GLOBAL WEAK SOLUTIONS FOR A NONLOCAL MULTISPECIES FOKKER–PLANCK–LANDAU SYSTEM

JINGWEI HU, ANSGAR JÜNGEL, AND NICOLA ZAMPONI

Abstract. The global-in-time existence of weak solutions to a spatially homogeneous multispecies Fokker–Planck–Landau system for plasmas in the three-dimensional whole space is shown. The Fokker–Planck–Landau system is a simplification of the Landau equations assuming a linearized, velocity-independent, and isotropic kernel. The resulting equations depend nonlocally and nonlinearly on the moments of the distribution functions via the multispecies local Maxwellians. The existence proof is based on a three-level approximation scheme, energy and entropy estimates, as well as compactness results, and it holds for both soft and hard potentials.

1. Introduction

The Fokker–Planck–Landau equations describe the local collisional relaxation process of the particle distribution functions in plasmas under binary collisions [1]. In this paper, we investigate a multispecies, linearized, spatially homogeneous version of these equations. More preciely, the distribution functions $f_i(v,t)$ of the ith species of the multicomponent plasma, depending on the velocity $v \in \mathbb{R}^3$ and time $t \geq 0$, are assumed to satisfy the initial-value problem

$$\partial_t f_i = \sum_{j=1}^{s} c_{ji} \operatorname{div}\left(\nabla f_i + m_i \frac{v - u_{ji}}{T_{ji}} f_i\right) \quad \text{in } \mathbb{R}^3,\ t > 0, \tag{1}$$

$$f_i(\cdot, 0) = f_i^0 \quad \text{in } \mathbb{R}^3,\ i = 1, \ldots, s, \tag{2}$$

where $s \in \mathbb{N}$ is the number of species and $m_i > 0$ the molar mass of the ith species. Before defining the quantities c_{ji}, u_{ji}, and T_{ji}, we introduce the first moments of f_i, namely the

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number density n_i, partial velocity u_i, and partial temperature T_i by

$$n_i = \int_{\mathbb{R}^3} f_i \mathrm{d}v, \quad u_i = \frac{1}{n_i}\int_{\mathbb{R}^3} f_i v \mathrm{d}v, \quad T_i = \frac{m_i}{3n_i}\int_{\mathbb{R}^3} f_i |v-u_i|^2 \mathrm{d}v, \tag{3}$$

as well as the partial mass density $\rho_i = m_i n_i$. Then the diffusion coefficients c_{ji} and "multispecies" velocities u_{ji} and temperatures T_{ji} are given by

$$c_{ji} = \frac{|\log\Lambda| q_i^2 q_j^2}{8\pi\varepsilon_0^2 m_i^2} n_j \left(\frac{T_j}{m_j}\right)^{\gamma/2}, \tag{4}$$

$$u_{ji} = \frac{c_{ji} m_i \rho_i u_i + c_{ij} m_j \rho_j u_j}{c_{ji} m_i \rho_i + c_{ij} m_j \rho_j}, \tag{5}$$

$$T_{ji} = \frac{c_{ji}\rho_i T_i + c_{ij}\rho_j T_j}{c_{ji}\rho_i + c_{ij}\rho_j} + \frac{c_{ji} m_i \rho_i c_{ij} m_j \rho_j |u_i - u_j|^2}{3(c_{ji}\rho_i + c_{ij}\rho_j)(c_{ji} m_i \rho_i + c_{ij} m_j \rho_j)}, \tag{6}$$

where $\log\Lambda > 0$ is the Coulomb logarithm, $\Lambda > 0$ being related to the Debye length, ε_0 is the vacuum permittivity, q_i is the charge of the ith species, and $\gamma \in \mathbb{R}$ models the interaction strength between particles (see Section 2 for details).

Note that c_{ji}, u_{ji}, and T_{ji} are functions of time only, and they depend in a nonlocal and nonlinear way on the distribution functions. We write $c_{ji}[f] = c_{ji}$, $u_{ji}[f] = u_{ji}$, and $T_{ji}[f] = T_{ji}$ with $f = (f_1, \ldots, f_s)$ to make this dependence clear. Observe that the symmetries $T_{ij} = T_{ji}$ and $u_{ji} = u_{ij}$ for $j \neq i$ hold as well as $T_{ii} = T_i$ and $u_{ii} = u_i$.

Single-species kinetic Fokker–Planck equations have been mathematically studied in the literature since the 1980s; see, e.g., [4]. One main interest was the proof of hypocoercivity [6, 14]. There are only a few works concerned with multispecies models. The diffusion limit of a kinetic Fokker–Planck system for charged particles towards the Nernst–Planck equations was proved in [15]. Furthermore, in [7, 11], the limit of vanishing electron–ion mass ratios for nonhomogeneous kinetic Fokker–Planck systems was investigated. The multispecies modeling in [7] is very close to ours, but the model of [7] also includes spatial and electric effects. However, an existence analysis of multispecies Fokker–Planck systems is missing in the literature. In this paper, up to our knowledge, we provide such an analysis for the first time.

