{\mathcal{L}_2(Y;H_N)} \\ &\le C_\sigma(1+\|u\|^2_{H_N}),\end{aligned}$$

where we recall that $P^{1/2}=\operatorname{diag}(\pi_1^{1/2},\ldots,\pi_n^{1/2})$. Therefore, the lemma follows after applying Theorem 22. □

2.3. **Uniform estimates.** We prove some energy-type estimates uniform in N.

Lemma 5 (A priori estimates)**.** *Let $T>0$ and let $u^{(N)}$ be the pathwise unique strong solution to (10)-(11) on $[0,T]$. Then there exists a constant $C_1>0$ which depends on $\mathbb{E}\|u^0\|^2_{L^2(\mathcal{O})}$, C_σ, and T but not on N such that*

$$\sup_{N\in\mathbb{N}}\mathbb{E}\bigg(\sup_{t\in(0,T)}\|u^{(N)}\|^2_{L^2(\mathcal{O})}\bigg)\le C_1, \tag{13}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}\bigg(\int_0^T\|\nabla u^{(N)}\|^2_{L^2(\mathcal{O})}dt\bigg)\le C_1, \tag{14}$$

$$\alpha\sup_{N\in\mathbb{N}}\mathbb{E}\bigg(\int_0^T\big\|\nabla(u^{(N)})^2\big\|^2_{L^2(\mathcal{O})}dt\bigg)\le C_1, \tag{15}$$

and $\alpha=\alpha_1$ if (7) holds, $\alpha=\alpha_2$ if (8) holds.

We remark that (13) shows that $(u^{(N)})$ is bounded in $L^2(\mathcal{O}\times(0,T)\times\Omega)$, so together with (14), we infer a uniform bound for $u^{(N)}$ in $L^2((0,T)\times\Omega;H^1(\mathcal{O}))$.

Proof. We apply the Itô formula (Theorem 19) to the process $X(t)=u^{(N)}(t)$, where $u^{(N)}$ solves (12):

$$\begin{aligned}&\frac12\|P^{1/2}u^{(N)}(t)\|^2_{L^2(\mathcal{O})}-\frac12\|\Pi_N(P^{1/2}u^0)\|^2_{L^2(\mathcal{O})}\\ &\quad=\sum_{i,j=1}^n\int_0^t\big(u_i^{(N)}(s),\Pi_N\operatorname{div}(\pi_iA_{ij}(u^{(N)}(s))\nabla u_j^{(N)}(s))\big)_{L^2(\mathcal{O})}ds\\ &\qquad+\frac12\int_0^t\big\|\Pi_N(P^{1/2}\sigma(u^{(N)}(s)))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds\\ &\qquad+\sum_{i,j=1}^n\int_0^t\big(u_i^{(N)}(s),\Pi_N(\pi_i\sigma_{ij}(u^{(N)}(s)))dW_j(s)\big)_{L^2(\mathcal{O})}\\ &\quad=-\sum_{i,j=1}^n\int_0^t\big(\nabla u_i^{(N)}(s),\pi_iA_{ij}(u^{(N)}(s))\nabla u_j^{(N)}(s)\big)_{L^2(\mathcal{O})}ds\end{aligned}$$

$$+\frac{1}{2}\int_0^t \big\|\Pi_N(P^{1/2}\sigma(u^{(N)}(s)))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds$$

$$+\sum_{i,j=1}^n\int_0^t \pi_i\big(u_i^{(N)}(s),\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}. \tag{16}$$

The first term on the right-hand side can be estimated by using Lemma 3:

$$\begin{aligned}
&\sum_{i,j=1}^n\big(\nabla u_i^{(N)}(s),\pi_iA_{ij}(u^{(N)}(s))\nabla u_j^{(N)}\big)_{L^2(\mathcal{O})} \\
&\quad\ge\sum_{i=1}^n\pi_ia_{i0}\int_{\mathcal{O}}|\nabla u_i^{(N)}|^2dx+3\alpha\sum_{i=1}^n\pi_i\int_{\mathcal{O}}|u_i^{(N)}|^2|\nabla u_i^{(N)}|^2dx \\
&\quad\ge C\|\nabla u^{(N)}\|^2_{L^2(\mathcal{O})}+C\alpha\|\nabla(u^{(N)})^2\|^2_{L^2(\mathcal{O})},
\end{aligned}$$

where $(u^{(N)})^2=((u_1^{(N)})^2,\ldots,(u_n^{(N)})^2)$ and here and in the following, $C>0$ is a generic constant independent of N with values changing from line to line. Therefore, (16) becomes

$$\begin{aligned}
&\frac{1}{2}\|P^{1/2}u^{(N)}(t)\|^2_{L^2(\mathcal{O})}+C\int_0^t\|\nabla u^{(N)}(s)\|^2_{L^2(\mathcal{O})}ds+C\alpha\int_0^t\|\nabla(u^{(N)}(s)^2)\|^2_{L^2(\mathcal{O})}ds \\
&\le\frac{1}{2}\|P^{1/2}u^0\|^2_{L^2(\mathcal{O})}+\frac{1}{2}\int_0^t\big\|P^{1/2}\sigma(u^{(N)}(s))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds \qquad (17)\\
&\quad+\sum_{i,j=1}^n\int_0^t\pi_i\big(u_i^{(N)}(s),\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}.
\end{aligned}$$

For the second integral on the right-hand side, we take into account assumption (5):

$$\begin{aligned}
\frac{1}{2}\int_0^t\big\|P^{1/2}\sigma(u^{(N)}(s))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds&\le C\int_0^t\big\|\sigma(u^{(N)}(s))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds \\
&\le C\int_0^t\big(1+\|u^{(N)}\|^2_{L^2(\mathcal{O})}\big)ds=Ct+C\int_0^t\|u^{(N)}\|^2_{L^2(\mathcal{O})}ds.
\end{aligned}$$

To estimate the last integral in (17), we observe that, since the process $u^{(N)}$ is H_N-valued and a solution to (10), the process

$$\mu^{(N)}(t)=\sum_{i,j=1}^n\int_0^t\pi_i\big(u_i^{(N)},\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})},\quad t\in[0,T],$$

is an $\mathcal{F}_t$-martingale. Then, by the Burkholder-Davis-Gundy inequality (see Proposition 21), we have

$$\mathbb{E}\bigg(\sup_{t\in(0,T)}\bigg|\sum_{i,j=1}^n\int_0^t\pi_i\big(u_i^{(N)},\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}\bigg|\bigg)$$

$$\leq C\mathbb{E}\bigg(\int_0^T \|u^{(N)}(s)\|_{L^2(\mathcal{O})}^2 \big\|\sigma(u^{(N)}(s))\big\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2\bigg)^{1/2},$$

and by the Hölder inequality, assumption (5) on σ, and the Young inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in(0,T)}\bigg|\sum_{i,j=1}^n\int_0^t \pi_i\big(u_i^{(N)},\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}\bigg| \\
&\leq C\mathbb{E}\bigg\{\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|_{L^2(\mathcal{O})}^2\bigg)^{1/2} C_\sigma^{1/2}\bigg(\int_0^T\big(1+\|u^{(N)}\|_{L^2(\mathcal{O})}^2\big)ds\bigg)^{1/2}\bigg\} \\
&\leq \frac{1}{4}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|_{L^2(\mathcal{O})}^2\bigg) + C\bigg(T+\mathbb{E}\int_0^T\|u^{(N)}\|_{L^2(\mathcal{O})}^2 ds\bigg). \qquad (18)
\end{aligned}$$

We take in (17) the supremum over $t\in(0,T)$ and the mathematical expectation and use the inequality $\|P^{1/2}u^{(N)}\|_{L^2(\mathcal{O})}\geq C\|u^{(N)}\|_{L^2(\mathcal{O})}$ for some constant $C>0$ only depending on $\pi_1,\ldots,\pi_n$ and the previous estimates to conclude that

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|_{L^2(\mathcal{O})}^2\bigg) + C\mathbb{E}\int_0^t\|\nabla u^{(N)}(s)\|_{L^2(\mathcal{O})}^2 ds + C\alpha\mathbb{E}\int_0^t\|\nabla(u^{(N)}(s)^2)\|_{L^2(\mathcal{O})}^2 ds \\
&\leq CT + C\mathbb{E}\big(\|u^0\|_{L^2(\mathcal{O})}^2\big) + C\int_0^T\mathbb{E}\bigg(\sup_{t\in[0,\tau]}\|u^{(N)}(\tau)\|_{L^2(\mathcal{O})}^2\bigg)ds. \qquad (19)
\end{aligned}$$

We infer from the Gronwall lemma that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|_{L^2(\mathcal{O})}^2\bigg)\leq C,$$

where $C>0$ depends on $\mathbb{E}\|u^0\|_{L^2(\mathcal{O})}^2$, C_σ, and T. This proves (13). Inserting the previous estimate into (19), we deduce immediately estimates (14) and (15). □

We need a higher-order moment estimate, which is proved in the following lemma.

Lemma 6. *Let $T>0$ and let $u^{(N)}$ be the pathwise unique strong solution to* (10)-(11) *on* $[0,T]$. *Furthermore, let $p>2$ and $\mathbb{E}\|u^0\|_{L^2(\mathcal{O})}^p<\infty$. Then there exists a constant $C_2>0$ which depends on p, $\mathbb{E}\|u^0\|_{L^2(\mathcal{O})}^p$, C_σ, and T but not on N such that*

$$\sup_{N\in\mathbb{N}}\mathbb{E}\bigg(\sup_{t\in(0,T)}\|u^{(N)}\|_{L^2(\mathcal{O})}^p\bigg)\leq C_2. \qquad (20)$$

Proof. We take the supremum over $t\in(0,T)$ in (17) and neglect the second and third terms on the left-hand side. Then, raising both sides to the the power $p/2$ and applying the Hölder inequality, we find that

$$\sup_{t\in(0,T)}\|u^{(N)}\|_{L^2(\mathcal{O})}^p\leq C\|u^0\|_{L^2(\mathcal{O})}^p + CT^{p/2-1}\int_0^T\big\|\sigma(u^{(N)}(s))\big\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^p ds$$

$$+ C\bigg(\sup_{t\in(0,T)}\sum_{i,j=1}^n\int_0^t \big(u_i^{(N)}(s),\pi_i\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}\bigg)^{p/2}.$$

Taking the mathematical expectation and using assumption (5), it follows that

$$\mathbb{E}\bigg(\sup_{t\in(0,T)}\|u^{(N)}\|^p_{L^2(\mathcal{O})}\bigg) \le C + C\mathbb{E}\|u^0\|^p_{L^2(\mathcal{O})} + C\mathbb{E}\int_0^T \|u^{(N)}(s)\|^p_{L^2(\mathcal{O})}ds$$

$$+ C\mathbb{E}\bigg(\sup_{t\in(0,T)}\sum_{i,j=1}^n\int_0^t \big(u_i^{(N)}(s),\pi_i\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}\bigg)^{p/2}. \tag{21}$$

