CROSS DIFFUSION PREVENTING BLOW UP IN THE TWO-DIMENSIONAL KELLER-SEGEL MODEL*

SABINE HITTMEIR AND ANSGAR JÜNGEL†

Abstract. A (Patlak-) Keller-Segel model in two space dimensions with an additional cross-diffusion term in the equation for the chemical signal is analyzed. The main feature of this model is that there exists a new entropy functional, yielding gradient estimates for the cell density and chemical substance. This allows one to prove, for arbitrarily small cross diffusion, the global existence of weak solutions to the parabolic-parabolic model as well as the global existence of bounded weak solutions to the parabolic-elliptic model, thus preventing blow up of the cell density. Furthermore, the long-time decay of the solutions to the parabolic-elliptic model is shown and finite-element simulations are presented illustrating the influence of the regularizing cross-diffusion term.

Key words. Chemotaxis, Keller-Segel model, cross-diffusion, global existence of solutions, long-time decay, blow-up of solutions.

AMS subject classifications. 35A05, 35B40, 35K55, 35Q80, 92C17.

1. Introduction. Chemotaxis, the directed movement of cells in response to chemical gradients, plays an important role in many biological fields, such as embryogenesis, immunology, cancer growth, and wound healing [19, 31]. The mathematical modeling of chemotaxis dates to the pioneering works of Patlak from 1953 [30] and Keller and Segel from 1970 [26]. The original model equations have been reduced to describe the evolution of the cell density $n(x,t)$ and the concentration of the chemical signal $S(x,t)$, and it is given, in its general form, by

$$\begin{aligned} n_t &= \operatorname{div}\big(D_1(n,S)\nabla n - \chi(n,S)n\nabla S\big) + R_1(n,S), \\ \alpha S_t &= D_2\Delta S + R_2(n,S), \quad x\in\Omega,\ t>0, \end{aligned}$$

where $\Omega\subset\mathbb{R}^d$ $(d\geq 1)$ is a bounded domain. This system of equations is supplemented by homogeneous Neumann boundary and initial conditions:

$$\nabla n\cdot\nu = \nabla S\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0, \tag{1.1}$$

$$n(\cdot,0)=n_0, \quad S(\cdot,0)=S_0 \quad \text{in } \Omega, \tag{1.2}$$

where the initial condition for S is only needed if $\alpha\neq 0$, and ν denotes the exterior unit normal to $\partial\Omega$. The positive coefficients D_1 and D_2 describe the diffusivity of the cells and the chemical substance, respectively. The nonlinear term $\chi(n,S)n\nabla S$ models the cell movement towards higher concentrations of the chemical signal, which is called positive chemotaxis. The coefficient χ is the chemotactic sensitivity. Furthermore, R_1 describes the cell growth and death and R_2 the production and degradation of the chemical signal. The parameter $\alpha\geq 0$ is a measure of the ratio of the time scales of the cell movement and the distribution of the chemical. When $\alpha=1$, we call the above system to be of parabolic-parabolic type, whereas in the case $\alpha=0$, it is

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†Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstr. 8-10, 1040 Wien, Austria (hittmeir@tuwien.ac.at, juengel@tuwien.ac.at)

called parabolic-elliptic. The classical Keller-Segel model corresponds to the choice $D_1 = \chi = 1$, $R_1 = 0$, and $R_2(n, S) = \mu n - S$, where $\mu > 0$ is the secretion or production rate at which the chemical substance is emitted by the cells. The rigorous derivation of the classical Keller-Segel model from an interacting stochastic many-particle system has been perfomed by Stevens [34].

The classical Keller-Segel model exhibits the phenomenon of cell aggregation. The more cells are aggregated, the more the attracting chemical signal is produced by the cells. This process is counter-balanced by cell diffusion, but if the cell density is sufficiently large, the nonlocal chemical interaction dominates diffusion and results – in two and three space dimensions – in a blow-up of the cell density, see, e.g., [21]. In two-dimensional domains, the critical threshold for blow-up is given by $M := \int_\Omega n_0 dx = 4\pi$ [29]. If $M < 4\pi$, solutions exist globally in time, whereas for $M > 4\pi$, solutions blow up in finite time. In the parabolic-elliptic whole space model, the critical value is 8π [4]. In the critical case $M = 8\pi$, a global solution exists, which becomes unbounded as $t \to \infty$ [3]. We mention that in dimensions $d \geq 3$, a related critical phenomenon occurs, where the $L^{d/2}$ norm of the initial density n_0 plays a similar role as the initial mass M in two dimensions, see, e.g., [10].

Motivated by numerical and modeling issues, the question how blow up of cells can be avoided has been investigated intensively the last years. In this paper, we show that an additional cross-diffusion term in the equation for the chemical substance leads to global solutions to the parabolic-parabolic model and to global *bounded* solutions to the parabolic-elliptic model. Before we explain our main results, we review the methods used in the literature to prevent overcrowding of cells. In the following, if not stated otherwise, we consider the case $D_1 = \chi = 1$, $R_1 = 0$, and $R_2 = \mu n - S$.

The first method is to modify the chemotactic sensitivity. If we suppose that aggregation stops when the cell density reaches the maximal value $n_\infty = 1$ (volume-filling effect), one may model the sensitivity by $\chi(n) = 1-n$. In this case, the cell density satisfies $0 \leq n \leq 1$ and the global existence of solutions can be proved, see [7] for the parabolic-elliptic equations and [15] for the parabolic-parabolic model. Furthermore, if the sensitivity is sublinear in two dimensions or, more generally, $\chi(n) = n^p$ with $0 < p < 2/d$, all solutions to the parabolic-parabolic model are global and bounded [22]. Global solutions are also obtained when the sensitivity depends on the chemical concentration. For instance, in the biased random walk approach, which leads to $\chi(n, S) = (1 + \beta)/(S + \beta)$ $(\beta > 0)$ [2], solutions to the two-dimensional parabolic-elliptic model exist globally in time. Finally, we mention the work by Hillen et al. [20] in which the gradient of the chemical is replaced by a nonlocal gradient yielding global bounded solutions.

A second way consists in modifying the cell diffusion. In the context of the volume-filling effect, Burger et al. [7] have altered the cell equation to $n_t = \operatorname{div}(n(1 - n)\nabla(n - S))$, which has the property that the mobility vanishes at the threshold value $n_\infty = 1$. The corresponding Keller-Segel system possesses a quadratic energy functional involving gradients of the chemical, which allows one to show the global existence of solutions to the parabolic-elliptic system [7]. Global existence results can be also achieved by employing the degenerate diffusion $D_1(n, S) = n^q$ with $q > 1$, which can be interpreted as a volume-filling effect on the level of cell diffusion, see [8, 27] for the parabolic-elliptic model and [28] for the parabolic-parabolic model (with $q > 2 - 4/d$). According to Hillen and Painter [18], the diffusion coefficient and the sensitivity are related by $D_1 = P(n) - P'(n)n$ and $\chi = P(n)r(n)$, where $P(n(x, t))$ is the probability that a cell at (x, t) finds space at its neighboring location.

This (parabolic-parabolic) model admits a global solution [40]. Furthermore, global solutions to the parabolic-elliptic model exist if $D_1 = n^q$, $n\chi = n^p$, and $p > q - 2/d$ [35]. On the other hand, if $D_1(n)/(n\chi(n))$ grows faster than $n^{2/d}$ as $n \to \infty$, then there exist unbounded solutions [39].

A third method is to consider nonvanishing growth-death models $R_1 \neq 0$. In [5], the rate $R_1(n,S) = n(1-n)(n-a)$ with $0 \le a \le 1$ is taken in the parabolic-elliptic model, and the global existence of solutions has been shown. In the logistic-growth model $R_1(n,S) = n(1-n^{p-1})$, a global weak solution to the parabolic-elliptic model exists for all $p > 2 - 1/d$ [38]. Moreover, if $p > \max\{d/2, 2-1/d\}$, the solution is bounded for all $t > 0$ even when the initial datum is in $L^1(\Omega)$ only.

In this paper we suggest a *fourth way* to prevent blow-up of the cell density by adding a cross-diffusion term. More precisely, we analyze the following system

$$n_t = \operatorname{div}(\nabla n - n\nabla S), \tag{1.3}$$
$$\alpha S_t = \Delta S + \delta\Delta n + \mu n - S, \quad x \in \Omega,\ t > 0, \tag{1.4}$$

with the initial and boundary conditions (1.1)-(1.2), where $\Omega \subset \mathbb{R}^2$ and $\delta > 0$. This choice is motivated by the fact that the augmented model possesses a new entropy functional allowing for global existence results and revealing some interesting structural properties of the system. Moreover, numerical results (see Section 5) indicate the existence of nonhomogeneous steady states for this system. For numerical approximations of the parabolic-parabolic model, we refer to [6, 13]; the parabolic-elliptic Keller-Segel model in $d = 2$ has been numerically solved in, e.g., [14, 17, 32]. We show below (Theorems 1.1 and 1.2) that the system (1.1)-(1.4) possesses a global weak solution, *for arbitrarily small* values of $\delta > 0$. Thus, discrete (finite-element) solutions are expected to exist for all times too. We confirm this statement by numerical simulations.

At first sight, the additional cross-diffusion term seems to cause a number of serious difficulties. Rewriting the system (1.3)-(1.4) in the form

$$\partial_t \begin{pmatrix} n \\ \alpha S \end{pmatrix} = \operatorname{div}\left(\begin{pmatrix} 1 & -n \\ \delta & 1 \end{pmatrix} \nabla \begin{pmatrix} n \\ S \end{pmatrix} \right) + \begin{pmatrix} 0 \\ \mu n - S \end{pmatrix},$$

we see that the diffusion matrix is neither symmetric nor positive definite. Moreover, we cannot employ anymore the maximum principle to the equation for the chemical substance, and it is not clear how to prove the nonnegativity of the cell density or the boundedness of the variables. All these difficulties can be resolved by the observation that the system (1.3)-(1.4) possesses a logarithmic entropy. Indeed, differentiating formally the entropy functional

$$E(t) = \int_\Omega \left(n(\log n - 1) + \frac{\alpha}{2\delta} S^2 \right)(x,t)dx$$

yields the entropy production equation

$$\frac{dE}{dt} + \int_\Omega \left(4|\nabla\sqrt{n}|^2 + \frac{1}{\delta}|\nabla S|^2 + \frac{1}{\delta} S^2 \right) dx = \frac{\mu}{\delta} \int_\Omega nSdx. \tag{1.5}$$

When the secretion rate vanishes, $\mu = 0$, E turns out to be a Lyapunov functional. The right-hand side can be bounded for $\mu > 0$ by use of the Hölder and Gagliardo-Nirenberg inequalities (see the estimates following (3.6)):

$$\frac{\mu}{\delta} \int_\Omega nSdx \le C(\mu,\delta)\|n\|_{L^1(\Omega)}^{5/2} + \int_\Omega \left(2|\nabla\sqrt{n}|^2 dx + \frac{1}{2\delta}|\nabla S|^2 + \frac{1}{2\delta} S^2 \right) dx.$$

Inserting this estimate into (1.5) provides H^1 bounds for $\sqrt{n}$ and S.

The existence of the entropy functional has another consequence: It allows us to reformulate the system (1.3)-(1.4) in such a way that the new diffusion matrix becomes positive definite. Introducing the new variable $y = \delta \log n$, the transformed system reads as

$$\partial_t \begin{pmatrix} n \\ \alpha S \end{pmatrix} = \operatorname{div}\left(\begin{pmatrix} \delta^{-1} e^{y/\delta} & -e^{y/\delta} \\ e^{y/\delta} & 1 \end{pmatrix} \nabla \begin{pmatrix} y \\ S \end{pmatrix} \right) + \begin{pmatrix} 0 \\ \mu n - S \end{pmatrix}. \tag{1.6}$$

It is well known that, in hyperbolic or parabolic systems, the existence of an entropy functional is equivalent to the existence of a change of unknowns which "symmetrizes" the system [11, 25]. (For parabolic systems, "symmetrization" means that the transformed diffusion matrix is symmetric and positive definite.) In the Keller-Segel system (1.6), we obtain a *nonsymmetric*, but still positive definite diffusion matrix.

We state now our existence results for the parabolic-parabolic system ($\alpha > 0$) as well as for the parabolic-elliptic system ($\alpha = 0$).