Equations (1)–(6) are a simplification of the Fokker–Planck–Landau system (see Section 2). In this context, the right-hand side of (1) can be interpreted as the collision operator

$$Q_{ji}(f_i) = \sum_{j=1}^{s} c_{ji} \operatorname{div}\left(\nabla f_i + m_i \frac{v - u_{ji}}{T_{ji}} f_i\right).$$

Our model satisfies some physical properties, like mass, momentum, and energy conservation (see Lemma 2 in Section 2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}(m_i f_i + m_j f_j)\mu(v)\mathrm{d}v = 0 \quad \text{for } \mu(v) = 1, v, |v|^2,$$

and it fulfills an H-theorem or the entropy decay (see Lemma 3 in Section 2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^s\int_{\mathbb{R}^3} f_i\log f_i\mathrm{d}v = -\sum_{i,j=1}^s\int_{\mathbb{R}^3} c_{ji}f_i\left|\nabla\log\frac{f_i}{M_{ij}}\right|^2\mathrm{d}v \le 0,$$

which follows from the gradient-flow-type formulation of (1),

$$\partial_t f_i = \sum_{j=1}^s c_{ji}\operatorname{div}\left(f_i\nabla\log\frac{f_i}{M_{ij}}\right) \quad \text{in } \mathbb{R}^3,\ t>0,\ i=1,\ldots,s, \tag{7}$$

where M_{ij} are the "multispecies" Maxwellians

$$M_{ij}(v) = n_i\left(\frac{m_i}{2\pi T_{ij}}\right)^{3/2}\exp\left(-\frac{m_i|v-u_{ij}|^2}{2T_{ij}}\right). \tag{8}$$

Based on these properties, we are able to prove the global existence of weak solutions to (1)–(6). To simplify the notation, we set $\langle v\rangle := (1+|v|^2)^{1/2}$.

Theorem 1. *Let $f_i^0\in L^1(\mathbb{R}^3;\langle v\rangle^2\mathrm{d}v)$ be nonnegative with $\int_{\mathbb{R}^3} f_i^0\log f_i^0\mathrm{d}v<\infty$, let $\gamma\in\mathbb{R}$, and let the constants $m_i, q_i, \Lambda, \varepsilon_0 > 0$ for $i=1,\ldots,s$. Then, for any $T>0$, there exists a nonnegative weak solution f_i to (1)–(6) satisfying for all $i=1,\ldots,s$,*

$$\begin{aligned} &f_i\in L^\infty(0,T;L^1(\mathbb{R}^3;\langle v\rangle^2\mathrm{d}v))\cap L^2(0,T;H^1(\mathbb{R}^3)),\\ &f_i\log f_i\in L^\infty(0,T;L^1(\mathbb{R}^3)),\quad \partial_t f_i\in L^1(0,T;W^{-1,1}(\mathbb{R}^3)). \end{aligned}$$

Moreover, there exists a constant $c>0$ such that $T_{ji}(t)\ge c>0$ for $t\in(0,T)$ and $c_{ji}\in L^\infty(0,T)$, $u_{ji}\in L^q(0,T)$ for any $q<\infty$.

For the proof, we show first the existence of solutions to an approximate problem, derive estimates uniform in the approximation parameters, and then pass to the limit of vanishing parameters using compactness arguments. The construction of the approximate scheme is surprisingly delicate, and we need three approximation levels. First, we solve a regularized version of (1) in the ball B_M around the origin with radius $M>0$ to avoid compactness issues due to the whole space $\mathbb{R}^3$. Second, we truncate the nonlocal terms with the parameter $\varepsilon>0$ in such a way that $c_{ji}[f]$ and $T_{ji}[f]$ are positive and bounded from above and $|u_{ji}[f]|$ is bounded from above. Third, we need an elliptic regularization yielding $W^{1,p}(\mathbb{R}^3)$ solutions with $p>3$ and a moment regularization yielding estimates for higher-order moments, both with the same parameter $\delta>0$. More precisely, we add to the right-hand side of the truncated system the expressions

$$E_1 = \delta\operatorname{div}(|\nabla f_i|^{p-2}\nabla f_i),\quad E_2 = -\delta\langle v\rangle^K f_i + \delta g(v)\int_{B_M}\langle v\rangle^K f_i\mathrm{d}v,$$