For the last term, we use the Burkholder-Davis-Gundy and Young inequalities,

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in(0,T)}\sum_{i,j=1}^n\int_0^t \big(u^{(N)}(s),\pi_i\sigma_{ij}(u^{(N)}(s))dW_j(s)\big)_{L^2(\mathcal{O})}\bigg)^{p/2} \\
&\quad\le C\mathbb{E}\bigg(\int_0^T \|u^{(N)}(s)\|^2_{L^2(\mathcal{O})}\big\|\sigma(u^{(N)}(s))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds\bigg)^{p/4} \\
&\quad\le C\mathbb{E}\bigg\{\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|^2_{L^2(\mathcal{O})}\bigg)^{p/4} C_\sigma^{p/4}\bigg(\int_0^T \big(1+\|u^{(N)}\|^2_{L^2(\mathcal{O})}\big)ds\bigg)^{p/4}\bigg\} \\
&\quad\le C\mathbb{E}\bigg\{\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|^p_{L^2(\mathcal{O})}\bigg)^{1/2}\bigg(\int_0^T \big(1+\|u^{(N)}\|^p_{L^2(\mathcal{O})}\big)ds\bigg)^{1/2}\bigg\} \\
&\quad\le \frac{1}{2}\mathbb{E}\bigg(\sup_{t\in[0,T]}\|u^{(N)}(t)\|^p_{L^2(\mathcal{O})}\bigg) + C\mathbb{E}\int_0^T \big(1+\|u^{(N)}\|^p_{L^2(\mathcal{O})}\big)ds.
\end{aligned}$$

Inserting this estimate into (21) and observing that the first term on the right-hand side of the previous inequality can be absorbed by the first term on the left-hand side of (21), we infer that

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{t\in(0,T)}\|u^{(N)}\|^p_{L^2(\mathcal{O})}\bigg) &\le C + C\mathbb{E}\|u^0\|^p_{L^2(\mathcal{O})} + C\mathbb{E}\int_0^T \sup_{\tau\in(0,s)}\|u^{(N)}(\tau)\|^p_{L^2(\mathcal{O})}ds \\
&\quad + C\mathbb{E}\int_0^T \big(1+\|u^{(N)}\|^p_{L^2(\mathcal{O})}\big)ds.
\end{aligned}$$

Then the Gronwall inequality implies that

$$\mathbb{E}\bigg(\sup_{t\in(0,T)}\|u^{(N)}\|^p_{L^2(\mathcal{O})}\bigg) \le C,$$

which concludes the proof. ☐

The previous lemma allows us to improve slightly the regularity of $u^{(N)}$.

Lemma 7. *Let $T>0$ and let $u^{(N)}$ be the pathwise unique strong solution to* (10)-(11) *on $[0,T]$. Then $(u_i^{(N)})^2 \in L^3((0,T)\times\Omega;L^2(\mathcal{O}))$ for $i=1,\ldots,N$ and, for some constant*

$C_3 > 0$,

$$\mathbb{E}\int_0^T \|(u^{(N)})^2\|_{L^2(\mathcal{O})}^3 dt \le C_3,$$

where $(u^{(N)})^2$ is the vector with the coefficients $(u_i^{(N)})^2$ for $i = 1, \ldots, N$.

Proof. By the Gagliardo-Nirenberg inequality with $\theta = d/(2+d)$ and the Hölder inequality with $q = 2(2+d)/(3d)$ and $q' = 2(2+d)/(4-d)$ (here, we need that $d \le 3$), we find that

$$\begin{aligned}
\mathbb{E}\int_0^T \|(u^{(N)})^2\|_{L^2(\mathcal{O})}^3 dt &\le C\mathbb{E}\int_0^T \|(u^{(N)})^2\|_{H^1(\mathcal{O})}^{3d/(2+d)} \|(u^{(N)})^2\|_{L^1(\mathcal{O})}^{6/(2+d)} dt \\
&\le C\mathbb{E}\bigg(\sup_{t\in(0,T)} \|u^{(N)}\|_{L^2(\mathcal{O})}^{12/(2+d)} \int_0^T \|(u^{(N)})^2\|_{H^1(\mathcal{O})}^{3d/(2+d)} dt\bigg) \\
&\le C\bigg\{\mathbb{E}\bigg(\sup_{t\in(0,T)} \|u^{(N)}\|_{L^2(\mathcal{O})}^{24/(4-d)}\bigg)\bigg\}^{1/q'} \bigg\{\mathbb{E}\int_0^T \|(u^{(N)})^2\|_{H^1(\mathcal{O})}^2 dt\bigg\}^{1/q}.
\end{aligned}$$

The first factor is uniformly bounded by (20) with $p = 24/(4-d)$ and the second factor is uniformly bounded as a consequence of (13) and (14). □

2.4. **Tightness.** The aim of this subsection is to prove that the sequence of laws of $u^{(N)}$ is tight on a certain topological space. For this, we introduce the following spaces:

- $C^0([0,T]; H^3(\mathcal{O})')$ is the space of continuous functions $u : [0,T] \to H^3(\mathcal{O})'$ with the topology $\mathcal{T}_1$ induced by the norm $\|u\|_{C^0([0,T];H^3(\mathcal{O})')} = \sup_{t\in(0,T)} \|u(t)\|_{H^3(\mathcal{O})'}$;
- $L_w^2(0,T; H^1(\mathcal{O}))$ is the space $L^2(0,T; H^1(\mathcal{O}))$ with the weak topology $\mathcal{T}_2$;
- $L^2(0,T; L^2(\mathcal{O}))$ is the space of square integrable functions $u : (0,T) \to L^2(\mathcal{O})$ with the topology $\mathcal{T}_3$ induced by the norm $\|\cdot\|_{L^2(0,T;L^2(\mathcal{O}))}$;
- $C^0([0,T]; L_w^2(\mathcal{O}))$ is the space of weakly continuous functions $u : [0,T] \to L^2(\mathcal{O})$ endowed with the weakest topology $\mathcal{T}_4$ such that for all $h \in L^2(\mathcal{O})$, the mappings

$$C^0([0,T]; L_w^2(\mathcal{O})) \to C^0([0,T]; \mathbb{R}), \quad u \mapsto (u(\cdot), h)_{L^2(\mathcal{O})},$$

are continuous.

In particular, convergence in $C^0([0,T]; L_w^2(\mathcal{O}))$ means the following: $u_n \to u$ in $C^0([0,T]; L_w^2(\mathcal{O}))$ as $n \to \infty$ holds if and only if

$$\lim_{n\to\infty} \sup_{t\in(0,T)} |(u_n(t) - u(t), h)_{L^2(\mathcal{O})}| = 0 \quad \text{for all } h \in L^2(\mathcal{O}).$$

We need another space: Let $r > 0$ and $B := \{u \in L^2(\mathcal{O}) : \|u\|_{L^2(\mathcal{O})} \le r\}$. Let q be the metric compatible with the weak topology on B. We define the following subspace of $C^0([0,T]; L_w^2(\mathcal{O}))$:

$$\begin{aligned}
C^0([0,T]; B_w) = {} & \text{set of all weakly continuous functions } u : [0,T] \to L^2(\mathcal{O}) \\
& \text{such that } \sup\nolimits_{t\in(0,T)} \|u(t)\|_{L^2(\mathcal{O})} \le r.
\end{aligned} \tag{22}$$

This space is metrizable with the metric $q^*(u,v) = \sup_{t\in(0,T)} q(u(t),v(t))$ [5, Theorem 3.29]. By the Banach-Alaoglu theorem, B_w is compact [5, Theorem 3.16], so, $(C^0([0,T];B_w),q^*)$ is a complete metric space.

The following lemma ensures that any sequence in $C^0([0,T];B)$ which converges in some space $C^0([0,T];U')$ with $U\subset H^1(\mathcal{O})$ is also convergent in $C^0([0,T];B_w)$. We apply this lemma with $U = H^3(\mathcal{O})$.

Lemma 8 (Lemma 2.1 in [8]). *Let $u_n:[0,T]\to L^2(\mathcal{O})$ ($n\in\mathbb{N}$) be functions satisfying*

$$\sup_{n\in\mathbb{N}}\sup_{t\in(0,T)}\|u_n(t)\|_{L^2(\mathcal{O})}\le r,$$
$$u_n\to u \quad\text{in } C^0([0,T];U') \quad \text{as } n\to\infty,$$

where $U\subset H^1(\mathcal{O})$ and U' is the dual space of U. Then u_n, $u\in C^0([0,T];B_w)$ and $u_n\to u$ in $C^0([0,T];B_w)$ as $n\to\infty$.

We define the space

$$Z_T := C^0([0,T];H^3(\mathcal{O})')\cap L^2_w(0,T;H^1(\mathcal{O}))\cap L^2(0,T;L^2(\mathcal{O}))\cap C^0([0,T];L^2_w(\mathcal{O})), \tag{23}$$

endowed with the topology $\mathcal{T}$ which is the maximum of the topologies $\mathcal{T}_i$, $i=1,2,3,4$, of the corresponding spaces. On this space, we can formulate a compactness criterion which is analogous to the result due to Mikulevcius and Rozowskii [32].

Lemma 9 (Compactness criterion). *Let $(Z_T,\mathcal{T})$ be as defined in (23). A set $K\subset Z_T$ is $\mathcal{T}$-relatively compact if the following three conditions hold:*

(1) $\sup_{u\in K}\sup_{t\in(0,T)}\|u(t)\|_{L^2(\mathcal{O})}<\infty$,
(2) K is bounded in $L^2(0,T;H^1(\mathcal{O}))$, and
(3) $\lim_{\delta\to0}\sup_{u\in K}\sup_{s,t\in(0,T),\,|s-t|\le\delta}\|u(s)-u(t)\|_{H^3(\mathcal{O})'}=0$.

We refer to [8, Lemma 2.3] for a proof. The result follows since the embeddings $H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})\hookrightarrow H^3(\mathcal{O})'$ are continuous and the embedding $H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$ is compact, such that we can apply Dubinskii's Theorem [18] (also see [37]) to a sequence $(u_n)_{n\in\mathbb{N}}\subset K$ to conclude that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ that is convergent in $C^0([0,T];H^3(\mathcal{O})')$. By Lemma 8, this subsequence is also convergent in $C^0([0,T];B_w)$.

The compactness criterion in Lemma 9 allows for a proof of the following tightness criterion taken from [8, Corollary 2.6].

Theorem 10 (Tightness criterion). *Let H, V, and U be separable Hilbert spaces such that the embeddings $U\hookrightarrow V\hookrightarrow H$ are dense and continuous and the embedding $V\hookrightarrow H$ is compact. Furthermore, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted U'-valued stochastic processes such that*

(1) there exists $C>0$ such that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\bigg(\sup_{t\in(0,T)}\|X_n(t)\|_H^2\bigg)\le C,$$

(2) there exists $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\bigg(\int_0^T \|X_n(t)\|_V^2 dt \bigg) \le C,$$

(3) $(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition in U' (see Definition 3 in the appendix). Furthermore, let $\mathbb{P}_n$ be the law of X_n on Z_T. Then $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight on Z_T.

The main result of this subsection is the tightness of the laws $\mathcal{L}(u^{(N)})$ of the solutions $u^{(N)}$ to (10)-(11).

Lemma 11. *The set of measures $\{\mathcal{L}(u^{(N)}) : N \in \mathbb{N}\}$ is tight on $(Z_T, \mathcal{T})$.*

Proof. The idea of the proof is to apply Theorem 10 with $U = H^3(\mathcal{O})$, $V = H^1(\mathcal{O})$, and $H = L^2(\mathcal{O})$. In view of estimates (13) and (14), conditions (1) and (2) of Theorem 10 are fulfilled. It remains to show that $(u^{(N)})_{N\in\mathbb{N}}$ satisfies the Aldous condition in $H^3(\mathcal{O})'$. To this end, let $(\tau_N)_{N\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-stopping times such that $0 \le \tau_N \le T$. Let $t \in [0, T]$ and $\phi \in H^3(\mathcal{O})$. Then (10) can be written as

$$\begin{aligned} \langle u_i^{(N)}(t), \phi\rangle &= \langle \Pi_N(u_i^0), \phi\rangle - \sum_{j=1}^n \int_0^t \big\langle A_{ij}(u^{(N)})\nabla u_j^{(N)}, \nabla \Pi_N\phi\big\rangle ds \\ &\quad + \sum_{j=1}^n \bigg\langle \int_0^t \Pi_N\big(\sigma_{ij}(u^{(N)}(s))\big) dW_j(s), \phi \bigg\rangle \\ &=: J_1^{(N)} + J_2^{(N)}(t) + J_3^{(N)}(t), \end{aligned} \tag{24}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing between $H^3(\mathcal{O})'$ and $H^3(\mathcal{O})$. We estimate each term on the right-hand side individually.