THEOREM 1.1 (Global existence). *Let $\alpha \geq 0$, $\delta > 0$, $\mu > 0$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^{0,1}$. Furthermore, let $0 \leq n_0 \in L^1(\Omega)$ be such that $n_0 \log n_0 \in L^1(\Omega)$, $S_0 \in L^2(\Omega)$. Then there exists a weak solution (n, S) to (1.1)-(1.4) satisfying $n \geq 0$ in $\Omega \times (0, \infty)$ and*

$$\begin{aligned} & n_t \in L^1_{\rm loc}(0,\infty; (W^{1,\infty}(\Omega))'), \quad \alpha S_t \in L^{4/3}_{\rm loc}(0,\infty; (W^{1,4}(\Omega))'), \\ & n \in L^2_{\rm loc}(0,\infty; W^{1,1}(\Omega)), \quad \sqrt{n} \in L^2_{\rm loc}(0,\infty; H^1(\Omega)), \\ & S \in L^2_{\rm loc}(0,\infty; H^1(\Omega)), \quad \alpha S \in L^\infty_{\rm loc}(0,\infty; L^2(\Omega)). \end{aligned}$$

Moreover, we have the following regularity results:

$$n^\beta \in L^2_{\rm loc}(0,\infty; H^1(\Omega)) \quad \text{for } 0 < \beta < 1/4, \tag{1.7}$$

$$\log n \in L^2_{\rm loc}(0,\infty; H^1(\Omega)) \quad \text{if } \log n_0 \in L^1(\Omega). \tag{1.8}$$

Estimate (1.8) shows that the cell density $n(\cdot, t)$ is positive a.e. in Ω but it does not exclude that $n(\cdot, t)$ vanishes at certain points. The existence proof is based on the construction of a problem which approximates (1.6). First, we replace (1.6) by a time-discrete system using the implicit Euler discretization with time step $\tau > 0$. Then (1.6) becomes a recursive sequence of elliptic problems. Second, we add the fourth-order operator $-\varepsilon(\Delta^2 y + y e^{y/\delta})$ with $\varepsilon > 0$ to the first component of (1.6), which guarantees coercivity of the elliptic system in $H^2(\Omega)$ with respect to y. Third, we add the regularization $\varepsilon \operatorname{div}(|\nabla y|^2 \nabla y)$ to the first component. This regularization, which was also used in [24], is not needed for the existence proof but for the derivation of the additional a priori estimates (1.7)-(1.8). The existence of weak solutions $(n_\varepsilon, S_\varepsilon)$ to this approximate problem is then proved by the Leray-Schauder fixed-point theorem. The a priori estimates from the (discrete) entropy inequality are sufficient to pass to the limit $(\varepsilon, \tau) \to 0$. More precisely, for given $T > 0$, we infer from the entropy estimate that $\nabla S_\varepsilon \rightharpoonup \nabla S$ weakly in $L^2(0,T; L^2(\Omega))$, as $(\varepsilon, \tau) \to 0$, and, by the Aubin compactness lemma, $n_\varepsilon \to n$ strongly in $L^2(0,T; L^p(\Omega))$ for all $p < 2$ (see Step 2 of the proof below). This does *not* imply that $(n_\varepsilon \nabla S_\varepsilon)$ converges weakly. A remedy of this difficulty is to exploit the $L^1 \log L^1$ bound for (n_ε) which allows us to improve the strong convergence of (n_ε) to the case $p = 2$ (see Proposition 2.1). As a consequence, $n_\varepsilon \nabla S_\varepsilon \to n \nabla S$ weakly in $L^1(0,T; L^1(\Omega))$.

The entropy estimate from (1.5) depends on the parameter $\delta > 0$ and does not allow for the limit $\delta \to 0$ in a direct way. This is not surprising since in the limit $\delta \to 0$, the limiting solution is expected to satisfy the classical Keller-Segel system for which finite-time blow up is possible. We expect that the limit $\delta \to 0$ leads to measure-valued solutions, see, e.g., [12]. The investigation of this limit is devoted to future work.

The proof of Theorem 1.1 covers the parabolic-elliptic system $\alpha = 0$. In fact, we are able to prove some regularity and the uniqueness of solutions, as stated in the following theorem.

Theorem 1.2 (Regularity and uniqueness for the parabolic-elliptic system). *Let $\alpha = 0$, $\delta > 0$, $\mu > 0$, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^{1,1}$. Furthermore, let $0 \le n_0 \in L^\infty(\Omega)$. Then there exists a unique weak solution (n, S) to (1.1)-(1.4) satisfying $n \ge 0$ in $\Omega \times (0,\infty)$ and*

$$n \in L^\infty(0,\infty; L^\infty(\Omega)), \quad n_t \in L^2_{\rm loc}(0,\infty; (H^1(\Omega))'), \quad n \in L^2_{\rm loc}(0,\infty; H^1(\Omega)),$$
$$S \in L^\infty(0,\infty; L^\infty(\Omega)), \quad S \in L^2_{\rm loc}(0,\infty; H^1(\Omega)).$$

Moreover, $S + \delta n \in L^\infty(0,\infty; W^{2,p}(\Omega))$ for all $p < \infty$.

The idea of the proof of the L^∞ bound for n is to introduce the new variable $v = S + \delta n$. Then (1.1)-(1.4) with $\alpha = 0$ can be written formally as

$$n_t = \operatorname{div}\big((1+\delta n)\nabla n - n\nabla v\big), \quad 0 = \Delta v + (\mu+\delta)n - v.$$

This corresponds to the Keller-Segel model with the nonlinear cell diffusivity $D_1 = 1 + \delta n$. Similar models have been investigated in the literature. For instance, in [27] the boundedness of n is shown by estimating the L^p norm of n and then by passing to the limit $p \to \infty$. This Moser-type strategy has been first applied in [23] to the Keller-Segel model. Global boundedness of n and S in space and time has been proved by Calvez and Carrillo [8]. Our proof is based on these techniques.

Theorem 1.2 shows that blow up of the solutions cannot occur. When the secretion rate $\mu > 0$ is sufficiently small or δ is sufficiently large, we are able to show the exponential time decay of the weak solution (n, S) to the homogeneous steady state (n^*, S^*), where $n^* = M/\text{meas}(\Omega)$, $S^* = \mu n^*$, and $M = \int_\Omega n_0 dx$. For this, we define the relative entropy

$$E^*(t) = \int_\Omega \Big(n \log\frac{n}{n^*}\Big)(x,t)dx.$$

Theorem 1.3 (Exponential time decay). *Let (n, S) be the weak solution constructed in Theorem 1.2. Then there exist constants $c_1(\Omega)$, $c_2(\Omega) > 0$ only depending on Ω such that if $\mu^2 M/\delta < c_1(\Omega)$ then*

$$E^*(t) \le E^*(0)e^{-\kappa t}, \quad t \ge 0,$$

where $\kappa = c_2(\Omega)(c_1(\Omega) - \mu^2 M/\delta) > 0$. Moreover, for $t \ge 0$,

$$\|n(\cdot,t) - n^*\|_{L^1(\Omega)} \le \sqrt{2E^*(0)}e^{-\kappa t/2}, \quad \|S(\cdot,t) - S^*\|_{L^2(\Omega)} \le c_3 e^{-\kappa t/4},$$

where $c_3 > 0$ depends on μ, δ, n^*, and the global L^∞ bound of n.

We expect that a largeness condition on δ is necessary to obtain convergence to the *homogeneous* steady state since the Keller-Segel system with $\delta = 0$ exhibits blow-up solutions for large initial mass.

The paper is organized as follows. A strong convergence result, which improves slightly the compactness resulting from the Aubin lemma, is shown in Section 2. Section 3 is devoted to the proof of Theorem 1.1. The parabolic-elliptic system is studied in Section 4, and Theorems 1.2 and 1.3 are proved. Finally, some numerical simulations are presented in Section 5 illustrating the smoothing effect of the cross-diffusion parameter $\delta > 0$.

2. A convergence result. In this section, we prove the strong convergence of a sequence of functions bounded in certain Sobolev spaces.

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $T > 0$, and $s \geq 0$. Furthermore, let (u_ε) be a sequence of nonnegative functions satisfying*

$$\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|u_\varepsilon \log u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t u_\varepsilon\|_{L^1(0,T;(H^s(\Omega))^*)} \leq C$$

for some $C > 0$ independent of ε. Then, up to a subsequence, as $\varepsilon \to 0$,

$$u_\varepsilon \to u \quad \text{strongly in } L^2(0,T;L^{d/(d-1)}(\Omega)).$$

The above uniform estimates are typical for solutions u_ε of nonlinear diffusion equations for which $\int_\Omega u_\varepsilon(\log u_\varepsilon - 1)dx$ is an entropy with $\int_\Omega |\nabla\sqrt{u_\varepsilon}|^2 dx$ as the corresponding entropy production. Notice that the estimate implies that $\nabla u_\varepsilon = 2\sqrt{u_\varepsilon}\nabla\sqrt{u_\varepsilon}$ is uniformly bounded in $L^2(0,T;L^1(\Omega))$. Hence, since the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $p < d/(d-1)$, we conclude from the Aubin lemma that there exists a subsequence of (u_ε), which is not relabeled, such that $u_\varepsilon \to u$ strongly in $L^2(0,T;L^p(\Omega))$ as $\varepsilon \to 0$. The additional estimate for $u_\varepsilon \log u_\varepsilon$ in L^1 allows us to extend this convergence result to $p = d/(d-1)$. This is the novelty of Proposition 2.1.

Proof. By the above application of the Aubin lemma, it holds $u_\varepsilon \to u$ a.e. First, we claim that this convergence and the bound for $(u_\varepsilon \log u_\varepsilon)$ imply that $\sqrt{u_\varepsilon} \to \sqrt{u}$ strongly in $L^\infty(0,T;L^2(\Omega))$ as $\varepsilon \to 0$. Indeed, by the Fatou lemma,

$$\sup_{(0,T)} \int_\Omega |u \log u| dx = \sup_{(0,T)} \int_\Omega \lim_{\varepsilon\to 0} |u_\varepsilon \log u_\varepsilon| dx \leq \liminf_{\varepsilon\to 0} \sup_{(0,T)} \int_\Omega |u_\varepsilon \log u_\varepsilon| dx \leq C.$$

Let $L > 1$ and $v_\varepsilon = \min\{u_\varepsilon, L\}$. Then $v_\varepsilon \to v = \min\{u, L\}$ a.e. By dominated convergence, it holds for sufficiently small $\varepsilon > 0$:

$$\sup_{(0,T)} \int_\Omega |v_\varepsilon - v| dx \leq \frac{1}{\log L}.$$

Then, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} \sup_{(0,T)} \int_\Omega |u_\varepsilon - u| dx &\leq \sup_{(0,T)} \int_\Omega |u_\varepsilon - v_\varepsilon| dx + \sup_{(0,T)} \int_\Omega |v_\varepsilon - v| dx + \sup_{(0,T)} \int_\Omega |v - u| dx \\ &\leq \sup_{(0,T)} \int_{\{u_\varepsilon \geq L\}} (u_\varepsilon - L) dx + \frac{1}{\log L} + \sup_{(0,T)} \int_{\{u \geq L\}} (u - L) dx \\ &\leq \sup_{(0,T)} \int_\Omega \frac{|u_\varepsilon \log u_\varepsilon|}{\log L} dx + \frac{1}{\log L} + \sup_{(0,T)} \int_\Omega \frac{u \log u}{\log L} dx \leq \frac{1 + 2C}{\log L}. \end{aligned}$$

Since $L > 1$ is arbitrary, this shows that $u_\varepsilon \to u$ strongly in $L^\infty(0,T;L^1(\Omega))$ and hence, $\sqrt{u_\varepsilon} \to \sqrt{u}$ strongly in $L^\infty(0,T;L^2(\Omega))$.

Next, we apply the Gagliardo-Nirenberg inequality (see p. 1034 in [41])

$$\begin{aligned}\|\sqrt{u_\varepsilon}-\sqrt{u}\|^4_{L^4(0,T;L^{2d/(d-1)}(\Omega))} &\le C_1\int_0^T \|\sqrt{u_\varepsilon}-\sqrt{u}\|^2_{H^1(\Omega)}\|\sqrt{u_\varepsilon}-\sqrt{u}\|^2_{L^2(\Omega)}dt \\ &\le C_2\big(\|\sqrt{u_\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))}+\|\sqrt{u}\|^2_{L^2(0,T;H^1(\Omega))}\big) \\ &\quad\times \|\sqrt{u_\varepsilon}-\sqrt{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\to 0 \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Hence, $\sqrt{u_\varepsilon} \to \sqrt{u}$ strongly in $L^4(0,T;L^{2d/(d-1)}(\Omega))$ and consequently, $u_\varepsilon \to u$ strongly in $L^2(0,T;L^{d/(d-1)}(\Omega))$. □

3. The parabolic-parabolic system. This section is devoted to the proof of Theorem 1.1. The proof is divided in several steps.