where $g(v)=\pi^{-3/2}e^{-|v|^2}$ satisfies $\int_{\mathbb{R}^3} g(v)\mathrm{d}v = 1$, and $p>3$ and $K>2$ are sufficiently large. Expression E_1 yields an estimate for ∇f_i in $L^p(\mathbb{R}^d)$, while expression E_2 provides an estimate for f_i in $L^1(\mathbb{R}^3;\langle v\rangle^K\mathrm{d}v)$. The latter term is constructed in such a way that the mass is controlled (and conserved when B_M is replaced by $\mathbb{R}^3$ in the limit $M\to\infty$), since

$$\int_{B_M}E_2\mathrm{d}v = -\delta\bigg(1-\int_{B_M}g(v)\mathrm{d}v\bigg)\int_{B_M}\langle v\rangle^K f_i\mathrm{d}v \le 0.$$

However, this regularization provides additional terms when using the test functions f_i, $\log f_i$, and $|v|^2$ to derive bounds for the $L^2(\mathbb{R}^3)$ norm, the entropy, and the energy. For instance, using the test function f_i in the approximated system (see (17) below), we infer from $c_{ji}[f]\ge\varepsilon$ after some computations, detailed in Section 3, that

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}f_i^2\mathrm{d}v + \delta\int_{\mathbb{R}^3}\langle v\rangle^K f_i^2\mathrm{d}v + \delta\int_{\mathbb{R}^3}|\nabla f_i|^p\mathrm{d}v + \varepsilon\int_{\mathbb{R}^3}|\nabla f_i|^2\mathrm{d}v \\ \le C(\varepsilon)\int_{\mathbb{R}^3}f_i^2\mathrm{d}v + \delta\int_{\mathbb{R}^3}\langle v\rangle^K f_i\mathrm{d}v.\end{aligned}$$

In order to bound the last term on the right-hand side, we use (a cutoff version of) the test function $\langle v\rangle^\theta$ for $0<\theta<1-3/p$, which gives bounds for higher-order moments depending on δ. This is sufficient to pass to the limit $M\to\infty$ and then $\varepsilon\to 0$. For the limit $\delta\to 0$, we derive uniform estimates for the entropy and energy as well as the higher-order moment bound $\delta\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i\mathrm{d}v\le C$, where the constant $C>0$ only depends on the initial entropy and energy. This is sufficient to show that $E_2\to 0$ as $\delta\to 0$.

Another issue is the limit $\delta\to 0$ in the collision operator since it requires uniform bounds for the nonlocal terms $c_{ji}[f]$, $T_{ji}[f]$, and $u_{ji}[f]$. The most delicate point is the proof of a uniform positive lower bound for the temperature $T_{ji}[f]$. The idea is to estimate $T_{ji}[f]\ge\min\{T_i,T_j\}$ and

$$T_i \ge C\int_{\{|v-u_i|>\lambda\}}f_i|v-u_i|\mathrm{d}v \ge C\lambda^2\int_{\{|v-u_i|>\lambda\}}f_i\mathrm{d}v \ge C\lambda^2\bigg(n_i - \int_{\{|v-u_i|<\lambda\}}f_i\mathrm{d}v\bigg),$$

where $\lambda>0$ is arbitrary. By the Fenchel–Young inequality, we can estimate the integral on the right-hand side in terms of the initial entropy plus a number, and a suitable choice of the parameters allows us to conclude a lower bound only depending on the initial entropy; see Lemma 10.

Because of the truncations, we need to perform the limits $M\to\infty$, $\varepsilon\to 0$, and $\delta\to 0$ separately. Indeed, the energy conservation property of the collision operator holds only on the level of the nontruncated quantities c_{ji}, T_{ji}, and u_{ji}. Therefore, we pass to the limit $\varepsilon\to 0$ before deriving the energy and entropy bounds that eventually allow us to perform the limit $\delta\to 0$.

Let us discuss some extensions of Theorem 1. Our existence result also holds in the d-dimensional space. In this case, we choose $p>d$ and adjust the parameters $\theta>0$ and $K>2$ in a suitable way. We may also assume more general functions $c_{ji}[f]$, $u_{ji}[f]$, and $T_{ji}[f]$. It is possible to generalize the dependency of $c_{ji}[f]$ on T_j, but a suitable growth condition is needed. The choice of $u_{ji}[f]$ and $T_{ji}[f]$ guarantees momentum and energy conservation (see Section 2.2), and their definitions need to be compatible with these conservation properties.

The paper is organized as follows. Some details on the physical assumptions leading to model (1)–(6) are given in Section 2. Section 3 is devoted to the proof of Theorem 1. A compactness result in $\mathbb{R}^3$ is shown in Appendix A, and the rigorous treatment of nonintegrable test functions is sketched in Appendix B.