First, consider the term involving the diffusion coefficients. Let $\theta > 0$. Then, using the (at most) quadratic dependence of A_{ij} on u_k and the continuous embedding $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$ (this is another instance where we use $d \le 3$), we find that

$$\begin{aligned} &\mathbb{E}\bigg| \int_{\tau_N}^{\tau_N+\theta} \big\langle A_{ij}(u^{(N)})\nabla u_j^{(N)}, \nabla\Pi_N\phi\big\rangle ds \bigg| \\ &\le \mathbb{E} \int_{\tau_N}^{\tau_N+\theta} \|A_{ij}(u^{(N)})\|_{L^2(\mathcal{O})} \|\nabla u_j^{(N)}\|_{L^2(\mathcal{O})} \|\nabla\phi\|_{L^\infty(\mathcal{O})} ds \\ &\le \mathbb{E}\bigg(\int_{\tau_N}^{\tau_N+\theta} \big(1 + \|(u^{(N)})^2\|_{L^2(\mathcal{O})}\big) \|\nabla u^{(N)}\|_{L^2(\mathcal{O})} ds \bigg) \|\phi\|_{H^3(\mathcal{O})} \\ &\le \mathbb{E}\Big(\big(\theta^{1/2} + \theta^{1/6} \|(u^{(N)})^2\|_{L^3(0,T;L^2(\mathcal{O}))}\big) \|\nabla u^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))} \Big) \|\phi\|_{H^3(\mathcal{O})} \\ &\le \bigg\{ \theta^{1/2} + \theta^{1/6} \bigg(\mathbb{E}\bigg(\int_0^T \|(u^{(N)})^2\|_{L^2(\mathcal{O})}^3 dt \bigg)^{2/3} \bigg)^{1/2} \end{aligned}$$

$$\times\left(\mathbb{E}\int_0^T\|\nabla u^{(N)}\|_{L^2(\mathcal{O})}^2dt\right)^{1/2}\Bigg\}\|\phi\|_{H^3(\mathcal{O})},$$

where in the last two inequalities we applied the Hölder inequality with respect to time and then with respect to the random variable. The vector $(u^{(N)})^2$ consists of elements $(u_i^{(N)})^2$ for $i=1,\ldots,N$. Taking into account the estimates from Lemmas 5 and 7, we deduce that

$$\mathbb{E}\left|\int_{\tau_N}^{\tau_N+\theta}\left\langle A_{ij}(u^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi\right\rangle ds\right|\le C\theta^{1/6}\|\phi\|_{H^3(\mathcal{O})}. \tag{25}$$

For the stochastic term, we use assumption (5) on σ, the Itô isometry (see Proposition 20), and the Hölder inequality to obtain

$$\begin{aligned}\mathbb{E}&\left|\left\langle\int_{\tau_N}^{\tau_N+\theta}\Pi_N(\sigma_{ij}(u^{(N)}(s)))dW_j(s),\phi\right\rangle\right|^2\\&\le\mathbb{E}\left(\int_{\tau_N}^{\tau_N+\theta}\|\sigma(u^{(N)})\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2dt\right)\|\phi\|_{L^2(\mathcal{O})}^2\\&\le C_\sigma\mathbb{E}\left(\int_{\tau_N}^{\tau_N+\theta}\left(1+\|u^{(N)}\|_{L^2(\mathcal{O})}^2\right)dt\right)\|\phi\|_{L^2(\mathcal{O})}^2\\&\le C\left(\theta+\theta^{1/3}\left(\mathbb{E}\int_0^T\|u^{(N)}\|_{L^2(\mathcal{O})}^3dt\right)^{2/3}\right)\|\phi\|_{L^2(\mathcal{O})}^2\le C\theta^{1/3}\|\phi\|_{L^2(\mathcal{O})}^2.\end{aligned}\tag{26}$$

Next, let $\kappa>0$ and $\varepsilon>0$. By the definition of the $H^3(\mathcal{O})'$ norm, the Chebyshev inequality, and estimate (25), we have

$$\begin{aligned}\mathbb{P}\Big\{\|J_2^{(N)}(\tau_N+\theta)-J_2^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\Big\}&\le\frac{1}{\kappa}\mathbb{E}\|J_2^{(N)}(\tau_N+\theta)-J_2^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\\=\frac{1}{\kappa}\sup_{\|\phi\|_{H^3(\mathcal{O})}=1}\mathbb{E}\big|\big\langle J_2^{(N)}(\tau_N+\theta)-J_2^{(N)}(\tau_N),\phi\big\rangle\big|&\le\frac{C\theta^{1/6}}{\kappa}.\end{aligned}$$

Thus, choosing $\delta_1=(\kappa\varepsilon/C)^6$, we infer that

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\delta_1}\mathbb{P}\Big\{\|J_2^{(N)}(\tau_N+\theta)-J_2^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\Big\}\le\varepsilon.$$

In a similar way, it follows that

$$\begin{aligned}\mathbb{P}\Big\{\|J_3^{(N)}(\tau_N+\theta)-J_3^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\Big\}&\le\frac{1}{\kappa^2}\mathbb{E}\|J_3^{(N)}(\tau_N+\theta)-J_3^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}^2\\&\le\frac{C_2\theta^{1/3}}{\kappa^2},\end{aligned}$$

and choosing $\delta_2=(\kappa^2\varepsilon/C)^3$ gives

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\delta_1}\mathbb{P}\Big\{\|J_3^{(N)}(\tau_N+\theta_2)-J_3^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\Big\}\le\varepsilon.$$

This shows that the Aldous condition holds for all three terms $J_i^{(N)}$, $i = 1, 2, 3$. Consequently, in view of (24), it also holds for $(u^{(N)})_{N\in\mathbb{N}}$. We conclude the proof by invoking Theorem 10. $\square$

2.5. **Convergence of the approximate solutions.** First, we show that the space Z_T, defined in (23), verifies the assumption of the Skorokhod-Jakubowski theorem (see Theorem 23 in the appendix). More precisely, we prove that on each space in definition (23) of Z_T, there exists a countable set of continuous real-valued functions separating points.

Lemma 12. *The topological space Z_T, defined in* (23)*, satisfies the assumption of Theorem 23.*

Proof. Since the spaces $C^0([0,T];H^3(\mathcal{O})')$ and $L^2(0,T;L^2(\mathcal{O}))$ are separable, metrizable, and complete, the assumption of Theorem 23 is satisfied; see [3, Exposé 8]. For the space $L^2_w(0,T;H^1(\mathcal{O}))$, it is sufficient to define

$$f_m(u) = \int_0^T (u(t), v_m(t))_{H^1(\mathcal{O})}dt \in \mathbb{R}, \quad \text{where } u \in L^2_w(0,T;H^1(\mathcal{O})),\ m\in\mathbb{N},$$

and $(v_m)_{m\in\mathbb{N}}$ is a dense subset of $L^2(0,T;H^1(\mathcal{O}))$.

It remains to consider the space $C^0([0,T];L^2_w(\mathcal{O}))$. Let $(w_m)_{m\in\mathbb{N}}$ be a dense subset of $L^2(0,T;L^2(\mathcal{O}))$ and let $\mathbb{Q}_T$ be the set of rational numbers from the interval $[0,T]$. Then the family $\{f_{m,t} : m\in\mathbb{N},\ t\in\mathbb{Q}_T\}$, defined by

$$f_{m,t}(u) = (u(t), w_m)_{L^2(\mathcal{O})} \in \mathbb{R}, \quad \text{where } u\in C^0([0,T];L^2_w(\mathcal{O})),\ m\in\mathbb{N},\ t\in\mathbb{Q}_T,$$

consists of continuous functions separating points in $C^0([0,T];L^2_w(\mathcal{O}))$. $\square$

In view of Lemma 12 and Theorem 23, we infer the following result.

Corollary 13. *Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of Z_T-valued random variables such that their laws $\mathcal{L}(\eta_n)$ on $(Z_T,\mathcal{T})$ form a tight sequence of probability measures. Then there exists a subsequence $(\eta_k)_{k\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, and Z_T-valued random variables $\widetilde\eta$, $\widetilde\eta_k$ with $k\in\mathbb{N}$ such that the variables η_k and $\widetilde\eta_k$ have the same laws on Z_T and $(\widetilde\eta_k)_{k\in\mathbb{N}}$ converges to $\widetilde\eta$ a.s. on $\widetilde\Omega$.*

By Lemma 11, the set of measures $\{\mathcal{L}(u^{(N)}) : N\in\mathbb{N}\}$ is tight on $(Z_T,\mathcal{T})$ and by Lemma 12, the space $Z_T\times C^0([0,T];Y_0)$ satisfies the assumption of Theorem 23. Therefore, we can apply Corollary 13 to deduce the existence of a subsequence of $(u^{(N)})_{N\in\mathbb{N}}$, which is not relabeled, a probability space $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, and, on this space, $Z_T\times C^0([0,T];Y_0)$-valued random variables $(\widetilde u,\widetilde W)$, $(\widetilde u^{(N)},\widetilde W^{(N)})$ with $N\in\mathbb{N}$ such that $(\widetilde u^{(N)},\widetilde W^{(N)})$ has the same law as $(u^{(N)},W)$ on $\mathcal{B}(Z_T\times C^0([0,T];Y_0))$ and

$$(\widetilde u^{(N)},\widetilde W^{(N)}) \to (\widetilde u,\widetilde W) \quad \text{in } Z_T\times C^0([0,T];Y_0),\ \widetilde{\mathbb{P}}\text{-a.s., as } N\to\infty.$$

Because of the definition of the space Z_T, this convergence means that $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned} \widetilde u^{(N)} &\to \widetilde u \quad \text{in } C^0([0,T];H^3(\mathcal{O})'), \\ \widetilde u^{(N)} &\rightharpoonup \widetilde u \quad \text{weakly in } L^2(0,T;H^1(\mathcal{O})), \end{aligned}$$

$$\begin{aligned} \widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } L^2(0,T;L^2(\mathcal{O})), \\ \widetilde{u}^{(N)} &\to \widetilde{u} \quad \text{in } C^0([0,T];L^2_w(\mathcal{O})), \\ \widetilde{W}^{(N)} &\to \widetilde{W} \quad \text{in } C^0([0,T];Y_0). \end{aligned} \tag{27}$$

Since $u^{(N)}$ is an element of $C^0([0,T];H_N)$ $\mathbb{P}$-a.s., $C^0([0,T];H_N)$ is a Borel set of $C^0([0,T];H^3(\mathcal{O})')\cap L^2(0,T;L^2(\mathcal{O}))$, and since $u^{(N)}$ and $\widetilde{u}^{(N)}$ have the same laws, we infer that

$$\mathcal{L}(\widetilde{u}^{(N)})\big(C^0([0,T];H_N)\big) = 1 \quad \text{for all } N\geq 1.$$

Note that, as $\mathcal{B}(Z_T\times C^0([0,T];Y_0))$ is a subset of $\mathcal{B}(Z_T)\times\mathcal{B}(C^0([0,T];Y_0))$, the function $\widetilde{u}$ is a Z_T-Borel random variable. Furthermore, in view of estimates (13)-(15) and (20) and the equivalence of the laws of $\widetilde{u}^{(N)}$ and $\widetilde{u}$ on $\mathcal{B}(Z_T)$, we have the uniform bounds

$$\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\Big(\sup_{t\in(0,T)}\|\widetilde{u}^{(N)}\|^2_{L^2(\mathcal{O})}\Big)\leq C_1, \tag{28}$$

$$\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\int_0^T\|\widetilde{u}^{(N)}\|^2_{H^1(\mathcal{O})}dt\bigg) + \alpha\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\int_0^T\big\|(\widetilde{u}^{(N)})^2\big\|^2_{H^1(\mathcal{O})}dt\bigg)\leq C_1, \tag{29}$$

$$\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\bigg(\sup_{t\in(0,T)}\|\widetilde{u}^{(N)}\|^p_{L^2(\mathcal{O})}\bigg)\leq C_2, \tag{30}$$

where $p\geq 2$ is any number.