Step 1: existence of a time-discrete problem. Let $T>0$ and $K\in\mathbb{N}$. We split the time interval in the subintervals

$$(0,T] = \bigcup_{k=1}^K ((k-1)\tau, k\tau], \qquad \tau = T/K.$$

Let $0<\varepsilon<\min\{1,1/(\delta\tau)\}$. For given functions y_{k-1} and S_{k-1}, we wish to solve the approximate elliptic problem

$$\begin{aligned}&\frac{1}{\tau}\begin{pmatrix} e^{y_k/\delta}-e^{y_{k-1}/\delta} \\ \alpha(S_k-S_{k-1})\end{pmatrix} - \operatorname{div}\left(D(y_k)\nabla\begin{pmatrix} y_k \\ S_k\end{pmatrix}\right) \\ &\quad = \varepsilon\begin{pmatrix} -\Delta^2 y_k + \delta^{-2}\operatorname{div}(|\nabla y_k|^2\nabla y_k) - y_k e^{y_k/\delta} \\ 0\end{pmatrix} + \begin{pmatrix} 0 \\ \mu e^{y_k/\delta}-S_k\end{pmatrix},\end{aligned} \tag{3.1}$$

subject to the boundary conditions

$$\nabla y_k\cdot\nu = \nabla\Delta y_k\cdot\nu = \nabla S_k\cdot\nu = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

where the diffusion matrix is given by

$$D(y_k) = \begin{pmatrix} \delta^{-1}e^{y_k/\delta} & -e^{y_k/\delta} \\ e^{y_k/\delta} & 1\end{pmatrix}.$$

Here, $n_k := \exp(y_k/\delta)$ is an approximation of the cell density at time $k\tau$. In the first time step, we do not replace n_0 by $\exp(y_0/\delta)$. If y_k is a bounded function, the cell density is strictly positive. Due to the fourth-order term, we need the additional boundary condition $\nabla\Delta y_k\cdot\nu = 0$ on $\partial\Omega$. In the limit $\varepsilon\to 0$, we will loose this condition. The following proposition is also valid for three-dimensional domains Ω.

Proposition 3.1. *Let y_{k-1} be a measurable function such that $\exp(y_{k-1}) \in L^1(\Omega)$ and let $S_{k-1}\in L^2(\Omega)$. Then there exists a solution $(y_k,S_k)\in H^2(\Omega)\times H^1(\Omega)$ to (3.1)-(3.2).*

Proof. First, we solve a linearized problem. For this, let $(\bar y,\bar S)\in H^{7/4}(\Omega)\times L^2(\Omega)$ be given. In view of the Sobolev embedding $H^{7/4}(\Omega)\hookrightarrow L^\infty(\Omega)$ for $\Omega\subset\mathbb{R}^d$ ($d\le 3$), we have $\exp(\bar y/\delta)\in L^\infty(\Omega)$. We show the existence of a unique solution $(y,S)\in H^2(\Omega)\times H^1(\Omega)$ to the linear problem

$$a((y,S),(z,R)) = F(z,R) \quad \text{for all } (y,S),(z,R)\in H^2(\Omega)\times H^1(\Omega), \tag{3.3}$$

where

$$\begin{aligned} a((y,S),(z,R)) &= \int_\Omega \begin{pmatrix} \nabla z \\ \nabla R \end{pmatrix}^\top D(\bar y) \begin{pmatrix} \nabla y \\ \nabla S \end{pmatrix} dx \\ &\quad + \varepsilon \int_\Omega \big(\Delta y \Delta z + \delta^{-2}|\nabla \bar y|^2 \nabla y\cdot\nabla z + y e^{\bar y/\delta} z\big)dx + \int_\Omega SRdx, \\ F(z,R) &= -\frac{1}{\tau}\int_\Omega \begin{pmatrix} e^{\bar y/\delta} - e^{y_{k-1}/\delta} \\ \alpha(\bar S - S_{k-1}) \end{pmatrix} \cdot \begin{pmatrix} z \\ R \end{pmatrix} dx + \mu \int_\Omega e^{\bar y/\delta} R dx. \end{aligned}$$

The diffusion coefficients are bounded. Moreover, in view of the embedding $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$ for $\Omega\subset\mathbb{R}^d$ ($d\le 3$), we can estimate

$$\int_\Omega |\nabla \bar y|^2 \nabla y\cdot\nabla z dx \le \|\nabla \bar y\|^2_{L^4(\Omega)}\|\nabla y\|_{L^4(\Omega)}\|\nabla z\|_{L^4(\Omega)} \le C(\bar y)\|y\|_{H^2(\Omega)}\|z\|_{H^2(\Omega)}.$$

This proves the continuity of the bilinear form $a : (H^2(\Omega)\times H^1(\Omega))^2 \to \mathbb{R}$. The functional $F : H^2(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is continuous, too. Moreover, a is coercive since

$$\begin{aligned} a((y,S),(y,S)) &= \int_\Omega \big(\delta^{-1}e^{\bar y/\delta}|\nabla y|^2 + |\nabla S|^2\big)dx \\ &\quad + \varepsilon\int_\Omega \big((\Delta y)^2 + \delta^{-2}|\nabla \bar y|^2|\nabla y|^2 + y^2 e^{\bar y/\delta}\big)dx + \int_\Omega S^2 dx \\ &\ge C(\varepsilon)\|y\|^2_{H^2(\Omega)} + \|S\|^2_{H^1(\Omega)} \end{aligned}$$

for some $C(\varepsilon) > 0$. By the Lax-Milgram lemma, there exists a unique solution $(y,S)\in H^2(\Omega)\times H^1(\Omega)$ to (3.3).

Now we turn to the nonlinear problem which we solve by applying the Leray-Schauder fixed-point theorem (see Theorem B.5 in [36]). For given $(\bar y,\bar S)\in H^{7/4}(\Omega)\times L^2(\Omega)$ and $\sigma\in[0,1]$, we define the fixed-point operator $B : H^{7/4}(\Omega)\times L^2(\Omega)\times[0,1]\to H^{7/4}(\Omega)\times L^2(\Omega)$ by $B(\bar y,\bar S;\sigma) = (y,S)$, where $(y,S)\in H^2(\Omega)\times H^1(\Omega)$ is the solution to the linear problem

$$a((y,S),(z,R)) = \sigma F(z,R) \quad \text{for all } (z,R)\in H^2(\Omega)\times H^1(\Omega). \tag{3.4}$$

We notice that $B(\bar y,\bar S;0) = (0,0)$ for all $(\bar y,\bar S)\in H^{7/4}(\Omega)\times L^2(\Omega)$. Standard arguments prove that B is continuous and, because of the compact embeddings $H^2(\Omega)\hookrightarrow H^{7/4}(\Omega)$ and $H^1(\Omega)\hookrightarrow L^2(\Omega)$ for $d\le 3$, also compact. It remains to show that there exists a constant $C>0$ such that for any $(y,S)\in H^{7/4}(\Omega)\times L^2(\Omega)$, $\sigma\in[0,1]$ satisfying $B(y,S;\sigma) = (y,S)$, the estimate $\|(y,S)\|_{H^{7/4}(\Omega)\times L^2(\Omega)}\le C$ holds.

In order to prove this bound, we use first the test function $(z,R) = (1,0)$ in (3.4) giving, with the elementary inequality $xe^x \ge e^x - 1$ for $x\ge 0$,

$$\begin{aligned} \int_\Omega e^{y/\delta}dx &= \int_\Omega e^{y_{k-1}/\delta}dx - \varepsilon\tau\int_\Omega y e^{y/\delta}dx \le \int_\Omega e^{y_{k-1}/\delta}dx + \varepsilon\tau\delta\int_\Omega (1-e^{y/\delta})dx \\ &\le \int_\Omega e^{y_{k-1}/\delta}dx + \varepsilon\tau\delta\,\mathrm{meas}(\Omega). \end{aligned}$$

Hence $\exp(y/\delta)$ is bounded in $L^1(\Omega)$ uniformly in $t>0$:

$$\int_\Omega e^{y/\delta}dx \le \int_\Omega e^{y_0/\delta}dx + \varepsilon\delta T\mathrm{meas}(\Omega). \tag{3.5}$$

Next, we employ the test function (y, S) in (3.4):

$$\frac{1}{\tau}\int_\Omega \big((e^{y/\delta} - e^{y_{k-1}/\delta})y + \alpha(S - S_{k-1})S\big)dx + 4\delta\int_\Omega |\nabla e^{y/2\delta}|^2 dx$$
$$+ \int_\Omega (|\nabla S|^2 + S^2)dx + \varepsilon\int_\Omega \big((\Delta y)^2 + \delta^{-2}|\nabla y|^4 + y^2 e^{y/\delta}\big)dx$$
$$= \sigma\mu\int_\Omega e^{y/\delta} S dx. \tag{3.6}$$

We estimate the right-hand side of (3.6) using the Hölder inequality and the Sobolev embedding theorem:

$$\mu\int_\Omega e^{y/\delta} S dx \le \mu\|e^{y/\delta}\|_{L^{6/5}(\Omega)}\|S\|_{L^6(\Omega)} \le \mu C_0\|e^{y/2\delta}\|^2_{L^{12/5}(\Omega)}\|S\|_{H^1(\Omega)}.$$

The Gagliardo-Nirenberg inequality (see p. 1034 in [41]) with $\theta = d/12$ and the Young inequality for $p_1 = 1/\theta$, $p_2 = 2/(1-2\theta)$, and $p_3 = 2$ imply that

$$\mu\int_\Omega e^{y/\delta} S dx \le \mu C_1\|e^{y/2\delta}\|^{2\theta}_{H^1(\Omega)}\|e^{y/2\delta}\|^{2(1-\theta)}_{L^2(\Omega)}\|S\|_{H^1(\Omega)}$$
$$\le \mu C_1\|e^{y/2\delta}\|^{2\theta}_{H^1(\Omega)}\|e^{y/\delta}\|^{1-\theta}_{L^1(\Omega)}\|S\|_{H^1(\Omega)}$$
$$\le 2\delta\|e^{y/2\delta}\|^2_{H^1(\Omega)} + C_2(\delta)\|e^{y/\delta}\|^{(2-2\theta)/(1-2\theta)}_{L^1(\Omega)} + \frac{1}{2}\|S\|^2_{H^1(\Omega)}.$$

The first and last term can be absorbed by the second and third integral in (3.6), respectively, using (3.5). The second term is bounded because of (3.5). Since $\phi(x) = x(\log x - 1)$ is convex, $\phi(x) - \phi(z) \le \phi'(x)(x - z)$ for all x, $z > 0$. This yields for $x = e^{y/\delta}$ and $z = e^{y_{k-1}/\delta}$

$$\int_\Omega (e^{y/\delta} - e^{y_{k-1}/\delta})y dx \ge \delta\int_\Omega \big(\phi(e^{y/\delta}) - \phi(e^{y_{k-1}/\delta})\big)dx.$$

Moreover,

$$\alpha\int_\Omega (S - S_{k-1})S dx \ge \frac{\alpha}{2}\int_\Omega (S^2 - S_{k-1}^2)dx.$$

Hence, (3.6) becomes

$$\frac{1}{\tau}(E_k - E_{k-1}) + 2\int_\Omega |\nabla e^{y/2\delta}|^2 dx + \frac{1}{2\delta}\|S\|^2_{H^1(\Omega)}$$
$$+ \frac{\varepsilon}{\delta}\int_\Omega \big((\Delta y)^2 + \delta^{-2}|\nabla y|^4 + y^2 e^{y/\delta}\big)dx \le C_3(\delta)\|e^{y/\delta}\|^{(2-2\theta)/(1-2\theta)}_{L^1(\Omega)}, \tag{3.7}$$

where

$$E_k = \int_\Omega \Big(\phi(e^{y_k/\delta}) + \frac{\alpha}{2\delta}S^2\Big)dx.$$

With the test function $(e^{-y/\delta}, 0)$ in (3.4), we obtain, after some elementary calculations,

$$\varepsilon\int_\Omega (-y)dx = -\frac{1}{\delta^2}\int_\Omega |\nabla y|^2 dx + \frac{\sigma}{\tau}\int_\Omega \big(1 - e^{(y_{k-1}-y)/\delta}\big)dx$$
$$- \frac{\varepsilon}{\delta}\int_\Omega e^{-y/\delta}\Big((\Delta y)^2 - \frac{1}{\delta}\Delta y|\nabla y|^2 + \frac{1}{\delta^2}|\nabla y|^4\Big)dx + \frac{1}{\delta}\int_\Omega \nabla S\cdot\nabla y dx.$$