2. Motivation of the model and some properties

In this section, we motivate the Fokker–Planck–Landau system (1) and detail the underlying physical assumptions leading to this model. Moreover, we discuss its conservation properties and the H-theorem (entropy decay).

2.1. **The homogeneous Fokker–Planck–Landau system.** Model (1)–(6) is a simplification of the spatially homogeneous multispecies Landau system by linearizing the Landau collision operator and assuming that the operator kernel is independent of the velocity. More precisely, let

$$\partial_t f_i = \sum_{j=1}^{s} \widehat{Q}_{ji}(f_j, f_i) \quad \text{in } \mathbb{R}^3,\ t > 0,\ i = 1\ldots, s, \tag{9}$$

be the spatially homogeneous Landau equation [3] for a plasma consisting of s species. The Landau collision operator $\widehat{Q}_{ji}(f_j, f_i)$ models binary collisions between species j and i:

$$\widehat{Q}_{ji}(f_j, f_i) = \widehat{c}_{ji} \operatorname{div}_v \left\{ \int_{\mathbb{R}^3} A(v - v_*)\left(f_j(v_*)\nabla_v f_i(v) - \frac{m_i}{m_j} f_i(v)\nabla_{v_*} f_j(v_*)\right) \mathrm{d}v_* \right\}, \tag{10}$$

where $\widehat{c}_{ji} = |\log\Lambda| q_i^2 q_j^2/(8\pi\varepsilon_0^2 m_i^2)$ is a constant and $A(z) = |z|^{\beta+2}(\mathbb{I} - z\otimes z/|z|^2)$ is the (positive semidefinite) kernel matrix with $\mathbb{I}$ being the 3×3 identity matrix. The parameter β refers to the case of hard potentials if $\beta > 0$, Maxwellian molecules if $\beta = 0$, and soft potentials if $\beta < 0$. The latter case includes Coulomb interactions with $\beta = -3$. The Landau equation is obtained as the grazing collisions limit of the Boltzmann equation [1, 5, 13]. A spectral-gap analysis for the multispecies Landau system was performed in [9]. We also refer to this reference for results on the well-posedness of the single-species equation.

The collision operator $\widehat{Q}_{ji}$ conserves mass, momentum, and energy. Indeed, it can be written in the weak form

$$\begin{aligned} \int_{\mathbb{R}^3} \widehat{Q}_{ji}(f_j, f_i)\phi \mathrm{d}v = -\widehat{c}_{ji} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} &\nabla_v \phi(v)^T A(v - v_*) \\ &\times \left(\nabla_v \log f_i(v) - \frac{m_i}{m_j}\nabla_{v_*} \log f_j(v_*)\right) f_i(v) f_j(v_*) \mathrm{d}v \mathrm{d}v_* \end{aligned} \tag{11}$$

for suitable test functions ϕ. We obtain mass conservation by choosing $\phi = 1$:

$$\int_{\mathbb{R}^3} \widehat{Q}_{ji}(f_j, f_i)\mathrm{d}v = 0, \quad i, j = 1, \ldots, s.$$

Using $\widehat{c}_{ji}m_i/m_j = \widehat{c}_{ij}m_j/m_i$ and exchanging v and v_*, a computation shows that

$$\begin{aligned}\int_{\mathbb{R}^3} \widehat{Q}_{ij}(f_i,f_j)\psi \mathrm{d}v = \widehat{c}_{ji}\frac{m_i}{m_j}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_{v_*}\psi(v_*)^T A(v-v_*) \\ \times \bigg(\nabla_v \log f_i(v) - \frac{m_i}{m_j}\nabla_{v_*}\log f_j(v_*)\bigg) f_i(v)f_j(v_*)\mathrm{d}v\mathrm{d}v_*\end{aligned} \tag{12}$$

for another test function ψ, and an addition of (11) and (12) gives

$$\begin{aligned}\int_{\mathbb{R}^3}\big(\widehat{Q}_{ji}(f_j,f_i)\phi + \widehat{Q}_{ij}(f_i,f_j)\psi\big)\mathrm{d}v = -\widehat{c}_{ji}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\bigg(\nabla_v\phi(v) - \frac{m_i}{m_j}\nabla_{v_*}\psi(v_*)\bigg)^T \\ \times A(v-v_*)\bigg(\nabla_v \log f_i(v) - \frac{m_i}{m_j}\nabla_{v_*}\log f_j(v_*)\bigg) f_i(v)f_j(v_*)\mathrm{d}v\mathrm{d}v_*.\end{aligned}$$