We deduce from (29) that there exists a subsequence of $(\widetilde{u}^{(N)})$ (not relabeled) which is weakly converging in $L^2((0,T)\times\widetilde{\Omega};H^1(\mathcal{O}))$ as $N\to\infty$. Since $\widetilde{u}^{(N)}\to\widetilde{u}$ $\widetilde{\mathbb{P}}$-a.s. in Z_T, we conclude that $\widetilde{u}\in L^2((0,T)\times\widetilde{\Omega};H^1(\mathcal{O}))$, i.e.

$$\widetilde{\mathbb{E}}\int_0^T\|\widetilde{u}(t)\|^2_{H^1(\mathcal{O})}dt<\infty. \tag{31}$$

Similarly, the bound (28) allows us to extract a subsequence which is weakly* convergent in $L^2(\widetilde{\Omega};L^\infty(0,T;L^2(\mathcal{O})))$ and

$$\widetilde{\mathbb{E}}\bigg(\sup_{t\in(0,T)}\|\widetilde{u}(t)\|^2_{L^2(\mathcal{O})}\bigg)<\infty. \tag{32}$$

The convergence $\widetilde{u}^{(N)}\to\widetilde{u}$ in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. implies, up to a subsequence, that

$$\widetilde{u}^{(N)}\to\widetilde{u} \quad \text{a.e. in } \mathcal{O}, \widetilde{\mathbb{P}}\text{-a.s.}$$

In particular, we have (componentwise) $(\widetilde{u}^{(N)})^2\to(\widetilde{u})^2$ a.e. in $\mathcal{O}$, $\widetilde{\mathbb{P}}$-a.s. On the other hand, by estimate (29), there exists a subsequence of $((\widetilde{u}^{(N)})^2)_{N\in\mathbb{N}}$ weakly converging to some function v in $L^2(\widetilde{\Omega};L^2(0,T;H^1(\mathcal{O})))$. The uniqueness of the limit function then implies that $v=\widetilde{u}^2$ and consequently,

$$(\widetilde{u}^{(N)})^2\rightharpoonup(\widetilde{u})^2 \quad \text{weakly in } L^2(\widetilde{\Omega};L^2(0,T;H^1(\mathcal{O}))).$$

It remains to show that the stochastic process $\widetilde{u}$ is a martingale solution to (1). The following lemmas are taken from [7, Lemma 5.2 and proof].

Lemma 14. *Suppose that the process* $(\widetilde{W}^{(N)}(t))_{t\in[0,T]}$*, defined on* $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$*, has the same law as the* Y*-valued cylindrical Wiener process* W*, defined on* $(\Omega,\mathcal{F},\mathbb{P})$*. Then* $\widetilde{W}^{(N)}$ *is also a* Y*-valued cylindrical Wiener process on* $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$*.*

Lemma 15. *The process* $(\widetilde{W}(t))_{t\in[0,T]}$ *is a* Y*-valued cylindrical Wiener process on* $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$*. If* $0\le s<t\le T$*, the increments* $\widetilde{W}(t)-\widetilde{W}(s)$ *are independent of the* σ*-algebra generated by* $\widetilde{u}(r)$ *and* $\widetilde{W}(r)$ *for* $r\in[0,s]$*.*

We denote by $\widetilde{\mathbb{F}}$ the filtration generated by $(\widetilde{u},\widetilde{W})$ and by $\widetilde{\mathbb{F}}^{(N)}$ the filtration generated by $(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$. Lemma 14 implies that $\widetilde{u}$ is progressively measurable with respect to $\widetilde{\mathbb{F}}$, and Lemma 15 shows that $\widetilde{u}^{(N)}$ is progressively measurable with respect to $\widetilde{\mathbb{F}}^{(N)}$.

The following lemma plays a significant role in establishing the existence of a martingale solution to (1).

Lemma 16. *It holds for all* $s,\ t\in[0,T]$ *with* $s\le t$ *and all* $\phi_1\in L^2(\mathcal{O})$ *and* $\phi_2\in H^3(\mathcal{O})$ *satisfying* $\nabla\phi_2\cdot\nu=0$ *on* $\partial\mathcal{O}$ *that*

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^T\big(\widetilde{u}^{(N)}(t)-\widetilde{u}(t),\phi_1\big)^2_{L^2(\mathcal{O})}dt=0, \tag{33}$$

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\big(\widetilde{u}^{(N)}(0)-\widetilde{u}(0),\phi_1\big)^2_{L^2(\mathcal{O})}=0, \tag{34}$$

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^T\bigg|\sum_{j=1}^n\int_0^t\Big\langle A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s)-A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s),\nabla\phi_2\Big\rangle ds\bigg|dt=0, \tag{35}$$

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^T\bigg|\sum_{j=1}^n\int_0^t\Big(\sigma_{ij}(\widetilde{u}^{(N)}(s))d\widetilde{W}_j^{(N)}(s)-\sigma_{ij}(\widetilde{u}(s))d\widetilde{W}_j(s),\phi_1\Big)_{L^2(\mathcal{O})}\bigg|^2dt=0. \tag{36}$$

Proof. Let $\phi_1\in L^2(\mathcal{O})$. We know that $\widetilde{u}^{(N)}\to\widetilde{u}$ in Z_T $\widetilde{\mathbb{P}}$-a.s. In particular, $\widetilde{u}^{(N)}\to\widetilde{u}$ in $C^0([0,T];L^2_w(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s., which means that for any $t\in[0,T]$,

$$\lim_{N\to\infty}(\widetilde{u}^{(N)}(t),\phi_1)_{L^2(\mathcal{O})}=(\widetilde{u}(t),\phi_1)_{L^2(\mathcal{O})}\quad\widetilde{\mathbb{P}}\text{-a.s.}$$

Estimate (28) provides a uniform bound for $(\widetilde{u}^{(N)}(t),\phi_1)^2_{L^2(\mathcal{O})}$ such that we can apply the dominated convergence theorem to conclude that

$$\lim_{N\to\infty}\int_0^T\big(\widetilde{u}^{(N)}(t)-\widetilde{u}(t),\phi_1\big)^2_{L^2(\mathcal{O})}dt=0\quad\widetilde{\mathbb{P}}\text{-a.s.} \tag{37}$$

We have for any $r>1$, by (30),

$$\widetilde{\mathbb{E}}\bigg(\bigg|\int_0^T\|\widetilde{u}^{(N)}(t)-\widetilde{u}(t)\|^2_{L^2(\mathcal{O})}dt\bigg|^r\bigg)\le C\widetilde{\mathbb{E}}\int_0^T\big(\|\widetilde{u}^{(N)}(t)\|^{2r}_{L^2(\mathcal{O})}+\|\widetilde{u}(t)\|^{2r}_{L^2(\mathcal{O})}\big)dt\le C.$$

This bound provides the equi-integrability of $\int_0^T\big(\widetilde{u}^{(N)}(t)-\widetilde{u}(t),\phi_1\big)^2_{L^2(\mathcal{O})}dt$. Taking into account the convergence (37), Vitali's convergence theorem (see the appendix) then shows that (33) holds.

Convergence (34) follows in a similar way. Indeed, since $\widetilde{u}^{(N)} \to \widetilde{u}$ in $C^0([0,T]; L^2_w(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and $\widetilde{u}$ is continuous at $t=0$, we infer that for any $\phi_1 \in L^2(\mathcal{O})$,

$$\lim_{N\to\infty} (\widetilde{u}^{(N)}(0), \phi_1)_{L^2(\mathcal{O})} = (\widetilde{u}(0), \phi_1)_{L^2(\mathcal{O})} \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

Then convergence (34) follows from (28) and Vitali's convergence theorem.

Next, we establish convergence (35) through several steps. Due to the structure of $A_{ij}(\widetilde{u}^{(N)})$, we need to show the following three convergences:

$$\lim_{N\to\infty} \int_0^t \big\langle \nabla \widetilde{u}_j^{(N)}(s) - \nabla \widetilde{u}_j(s), \nabla\phi \big\rangle ds = 0, \tag{38}$$

$$\lim_{N\to\infty} \int_0^t \Big\langle \widetilde{u}_j^{(N)}(s)\widetilde{u}_k^{(N)}(s)\nabla \widetilde{u}_k^{(N)}(s) - \widetilde{u}_j(s)\widetilde{u}_k(s)\nabla\widetilde{u}_k(s), \nabla\phi \Big\rangle ds = 0, \tag{39}$$

$$\lim_{N\to\infty} \int_0^t \Big\langle (\widetilde{u}_k^{(N)}(s))^2 \nabla \widetilde{u}_j^{(N)}(s) - (\widetilde{u}_k(s))^2\nabla\widetilde{u}(s), \nabla\phi \Big\rangle ds = 0, \tag{40}$$

for $j \neq k$ and suitable test functions ϕ. We deduce from convergence (27) that (38) follows for all $\phi \in H^1(\mathcal{O})$. The second convergence (39) is proved as follows:

$$\begin{aligned}
&\bigg| \int_0^t \Big\langle \widetilde{u}_j^{(N)}(s)\widetilde{u}_k^{(N)}(s)\nabla \widetilde{u}_k^{(N)}(s) - \widetilde{u}_j(s)\widetilde{u}_k(s)\nabla\widetilde{u}_k(s), \nabla\phi \Big\rangle ds \bigg| \\
&= \frac{1}{2}\bigg| \int_0^t \Big\langle \widetilde{u}_j^{(N)}(s)\nabla\big(\widetilde{u}_k^{(N)}(s)\big)^2 - \widetilde{u}_j(s)\nabla\big(\widetilde{u}_k(s)\big)^2, \nabla\phi \Big\rangle ds \bigg| \\
&= \frac{1}{2}\bigg| \int_0^t \Big\langle \big(\widetilde{u}_j^{(N)}(s) - \widetilde{u}_j(s)\big)\nabla\big(\widetilde{u}_k^{(N)}(s)\big)^2 + \widetilde{u}_j(s)\nabla\big\{\big(\widetilde{u}_k^{(N)}(s)\big)^2 - \big(\widetilde{u}_k(s)\big)^2\big\}, \nabla\phi \Big\rangle ds \bigg| \\
&\le \frac{1}{2}\int_0^t \|\widetilde{u}_j^{(N)}(s) - \widetilde{u}_j(s)\|_{L^2(\mathcal{O})} \|\nabla(\widetilde{u}_k^{(N)})^2\|_{L^2(\mathcal{O})} \|\nabla\phi\|_{L^\infty(\mathcal{O})} ds \\
&\quad + \frac{1}{2}\bigg| \int_0^t \Big(\widetilde{u}_j(s)\nabla\big\{\big(\widetilde{u}_k^{(N)}(s)\big)^2 - \big(\widetilde{u}_k(s)\big)^2\big\}, \nabla\phi \Big)_{L^2(\mathcal{O})} ds \bigg| \\
&=: I_1^{(N)} + I_2^{(N)}.
\end{aligned}$$

Let $\phi \in H^3(\mathcal{O})$. Then the embedding $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$ is continuous for $d \le 3$ and, using the Cauchy-Schwarz inequality,

$$I_1^{(N)} \le \frac{1}{2}\|\phi\|_{H^3(\mathcal{O})} \|\widetilde{u}_j^{(N)} - \widetilde{u}_j\|_{L^2(0,T;L^2(\mathcal{O}))} \|\nabla(\widetilde{u}_k^{(N)})^2\|_{L^2(0,T;L^2(\mathcal{O}))}.$$

Since $\widetilde{u}^{(N)} \to \widetilde{u}$ in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and $\nabla(\widetilde{u}^{(N)})^2$ is uniformly bounded in $L^2(0,T;$ $L^2(\mathcal{O}))$, it follows that $I_1^{(N)} \to 0$ as $N \to \infty$. For the second integral, we observe that $\widetilde{u}_j\nabla\phi \in L^2(0,T;L^2(\mathcal{O}))$ (using (29)) and $(\widetilde{u}^{(N)})^2 \rightharpoonup (\widetilde{u})^2$ weakly in $L^2(0,T;H^1(\mathcal{O}))$ (by (27)). This implies that $I_2^{(N)} \to 0$ as $N \to \infty$, and we have proved (39).