The first term on the right-hand side is nonpositive. The second term is estimated from above by $\mathrm{meas}(\Omega)/\tau$, and the third integral can be written as the sum of two squares. Finally, we apply the Cauchy-Schwarz inequality to the the last term, showing that it is bounded. Observing that $\int_\Omega y dx$ is bounded (as a consequence of the discrete entropy estimate (3.7)), we have shown that y_k is uniformly bounded in $L^1(\Omega)$ (with a constant that depends on ε). Hence, by (3.5) and (3.7), this gives a uniform H^2-bound for y_k and an H^1-bound for S_k. The proposition is proved. □

Step 2: limit of vanishing approximation parameters. From now on, we require that $d = 2$. Let (y_k, S_k) be a solution to (3.1)-(3.2) and define $y^{(\tau)}(x,t) = y_k(x)$, $n^{(\tau)}(x,t) = n_k(x) = \exp(y_k(x)/\delta) > 0$, and $S^{(\tau)}(x,t) = S_k(x)$ for $x \in \Omega$ and $t \in ((k-1)\tau, k\tau]$. Then $(n^{(\tau)}, S^{(\tau)})$ solves the following problem:

$$\begin{aligned} D_\tau n^{(\tau)} &= \mathrm{div}(\nabla n^{(\tau)} - n^{(\tau)}\nabla S^{(\tau)}) - \varepsilon\big(\Delta^2 y^{(\tau)} \\ &\quad - \delta^{-2}\mathrm{div}(|\nabla y^{(\tau)}|^2\nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}\big), \qquad (3.8) \\ \alpha D_\tau S^{(\tau)} &= \Delta S^{(\tau)} + \delta\Delta n^{(\tau)} + \mu n^{(\tau)} - S^{(\tau)}, \qquad (3.9) \end{aligned}$$

subject to the boundary conditions

$$\nabla n^{(\tau)}\cdot\nu = \nabla\Delta y^{(\tau)}\cdot\nu = \nabla S^{(\tau)}\cdot\nu = 0 \quad \text{on } \partial\Omega,$$

where $D_\tau n^{(\tau)} = (n^{(\tau)} - \sigma_\tau n^{(\tau)})/\tau$ is the discrete time derivative and σ_τ denotes the shift operator $(\sigma_\tau n^{(\tau)})(t) = n^{(\tau)}(\cdot, t-\tau)$ for $\tau \le t < T$. We collect some a priori estimates uniform in ε and τ.

Estimate (3.5) can be reformulated as

$$\|n_k\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)} + \varepsilon\delta T\mathrm{meas}(\Omega), \quad k \in \{1,\ldots,K\}.$$

Using this estimate, we can solve the discrete entropy inequality (3.7) recursively:

$$\begin{aligned} &E_k + 2\tau\sum_{j=1}^k \|\nabla\sqrt{n_j}\|^2_{L^2(\Omega)} + \frac{\tau}{2\delta}\sum_{j=1}^k \|S_j\|^2_{H^1(\Omega)} \\ &\quad + \frac{\varepsilon\tau}{\delta}\sum_{j=1}^k\int_\Omega\big((\Delta y_j)^2 + \delta^{-2}|\nabla y_j|^4 + y_j^2 e^{y_j/\delta}\big)dx \le E_0 + \tau kC \le E_0 + TC, \end{aligned}$$

where $C > 0$ depends on δ, T, and the L^1-norm of n_0 (notice that $0 < \varepsilon < 1$). Hence, we have proved:

LEMMA 3.2. *The following bounds hold:*

$$\begin{aligned} &\|n^{(\tau)}\log n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|\sqrt{n^{(\tau)}}\|_{L^2(0,T;H^1(\Omega))} \\ &\qquad + \|S^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le C, \qquad (3.10) \\ &\sqrt{\varepsilon}\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)} + \sqrt[4]{\varepsilon}\|\nabla y^{(\tau)}\|_{L^4(\Omega_T)} + \sqrt{\varepsilon}\big\|y^{(\tau)}\sqrt{n^{(\tau)}}\big\|_{L^2(\Omega_T)} \le C, \qquad (3.11) \end{aligned}$$

where $C > 0$ is independent of ε and τ.

The following lemma is a consequence of the above uniform estimates.

LEMMA 3.3. *The following bound holds:*

$$\|n^{(\tau)}\|_{L^2(0,T;W^{1,1}(\Omega))} + \|n^{(\tau)}\|_{L^{4/3}(0,T;W^{1,4/3}(\Omega))} \le C, \qquad (3.12)$$

where $C > 0$ is independent of ε and τ.

Proof. We employ the Cauchy-Schwarz inequality:

$$\begin{aligned}\|\nabla n^{(\tau)}\|^2_{L^2(0,T;L^1(\Omega))} &= 4\int_0^T \|\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}\|^2_{L^1(\Omega)}dt \\ &\le 4\int_0^T \|\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)}\|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(\Omega)}dt \\ &\le 4\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}\|\nabla\sqrt{n^{(\tau)}}\|^2_{L^2(0,T;L^2(\Omega))} \le C,\end{aligned}$$

using (3.10). This shows the first estimate. Notice that this bound implies, because of the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ for $d = 2$, that $(n^{(\tau)})$ is bounded in $L^2(\Omega_T)$, where $\Omega_T = \Omega \times (0, T)$. Then the second bound follows from

$$\begin{aligned}\|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)} &= 2\|\sqrt{n^{(\tau)}}\nabla\sqrt{n^{(\tau)}}\|_{L^{4/3}(\Omega_T)} \\ &\le 2\|\sqrt{n^{(\tau)}}\|_{L^4(\Omega_T)}\|\nabla\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)} \le C,\end{aligned}$$

which finishes the proof. □

Next, we show that the discrete time derivatives of $n^{(\tau)}$ and $S^{(\tau)}$ are uniformly bounded.

Lemma 3.4. *It holds for any $\eta > 0$:*

$$\|D_\tau n^{(\tau)}\|_{L^1(0,T;(H^{2+\eta}(\Omega))')} + \alpha\|D_\tau S^{(\tau)}\|_{L^{4/3}(0,T;(W^{1,4}(\Omega))')} \le C, \tag{3.13}$$

where $C > 0$ does not depend on ε and τ.

Proof. Let $\eta > 0$ and $\phi \in L^\infty(0, T; H^{2+\eta}(\Omega))$. By Sobolev embedding, it holds $\phi \in L^\infty(0, T; W^{1,\infty}(\Omega))$. Then, by (3.8) and Hölder's inequality,

$$\begin{aligned}\int_0^T \langle D_\tau n^{(\tau)}, \phi\rangle dt &= -\int_0^T\int_\Omega \big(\nabla n^{(\tau)}\cdot\nabla\phi - n^{(\tau)}\nabla S^{(\tau)}\cdot\nabla\phi\big)dx\,dt \\ &\quad + \varepsilon\int_0^T\int_\Omega \big(\Delta y^{(\tau)}\Delta\phi + \delta^{-2}|\nabla y^{(\tau)}|^2\nabla y^{(\tau)}\cdot\nabla\phi + y^{(\tau)}n^{(\tau)}\phi\big)dx\,dt \\ &\le \|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\ &\quad + \|n^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla S^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla\phi\|_{L^\infty(\Omega_T)} \\ &\quad + \varepsilon\|\Delta y^{(\tau)}\|_{L^2(\Omega_T)}\|\Delta\phi\|_{L^2(\Omega_T)} + \varepsilon\delta^{-2}\|\nabla y^{(\tau)}\|^3_{L^4(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\ &\quad + \varepsilon\big\|y^{(\tau)}\sqrt{n^{(\tau)}}\big\|_{L^2(\Omega_T)}\big\|\sqrt{n^{(\tau)}}\big\|_{L^4(\Omega_T)}\|\phi\|_{L^4(\Omega_T)} \\ &\le C\big(\|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|\phi\|_{L^\infty(0,T;H^2(\Omega))}\big) \\ &\le C\|\phi\|_{L^\infty(0,T;(H^{2+\eta}(\Omega))')},\end{aligned}$$

using (3.10)-(3.12). Here, the brackets $\langle\cdot,\cdot\rangle$ denote the dual product in the corresponding spaces. In a similar way, using (3.9),

$$\begin{aligned}\alpha\int_0^T \langle D_\tau S^{(\tau)}, \phi\rangle dt &= -\int_0^T\int_\Omega \big(\nabla S^{(\tau)}\cdot\nabla\phi + \delta\nabla n^{(\tau)}\cdot\nabla\phi - \mu n^{(\tau)}\phi + S^{(\tau)}\phi\big)dx\,dt \\ &\le \|\nabla S^{(\tau)}\|_{L^2(\Omega_T)}\|\nabla\phi\|_{L^2(\Omega_T)} + \delta\|\nabla n^{(\tau)}\|_{L^{4/3}(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\ &\quad + \big(\mu\|n^{(\tau)}\|_{L^2(\Omega_T)} + \|S^{(\tau)}\|_{L^2(\Omega_T)}\big)\|\phi\|_{L^2(\Omega_T)} \\ &\le C\|\phi\|_{L^4(0,T;W^{1,4}(\Omega))},\end{aligned}$$

which shows the lemma. □

Taking into account (3.10), (3.12), and (3.13), Aubin's lemma [33, Thm. 6] (or using Lemma A.2 in [9]) provides the existence of subsequences of $(n^{(\tau)})$ and $(S^{(\tau)})$, which are not relabeled, such that, as $(\varepsilon, \tau) \to 0$,

$$\begin{aligned} n^{(\tau)} \to n \quad & \text{strongly in } L^2(0,T;L^p(\Omega)),\ p<2, \\ S^{(\tau)} \to S \quad & \text{strongly in } L^2(0,T;L^q(\Omega)),\ q<\infty,\ \text{if } \alpha \neq 0. \end{aligned}$$

Here, we have used the compactness of the embeddings $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \le p < 2$ and $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$ in two-dimensional domains. Notice that the positivity of $n^{(\tau)} = \exp(y^{(\tau)}/\delta)$ implies the nonnegativity of the limit function n. The estimates (3.10) and (3.12) show that, up to subsequences,

$$\begin{aligned} \nabla n^{(\tau)} \rightharpoonup \nabla n \quad & \text{weakly in } L^{4/3}(0,T;L^{4/3}(\Omega)), \\ \nabla S^{(\tau)} \rightharpoonup \nabla S \quad & \text{weakly in } L^2(0,T;L^2(\Omega)), \end{aligned}$$

and (3.13) implies that, if $\alpha \neq 0$,

$$D_\tau S^{(\tau)} \rightharpoonup S_t \quad \text{weakly in } L^{4/3}(0,T;(H^2(\Omega))').$$

Unfortunately, these results do not allow us to pass to the limit in the term $n^{(\tau)}\nabla S^{(\tau)}$. However, we are able to exploit the boundedness of $n^{(\tau)} \log n^{(\tau)}$ in L^1. Indeed, Proposition (2.1) shows that, up to a subsequence,

$$n^{(\tau)} \to n \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Hence, we find that

$$n^{(\tau)}\nabla S^{(\tau)} \rightharpoonup n\nabla S \quad \text{weakly in } L^1(0,T;L^1(\Omega)). \tag{3.14}$$

Since, by (3.11), for all $\phi \in L^4(0,T;H^2(\Omega))$,

$$\begin{aligned} &\left|\varepsilon\int_0^T \langle \Delta^2 y^{(\tau)} - \delta^{-2}\mathrm{div}(|\nabla y^{(\tau)}|^2\nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}, \phi\rangle dt\right| \\ &\quad \le \varepsilon\big(\|y^{(\tau)}\|_{L^2(0,T;H^2(\Omega))}\|\phi\|_{L^2(0,T;H^2(\Omega))} + \delta^{-2}\|\nabla y^{(\tau)}\|^3_{L^4(\Omega_T)}\|\nabla\phi\|_{L^4(\Omega_T)} \\ &\qquad + \big\|y^{(\tau)}\sqrt{n^{(\tau)}}\big\|_{L^2(\Omega_T)}\big\|\sqrt{n^{(\tau)}}\big\|_{L^4(\Omega_T)}\|\phi\|_{L^4(\Omega_T)}\big) \\ &\quad \le C(\varepsilon^{1/2} + \varepsilon^{1/4})\|\phi\|_{L^4(0,T;H^2(\Omega))}, \end{aligned}$$

we infer that

$$\varepsilon\big(\Delta^2 y^{(\tau)} - \mathrm{div}(|\nabla y^{(\tau)}|^2\nabla y^{(\tau)}) + y^{(\tau)}n^{(\tau)}\big) \rightharpoonup 0 \quad \text{weakly in } L^{4/3}(0,T;(H^2(\Omega))').$$

Observing (3.8), this result and (3.14) imply

$$D_\tau n^{(\tau)} \rightharpoonup \mathrm{div}(\nabla n - n\nabla S) \quad \text{weakly in } L^1(0,T;(H^{2+\eta}(\Omega))').$$

Since $D_\tau n^{(\tau)} \to n_t$ in the sense of distributions, this gives

$$n_t = \mathrm{div}(\nabla n - n\nabla S) \quad \text{in } L^1(0,T;(H^{2+\eta}(\Omega))').$$

Due to the boundedness of the right-hand side we can now employ a density argument to extend this equation to the space $L^1(0,T;(W^{1,\infty}(\Omega))')$.