Then conservation of momentum follows by choosing $\phi(v) = m_i v$ and $\psi(v) = m_j v$,

$$\int_{\mathbb{R}^3}\widehat{Q}_{ji}(f_j,f_i)m_i v\mathrm{d}v + \int_{\mathbb{R}^3}\widehat{Q}_{ij}(f_i,f_j)m_j v\mathrm{d}v = 0;$$

conservation of energy follows after the choice $\phi(v) = m_i|v|^2$ and $\psi(v) = m_j|v|^2$,

$$\int_{\mathbb{R}^3}\widehat{Q}_{ji}(f_j,f_i)m_i|v|^2\mathrm{d}v + \int_{\mathbb{R}^3}\widehat{Q}_{ij}(f_i,f_j)m_j|v|^2\mathrm{d}v = 0;$$

and we obtain entropy decay after choosing $\phi(v) = \log f_i(v)$ and $\psi(v) = \log f_j(v)$:

$$\int_{\mathbb{R}^3}\widehat{Q}_{ji}(f_j,f_i)\log f_i\mathrm{d}v + \int_{\mathbb{R}^3}\widehat{Q}_{ij}(f_i,f_j)\log f_j\mathrm{d}v \le 0, \quad i,j = 1,\ldots,s.$$

2.2. **The homogeneous linearized Fokker–Planck–Landau system.** In this section, we derive model (1)–(6) from the full multi-species Landau system presented in the previous section. Our derivation is motivated by [10], where a multi-species BGK model is obtained from the multi-species Boltzmann equation. We make two simplifications in model (9)–(10). First, we replace f_j in $\widehat{Q}_{ji}(f_j,f_i)$ by the Maxwellian

$$M_{ji} = n_j\bigg(\frac{m_j}{2\pi T_{ji}}\bigg)^{3/2}\exp\bigg(-\frac{m_j|v-u_{ji}|^2}{2T_{ji}}\bigg),$$

where n_j, u_{ji}, and T_{ji} are given by (3), (5), and (6), respectively. Then the collision operator becomes

$$\widehat{Q}_{ji}(M_{ji},f_i) = \widehat{c}_{ji}\,\mathrm{div}\,\bigg\{\widehat{A}_{ji}(v)\bigg(\nabla f_i + m_i\frac{v-u_{ji}}{T_{ji}}f_i\bigg)\bigg\},$$
$$\text{where } \widehat{A}_{ji}(v) = \int_{\mathbb{R}^3} A(v-v_*)M_{ji}(v_*)\mathrm{d}v_*.$$

In this step, we used the fact $A(z)z = 0$ for $z \in \mathbb{R}^3$ and from now on, all derivatives are with respect to v. Second, we suppose that the matrix $\widehat{A}_{ji}$ is independent of the velocity v (otherwise, the computation of the moments becomes awkward) and that $\widehat{A}_{ji}$ is diagonal (i.e., we neglect anisotropic diffusion). This leads to the Dougherty operator (see [8] for a similar model)

$$Q_{ji}(f_i) = c_{ji}\,\mathrm{div}\left(\nabla f_i + m_i\frac{v-u_{ji}}{T_{ji}}f_i\right), \tag{13}$$

where the coefficients c_{ji} should be a reasonable approximation of the exact expression

$$\widehat{c}_{ji}\widehat{A}_{ji}(v) = \frac{|\log\Lambda|q_i^2q_j^2}{8\pi\varepsilon_0^2m_i^2}\int_{\mathbb{R}^3}A(v-v_*)M_{ji}(v_*)\mathrm{d}v_*.$$

Assuming that the kinetic energy $m_j|v-v_*|^2$ is of the order of the thermal energy T_j (we neglected the Boltzmann constant), we may approximate $A(v-v_*)$ by $(T_j/m_j)^{(\beta+2)/2}$, such that we can replace $\widehat{c}_{ji}\widehat{A}_{ji}$ by

$$c_{ji} := \frac{|\log\Lambda|q_i^2q_j^2}{8\pi\varepsilon_0^2m_i^2}n_j\left(\frac{T_j}{m_j}\right)^{(\beta+2)/2},$$

and the definition for c_{ji} is exactly (4) after setting $\gamma := \beta+2$.

To determine u_{ji} and T_{ji}, we assume that the operator (13) conserves the momentum und energy (mass is automatically preserved):

$$\int_{\mathbb{R}^3}Q_{ji}(f_i)m_iv\mathrm{d}v + \int_{\mathbb{R}^3}Q_{ij}(f_j)m_jv\mathrm{d}v = 0, \tag{14}$$

$$\int_{\mathbb{R}^3}Q_{ji}(f_i)m_i|v|^2\mathrm{d}v + \int_{\mathbb{R}^3}Q_{ij}(f_j)m_j|v|^2\mathrm{d}v = 0, \quad i,j=1,\ldots,s. \tag{15}$$

Then a straightforward computation leads to the expressions (5) and (6). We summarize:

Lemma 2 (Conservation properties)**.** *Let u_{ji} and T_{ji} be given by* (5) *and* (6)*, respectively. Then Q_{ji} conserves the mass, momentum, and energy in the sense of* (14) *and* (15)*.*

The collision operator Q_{ji} also fulfills an H-theorem.