We turn to the proof of (40). Let $\phi \in H^3(\mathcal{O})$ be such that $\nabla\phi\cdot\nu = 0$ on $\partial\mathcal{O}$. An integration by parts leads to

$$\begin{aligned}
&\int_0^t \Big\langle (\widetilde{u}_k^{(N)}(s))^2\nabla\widetilde{u}_j^{(N)}(s) - (\widetilde{u}_k(s))^2\nabla\widetilde{u}_j(s), \nabla\phi\Big\rangle ds \\
&\quad= \int_0^t\int_{\mathcal{O}} \Big((\widetilde{u}_k^{(N)}(s))^2\nabla\widetilde{u}_j^{(N)}(s) - (\widetilde{u}_k(s))^2\nabla\widetilde{u}_j(s)\Big)\cdot\nabla\phi dxds \\
&\quad= -\int_0^t\int_{\mathcal{O}} \Big((\widetilde{u}_k^{(N)}(s))^2\widetilde{u}_j^{(N)}(s) - (\widetilde{u}_k(s))^2\widetilde{u}_j(s)\Big)\Delta\phi dxds \\
&\qquad -\int_0^t\int_{\mathcal{O}} \Big(\widetilde{u}_j^{(N)}(s)\nabla(u_k^{(N)}(s))^2 - \widetilde{u}_j(s)\nabla(\widetilde{u}_k(s))^2\Big)\cdot\nabla\phi dxds \\
&\quad=: I_3^{(N)} + I_4^{(N)}.
\end{aligned}$$

The estimates for $I_1^{(N)} + I_2^{(N)}$ show that $I_4^{(N)} \to 0$ as $N\to\infty$. We estimate $I_3^{(N)}$ as follows, using the continuous embeddings $H^3(\mathcal{O}) \hookrightarrow W^{2,4}(\mathcal{O})$ and $H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$ (for $d\le 3$):

$$\begin{aligned}
I_3^{(N)} &= -\int_0^t\int_{\mathcal{O}} \big(\widetilde{u}_j^{(N)}(s) - \widetilde{u}_j(s)\big)\big(\widetilde{u}_k^{(N)}(s)\big)^2\Delta\phi dxds \\
&\qquad + \int_0^t\int_{\mathcal{O}} \big((\widetilde{u}_k^{(N)}(s))^2 - (\widetilde{u}_k(s))^2\big)\widetilde{u}_j(s)\Delta\phi dxds \\
&\le \int_0^t \big\|\widetilde{u}_j^{(N)}(s) - \widetilde{u}_j(s)\big\|_{L^2(\mathcal{O})}\big\|(\widetilde{u}_k^{(N)}(s))^2\big\|_{L^4(\mathcal{O})}\|\Delta\phi\|_{L^4(\mathcal{O})}ds \\
&\qquad + \int_0^t\int_{\mathcal{O}} \big((\widetilde{u}_k^{(N)}(s))^2 - (\widetilde{u}_k(s))^2\big)\widetilde{u}_j(s)\Delta\phi dxds \\
&\le \big\|\widetilde{u}_j^{(N)} - \widetilde{u}_j\big\|_{L^2(0,T;L^2(\mathcal{O}))}\big\|(\widetilde{u}_k^{(N)})^2\big\|_{L^2(0,T;H^1(\mathcal{O}))}\|\phi\|_{H^3(\mathcal{O})} \\
&\qquad + \int_0^t\int_{\mathcal{O}} \big((\widetilde{u}_k^{(N)}(s))^2 - (\widetilde{u}_k(s))^2\big)\widetilde{u}_j(s)\Delta\phi dxds.
\end{aligned}$$

The convergences (27) and $\widetilde{u}_j\Delta\phi \in L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. imply that $I_3^{(N)} \to 0$ as $N\to\infty$.

Convergences (38)-(40) imply that $\widetilde{\mathbb{P}}$-a.s.

$$\lim_{N\to\infty}\int_0^t \big(A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s), \nabla\phi_2\big)_{L^2(\mathcal{O})}ds = \int_0^t \big(A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s), \nabla\phi_2\big)_{L^2(\mathcal{O})}ds \tag{41}$$

for all $\phi_2 \in H^3(\mathcal{O})$ with $\nabla\phi_2\cdot\nu = 0$ on $\partial\mathcal{O}$. Furthermore, employing the structure of $A_{ij}(u^{(N)})$, the continuous embedding $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$ (again for $d\le 3$ only), and estimates (29)-(30), we find that

$$\begin{aligned}
&\widetilde{\mathbb{E}}\bigg(\bigg|\int_0^t \big(A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s), \nabla\phi_2\big)_{L^2(\mathcal{O})}ds\bigg|^2\bigg) \\
&\quad\le \|\nabla\phi_2\|_{L^\infty(\mathcal{O})}^2\widetilde{\mathbb{E}}\bigg(\bigg|\int_0^t \big\|A_{ij}(u^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s)\big\|_{L^1(\mathcal{O})}ds\bigg|^2\bigg)
\end{aligned}$$

$$\begin{aligned}
&\le C\|\phi_2\|_{H^3(\mathcal{O})}^2\widetilde{\mathbb{E}}\bigg(\bigg|\int_0^t \big(1+\|\widetilde{u}^{(N)}(s)^2\|_{L^2(\mathcal{O})}\big)\|\nabla\widetilde{u}^{(N)}(s)\|_{L^2(\mathcal{O})}ds\bigg|^2\bigg) \\
&\le C\|\phi_2\|_{H^3(\mathcal{O})}^2\Big(T^{1/2}\big(\widetilde{\mathbb{E}}\|\widetilde{u}^{(N)}\|_{L^2(0,T;H^1(\mathcal{O}))}^2\big)^{1/2} \\
&\quad + T^{1/6}\big(\widetilde{\mathbb{E}}\|(\widetilde{u}^{(N)})^2\|_{L^3(0,T;L^2(\mathcal{O}))}^3\big)^{1/3}\big(\widetilde{\mathbb{E}}\|\widetilde{u}^{(N)}\|_{L^2(0,T;H^1(\mathcal{O}))}^2\big)^{1/2}\Big) \le C.
\end{aligned}$$

This bound and the $\widetilde{\mathbb{P}}$-a.s. convergence (41) allow us to apply the Vitali convergence theorem to infer that (35) holds.

It remains to prove convergence (36). Since $\widetilde{W}^{(N)} \to \widetilde{W}$ in $C^0([0,T];Y_0)$, it is sufficient to show that $\sigma_{ij}(\widetilde{u}^{(N)}) \to \sigma_{ij}(\widetilde{u})$ in $L^2(0,T;\mathcal{L}_2(Y;L^2(\mathcal{O})))$ $\mathbb{P}$-a.s. We estimate for $\phi \in L^2(\mathcal{O})$:

$$\begin{aligned}
&\int_0^t \big\|\big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s)),\phi\big)_{L^2(\mathcal{O})}\big\|_{\mathcal{L}_2(Y;\mathbb{R})}^2 ds \\
&\quad \le \int_0^t \big\|\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s))\big\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2\|\phi\|_{L^2(\mathcal{O})}^2 ds \\
&\quad \le C_\sigma\|\widetilde{u}^{(N)} - \widetilde{u}\|_{L^2(0,T;L^2(\mathcal{O}))}^2\|\phi\|_{L^2(\mathcal{O})}^2.
\end{aligned}$$

Since $\widetilde{u}^{(N)} \to \widetilde{u}$ in $L^2(0,T;L^2(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s., by (27), we infer that for $t \in [0,T]$, $\omega \in \widetilde{\Omega}$, and $\phi \in L^2(\mathcal{O})$,

$$\lim_{N\to\infty}\int_0^t \big\|\big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s)),\phi\big)_{L^2(\mathcal{O})}\big\|_{\mathcal{L}_2(Y;\mathbb{R})}^2 ds = 0. \tag{42}$$

We conclude from (30) and (31) that

$$\begin{aligned}
&\widetilde{\mathbb{E}}\bigg|\int_0^t \big\|(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s)),\phi\big)_{L^2(\mathcal{O})}\big\|_{\mathcal{L}_2(Y;\mathbb{R})}^2 ds\bigg|^2 \\
&\quad \le C\widetilde{\mathbb{E}}\bigg(\|\phi\|_{L^2(\mathcal{O})}^4\int_0^t \big(\|\sigma_{ij}(\widetilde{u}^{(N)}(s))\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^4 + \|\sigma_{ij}(\widetilde{u}(s))\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^4\big)ds\bigg) \\
&\quad \le C\bigg(1+\widetilde{\mathbb{E}}\Big(\sup_{t\in(0,T)}\|\widetilde{u}^{(N)}(t)\|_{L^2(\mathcal{O})}^4 + \sup_{t\in(0,T)}\|\widetilde{u}(t)\|_{L^2(\mathcal{O})}^4\Big)\bigg) \le C.
\end{aligned}$$

With this bound, convergence (42), and the Vitali convergence theorem we obtain for all $\phi \in L^2(\mathcal{O})$,

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\int_0^t \big\|\big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s)),\phi\big)_{L^2(\mathcal{O})}\big\|_{\mathcal{L}_2(Y;\mathbb{R})}^2 ds = 0.$$

Hence, by the Itô isometry (Proposition 20) for $t \in [0,T]$ and $\phi \in L^2(\mathcal{O})$,

$$\lim_{N\to\infty}\widetilde{\mathbb{E}}\bigg|\bigg(\int_0^t \big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s))\big)d\widetilde{W}_j(s),\phi\bigg)_{L^2(\mathcal{O})}\bigg|^2 = 0. \tag{43}$$