The limit $(\varepsilon, \tau) \to 0$ in the linear equation (3.9) leads to

$$\alpha S_t = \Delta S + \delta \Delta n + \mu n - S \quad \text{in } L^2(0, T; (W^{1,\infty}(\Omega))').$$

Since $(D_\tau n^{(\tau)})$ is bounded in $W^{1,1}(0, T; (W^{1,\infty}(\Omega))') \hookrightarrow C^0([0, T]; (W^{1,\infty}(\Omega))')$, the initial datum $n(\cdot, 0) = n_0$ holds in the sense of $(W^{1,\infty}(\Omega))'$.

The above proof also works in the case $\alpha = 0$. Indeed, we loose the estimate for $(S^{(\tau)})$ in $L^\infty(0, T; L^2(\Omega))$ and the estimate on the time derivative of $S^{(\tau)}$. Both bounds are not required in the convergence arguments since we only need weak convergence for $(S^{(\tau)})$. This proves the existence result for $\alpha \geq 0$.

Step 3: proof of the regularity results. We show (1.7) and (1.8). We employ $(n^{(\tau)})^{\gamma-1}$ with $0 < \gamma < 1/2$ as a test function in the weak formulation of (3.8). This is possible since $n^{(\tau)} = \exp(y^{(\tau)}/\delta)$ is strictly positive and, on each subinterval $((k-1)\tau, k\tau]$, an element of $H^1(\Omega)$. Then

$$\begin{aligned} \frac{4(1-\gamma)}{\gamma^2} \|\nabla (n^{(\tau)})^{\gamma/2}\|^2_{L^2(\Omega_T)} &= (1-\gamma) \int_0^T \int_\Omega |\nabla n^{(\tau)}|^2 (n^{(\tau)})^{\gamma-2} dx\, dt \\ &= \int_0^T \int_\Omega \Big(D_\tau n^{(\tau)} (n^{(\tau)})^{\gamma-1} + \frac{1-\gamma}{\gamma} \nabla (n^{(\tau)})^\gamma \cdot \nabla S^{(\tau)} \Big) dx\, dt \\ &\quad + \varepsilon \int_0^T \int_\Omega \big(\Delta y^{(\tau)} \Delta (n^{(\tau)})^{\gamma-1} + \delta^{-2} |\nabla y^{(\tau)}|^2 \nabla y^{(\tau)} \cdot \nabla (n^{(\tau)})^{\gamma-1} \big) dx\, dt \\ &\quad + \varepsilon \int_0^T \int_\Omega y^{(\tau)} (n^{(\tau)})^\gamma dx\, dt. \end{aligned} \tag{3.15}$$

Using the concavity of the function $x \mapsto x^\gamma$, we infer that

$$\begin{aligned} \int_0^T \int_\Omega D_\tau n^{(\tau)} (n^{(\tau)})^{\gamma-1} dx\, dt &= \sum_{k=1}^K \int_\Omega (n_k - n_{k-1}) n_k^{\gamma-1} dx \\ &\leq \frac{1}{\gamma} \sum_{k=1}^K \int_\Omega (n_k^\gamma - n_{k-1}^\gamma) dx = \frac{1}{\gamma} \int_\Omega \big((n^{(\tau)}(\cdot, T))^\gamma - n_0^\gamma \big) dx, \end{aligned}$$

which is uniformly bounded by (3.10). We know from (3.10) that $(\nabla S^{(\tau)})$ is bounded in $L^2(\Omega_T)$. Hence, in order to show that the integral

$$\int_0^T \int_\Omega \nabla (n^{(\tau)})^\gamma \cdot \nabla S^{(\tau)} dx\, dt \leq \frac{1}{2} \|\nabla (n^{(\tau)})^\gamma\|^2_{L^2(\Omega_T)} + \frac{1}{2} \|\nabla S^{(\tau)}\|^2_{L^2(\Omega_T)}$$

is uniformly bounded, it remains to control $(\nabla (n^{(\tau)})^\gamma)$ in $L^2(\Omega_T)$. For this, we observe that, using the elementary inequality $x^{2\gamma-1} \leq \eta x^{\gamma-1} + C(\eta)$ for all $x > 0$ and $\eta > 0$ (here we need the assumption $\gamma < 1/2$),

$$\begin{aligned} |\nabla (n^{(\tau)})^\gamma|^2 &= \gamma^2 |\nabla n^{(\tau)}|^2 (n^{(\tau)})^{-1} (n^{(\tau)})^{2\gamma-1} \\ &\leq \gamma^2 |\nabla n^{(\tau)}|^2 (n^{(\tau)})^{-1} \big(\eta (n^{(\tau)})^{\gamma-1} + C(\eta) \big) \\ &= 4\eta\gamma^2 |\nabla (n^{(\tau)})^{\gamma/2}|^2 + 4C(\eta)\gamma^2 |\nabla \sqrt{n^{(\tau)}}|^2. \end{aligned}$$

For sufficiently small $\eta > 0$, the first term can be absorbed by the left-hand side of (3.15). Furthermore, the integral over the second term is uniformly bounded, by (3.10). This shows that the first integral on the right-hand side of (3.15) is uniformly bounded. We proceed by estimating the second integral in (3.15). To this end, we rewrite this integral as

$$
\begin{aligned}
&\frac{\varepsilon(\gamma-1)}{\delta}\int_0^T\int_\Omega (n^{(\tau)})^{\gamma-1}\Big((\Delta y^{(\tau)})^2+\frac{\gamma-1}{\delta}|\nabla y^{(\tau)}|^2\Delta y^{(\tau)}+\frac{1}{\delta^2}|\nabla y^{(\tau)}|^4\Big)dx\,dt \\
&\quad = -\frac{\varepsilon(1-\gamma)}{\delta}\int_0^T\int_\Omega (n^{(\tau)})^{\gamma-1}\bigg(\Big(\frac12(\gamma-1)\Delta y^{(\tau)}+\frac1\delta|\nabla y^{(\tau)}|^2\Big)^2 \\
&\qquad + \Big(1-\frac14(1-\gamma)^2\Big)(\Delta y^{(\tau)})^2\bigg)\,dx\,dt\le 0.
\end{aligned}
$$

The last integral in (3.15) is estimated as follows:

$$
\delta\varepsilon\int_0^T\int_\Omega \log(n^{(\tau)})(n^{(\tau)})^\gamma dx \le \delta\varepsilon C(1+\|n^{(\tau)}\|^2_{L^2(\Omega_T)})\le \delta\varepsilon C.
$$

This shows that

$$
\|\nabla(n^{(\tau)})^{\gamma/2}\|_{L^2(\Omega_T)}\le C(\gamma)\big(\|\nabla\sqrt{n^{(\tau)}}\|_{L^2(\Omega_T)}+\|\nabla S^{(\tau)}\|_{L^2(\Omega_T)}+1\big).
$$

The regularity (1.7) follows with $\beta=\gamma/2<1/4$.

Next, we choose the (admissible) test function $1/n^{(\tau)}$ in the weak formulation of (3.8):

$$
\begin{aligned}
\frac{1}{\delta^2}\|\nabla y^{(\tau)}\|^2_{L^2(\Omega)} &= \int_0^T\int_\Omega \frac{|\nabla n^{(\tau)}|^2}{(n^{(\tau)})^2}dx\,dt \\
&= \int_0^T\int_\Omega \Big(\frac{D_\tau n^{(\tau)}}{n^{(\tau)}}+\frac{1}{n^{(\tau)}}\nabla S^{(\tau)}\cdot\nabla n^{(\tau)}\Big)dx\,dt \\
&\quad + \varepsilon\int_0^T\int_\Omega\Big(\Delta y^{(\tau)}\Delta\frac{1}{n^{(\tau)}}+\delta^{-2}|\nabla y^{(\tau)}|^2\nabla y^{(\tau)}\cdot\nabla\frac{1}{n^{(\tau)}}\Big)dx\,dt \\
&\quad + \varepsilon\int_0^T\int_\Omega y^{(\tau)}dx\,dt.
\end{aligned}
\tag{3.16}
$$

We estimate the right-hand side term by term. The elementary inequality $1-x\le -\log x$ leads to

$$
\begin{aligned}
\int_0^T\int_\Omega \frac{D_\tau n^{(\tau)}}{n^{(\tau)}}dx\,dt &= \sum_{k=1}^K\int_\Omega\Big(1-\frac{n_{k-1}}{n_k}\Big)dx\le\sum_{k=1}^K\int_\Omega\big(\log n_{k-1}-\log n_k\big)dx \\
&= \int_\Omega\big(\log n^{(\tau)}(x,T)-\log n_0(x)\big)dx \\
&\le \int_\Omega (n^{(\tau)}(x,T)-1-\log n_0(x))dx\le C,
\end{aligned}
$$

since $n^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$, by (3.10). Furthermore,

$$
\int_0^T\int_\Omega \frac{1}{n^{(\tau)}}\nabla S^{(\tau)}\cdot\nabla n^{(\tau)}dx\,dt\le\frac{1}{2\delta}\|\nabla y^{(\tau)}\|^2_{L^2(\Omega_T)}+\frac12\|\nabla S^{(\tau)}\|^2_{L^2(\Omega_T)}.
$$

The first term can be absorbed by the left-hand side of (3.16), the second term is uniformly bounded, by (3.10). This shows that the first integral on the right-hand side of (3.16) can be controlled. The second integral is nonnegative since it is equal to

$$-\frac{\varepsilon}{\delta}\int_0^T\int_\Omega\frac{1}{n^{(\tau)}}\Big((\Delta y^{(\tau)})^2-\frac{1}{\delta}\Delta y^{(\tau)}|\nabla y^{(\tau)}|^2+\frac{1}{\delta^2}|\nabla y^{(\tau)}|^4\Big)dx\,dt$$
$$=-\frac{\varepsilon}{\delta}\int_0^T\int_\Omega\frac{1}{n^{(\tau)}}\left(\Big(\frac12\Delta y^{(\tau)}-\frac{1}{\delta}|\nabla y^{(\tau)}|^2\Big)^2+\frac34(\Delta y^{(\tau)})^2\right)dx\,dt\le 0.$$

Finally, the last integral is estimated as follows:

$$\varepsilon\int_0^T\int_\Omega y^{(\tau)}dx\,dt=\varepsilon\delta\int_0^T\int_\Omega\log n^{(\tau)}dx\,dt\le\varepsilon\delta\|n^{(\tau)}\|_{L^1(\Omega_T)}.$$

Thus, $\nabla\log n^{(\tau)}$ is uniformly bounded in $L^2(\Omega_T)$ which proves (1.8).

4. The parabolic-elliptic system.

In this section, we prove Theorem 1.2 (Section 4.1) and Theorem 1.3 (Section 4.2).

4.1. Existence of solutions.

Taking $\alpha=0$ in the Keller-Segel system (1.3)-(1.4), we obtain the parabolic-elliptic system

$$n_t=\operatorname{div}(\nabla n-n\nabla S), \tag{4.1}$$
$$0=\Delta S+\delta\Delta n+\mu n-S,\quad x\in\Omega,\ t>0, \tag{4.2}$$

supplemented by the Neumann boundary conditions and the initial condition (1.1)-(1.2) with $\alpha=0$. The proof of Theorem 1.1 covers the case $\alpha=0$. However, this yields only low regularity. Therefore, we present an alternative proof of the existence of solutions to (1.1)-(1.2), (4.1)-(4.2) with bounded cell densities, preventing overcrowding of cells. As mentioned in the introduction, the idea is to define the new variable $v=S+\delta n$. Then the parabolic-elliptic system can be written as

$$n_t=\operatorname{div}\big((1+\delta n)\nabla n-n\nabla v\big), \tag{4.3}$$
$$0=\Delta v+(\mu+\delta)n-v,\quad x\in\Omega,\ t>0, \tag{4.4}$$

together with the boundary and initial conditions

$$\nabla n\cdot\nu=\nabla v\cdot\nu=0\quad\text{on }\partial\Omega,\ t>0,\quad n(\cdot,0)=n_0\quad\text{in }\Omega. \tag{4.5}$$

We consider the regularized system (3.1) with $\alpha=0$. Written in the variables $n_k=\exp(y_k/\delta)\in H^2(\Omega)$ and $v_k=S_k+\delta n_k\in H^1(\Omega)$, this system reads as

$$\frac{1}{\tau}(n_k-n_{k-1})=\operatorname{div}\big((1+\delta n_k)\nabla n_k-n_k\nabla v_k\big)$$
$$-\varepsilon\big(\Delta^2y_k-\delta^{-2}\operatorname{div}(|\nabla y_k|^2\nabla y_k)+y_kn_k\big), \tag{4.6}$$
$$0=\Delta v_k+(\mu+\delta)n_k-v_k\quad\text{in }\Omega. \tag{4.7}$$

The existence of solutions (n_k,v_k) is guaranteed by the proof of Theorem 1.1.