Lemma 3 (Entropy decay)**.** *It holds formally that*

$$\int_{\mathbb{R}^3}Q_{ji}(f_i)\log f_i\mathrm{d}v + \int_{\mathbb{R}^3}Q_{ij}(f_j)\log f_j\mathrm{d}v \le 0, \quad i,j=1,\ldots,s.$$

Proof. We use definition (8) of the Maxwellian and the conservation properties of Q_{ji}:

$$\begin{aligned}\int_{\mathbb{R}^3}Q_{ji}(f_i)\log M_{ij}\mathrm{d}v &= \int_{R^3}Q_{ji}(f_i)\left(\log n_i + \frac{3}{2}\log\frac{m_i}{2\pi T_{ji}} - \frac{m_i}{2T_{ji}}|v-u_{ji}|^2\right)\mathrm{d}v \\ &= -\frac{m_i}{2T_{ji}}\int_{\mathbb{R}^3}Q_{ji}(f_i)|v-u_{ji}|^2\mathrm{d}v \\ &= \frac{u_{ji}}{T_{ji}}\int_{\mathbb{R}^3}Q_{ji}(f_i)m_iv\mathrm{d}v - \frac{1}{2T_{ji}}\int_{\mathbb{R}^3}Q_{ji}(f_i)m_i|v|^2\mathrm{d}v \\ &= -\frac{u_{ij}}{T_{ij}}\int_{\mathbb{R}^3}Q_{ij}(f_j)m_jv\mathrm{d}v + \frac{1}{2T_{ij}}\int_{\mathbb{R}^3}Q_{ij}(f_j)m_j|v|^2\mathrm{d}v\end{aligned}$$

$$= \frac{m_j}{2T_{ij}} \int_{\mathbb{R}^3} Q_{ij}(f_j)|v - u_{ji}|^2 \mathrm{d}v = -\int_{\mathbb{R}^3} Q_{ij}(f_j) \log M_{ji} \mathrm{d}v,$$

where we also used the symmetry of u_{ji} and T_{ji}. Therefore, (7) yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q_{ji}(f_i) \log f_i \mathrm{d}v + \int_{\mathbb{R}^3} Q_{ij}(f_j) \log f_j \mathrm{d}v \\
&\quad = \int_{\mathbb{R}^3} Q_{ji}(f_i) \log \frac{f_i}{M_{ij}} \mathrm{d}v + \int_{\mathbb{R}^3} Q_{ij}(f_j) \log \frac{f_j}{M_{ji}} \mathrm{d}v \\
&\quad = -\int_{\mathbb{R}^3} c_{ji} f_i \left|\nabla \log \frac{f_i}{M_{ij}}\right|^2 \mathrm{d}v - \int_{\mathbb{R}^3} c_{ij} f_j \left|\nabla \log \frac{f_j}{M_{ij}}\right|^2 \mathrm{d}v \le 0,
\end{aligned}$$

ending the proof. □

Remark 4. For later use, we note that it holds formally that

$$\begin{aligned}
0 &= -\frac{1}{2} \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} \big(Q_{ji}(f_i) \log M_{ij} \mathrm{d}v + Q_{ij}(f_j) \log M_{ji}\big) \mathrm{d}v \\
&= -\sum_{i,j=1}^{s} \int_{\mathbb{R}^3} Q_{ji}(f_i) \log M_{ij} \mathrm{d}v = \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} c_{ji} f_i \nabla \log \frac{f_i}{M_{ij}} \cdot \nabla \log M_{ji} \mathrm{d}v.
\end{aligned} \tag{16}$$

3. Proof of Theorem 1

We prove the existence of weak solutions by introducing an approximate scheme, deriving suitable estimates uniform in the approximation parameters, and then passing to the limit of vanishing approximation parameters. Recall that $\langle v \rangle := (1 + |v|^2)^{1/2}$ and $g(v) = \pi^{-3/2} e^{-|v|^2}$ for $v \in \mathbb{R}^3$. We set $z^+ = \max\{0, z\}$ for $z \in \mathbb{R}$, and we choose the parameters $p > 3$ and $K > 0$ sufficiently large (to be specified later). Our approximated system is based on three approximation levels: the truncated domain size $M > 0$, the truncation parameter $0 < \varepsilon < 1$, and the regularization parameter $0 < \delta < 1$:

