ERRATUM: ANALYSIS OF DEGENERATE CROSS-DIFFUSION POPULATION MODELS WITH VOLUME FILLING

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Abstract. This note corrects Lemma 7 in [1] on the positive (semi-) definiteness of a certain matrix product, which yields a priori estimates for the cross-diffusion system.

1. Introduction

In our paper [1], we proved the global-in-time existence of bounded weak solutions to a certain class of degenerate cross-diffusion systems for the particle densities $u(x,t) = (u_1,\ldots,u_n)$, where $x \in \Omega \subset \mathbb{R}^d$ is the spatial variable and $t \geq 0$ is the time. The proof is based on an entropy method, i.e., we introduced a scalar functional $H[u] = \int_\Omega h(u)dx$ (called an entropy), which turns out to be not only a Lyapunov functional along the solutions but it also provides gradient estimates. A crucial step of the proof is the observation that the product between the Hessian $H := h''(u) \in \mathbb{R}^{n\times n}$ and the diffusion matrix $A = A(u) \in \mathbb{R}^{n\times n}$ is positive definite (non-uniformly in u). The proof of this observation (Lemma 7 in [1]) is wrong. In this note, we will correct the proof.

We introduce the hypertriangle

$$\mathscr{D} = \left\{ u \in \mathbb{R}^n : u_i > 0 \text{ for } i = 1,\ldots,n,\ \sum_{j=1}^n u_j < 1 \right\}.$$

The matrix coefficients of $A(u)$ contain nonlinear functions (see (3) in [1]) for which the following structural hypotheses have been imposed: There exist functions $q : [0,1] \to \mathbb{R}$, $\chi : \overline{\mathscr{D}} \to \mathbb{R}$ and a number $\gamma > 0$ such that for all $i = 1,\ldots,n$,

$$q(s) := q_i(s) > 0, \quad q'(s) \geq \gamma q(s) \text{ for } s \in (0,1), \quad q(0) = 0, \quad q \in C^3([0,1]), \tag{1}$$

$$p_i(u) = \exp\left(\frac{\partial \chi(u)}{\partial u_i}\right) \text{ for } u \in \mathscr{D}, \quad \chi \geq 0 \text{ is convex on } \overline{\mathscr{D}}, \quad \chi \in C^3(\overline{\mathscr{D}}), \tag{2}$$

and p_i is assumed to be positive on $\overline{\mathscr{D}}$. We introduce the following nonnegative number:

$$\kappa = \sup_{u\in\mathscr{D}} \sup_{\substack{z\in\mathbb{R}^n,\\ |z|=1}} \left(\sum_{i,j=1}^n \sqrt{u_i u_j} \frac{\partial^2 \chi}{\partial u_i \partial u_j} z_i z_j \right)^2. \tag{3}$$

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The following result replaces Lemma 7 in [1].

Lemma 1. *Assume that* (1)-(2) *hold. Let $\eta \in (0,1]$ be any number such that $\eta\kappa < 1$, where κ is defined in* (3)*. Then it holds for all $u \in \mathscr{D}$ and $v \in \mathbb{R}^n$ that*

$$v^\top (HA)v \geq p_0 c_1 q(u_{n+1}) \sum_{i=1}^n \frac{v_i^2}{u_i} + p_0 c_2 \frac{q'(u_{n+1})^2}{q(u_{n+1})} \bigg(\sum_{i=1}^n v_i \bigg)^2,$$

where $p_0 := \min_{i=1,\ldots,n} \inf_{u\in\mathscr{D}} p_i(u) > 0$,

$$c_1 = 1 - \eta\kappa > 0, \quad c_2 = \min\left\{ \frac{\eta}{4q(1/2)}, \frac{2}{\sup_{1/2\leq\sigma\leq 1} q'(\sigma)} \right\} > 0.$$

2. Proof of Lemma 1

Let $u = (u_i) \in \mathscr{D}$ and set $\varphi = q'/q$. It is shown in the proof of Lemma 7 in [1] that

$$\begin{aligned} \frac{1}{q}(HA)_{ij} &= \delta_{ij}\frac{p_i}{u_i} + \frac{\partial p_i}{\partial u_j} + \frac{\partial p_j}{\partial u_i} + \sum_{k=1}^n \frac{u_k}{p_k}\frac{\partial p_k}{\partial u_i}\frac{\partial p_k}{\partial u_j} \\ &\quad + \varphi\bigg(p_i + p_j + \sum_{k=1}^n u_k \bigg(\frac{\partial p_k}{\partial u_i} + \frac{\partial p_k}{\partial u_j} \bigg)\bigg) + \varphi^2 \sum_{k=1}^n u_k p_k. \end{aligned}$$

Observing that $\partial p_i/\partial u_j = p_i \partial^2\chi/\partial u_i \partial u_j$ and setting $\chi_{ij} = \partial^2\chi/\partial u_i\partial u_j$, the previous identity can be formulated as

$$\begin{aligned} \frac{1}{q}(HA)_{ij} &= \delta_{ij}\frac{p_i}{u_i} + (p_i + p_j)\chi_{ij} + \sum_{k=1}^n u_k p_k \chi_{ki}\chi_{kj} \\ &\quad + \varphi\bigg(p_i + p_j + \sum_{k=1}^n u_k p_k (\chi_{ki} + \chi_{kj}) \bigg) + \varphi^2 \sum_{k=1}^n u_k p_k \\ &=: I_{ij} + \varphi J_{ij} + \varphi^2 K_{ij}. \end{aligned}$$