We use the Itô isometry again and estimates (28) and (32) for $N \in \mathbb{N}$, $t \in [0,T]$, and $\phi \in L^2(\mathcal{O})$ to infer that

$$\begin{aligned}
\widetilde{\mathbb{E}}\bigg|\bigg(\int_0^t \big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s))\big)d\widetilde{W}_j(s), \phi\bigg)_{L^2(\mathcal{O})}\bigg|^2 \\
&= \widetilde{\mathbb{E}}\bigg(\int_0^t \big\|\big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s)), \phi\big)_{L^2(\mathcal{O})}\big\|^2_{\mathcal{L}_2(Y;\mathbb{R})} ds\bigg) \\
&\le \widetilde{\mathbb{E}}\bigg(\|\phi\|^2_{L^2(\mathcal{O})}\int_0^t \big\|\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s))\big\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))} ds\bigg) \\
&\le C\widetilde{\mathbb{E}}\bigg(t \sup_{t\in(0,T)}\|\widetilde{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})} + t\sup_{t\in(0,T)}\|\widetilde{u}(s)\|^2_{L^2(\mathcal{O})}\bigg) \le C.
\end{aligned}$$

This bound and convergence (43) allow us to apply the dominated convergence theorem to conclude that for all $\phi \in L^2(\mathcal{O})$,

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}\int_0^T \bigg|\bigg(\int_0^t \big(\sigma_{ij}(\widetilde{u}^{(N)}(s)) - \sigma_{ij}(\widetilde{u}(s))\big)d\widetilde{W}_j(s), \phi\bigg)_{L^2(\mathcal{O})}\bigg|^2 dt = 0.$$

This shows (36) and finishes the proof. □

Let us define

$$\begin{aligned}
\Lambda_i^{(N)}(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}, \phi)(t) &:= (\Pi_N(\widetilde{u}_i(0)), \phi)_{L^2(\mathcal{O})} \\
&\quad + \sum_{j=1}^n \int_0^t \Big\langle \Pi_N \operatorname{div}\big(A_{ij}(\widetilde{u}^{(N)}(s))\nabla\widetilde{u}_j^{(N)}(s)\big), \phi\Big\rangle ds \\
&\quad + \bigg(\sum_{j=1}^n \int_0^t \Pi_N\sigma_{ij}(\widetilde{u}^{(N)}(s))d\widetilde{W}_j^{(N)}, \phi\bigg)_{L^2(\mathcal{O})}, \\
\Lambda_i(\widetilde{u}, \widetilde{W}, \phi)(t) &:= (\widetilde{u}_i(0), \phi)_{L^2(\mathcal{O})} + \sum_{j=1}^n \int_0^t \Big\langle \operatorname{div}\big(A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s)\big), \phi\Big\rangle ds \\
&\quad + \bigg(\sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{u}(s))d\widetilde{W}_j(s), \phi\bigg)_{L^2(\mathcal{O})},
\end{aligned}$$

for $t \in [0,T]$ and $i = 1,\ldots,n$. The following corollary is essentially a consequence of Lemma 16.

Corollary 17. *It holds for any $\phi_1 \in L^2(\mathcal{O})$ and any $\phi_2 \in H^3(\mathcal{O})$ satisfying $\nabla\phi_2 \cdot \nu = 0$ on $\partial\mathcal{O}$ that*

$$\begin{aligned}
\lim_{N\to\infty}\big\|(\widetilde{u}^{(N)}, \phi_1)_{L^2(\mathcal{O})} - (\widetilde{u}, \phi_1)_{L^2(\mathcal{O})}\big\|_{L^2(\widetilde{\Omega}\times(0,T))} = 0, \\
\lim_{N\to\infty}\big\|\Lambda_i^{(N)}(\widetilde{u}^{(N)}, \widetilde{W}^{(N)}, \phi_2) - \Lambda_i(\widetilde{u}, \widetilde{W}, \phi_2)\big\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.
\end{aligned}$$

Proof. The first convergence follows immediately from the identity

$$\big\|(\widetilde{u}^{(N)},\phi_1)_{L^2(\mathcal{O})}-(\widetilde{u},\phi_1)_{L^2(\mathcal{O})}\big\|_{L^2(\widetilde\Omega\times(0,T)}=\widetilde{\mathbb{E}}\int_0^T\big|\big(\widetilde{u}^{(N)}(t)-\widetilde{u}(t),\phi_1\big)_{L^2(\mathcal{O})}\big|^2dt$$

and convergence (33). For the second convergence, let $\phi_2\in H^3(\mathcal{O})$ satisfying $\nabla\phi_2\cdot\nu=0$ on $\partial\mathcal{O}$. Fubini's theorem implies that

$$\begin{aligned}\big\|\Lambda_i^{(N)}(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi_2)-\Lambda_i(\widetilde{u},\widetilde{W},\phi_2)\big\|_{L^1(\widetilde\Omega\times(0,T))}\\ =\int_0^T\widetilde{\mathbb{E}}\Big|\Lambda_i^{(N)}(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi_2)-\Lambda_i(\widetilde{u},\widetilde{W},\phi_2)\Big|dt.\end{aligned}$$

Convergences (34)-(36) show that each term in the definition of $\Lambda_i^{(N)}(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi_2)$ tends to the corresponding terms in $\Lambda_i(\widetilde{u},\widetilde{W},\phi_2)$ at least in the space $L^1(\widetilde\Omega\times(0,T))$. ☐

Since $u^{(N)}$ is a strong solution to (10)-(11), it satisfies the identity

$$(u_i^{(N)}(t),\phi)_{L^2(\mathcal{O})}=\Lambda_i^{(N)}(u^{(N)},W,\phi)(t)\quad\mathbb{P}\text{-a.s.}$$

for all $t\in[0,T]$, $i=1,\ldots,n$, and $\phi\in H^1(\mathcal{O})$ and in particular, we have

$$\int_0^T\mathbb{E}\Big|(u_i^{(N)}(t),\phi)_{L^2(\mathcal{O})}-\Lambda_i^{(N)}(u^{(N)},W,\phi)(t)\Big|dt=0.$$

Since the laws $\mathcal{L}(u^{(N)},W)$ and $\mathcal{L}(\widetilde{u}^{(N)},\widetilde{W}^{(N)})$ coincide, we find that

$$\int_0^T\widetilde{\mathbb{E}}\Big|(\widetilde{u}_i^{(N)}(t),\phi)_{L^2(\mathcal{O})}-\Lambda_i^{(N)}(\widetilde{u}^{(N)},\widetilde{W}^{(N)},\phi)(t)\Big|dt=0.$$

By Corollary 17, the limit $N\to\infty$ in this equation yields

$$\int_0^T\widetilde{\mathbb{E}}\Big|(\widetilde{u}_i(t),\phi)_{L^2(\mathcal{O})}-\Lambda_i(\widetilde{u},\widetilde{W},\phi)(t)\Big|dt=0,\quad i=1,\ldots,n.$$

This identity holds for all $\phi\in H^3(\mathcal{O})$ satisfying $\nabla\phi\cdot\nu=0$ on $\partial\mathcal{O}$. By a density argument, it also holds for all $\phi\in H^1(\mathcal{O})$. Hence, for Lebesgue-a.e. $t\in(0,T]$ and $\widetilde{\mathbb{P}}$-a.e. $\omega\in\widetilde\Omega$, we deduce that

$$(\widetilde{u}_i(t),\phi)_{L^2(\mathcal{O})}-\Lambda_i(\widetilde{u},\widetilde{W},\phi)(t)=0,\quad i=1,\ldots,n.$$

By definition of Λ_i, this means that for Lebesgue-a.e. $t\in(0,T]$ and $\widetilde{\mathbb{P}}$-a.e. $\omega\in\widetilde\Omega$,

$$\begin{aligned}(\widetilde{u}_i(t),\phi)_{L^2(\mathcal{O})}&=(\widetilde{u}(0),\phi)_{L^2(\mathcal{O})}+\sum_{j=1}^n\int_0^t\big\langle\operatorname{div}\big(A_{ij}(\widetilde{u}(s))\nabla\widetilde{u}_j(s)\big),\phi\big\rangle ds\\ &\quad+\bigg(\sum_{j=1}^n\int_0^t\sigma_{ij}(\widetilde{u}(s))d\widetilde{W}_j(s),\phi\bigg)_{L^2(\mathcal{O})}.\end{aligned}$$

Setting $\widetilde{U}:=(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}},\widetilde{\mathbb{F}})$, we infer that the system $(\widetilde{U},\widetilde{W},\widetilde{u})$ is a martingale solution to (1) and the stochastic process $\widetilde{u}$ satisfies estimates (31) and (32).

Remark 18 (Random initial data)**.** The initial data may be chosen to be random, i.e., we prescribe an inital probability measure μ^0 on $L^2(\mathcal{O})$ instead of a given initial data. We assume that

$$\int_{L^2(\mathcal{O})} \|x\|^p_{L^2(\mathcal{O})} d\mu^0(x) < \infty \quad \text{for } p = \frac{24}{4-d}. \tag{44}$$

Now, in principle, we can carry out the whole analysis also in this case. Since for the given initial distribution μ^0 and a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we have an $\mathcal{F}_0$-measurable random variable, which we will denote by u^0 and whose distribution is μ^0. Because of assumption (44), we have $\mathbb{E}\|u^0\|^p_{L^2(\mathcal{O})} < \infty$ and consequently, the a priori estimates obtained in section 2.3 still hold true. As before we can show that the set of measure $\{\mathcal{L}(u^{(N)}) : N \in \mathbb{N}\}$ is tight on Z_T and therefore, by the Skorohod-Jakubowski theorem, we obtain a sequence of new random variables $(\widetilde{u}^{(N)})_{N\in\mathbb{N}}$ (and also a sequence of new Wiener processes) which have the same law as the old random variables $u^{(N)}$ on Z_T. In particular, $\mathcal{L}(\tilde{u}^{(N)}(0)) = \mathcal{L}(u^{(N)}(0))$ in $L^2(\mathcal{O})$ as well as $\widetilde{u}^{(N)} \to \widetilde{u}$ in $C^0([0,T]; L^2_w(\mathcal{O}))$ $\widetilde{\mathbb{P}}$-a.s. and $\widetilde{u}^{(N)}(0) \to \widetilde{u}(0)$ in $L^2(\mathcal{O})$ weakly $\widetilde{\mathbb{P}}$-a.s. We conclude that $\mathcal{L}(\tilde{u}(0)) = \mathcal{L}(\tilde{u}^{(N)}(0)) = \mathcal{L}(u^0) = \mu^0$. Thus, we have shown that the process $\widetilde{u}$ has the initial measure μ^0 and therefore is the required martingale solution of (1). □

2.6. **Nonnegativity of the solutions.** We show that if $u_i^0 \geq 0$ in $\mathcal{O}$ for $i = 1, \ldots, n$ and condition (9) on σ holds then $\widetilde{u}_i$ is nonnegative $\mathbb{P}$-a.s. For this, we employ the technique of [11]. The idea is to approximate the test function $f(z) = z^- = \max\{0, -z\}$ for $z \in \mathbb{R}$ and to use Itô's formula. We define as in [11, Section 2.4] the following functions:

$$f_\varepsilon(z) = \begin{cases} -z & \text{if } z \leq -\varepsilon, \\ -3\left(\frac{z}{\varepsilon}\right)^4 z - 8\left(\frac{z}{\varepsilon}\right)^3 z - 6\left(\frac{z}{\varepsilon}\right)^2 z & \text{if } -\varepsilon \leq z \leq 0 \\ 0 & \text{if } z \geq 0 \end{cases}$$

for $\varepsilon > 0$. Then f_ε has at most linear growth, i.e. $|f_\varepsilon(z)| \leq C|z|$ for all $z \in \mathbb{R}$, and the functions f'_ε and $\psi_\varepsilon := f_\varepsilon f''_\varepsilon + (f'_\varepsilon)^2$ are bounded in $\mathbb{R}$. We set

$$F_\varepsilon(v) = \int_{\mathcal{O}} f_\varepsilon(v(x))^2 dx, \quad F(v) = \int_{\mathcal{O}} f(v(x))^2 dx$$

for square-integrable functions $v : \mathcal{O} \to \mathbb{R}$.