$$\begin{aligned}
&\partial_t f_i + \delta\bigg(\langle v \rangle^K f_i - g(v) \int_{B_M} \langle v \rangle^K f_i^+ \mathrm{d}v\bigg) - \delta \operatorname{div}\big(|\nabla f_i|^{p-2} \nabla f_i\big) \\
&\quad = \sum_{j=1}^{s} c_{ji}^{\varepsilon}[f] \operatorname{div}\bigg(\nabla f_i + \frac{m_i f_i}{T_{ji}^{\varepsilon}[f]}(v - u_{ji}[f])\bigg) \quad \text{in } B_M,\ t > 0,
\end{aligned} \tag{17}$$

with the initial conditions (2) and the no-flux boundary conditions

$$\bigg\{\delta |\nabla f_i|^{p-2} \nabla f_i + \sum_{j=1}^{s} c_{ji}^{\varepsilon}[f]\bigg(\nabla f_i + \frac{m_i f_i}{T_{ji}^{\varepsilon}[f]}(v - u_{ji}[f])\bigg)\bigg\} \cdot \nu = 0 \quad \text{on } \partial B_M,\ t > 0, \tag{18}$$

where $f=(f_1,\ldots,f_s)$, $B_M\subset\mathbb{R}^3$ is the ball around the origin with radius M, and ν is the exterior unit normal vector to ∂B_M. The nonlinear coefficients are approximated by

$$c_{ji}^\varepsilon[f]=\begin{cases}\dfrac{|\log\Lambda|q_i^2q_j^2}{8\pi\varepsilon_0^2m_i^2}n_j\left(\dfrac{T_j^{\varepsilon,\uparrow}[f]}{m_j}\right)^{\gamma/2}+\varepsilon, & \gamma\ge 0\\ \dfrac{|\log\Lambda|q_i^2q_j^2}{8\pi\varepsilon_0^2m_i^2}n_j\left(\dfrac{T_j^{\varepsilon,\downarrow}[f]}{m_j}\right)^{\gamma/2}+\varepsilon, & \gamma< 0\end{cases}$$

$$u_{ji}^\varepsilon[f]=\frac{c_{ji}^\varepsilon[f]m_i\rho_iu_i^\varepsilon[f]+c_{ij}^\varepsilon[f]m_j\rho_ju_j^\varepsilon[f]}{c_{ji}^\varepsilon[f]m_i\rho_i+c_{ij}^\varepsilon[f]m_j\rho_j}, \tag{19}$$

$$\begin{aligned}T_{ji}^\varepsilon[f]&=\frac{c_{ji}^\varepsilon[f]\rho_iT_i^{\varepsilon,\downarrow}[f]+c_{ij}^\varepsilon[f]\rho_jT_j^{\varepsilon,\downarrow}[f]}{c_{ji}^\varepsilon[f]\rho_i+c_{ij}^\varepsilon[f]\rho_j}\\&\quad+\frac{c_{ji}^\varepsilon[f]m_i\rho_ic_{ij}^\varepsilon[f]m_j\rho_j|u_i^\varepsilon[f]-u_j^\varepsilon[f]|^2}{3(c_{ji}^\varepsilon[f]\rho_i+c_{ij}^\varepsilon[f]\rho_j)(c_{ji}^\varepsilon[f]m_i\rho_i+c_{ij}^\varepsilon[f]m_j\rho_j)},\end{aligned}$$

and the (truncated) moments are defined according to

$$\begin{aligned}n_i&=\int_{\mathbb{R}^3}f_i^0\mathrm{d}v,\quad \rho_i=m_in_i,\\ u_i^\varepsilon[f]&=\frac{1}{n_i}\int_{B_M}\min\left\{f_i^+,\frac{g(v)}{\varepsilon}\right\}v\mathrm{d}v,\\ T_j^{\varepsilon,\uparrow}[f]&=\frac{m_i}{3n_i}\int_{B_M}\min\left\{f_i^+,\frac{g(v)}{\varepsilon}\right\}|v-u_i^\varepsilon[f]|^2\mathrm{d}v,\\ T_j^{\varepsilon,\downarrow}[f]&=\frac{m_i}{3n_i}\int_{B_M}\max\{f_i,\varepsilon g(v)\}|v-u_i^\varepsilon[f]|^2\mathrm{d}v.\end{aligned}$$

Note that n_i is given by the initial datum f_i^0 because of mass conservation. The truncations guarantee that for all $f_1,\ldots,f_s\in L^1(\mathbb{R}^3;\langle v\rangle^2\mathrm{d}v)$, the integrals $u_i^\varepsilon[f]$, $T_j^{\varepsilon,\uparrow}[f]$, and $T_j^{\varepsilon,\downarrow}[f]$ are well defined and

$$\varepsilon\le c_{ji}^\varepsilon[f]\le C(\varepsilon),\quad |u_{ji}^\varepsilon[f]|\le C(\varepsilon),\quad c\varepsilon\le T_{ji}^\varepsilon[f]<\infty \tag{20}$$

for some constants $c>0$ and $C(\varepsilon)>0$ which are independent of M.