Step 1: nonnegativity of v. The test function $v_k^-=\min\{0,v_k\}$ in (4.7) leads to

$$\int_\Omega|\nabla v_k^-|^2dx=(\mu+\delta)\int_\Omega n_kv_k^-dx-\int_\Omega(v_k^-)^2dx\le 0,$$

since $n_k>0$. Hence, $v_k\ge 0$ in Ω and $v^{(\tau)}\ge 0$ in Ω_T. Performing the limit $(\varepsilon,\tau)\to 0$, this implies that $v\ge 0$ in Ω_T.

Step 2: regularity $n \in L^2(0,T;H^1(\Omega))$. Next, we use $\log n_k = y_k/\delta$ as a test function in the weak formulation of (4.6) and n_k as a test function in the weak formulation of (4.7). Summing both equations, the nonlinear term in $\nabla n_k \cdot \nabla v_k$ cancels and we end up with

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})\log n_k dx + \int_\Omega (1+\delta n_k)\frac{|\nabla n_k|^2}{n_k}dx \\ = -\frac{\varepsilon}{\delta}\int_\Omega \big((\Delta y_k)^2 + \delta^{-2}|\nabla y_k|^4 + y_k^2 n_k\big)dx + (\mu+\delta)\int_\Omega n_k^2 dx - \int_\Omega n_k v_k dx \\ \le (\mu+\delta)\int_\Omega n_k^2 dx.\end{aligned}$$

As in the proof of Theorem 1.1, we can write this inequality as

$$\frac{1}{\tau}\int_\Omega \big(n_k(\log n_k - 1) - n_{k-1}(\log n_{k-1} - 1)\big)dx + \delta\int_\Omega |\nabla n_k|^2 dx \le (\mu+\delta)\int_\Omega n_k^2 dx.$$

Solving this recursive inequality gives, with the notation of Section 3,

$$\begin{aligned}\int_\Omega n^{(\tau)}(\cdot,t)\big(\log n^{(\tau)}(\cdot,t) - 1\big)dx + \int_0^t\int_\Omega |\nabla n^{(\tau)}|^2 dx\, ds \\ \le \int_\Omega n_0(\log n_0 - 1)dx + (\mu+\delta)\int_0^t\int_\Omega (n^{(\tau)})^2 dx\, ds.\end{aligned}$$

In view of (3.12) and the continuous embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ for $d \le 2$, the right-hand side is uniformly bounded. This shows (after performing the limit $(\varepsilon,\tau) \to 0$) that $n \in L^2(0,T;H^1(\Omega))$.

Step 3: regularity $n \in L^\infty(0,T;L^2(\Omega))$. We employ n_k in the weak formulation of (4.6),

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k - n_{k-1})n_k dx + \int_\Omega (1+\delta n_k)|\nabla n_k|^2 dx \\ = \frac{1}{2}\int_\Omega \nabla v_k \cdot \nabla n_k^2 dx - \frac{\varepsilon}{\delta}\int_\Omega n_k\big((\Delta y_k)^2 + \delta^{-2}|\nabla y_k|^4 + \delta^{-1}\Delta y_k|\nabla y_k|^2\big)dx \\ - \varepsilon\int_\Omega y_k n_k^2 dx,\end{aligned}$$

and $n_k^2/2$ in the weak formulation of (4.7),

$$\frac{1}{2}\int_\Omega \nabla v_k \cdot \nabla n_k^2 dx = \frac{1}{2}(\mu+\delta)\int_\Omega n_k^3 dx - \frac{1}{2}\int_\Omega v_k n^2 dx.$$

Summing both equations, the expression in $\nabla v_k \cdot \nabla n_k^2$ cancels and, using the nonnegativity of v_k and the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (n_k^2 - n_{k-1}^2)dx + \frac{4\delta}{9}\int_\Omega |\nabla n_k^{3/2}|^2 dx \\ \le -\frac{\varepsilon}{2\delta}\int_\Omega n_k\big((\Delta y_k)^2 + \delta^{-2}|\nabla y_k|^4\big)dx - \varepsilon\delta\int_\Omega n_k^2\log n_k dx + \frac{1}{2}(\mu+\delta)\int_\Omega n_k^3 dx.\end{aligned}$$

Solving this recursive inequality, we end up with

$$\int_\Omega (n^{(\tau)})^2(\cdot,t)dx + \frac{4}{9}\delta\int_0^t\int_\Omega |\nabla(n^{(\tau)})^{3/2}|^2 dx\,ds \\ \le \int_\Omega n_0^2 dx - \varepsilon\delta\int_0^t\int_\Omega (n^{(\tau)})^2\log n^{(\tau)}dx\,ds + \frac{1}{2}(\mu+\delta)\int_0^t\int_\Omega (n^{(\tau)})^3 dx\,ds.$$

The second integral on the right-hand side is uniformly bounded since $x\mapsto -x^2\log x$ has an upper bound in $[0,\infty)$. The third integral is estimated by using the Gagliardo-Nirenberg inequality with $\theta = 2/3$:

$$\int_0^T\int_\Omega (n^{(\tau)})^3 dx = \int_0^T \|n^{(\tau)}\|^3_{L^3(\Omega)}dt \le C\int_0^T \|n^{(\tau)}\|^{3\theta}_{H^1(\Omega)}\|n^{(\tau)}\|^{3(1-\theta)}_{L^1(\Omega)}dt \\ \le C\|n^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}\|n^{(\tau)}\|^2_{L^2(0,T;H^1(\Omega))},$$

and this is bounded by (3.10) and Step 2. Hence, letting $(\varepsilon,\tau)\to 0$, we have $n\in L^\infty(0,T;\,L^2(\Omega))$.

Step 4: regularity $n\in L^\infty(0,T;L^3(\Omega))$. We employ the test function n_k^2 in the weak formulation of (4.6) and $2n_k^3/3$ in the weak formulation of (4.7) and sum both equations:

$$\frac{1}{\tau}\int_\Omega (n_k-n_{k-1})n_k^2 dx + 2\int_\Omega (1+\delta n_k)n_k|\nabla n_k|^2 dx \\ = -\frac{2\varepsilon}{\delta}\int_\Omega n_k^2\Big((\Delta y_k)^2 + \frac{2}{\delta}\Delta y_k|\nabla y_k|^2 + \frac{1}{\delta^2}|\nabla y_k|^4\Big)dx - \varepsilon\int_\Omega y_k n_k^3 dx \\ + \frac{2}{3}(\mu+\delta)\int_\Omega n_k^4 dx - \frac{2}{3}\int_\Omega v_k n_k^3 dx.$$

The first term on the left-hand side is bounded from above by

$$\frac{1}{3\tau}\int_\Omega (n_k^3 - n_{k-1}^3)dx.$$

The first integral on the right-hand side is nonnegative since it is equal to

$$-\frac{2\varepsilon}{\delta}\int_\Omega n_k^2\Big((\Delta y_k)^2 + \frac{1}{\delta}|\nabla y_k|^2\Big)^2 dx.$$

We estimate the second integral on the right-hand side as above. Furthermore, the fourth integral is nonnegative. Hence, we obtain

$$\frac{1}{3}\int_\Omega (n^{(\tau)})^3(\cdot,t)dx + \frac{\delta}{2}\int_0^t\int_\Omega |\nabla n_k^2|^2 dx\,ds \\ \le \frac{1}{3}\int_\Omega n_0^3 dx + \varepsilon C + \frac{2}{3}(\mu+\delta)\int_0^t \|n^{(\tau)}\|^4_{L^4(\Omega)}ds.$$

Estimate (3.11) and Step 3 show that the second integral on the right-hand side is uniformly bounded. In order to estimate the last integral, we employ the Gagliardo-Nirenberg inequality with $\theta = 1/2$:

$$\int_0^T \|n^{(\tau)}\|^4_{L^4(\Omega)}dt \le C\int_0^T \|n^{(\tau)}\|^{4\theta}_{H^1(\Omega)}\|n^{(\tau)}\|^{4(1-\theta)}_{L^2(\Omega)}dt \\ \le C\|n^{(\tau)}\|^2_{L^2(0,T;H^1(\Omega))}\|n^{(\tau)}\|^2_{L^\infty(0,T;L^2(\Omega))} \le C,$$

by Steps 2 and 3. This proves $n \in L^\infty(0,T;L^3(\Omega))$.

Step 5: regularity $v \in L^\infty(0,T;W^{1,\infty}(\Omega))$. In view of the regularity $n^{(\tau)} \in L^\infty(0,T;\,L^3(\Omega))$, standard elliptic theory implies that the problem

$$-\Delta v^{(\tau)} + v^{(\tau)} = (\mu+\delta)n^{(\tau)} \quad \text{in } \Omega, \quad \nabla v^{(\tau)} \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

has a unique solution satisfying $v^{(\tau)} \in L^\infty(0,T;W^{2,3}(\Omega))$. Here, we need the assumption $\partial\Omega \in C^{1,1}$. By Sobolev embedding, we find that $v^{(\tau)} \in L^\infty(0,T;W^{1,\infty}(\Omega))$.

Step 6: regularity $L^\infty(0,T;L^\infty(\Omega))$. This step is a consequence of the following proposition.

PROPOSITION 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $T>0$, $u_0 \in L^\infty(\Omega)$, $\delta > 0$, and $V \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Then there exists a unique weak solution u to*

$$u_t - \operatorname{div}((1+\delta u)\nabla u) = -\operatorname{div}(u\nabla V) \quad in\ \Omega,\ t>0, \tag{4.8}$$
$$((1+\delta u)\nabla u - u\nabla V)\cdot\nu = 0 \quad on\ \partial\Omega, \quad u(\cdot,0)=u_0 \quad in\ \Omega, \tag{4.9}$$

such that, for some constant $C>0$, depending on Ω and the L^∞ norm of ∇V,

$$\|u\|_{L^\infty(\Omega_T)} \le C\max\big\{1, \|u_0\|_{L^\infty(\Omega)}\big\}, \quad 0<t\le T.$$

A proof of the above result is sketched in [20] for $\delta = 0$ and with an L^∞ bound which depends on the L^1 norm of u_0.

Proof. Let $u_R = \max\{u,R\}$, where $R>0$ is a constant to be determined. Then a standard fixed-point argument proves that the quasilinear problem with bounded coefficients

$$u_t - \operatorname{div}\big((1+\delta u_R)\nabla u\big) = -\operatorname{div}(u\nabla V) \quad \text{in } \Omega,\ t>0, \tag{4.10}$$

with boundary and initial conditions (4.9) possesses a weak solution in $L^2(0,T;\,H^1(\Omega))$. Notice that u conserves mass.