We replace the diffusion coefficients $A_{ij}(u^{(N)})$ in (10) by the modified coefficients

$$A^+_{ij}(u^{(N)}) = \delta_{ij}\left(a_{i0} + \sum_{k=1}^n a_{ik} u_k^2\right) + 2a_{ij} u_i^+ u_j, \quad i, j = 1, \ldots, n,$$

where $z^+ = \min\{0, z\}$ is the positive part of $z \in \mathbb{R}$. Observe that generally, $A^+_{ij}(u) \neq A_{ij}(u)$ but if $u_i \geq 0$ for all $i = 1, \ldots, n$ then we obtain the original coefficients, $A^+_{ij}(u) = A_{ij}(u)$. The proof of Lemma 4 provides the existence of a pathwise unique strong solution $u^{(N)}$ to this truncated problem. The Itô formula in finite dimensions gives [11, Formula (3.3)]

$$F_\varepsilon(u_i^{(N)}(t)) = F_\varepsilon(u_i^{(N)}(0))$$

$$
\begin{aligned}
&+ 2\int_0^t\int_{\mathcal{O}} f_\varepsilon(u_i^{(N)}(s))f_\varepsilon'(u_i^{(N)}(s))\Pi_N\bigg(\sum_{j=1}^n \sigma_{ij}(u^{(N)}(s))\bigg)dxdW_j(s) \\
&- 2\int_0^t\int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}(s))\sum_{j=1}^n A_{ij}^+(u^{(N)}(s))\nabla u_i^{(N)}(s)\cdot\nabla u_j^{(N)}(s)dxds \\
&+ \int_0^t\int_{\mathcal{O}}\sum_{j=1}^n\sum_{k,\ell=1}^N\sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}(s))e_ke_\ell\sigma_{ij}^{mk}(u^{(N)}(s))\sigma_{ij}^{m\ell}(u^{(N)}(s))dxds \\
&=: I_{\varepsilon,0}^{(N)} + I_{\varepsilon,1}^{(N)} + I_{\varepsilon,2}^{(N)} + I_{\varepsilon,3}^{(N)},
\end{aligned} \tag{45}
$$

where σ_{ij}^{km} is defined in (6). We claim that the integral $I_{\varepsilon,1}^{(N)}$ is nonpositive. Indeed, we write

$$
\begin{aligned}
I_{\varepsilon,1}^{(N)} &= -2\int_0^t\int_{\mathcal{O}}\psi_\varepsilon(u_i^{(N)})A_{ii}^+(u^{(N)})|\nabla u_i^{(N)}|^2dxds \\
&\quad - 2\int_0^t\int_{\mathcal{O}}\psi_\varepsilon(u_i^{(N)})\sum_{j\neq i}A_{ij}^+(u^{(N)})\nabla u_i^{(N)}\cdot\nabla u_j^{(N)}dxds.
\end{aligned}
$$

The first term on the right-hand side is clearly nonpositive; the second term vanishes since $\psi_\varepsilon(u_i^{(N)}) = 0$ in $\{u_i^{(N)}\geq 0\}$ and $A_{ij}^+(u^{(N)}) = 0$ in $\{u_i^{(N)}\leq 0\}$. This shows that $I_{1,\varepsilon}^{(N)}\leq 0$. By (27), we know that $u^{(N)}\to u$ in $L^2(0,T;L^2(\mathcal{O}))$ as $N\to\infty$. (To be precise, we should work with the new processes $\widetilde{u}^{(N)}$ but we omit the tilde.) Therefore, up to a subsequence which is not relabeled, $u^{(N)}\to u$ for a.e. $(x,t,\omega)\in\mathcal{O}\times(0,T)\times\Omega$. Following the steps of [11, Section 3.2], we can show the following $\mathbb{P}$-a.s. convergence results as $N\to\infty$:

$$
\begin{aligned}
&F_\varepsilon(u_i^{(N)}(t))\to F_\varepsilon(u_i(t)), \quad I_{\varepsilon,0}^{(N)}\to F_\varepsilon(u_i^0), \\
&I_{\varepsilon,2}^{(N)}\to 2\int_0^t\int_{\mathcal{O}} f_\varepsilon(u_i(s))f_\varepsilon'(u_i(s))\sum_{j=1}^n\sigma_{ij}(u(s))dxdW_j(s), \\
&I_{\varepsilon,3}^{(N)}\to \int_0^t\int_{\mathcal{O}}\sum_{j=1}^n\sum_{k,\ell=1}^\infty\sum_{m=1}^\infty\psi_\varepsilon(u_i(s))e_ke_\ell\sigma_{ij}^{mk}(u(s))\sigma_{ij}^{m\ell}(u(s))dxds.
\end{aligned}
$$

Passing to the limit $N\to\infty$ in (45) then leads to

$$
\begin{aligned}
F_\varepsilon(u_i(t)) &\leq F_\varepsilon(u_i^0) + 2\int_0^t\int_{\mathcal{O}} f_\varepsilon(u_i(s))f_\varepsilon'(u_i(s))\sum_{j=1}^n\sigma_{ij}(u(s))dW_j(s)dx \\
&\quad + \int_0^t\int_{\mathcal{O}}\psi_\varepsilon(u_i(s))\sum_{j=1}^\infty\sum_{m=1}^\infty\big(\sigma_{ij}(u(s))\eta_m\big)^2dxds.
\end{aligned}
$$

Taking the mathematical expectation, the stochastic integral vanishes:

$$\mathbb{E}F_\varepsilon(u_i(t)) \le \mathbb{E}F_\varepsilon(u_i^0) + \mathbb{E}\int_0^t\int_{\mathcal{O}}\psi_\varepsilon(u_i(s))\sum_{j=1}^n\sum_{m=1}^\infty\Big(\sigma_{ij}(u(s))\eta_m\Big)^2 dxds. \tag{46}$$

It is shown in [11, Section 3.4] that in the limit $\varepsilon\to 0$, $\mathbb{P}$-a.s.[1]

$$\mathbb{E}F_\varepsilon(u_i(t)) \to \mathbb{E}\|u_i^-(t)\|_{L^2(\mathcal{O})}^2, \quad \mathbb{E}F_\varepsilon(u_i^0) \to \mathbb{E}\|(u_i^0)^-\|_{L^2(\mathcal{O})}^2,$$

$$\mathbb{E}\int_0^t\int_{\mathcal{O}}\psi_\varepsilon(u_i)\sum_{j=1}^n\sum_{m=1}^\infty\Big(\sigma_{ij}(u)\eta_m\Big)^2 dxds \to \mathbb{E}\int_0^t\sum_{j=1}^n\|\sigma_{ij}(-u^-)\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2 ds.$$

Thus, the limit $\varepsilon\to 0$ in (46) gives

$$\mathbb{E}\|u_i^-(t)\|_{L^2(\mathcal{O})}^2 \le \mathbb{E}\|(u_i^0)^-\|_{L^2(\mathcal{O})}^2 + \mathbb{E}\int_0^t\sum_{j=1}^n\|\sigma_{ij}(-u_i^-(s))\|_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}^2 ds.$$

The first term on the right-hand side vanishes since $u_i^0\ge 0$. For the second term, we employ the linear growth (9) of σ_{ij}, showing that

$$\mathbb{E}\|u_i^-(t)\|_{L^2(\mathcal{O})}^2 \le \mathbb{E}\int_0^t\|u_i^-(s)\|_{L^2(\mathcal{O})}^2 ds.$$

Gronwall's lemma implies that $\mathbb{E}\|u_i^-(t)\|_{L^2(\mathcal{O})}^2 = 0$ for $t\in(0,T)$ and consequently, $u_i(t)\ge 0$ in $\mathcal{O}$, $\mathbb{P}$-a.s. for a.e. $t\in[0,T]$ and all $i=1,\ldots,n$. This finishes the proof.

Appendix A. Some results from stochastic analysis

A.1. **Results for stochastic processes.** The following particular Itô formula is proved in [35, Theorem 4.2.5].

Theorem 19 (Itô formula)**.** *Let $V\subset H\subset V'$ be a Gelfand triple and U be a separable Hilbert space, $X_0\in L^2(\Omega;H)$, and let $a\in L^2(\Omega\times(0,T);V')$, $b\in L^2(\Omega\times(0,T);\mathcal{L}_2(U,H))$ be progressively measurable. Define the stochastic process*

$$X(t) = X_0 + \int_0^t a(s)ds + \int_0^t b(s)dW(s), \quad t\in(0,T).$$

Then

$$\frac12\|X(t)\|_H^2 = \frac12\|X_0\|_H^2 + \int_0^t\langle a(s),X(s)\rangle_{V',V}ds + \frac12\int_0^t\|b(s)\|_{\mathcal{L}_2(U,H)}^2 ds + \int_0^t(X(s),b(s)dW(s))_H \quad \text{for } t\in(0,T),$$

where $\langle\cdot,\cdot\rangle_{V',V}$ is the duality pairing between V' and V, $(\cdot,\cdot)_H$ is the inner product in H, and $X(s)\in L^2(\Omega\times(0,T);V)$ in $\langle a(s),X(s)\rangle_{V',V}$ is any V-valued progressively measurable $dt\otimes\mathbb{P}$ version of the equivalence class represented by $X(s)$.

[1]Observe that there is a typo in [11, formulas (3.21)-(3.24)]: The sum from $l=1$ to ∞ should be outside the brackets.

The next proposition can be found in [26, Prop. 2.10].

Proposition 20 (Itô isometry)**.** *Let $\sigma(u) \in L^2((0,T)\times\Omega;\mathcal{L}_2(Y;L^2(\mathcal{O})))$ be a predictable stochastic process. Then*

$$\mathbb{E}\bigg(\int_0^T \sigma(u(s))dW(s)\bigg)^2 = \mathbb{E}\int_0^T \|\sigma(u)\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds.$$

This result can be generalized in the following sense; see [26, Prop. 2.12].