Let $v \in \mathbb{R}^n$ and define $w_i = v_i/\sqrt{u_i}$. First, we reformulate the quadratic forms associated to $I = (I_{ij})$, $J = (J_{ij})$, and $K = (K_{ij})$:

$$\begin{aligned} v^\top I v &= \sum_{i=1}^n \frac{p_i}{u_i} v_i^2 + 2\sum_{i,j=1}^n p_i \chi_{ij} v_i v_j + \sum_{k=1}^n u_k p_k \bigg(\sum_{i=1}^n \chi_{ki} v_i \bigg)^2 \\ &= \sum_{i=1}^n p_i w_i^2 + 2\sum_{i,j=1}^n p_i \sqrt{u_i u_j}\chi_{ij} w_i w_j + \sum_{i=1}^n p_i \bigg(\sum_{j=1}^n \sqrt{u_i u_j}\chi_{ij} w_j \bigg)^2 \\ &= \sum_{i=1}^n p_i \bigg(w_i + \sum_{j=1}^n \sqrt{u_i u_j}\chi_{ij} w_j \bigg)^2, \end{aligned}$$

$$v^\top J v = 2\bigg(\sum_{k=1}^n v_k \bigg)\bigg(\sum_{i=1}^n p_i v_i + \sum_{i,j=1}^n u_i p_i \chi_{ij} v_j \bigg) = 2\sum_{i,k=1}^n p_i v_i v_k + 2\sum_{i,j,k=1}^n u_i p_i \chi_{ij} v_j v_k$$

$$= \sum_{i=1}^n p_i \bigg\{ 2\sum_{k=1}^n \sqrt{u_iu_k} w_k \bigg(w_i + \sum_{j=1}^n \sqrt{u_iu_j}\chi_{ij}w_j \bigg) \bigg\},$$
$$v^\top K v = \sum_{i=1}^n p_i u_i \bigg(\sum_{j=1}^n v_j \bigg)^2 = \sum_{i=1}^n p_i \bigg(\sum_{j=1}^n \sqrt{u_iu_j} w_j \bigg)^2.$$

By definition of p_0, we deduce that

$$\frac{1}{p_0 q} v^\top (HA) v \geq \sum_{i=1}^n \bigg(w_i + \sum_{j=1}^n \sqrt{u_iu_j}\chi_{ij}w_j + \varphi \sum_{j=1}^n \sqrt{u_iu_j} w_j \bigg)^2.$$

This shows that HA is positive semidefinite.

Next, we set $M_{ij} = \sqrt{u_iu_j}\chi_{ij}$ and $N_{ij} = \varphi\sqrt{u_iu_j}$. Then

$$(p_0q)^{-1} v^\top (HA) v \geq |w + Mw + Nw|^2,$$

where $w = (w_i)$, $M = (M_{ij})$, $N = (N_{ij})$. We employ the fact that M is symmetric positive semidefinite:

$$\begin{aligned}(p_0q)^{-1} v^\top (HA) v &= |w|^2 + 2w^\top (M+N) w + |Mw + Nw|^2 \\ &\geq |w|^2 + 2w^\top N w + \eta |Mw + Nw|^2,\end{aligned}$$

where $\eta \in (0,1]$ is arbitrary. By definition of κ, $|Mw|^2 \leq \kappa |w|^2$, and thus, $|Mw + Nw|^2 \geq \frac{1}{2}|Nw|^2 - |Mw|^2 \geq \frac{1}{2}|Nw|^2 - \kappa|w|^2$. We conclude that

$$(p_0q)^{-1} v^\top (HA) v \geq (1 - \eta\kappa)|w|^2 + 2w^\top N w + \frac{\eta}{2}|Nw|^2.$$

Since $\sum_{i=1}^n u_i = 1 - u_{n+1}$, we have

$$|w|^2 = \sum_{i=1}^n \frac{v_i^2}{u_i}, \quad w^\top N w = \varphi \bigg(\sum_{j=1}^n v_j \bigg)^2, \quad |Nw|^2 = \varphi^2 (1 - u_{n+1}) \bigg(\sum_{j=1}^n v_j \bigg)^2,$$

and consequently,

$$(p_0q)^{-1} v^\top (HA) v \geq (1 - \eta\kappa) \sum_{i=1}^n \frac{v_i^2}{u_i} + \varphi \bigg(2 + \frac{\eta}{2}(1 - u_{n+1})\varphi \bigg) \bigg(\sum_{j=1}^n v_j \bigg)^2.$$

This estimate replaces (25) in [1].

Now, we proceed similarly as in the proof of Lemma 7 in [1]. The inequalities

$$\begin{aligned} &2 + \frac{\eta}{2}(1-s)\varphi(s) \geq \frac{\eta}{2}(1-s)\varphi(s) \geq \frac{\eta}{4}\frac{q'(s)}{q(1/2)} && \text{for } 0 \leq s \leq \frac{1}{2}, \\ &2 + \frac{\eta}{2}(1-s)\varphi(s) \geq 2 \geq \frac{2q'(s)}{\sup_{1/2 \leq \sigma \leq 1} q'(\sigma)} && \text{for } \frac{1}{2} \leq s \leq 1 \end{aligned}$$

imply that $2 + \frac{\eta}{2}(1 - u_{n+1})\varphi(u_{n+1}) \geq c_2 q'(u_{n+1})$, which shows the conclusion.

References

[1] N. Zamponi and A. Jüngel. Analysis of degenerate cross-diffusion population models with volume filling. *Arch. Rat. Mech. Anal.*, 2016.

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