It remains to prove the L^∞ bound independent of R such that we can remove the index R in (4.10). This can be done by using (a variant of) u^p as a test function and deriving a recursive differential inequality. Since u^p is not an admissible test function, we have to truncate. Let $0<k<K$ with $k=\|u_0\|_{L^\infty(\Omega)}$, $p\ge 1$, and set $u_K = \min\{u,K\}$. Then $\phi(u) = [(u_K-k)^+]^p \in L^2(0,T;H^1(\Omega))$ since this function is bounded and $u \in L^2(0,T;H^1(\Omega))$. Hence, $\phi(u)$ is an admissible test function in the weak formulation of (4.10). Observing that

$$\Phi(u) = \int_0^u \phi(z)dz \ge \frac{1}{p+1}[(u_K-k)^+]^{p+1}$$

and, using $\phi(u_0)=0$,

$$\int_0^t \langle u_t, \phi(u)\rangle ds = \int_\Omega \big(\Phi(u(\cdot,t)) - \Phi(u_0)\big)dx \ge \frac{1}{p+1}\int_\Omega [(u_K(\cdot,t)-k)^+]^{p+1}dx,$$

we conclude from (4.10) that

$$\begin{aligned}\frac{1}{p+1}\int_\Omega[(u_K(\cdot,t)-k)^+]^{p+1}dx&\\ +p\int_0^t\int_\Omega(1+\delta u_R)[(u_K-k)^+]^{p-1}|\nabla(u_K-k)^+|^2dx\,ds&\\ =p\int_0^t\int_\Omega(u-k)[(u_K-k)^+]^{p-1}\nabla V\cdot\nabla(u_K-k)^+dx\,ds&\\ +kp\int_0^t\int_\Omega[(u_K-k)^+]^{p-1}\nabla V\cdot\nabla(u_K-k)^+dx\,ds.&\end{aligned}$$

Since $u-k=(u_K-k)^+$ on $\{k<u<K\}$, we can write

$$\begin{aligned}\frac{1}{p+1}\int_\Omega[(u_K(\cdot,t)-k)^+]^{p+1}dx&\\ +\frac{4p}{(p+1)^2}\int_0^t\int_\Omega|\nabla[(u_K-k)^+]^{(p+1)/2}|^2dx\,ds&\\ \le\frac{2p}{p+1}\int_0^t\int_\Omega[(u_K-k)^+]^{(p+1)/2}\nabla V\cdot\nabla[(u_K-k)^+]^{(p+1)/2}dx\,ds&\\ +\frac{2kp}{p+1}\int_0^t\int_\Omega[(u_K-k)^+]^{(p-1)/2}\nabla V\cdot\nabla[(u_K-k)^+]^{(p+1)/2}dx\,ds.&\end{aligned}$$

Applying the Cauchy-Schwarz inequality and absorbing the gradient terms by the second integral on the left-hand side, we infer that

$$\begin{aligned}\frac{1}{p+1}\int_\Omega[(u_K(\cdot,t)-k)^+]^{p+1}dx+\frac{2p}{(p+1)^2}\int_0^t\int_\Omega|\nabla[(u_K-k)^+]^{(p+1)/2}|^2dx\,ds&\\ \le p\|\nabla V\|^2_{L^\infty(\Omega_T)}\int_0^t\int_\Omega[(u_K-k)^+]^{p+1}dx\,ds&\\ +k^2p\|\nabla V\|^2_{L^\infty(\Omega_T)}\int_0^t\int_\Omega[(u_K-k)^+]^{p-1}dx\,ds.&\end{aligned}$$

We apply $a^{p-1}\le(p-1)a^{p+1}/(p+1)+2/(p+1)$ (which follows from the Young inequality) to the last integral:

$$\begin{aligned}k^2p\|\nabla V\|^2_{L^\infty(\Omega_T)}\int_0^t\int_\Omega[(u_K-k)^+]^{p-1}dx\,ds&\\ \le\frac{p(p-1)}{p+1}C_1(V)\int_0^t\int_\Omega[(u_K-k)^+]^{p+1}dx\,ds+\frac{2p}{p+1}C_2,&\end{aligned}$$

and hence,

$$\int_\Omega[(u_K(\cdot,t)-k)^+]^{p+1}dx\le p(p+1)C_1(V)\int_0^t\int_\Omega[(u_K-k)^+]^{p+1}dx\,ds+pC_2,$$

where $C_1>0$ depends on k and the L^∞ norm of ∇V, and $C_2>0$ depends on k, T, and Ω. By the Gronwall lemma,

$$\int_\Omega[(u_K(\cdot,t)-k)^+]^{p+1}dx\le pC_2e^{p(p+1)C_1(V)t},\quad t\ge0.$$

Since the right-hand side does not depend on $K > 0$, we can perform the limit $K \to \infty$ and obtain

$$\|(u(\cdot,t)-k)^+\|_{L^{p+1}(\Omega)} \le (pC_2)^{1/(p+1)} e^{pC_1(V)t}, \quad t \ge 0.$$

This shows that $u \in L^\infty(0,T;L^p(\Omega))$ for all $p < \infty$. Unfortunately, we cannot perform the limit $p \to \infty$ since the right-hand side is not unformly bounded in p.

Since $u_t - \mathrm{div}((1+\delta u_R)\nabla u) = -\mathrm{div}(u\nabla V) \in L^\infty(0,T;(W^{1,p}(\Omega))')$ for all $p < \infty$, maximal regularity implies that $u \in L^p(0,T;W^{1,p}(\Omega))$ (see, e.g., [16]). This, together with the regularity result $u \in L^\infty(0,T;L^p(\Omega))$ for all $p < \infty$, proves that $u^p \in L^2(0,T;H^1(\Omega))$ is an admissible test function in the weak formulation of (4.10). Estimating as above, we find that

$$\frac{d}{dt}\int_\Omega u^{p+1}dx + \frac{2p}{p+1}\int_\Omega |\nabla u^{(p+1)/2}|^2 dx \le p(p+1)C_1(V)\int_\Omega u^{p+1}dx,$$

where $C_1(V) > 0$ depends on the L^∞ norms of ∇V and n_0 and on T and Ω. Now, we can proceed exactly as in [20] (and similarly as in [27]) to deduce a set of recursive inequalities in the spirit of Alikakos [1] to prove the theorem. □

Next, we prove the global boundedness of n and v. We introduce the functional

$$F(t) = \int_\Omega \Big(h(n) - nv + \frac{1}{2(\mu+\delta)}(|\nabla v|^2 + v^2)\Big)dx,$$

where $h(n) = n(\log n - 1) + \delta n^2/2$. A straightforward computation shows that F is decreasing along the solution trajectories, $F(t) \le F(0)$ for $t \ge 0$. Proceeding as in the proof of Theorem 4.1 in [8], the entropy estimate implies that $\int_\Omega h(n)dx$ and $\int_\Omega |\nabla v|^2 dx$ are uniformly bounded in time. The superlinearity of h shows the time equi-integrability of n which implies $L^\infty(0,\infty;L^p(\Omega))$ estimates for n for all $p < \infty$. By elliptic regularity, we deduce that $v \in L^\infty(0,\infty;W^{1,\infty}(\Omega))$. From Lemma 4.1 in [27], which is based on the iteration technique of Alikakos, we find that $n \in L^\infty(0,\infty;L^\infty(\Omega))$.

Finally, the uniqueness of solutions to (1.1)-(1.2), (4.3)-(4.4) can be proved using the standard dual method. This completes the proof of Theorem 1.2.

4.2. Long-time behavior of solutions. We begin by recalling that a homogeneous steady state of (1.1)-(1.2) and (4.1)-(4.2) is given by $n^* = M/\mathrm{meas}(\Omega)$ and $S^* = \mu n^*$, where $M = \int_\Omega n_0 dx$ is the total mass. We define the relative entropy

$$E^*(t) = \int_\Omega \Big(n\log\frac{n}{n^*}\Big)(x,t)dx \ge 0.$$

The following lemma is a consequence of the relative entropy inequality.

Lemma 4.2. *Under the assumptions of Theorem 1.3, it holds*

$$\nabla\sqrt{n} \in L^2(0,\infty;L^2(\Omega)), \quad S - S^* \in L^2(0,\infty;H^1(\Omega)).$$

Proof. Since $\log n$ may be not integrable, we define the regularized entropy

$$E_\varepsilon(t) = \int_\Omega \Big((n+\varepsilon)\log\frac{n+\varepsilon}{n^*}\Big)(x,t)dx, \quad \varepsilon > 0.$$

We employ the test function $\log(n+\varepsilon) \in L^2(0,T;H^1(\Omega))$ in the weak formulation of (1.3):

$$\frac{dE_\varepsilon}{dt} = \langle n_t, \log(n+\varepsilon)\rangle = -4\int_\Omega |\nabla\sqrt{n+\varepsilon}|^2 dx + 2\int_\Omega \frac{n}{\sqrt{n+\varepsilon}}\nabla S\cdot\nabla\sqrt{n+\varepsilon}dx. \quad (4.11)$$

Furthermore, employing the test function $(S-S^*)/\delta$ in the equation

$$0 = \Delta(S-S^*) + \delta\Delta(n-n^*) + \mu(n-n^*) - (S-S^*),$$

we find that

$$\begin{aligned} 0 &= -\frac{1}{\delta}\int_\Omega |\nabla(S-S^*)|^2 dx - 2\int_\Omega \sqrt{n+\varepsilon}\nabla S\cdot\nabla\sqrt{n+\varepsilon}dx \\ &\quad + \frac{\mu}{\delta}\int_\Omega (n-n^*)(S-S^*)dx - \frac{1}{\delta}\int_\Omega (S-S^*)^2 dx. \end{aligned} \quad (4.12)$$

Adding (4.11) and (4.12) gives

$$\begin{aligned} &\frac{dE_\varepsilon}{dt} + 4\|\nabla\sqrt{n+\varepsilon}\|^2_{L^2(\Omega)} + \frac{1}{\delta}\|S-S^*\|^2_{H^1(\Omega)} \\ &\quad = \frac{\mu}{\delta}\int_\Omega (n-n^*)(S-S^*)dx - 2\int_\Omega \frac{\varepsilon}{\sqrt{n+\varepsilon}}\nabla S\cdot\nabla\sqrt{n+\varepsilon}dx. \end{aligned} \quad (4.13)$$

Since $\int_\Omega((n+\varepsilon)-(n^*+\varepsilon))dx = 0$, we can apply the Poincaré inequality

$$\|n-n^*\|_{L^{4/3}(\Omega)} = \|(n+\varepsilon)-(n^*+\varepsilon)\|_{L^{4/3}(\Omega)} \le C(\Omega)\|\nabla(n+\varepsilon)\|_{L^1(\Omega)}$$

to the first integral at the right-hand side of (4.13):

$$\begin{aligned} \frac{\mu}{\delta}\int_\Omega (n-n^*)(S-S^*)dx &\le \frac{\mu}{\delta}\|n-n^*\|_{L^{4/3}(\Omega)}\|S-S^*\|_{L^4(\Omega)} \\ &\le \frac{\mu}{\delta}C(\Omega)\|\nabla(n+\varepsilon)\|_{L^1(\Omega)}\|S-S^*\|_{H^1(\Omega)}. \end{aligned}$$

Using

$$\begin{aligned} \|\nabla(n+\varepsilon)\|_{L^1(\Omega)} &\le 2\|\sqrt{n+\varepsilon}\|_{L^2(\Omega)}\|\nabla\sqrt{n+\varepsilon}\|_{L^2(\Omega)} \\ &= 2\sqrt{M+\varepsilon\mathrm{meas}(\Omega)}\|\nabla\sqrt{n+\varepsilon}\|_{L^2(\Omega)}, \end{aligned}$$

we proceed by applying the Young inequality:

$$\begin{aligned} \frac{\mu}{\delta}\int_\Omega (n-n^*)(S-S^*)dx &\le C(\Omega)\frac{\mu^2}{\delta}(M+\varepsilon\mathrm{meas}(\Omega))\|\nabla\sqrt{n+\varepsilon}\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{2\delta}\|S-S^*\|^2_{H^1(\Omega)}. \end{aligned}$$

Thus, (4.13) becomes

$$\begin{aligned} &\frac{dE_\varepsilon}{dt} + \Big(4 - C(\Omega)\frac{\mu^2}{\delta}(M+\varepsilon\mathrm{meas}(\Omega))\Big)\|\nabla\sqrt{n+\varepsilon}\|^2_{L^2(\Omega)} + \frac{1}{2\delta}\|S-S^*\|^2_{H^1(\Omega)} \\ &\quad \le 2\int_\Omega \frac{\varepsilon}{\sqrt{n+\varepsilon}}\nabla S\cdot\nabla\sqrt{n+\varepsilon}dx \le 2\sqrt{\varepsilon}\|\nabla S\|_{L^2(\Omega)}\|\nabla\sqrt{n+\varepsilon}\|_{L^2(\Omega)}. \end{aligned}$$

The limit $\varepsilon \to 0$ gives

$$\frac{dE^*}{dt} + \Big(4 - C(\Omega)\frac{\mu^2}{\delta}M\Big)\|\nabla\sqrt{n}\|_{L^2(\Omega)}^2 + \frac{1}{2\delta}\|S - S^*\|_{H^1(\Omega)}^2 \le 0. \tag{4.14}$$

This limit can be justified by first integrating the above inequality in $(t, t+h)$, performing the limit $\varepsilon \to 0$, and then passing to the limit $h \to 0$ to obtain the differentiated expression. The lemma follows if $C(\Omega)\mu^2 M/\delta < 4$. □

In order to prove the first part of Theorem 1.3, we choose $\mu > 0$ and $\delta > 0$ such that $C(\Omega)\mu^2 M/\delta < 4$ and use the logarithmic Sobolev inequality

$$E^* = \int_\Omega n \log\frac{n}{n^*}dx \le K(\Omega)\int_\Omega |\nabla\sqrt{n}|^2 dx.$$

This shows that

$$\frac{dE^*}{dt} + \Big(4 - C(\Omega)\frac{\mu^2}{\delta}M\Big)K(\Omega)^{-1}E^* \le 0, \quad t > 0.$$

Gronwall's lemma concludes the exponential decay of $E^*(t)$. The L^1 decay of $n - n^*$ is an immediate consequence of the Csiszár-Kullback inequality [37]. For the decay of $S - S^*$, we observe first that, by interpolation and the global L^∞ bound for n,

$$\|n(\cdot,t) - n^*\|_{L^2(\Omega)} \le \|n(\cdot,t) - n^*\|_{L^\infty(\Omega)}^{1/2}\|n(\cdot,t) - n^*\|_{L^1(\Omega)}^{1/2} \le C_1 e^{-\kappa t/4}.$$

Hence, using $n - n^*$ as a test function in the weak formulation of (4.4) and applying Young's inequality to the right-hand side,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla(v - v^*)|^2 dx + \int_\Omega (v - v^*)^2 dx &\le \frac{1}{2}(\mu+\delta)^2\int_\Omega (n - n^*)^2 dx \\ &\le \frac{1}{2}(\mu+\delta)^2 C_1^2 e^{-\kappa t/2}.\end{aligned}$$

Since $S - S^* = (v - v^*) - \delta(n - n^*)$, the theorem follows.