Proposition 21 (Burkholder-Davis-Gundy inequality)**.** *Let $p \geq 2$ and let $\sigma : L^2(\mathcal{O}) \times [0,T]\times\Omega \to \mathcal{L}_2(Y;L^2(\mathcal{O}))$ be a predictible stochastic process such that*

$$\mathbb{E}\bigg(\int_0^T \|\sigma(u(s))\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds\bigg)^{p/2} < \infty.$$

Then, for some $C > 0$ depending on p,

$$\mathbb{E}\bigg|\int_0^T \sigma(u(s))dW(s)\bigg|^p \leq C\mathbb{E}\bigg(\int_0^T \|\sigma(u(s))\|^2_{\mathcal{L}_2(Y;L^2(\mathcal{O}))}ds\bigg)^{p/2}.$$

A.2. **Finite-dimensional stochastic differential equations.** We state a result on the existence of the pathwise unique strong solution to the stochastic differential equation on $\mathbb{R}^n$ (essentially taken from [35, Theorem 3.1.1]; originally from [27]),

$$\pi\cdot dX(t) = a(X,t)dt + b(X,t)dW(t), \quad t>0, \quad X(0) = X_0. \tag{47}$$

Here, $\pi = (\pi_1,\ldots,\pi_n)\in(0,\infty)^n$, $a:\mathbb{R}^n\times[0,\infty)\times\Omega\to\mathbb{R}^n$ and $b:\mathbb{R}^n\times[0,\infty)\times\Omega\to\mathbb{R}^{n\times m}$ are both continuous in $x\in\mathbb{R}^n$ for fixed $t\in[0,\infty)$, $\omega\in\Omega$, progressively measurable, and satisfy for all R, $T>0$,

$$\int_0^T \sup_{|x|\leq R}\big(|a(x,t)|^2 + |b(x,t)|^2\big)dt < \infty \quad \text{in } \Omega, \tag{48}$$

where $|a(x,t)|$ is the Euclidean norm on $\mathbb{R}^n$ and $|b(x,t)|$ is the Frobenius norm on $\mathbb{R}^{n\times m}$. Furthermore, we assume that for all R, $t>0$, and x, $y\in\mathbb{R}^n$ with $|x|$, $|y|\leq R$,

$$\begin{aligned} 2\big(a(x,t)-a(y,t),x-y\big) + \big|b(x,t)-b(y,t)\big|^2 &\leq K_R(t)|x-y|^2, \\ 2(a(x,t),x) + |b(x,t)|^2 &\leq K_1(t)(1+|x|^2), \end{aligned} \tag{49}$$

where for every $R>0$, $K_R(t)$ is an $\mathbb{R}_+$-valued $\mathcal{F}_t$-adapted process satisyfing $\int_0^T K_R(t)dt < \infty$ in Ω for all R, $T>0$. We call X the pathwise strong solution to (47) if $X(t) = (X_1(t),\ldots,X_n(t))$ for $t\geq 0$ is a $\mathbb{P}$-a.s. continuous $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted process such that $\mathbb{P}$-a.s. for all $t\geq 0$,

$$\pi_i X_i(t) = \pi_i X_{0i} + \int_0^t a_i(X(s),s)ds + \int_0^t \sum_{j=1}^m b_{ij}(X(s),s)dW_j(s), \quad i=1,\ldots,n. \tag{50}$$

Theorem 22 (Existence of solutions)**.** *Let Assumptions* (48)-(49) *hold and let $X_0:\Omega\to\mathbb{R}^n$ be $\mathcal{F}_0$-measurable. Then there exists a (up to $\mathbb{P}$-indistinguishability) pathwise unique strong solution to* (47).

The proof is the same as in [35, Theorem 3.1.1]. The difference to this theorem is the appearance of the constant vector π on the left-hand side of (47). As the proof in [35] is based on the Euler method and the vector is constant, this appearance does not change the arguments. We just have to take into account that $\min_{i=1,\ldots,n}\pi_i$ is positive.

A.3. **Tightness.** We recall some definitions and results on the tightness of families of probability measures. Let E be a separable Banach space with norm $\|\cdot\|_E$ and associated Borel σ-field $\mathcal{B}(E)$.

Definition 2 (Tightness)**.** *The family Λ of probability measures on $(E,\mathcal{B}(E))$ is said to be* tight *if and only if for any $\varepsilon>0$, there exists a compact set $K_\varepsilon\subset E$ such that*

$$\mu(K_\varepsilon)\geq 1-\varepsilon\quad\text{for all } \mu\in\Lambda.$$

The theorem of Skorokhod allows for the representation of the limit measure of a weakly convergent sequence of probability measures on a metric space as the law of a pointwise convergent sequence of random variables defined on a common probability space. Since our space Z_T, defined in (23), is not a metric space, we use Jakubowski's generalization of the Skorokhod Theorem, in the form given in [9, Theorem C.1] (see the original theorem in [22]). This version is valid for topological spaces.

Theorem 23 (Skorokhod-Jakubowski)**.** *Let Z be a topological space such that there exists a sequence $(f_m)_{m\in\mathbb{N}}$ of continuous functions $f_m:Z\to\mathbb{R}$ that separate points of Z. Let S be the σ-algebra generated by $(f_m)_{m\in\mathbb{N}}$. Then*

(1) Every compact subset of Z is metrizable.

(2) If $(\mu_m)_{m\in\mathbb{N}}$ is a tight sequence of probability measures on (Z,S), then there exists a subquence $(\mu_{m_k})_{k\in\mathbb{N}}$, a probability space $(\widetilde\Omega,\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, and Z-valued Borel measurable random variables ξ_k and ξ such that (i) *μ_{m_k} is the law of ξ_k and* (ii) *$\xi_k\to\xi$ almost surely on Ω.*

The Aldous condition is mentioned in the tightness criterion of Theorem 10, and therefore we recall its definition.

Definition 3 (Aldous condition)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes on a complete separable metric space S, defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with filtration $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$. We say that $(X_n)_{n\in\mathbb{N}}$ satisfies the* Aldous condition *if and only if for any $\varepsilon>0$, there exists $\eta>0$ such that for any $\delta>0$ and any sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n\leq T$, it holds that*

$$\sup_{n\in\mathbb{N}}\sup_{0<\theta<\delta}\mathbb{P}\big\{d\big(X_n(\tau_n+\theta),X_n(\tau_n)\big)\geq\eta\big\}\leq\varepsilon.$$

A.4. **Vitali's convergence theorem.** We use the following version of Vitali's convergence theorem (which can be seen as a special version of the theorem of De la Vallée-Poussin).

Theorem 24 (Vitali)**.** *Let (a_N) be a sequence of integrable functions on some probability space $(\Omega,\mathcal{B}(\Omega),\mathbb{P})$ such that $a_N\to a$ a.e. as $N\to\infty$ (or $a_N\to a$ in measure) for some* *integrable function a and there exist* $r > 1$ *and a constant* $C >$ *such that* $\mathbb{E}|a_N|^r \leq C$ *for all* $N \in \mathbb{N}$. *Then* $\mathbb{E}|a_N| \to \mathbb{E}|a|$ *as* $N \to \infty$.

References

[1] H. Amann. Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems. *Diff. Int. Eqs.* 3 (1990), 13-75.

[2] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: H. J. Schmeisser and H. Triebel (editors), *Function Spaces, Differential Operators and Nonlinear Analysis*, pp. 9-126. Teubner, Stuttgart, 1993.

[3] A. Badrikian. *Séminaire sur les fonctions aléatoires linéaires et les mesures cylindriques*. Lecture Notes Math. 139. Springer, Berlin, 1970.

[4] V. Barbu and M. Röckner. Variational solutions to nonlinear stochastic differential equations in Hilbert spaces. Submitted for publication, 2018. `arXiv:1802.07533.`

[5] H. Brézis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York, 2011.

[6] Z. Brzeźniak and D. Gatarek. Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces. *Stoch. Process. Appl.* 84 (1999), 187-225.

[7] Z. Brzeźniak, B. Goldys, and T. Jegaraj. Weak solutions of a stochastic Landau-Lifshitz-Gilbert equation. *Appl. Math. Research eXpress* 2013 (2013), no. 1, 1-33.

[8] Z. Brzeźniak and E. Motyl. The existence of martingale solutions to the stochastic Boussinesq equations. *Global Stoch. Anal.* 1 (2014), 175-216.

[9] Z. Brzeźniak and M. Ondreját. Stochastic wave equations with values in Riemanninan manifolds. *Stochastic partial differential equations and applications, Quaderni di Matematica.* 25 (2010), 65-97.

[10] S. Cerrai. Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *Probab. Theory Relat. Fields* 125 (2003), 271-304.

[11] M. Chekroun, E. Park, and R. Temam. The Stampacchia maximum principle for stochastic partial differential equations and applications. *J. Diff. Eqs.* 260 (2016), 2926-2972.

[12] L. Chen and A. Jüngel. Analysis of a multi-dimensional parabolic population model with strong cross-diffusion. *SIAM J. Math. Anal.* 36 (2004), 301-322.

[13] X. Chen, E. Daus, and A. Jüngel. Global existence analysis of cross-diffusion population systems for multiple species. *Archive Rat. Mech. Anal.* 227 (2018), 715-747.

[14] A. Chojnowska-Michalik and B. Goldys. Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces. *Probab. Theory Relat. Fields* 102 (1995), 331-356.

[15] D. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Second edition. Cambridge University Press, Cambridge, 2014.

[16] A. Debussche, M. Martinová, and J. Vovelle. Degenerate parabolic stochastic differential equations: quasilinear case. *Ann. Probab.* 44 (2016), 1916-1955.

[17] L. Denia and L. Stoica. A general analytical result for non-linear SPDEs and applications. *Electr. J. Probab.* 9 (2004), 674-709.

[18] Y. A. Dubinskii. Weak convergence for nonlinear elliptic and parabolic equations. *Mat. Sb.* 67(109) (1965), 609-642 (in Russian).

[19] F. Flandoli. A stochastic reaction-diffusion equations with multiplicative noise. *Appl. Math. Lett.* 4 (1991), 45-48.

[20] A. Förster and A. S. Michailov. Application of path integrals to stochastic reaction-diffusion equations. In: *Self-organization by Nonlinear Irreversible Processes* (Kühlungsborn, 1985). Springer Ser. Synergetics 33, pp. 89-94. Springer, Berlin, 1986.

[21] M. Hofmanová and T. Zhang. Quasilinear parabolic stochastic partial differential equations: Existence, uniqueness. *Stoch. Process. Appl.* 127 (2017), 3354-3371.

[22] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen.* 42 (1997), 209-216. English translation in *Theory Probab. Appl.* 42 (1997), 167-175.

[23] A. Jüngel. *Entropy Methods for Diffusive Partial Differential Equations.* BCAM SpringerBriefs, 2016.

[24] A. Jüngel and N. Zamponi. Qualitative behavior of solutions to cross-diffusion systems from population dynamics. *J. Math. Anal. Appl.* 440 (2016), 794-809.

[25] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Graduate Texts Math. 113, Springer, New York, 1988.

[26] R. Kruse. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations.* Lecture Notes Math. 2093. Springer, Cham, 2014.

[27] N. V. Krylov. On Kolmogorovs equations for finite dimensional diffusions. In: N. V. Krylov, M. Röckner, and J. Zabczyk. *Stochastic PDEs and Kolmogorov Equations in Infinite Dimensions* (Cetraro, 1998). Lecture Notes Math. 1715, pp. 1-63. Springer, Berlin, 1999.

[28] C. Kuehn and A. Neamtu. Pathwise mild solutions for quasilinear stochastic partial differential equations. Submitted for publication, 2018. `arXiv:1802.10016.`

[29] M. Kunze. Stochastic reaction-diffusion equations with Hölder continuous multiplicative noise. *Stoch. Anal. Appl.* 33 (2015), 331-355.

[30] W. Liu and M. Röckner. Local and global well-posedness of SPDE with generalized coercivity conditions. *J. Diff. Eqs.* 254 (2013), 725-755.

[31] R. Manthey and B. Maslowski. A random continuous model for two interacting populations. *Appl. Math. Optim.* 45 (2002), 213-236.

[32] R. Mikulevicius and B. Mikulevicius. Global L^2-solutions of stochastic Navier-Stokes equations. *Ann. Probab.* 33 (2005), 137-176.

[33] P. Nguyen and D. Pham. Stochastic systems of diffusion equations with polynomial reaction terms. *Asympt. Anal.* 33 (2016), 125-161.

[34] E. Pardoux. *Integrales Stochastiques Hilbertiennes.* Cahiers Mathématiques de Décision, no. 7617, Université Paris Dauphine, 1976.

[35] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations.* Lecture Notes Math. 1905. Springer, Berlin, 2007.

[36] N. Shigesada, K. Kawasaki, and E. Teramoto. Spatial segregation of interacting species. *J. Theor. Biol.* 79 (1979), 83-99.

[37] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* 146 (1987), 65-96.

[38] N. Zamponi and A. Jüngel. Analysis of degenerate cross-diffusion population models with volume filling. *Ann. Inst. H. Poincaré – Anal. Non Lin.* 34 (2017), 1–29. (Erratum: 34 (2017), 789–792.)

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