5. Numerical simulation. We compare the solutions of the classical Keller-Segel model with those from the augmented Keller-Segel system with additional cross diffusion. The simulations have been carried out using the COMSOL Multiphysics package with quadratic finite elements. The numerical solutions are for illustration only; a more detailed comparison is the subject of future work. We choose a circular domain Ω with radius $r = 2$ and

$$\begin{aligned}&\mu = 1, \quad M = 9\pi, \\ &n_0(x,y) = 60\big((x^3 - 0.3y^2)(x^2 + y^2 - 1)^2 + 0.15\big), \quad S_0(x,y) = 0.\end{aligned}$$

The initial cell density n_0 is illustrated in Figure 5.1. In all figures, we report the maximal numerical values for the cell density $n_{\max}$ for comparison. Since $M > 8\pi$ and the domain is radially symmetric, we expect that the cell density blows up in finite time and that blow up happens at the domain boundary; see [21, Thm. 4.9] for the parabolic-elliptic case.

First, we present simulations using the parabolic-parabolic model. Figure 5.2 shows the cell density at time $t = 1.4$ for the classical Keller-Segel model and at times $t = 1.4$ and $t = 1000$ for the augmented Keller-Segel model with two different

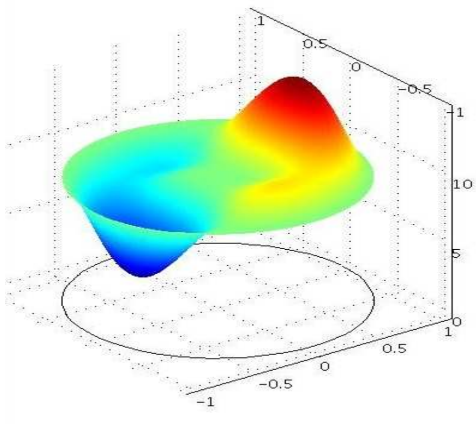

FIG. 5.1. *Initial cell density,* $n_{\max} = 15$.

values of δ. The cell density of the classical model can be computed numerically up to time $t = 1.4$ at which a peak forms. For $t > 1.4$, the numerical scheme breaks down (and the continuous solution blows up). In contrast, the solution of the augmented Keller-Segel model exists for all time as predicted by the theoretical results. At $t = 1000$, the cell density corresponds to a steady state solution. The effect of the cross-diffusion parameter δ is a smoothing of the density; the value of the cell density peak decreases with increasing values of δ. The numerical results indicate that the cell density remains globally bounded; the proof of the global boundedness in the parabolic-parabolic model is an open problem.

For comparison, the numerical solutions to the parabolic-elliptic model are shown in Figure 5.3. The blow-up time in the classical model is approximately $t = 0.47$. Again, we observe the smoothing effect of the additional cross diffusion term. At $t = 1000$, the maximal values of the cell density computed from the parabolic-parabolic and the parabolic-elliptic model almost coincide. This is clear since the steady states of the parabolic-parabolic and parabolic-elliptic models are the same. Furthermore, it seems that the augmented Keller-Segel model allows for smooth nonhomogeneous steady states.

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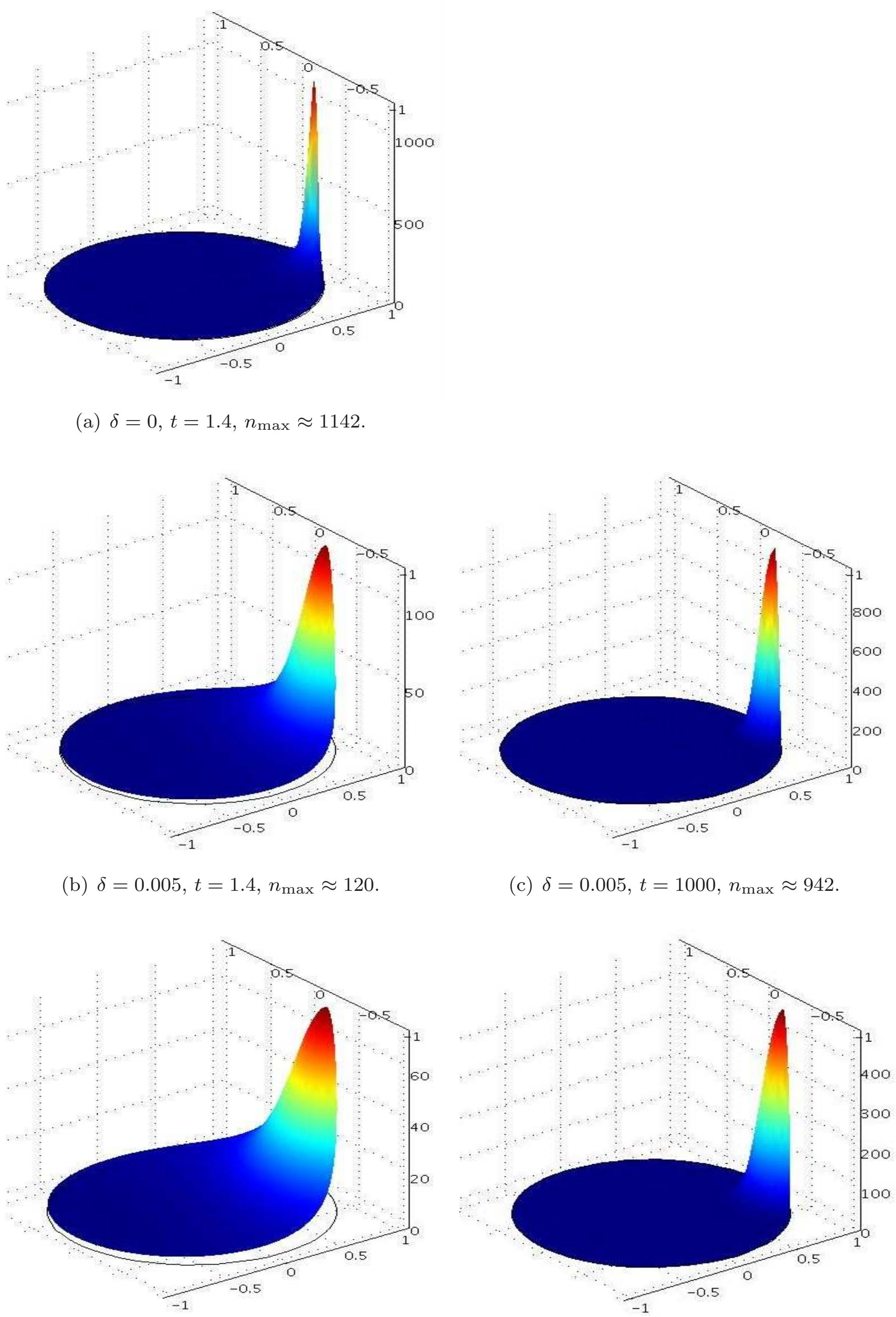


(a) $\delta = 0$, $t = 1.4$, $n_{\max} \approx 1142$.

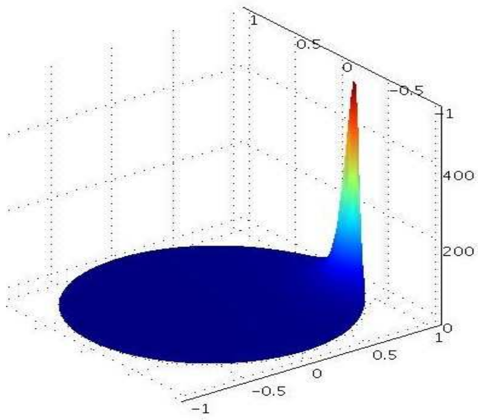
(b) $\delta = 0.005$, $t = 1.4$, $n_{\max} \approx 120$.

(c) $\delta = 0.005$, $t = 1000$, $n_{\max} \approx 942$.

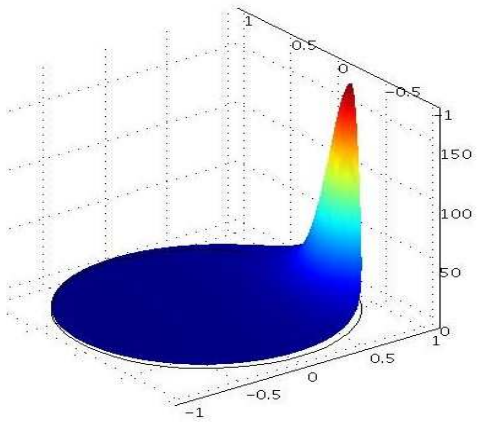
(d) $\delta = 0.01$, $t = 1.4$, $n_{\max} \approx 71$.

(e) $\delta = 0.01$, $t = 1000$, $n_{\max} \approx 468$.

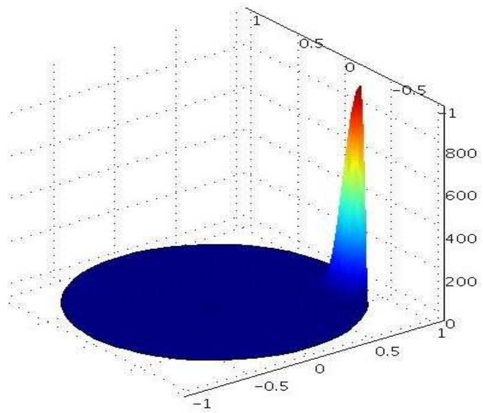
FIG. 5.2. *Cell density computed from the parabolic-parabolic Keller-Segel model.*



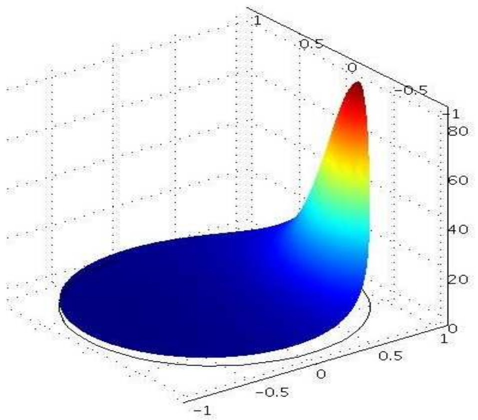
(a) $\delta = 0$, $t = 0.47$, $n_{\max} \approx 535$.



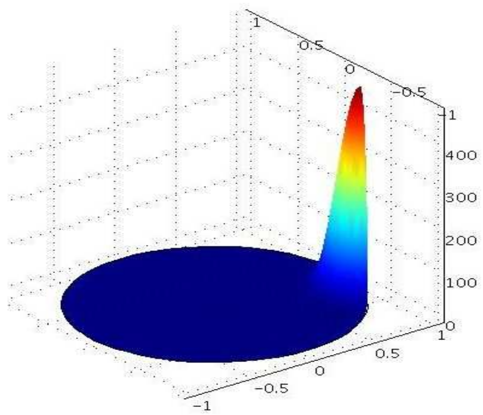
(b) $\delta = 0.005$, $t = 0.47$, $n_{\max} \approx 175$.



(c) $\delta = 0.005$, $t = 1000$, $n_{\max} \approx 942$.



(d) $\delta = 0.01$, $t = 0.47$, $n_{\max} \approx 83$.



(e) $\delta = 0.01$, $t = 1000$, $n_{\max} \approx 470$.

FIG. 5.3. *Cell density computed from the parabolic-elliptic Keller-Segel model.*

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