3.1. **Existence of solutions to the approximated system.** We show that there exists a weak solution f_i to (2), (17), and (18) by reformulating the equations as a fixed-point problem for a suitable mapping. For this, we introduce the space $X=L^p(0,T;L^p(B_M))$ recalling that $p>3$. Let $\sigma\in[0,1]$ and $\widehat{f}_i\in X$, $i=1,\ldots,s$, be given. We consider first the partially linearized equations

$$\begin{aligned}&\partial_tf_i+\delta\left(\langle v\rangle^Kf_i-\sigma g(v)\int_{B_M}\langle v\rangle^K\widehat{f}_i^+\mathrm{d}v\right)+\delta|f_i|^{p-2}f_i-\delta\operatorname{div}\left(|\nabla f_i|^{p-2}\nabla f_i\right)\\&\quad-\sigma\sum_{j=1}^sc_{ji}^\varepsilon[\widehat{f}]\operatorname{div}\left(\nabla f_i+\frac{m_if_i}{T_{ji}^\varepsilon[\widehat{f}]}(v-u_{ji}^\varepsilon[\widehat{f}])\right)=\sigma\delta|\widehat{f}_i|^{p-2}\widehat{f}_i,\end{aligned} \tag{21}$$

where $i=1,\ldots,s$, with initial and no-flux boundary conditions. This system can be formulated as the evolution equation $\partial_t f_i + A[f]f_i = b_i$ for $t>0$, where

$$\begin{aligned} A[f] &= \delta\langle v\rangle^K f_i + \delta|f_i|^{p-2}f_i - \delta\operatorname{div}\big(|\nabla f_i|^{p-2}\nabla f_i\big) \\ &\quad - \sigma\sum_{j=1}^s c_{ji}^\varepsilon[\widehat{f}]\operatorname{div}\bigg(\nabla f_i + \frac{m_i f_i}{T_{ji}^\varepsilon[\widehat{f}]}(v-u_{ji}^\varepsilon[\widehat{f}])\bigg), \\ b_i &= \sigma g(v)\int_{B_M}\langle v\rangle^K\widehat{f}_i^+\mathrm{d}v + \sigma\delta|\widehat{f}_i|^{p-2}\widehat{f}_i. \end{aligned}$$

The operator $A[f]:V\to V'$ with $V=W^{1,p}(B_M)$ and its dual space V' is monotone, hemicontinuous, and coercive. We conclude from [16, Theorem 30.A] that (21) possesses a unique solution $f_i\in L^p(0,T;V)$ with $\partial_t f_i\in L^{p/(p-1)}(0,T;V')$, $i=1,\ldots,s$.

Next, we use the test function f_i in the weak formulation of (21):

$$\begin{aligned} (22)\qquad & \frac12\int_{B_M} f_i(t)^2\mathrm{d}v - \frac12\int_{B_M}(f_i^0)^2\mathrm{d}v + \delta\int_0^t\int_{B_M}|f_i|^p\mathrm{d}v\mathrm{d}s + \delta\int_0^t\int_{B_M}|\nabla f_i|^p\mathrm{d}v\mathrm{d}s \\ &= -\delta\int_0^t\int_{B_M}\langle v\rangle^K f_i^2\mathrm{d}v\mathrm{d}s + \sigma\int_0^t\bigg(\int_{B_M} f_i g(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^K\widehat{f}_i^+\mathrm{d}v\bigg)\mathrm{d}s \\ &\quad - \sigma\sum_{j=1}^n\int_0^t\int_{B_M} c_{ji}^\varepsilon[\widehat{f}]\bigg(|\nabla f_i|^2 + \frac{m_i f_i}{T_{ji}^\varepsilon[\widehat{f}]}(v-u_{ji}^\varepsilon[\widehat{f}])\cdot\nabla f_i\bigg)\mathrm{d}v\mathrm{d}s \\ &\quad + \sigma\delta\int_0^t\int_{B_M}|\widehat{f}_i|^p\mathrm{d}v\mathrm{d}s. \end{aligned}$$

Taking into account that we integrate over a bounded domain, and in particular that $\langle v\rangle^K$ is bounded, we estimate the second term on the right-hand side as follows, using Hölder's inequality as well as the embeddings $L^p(B_M)\hookrightarrow L^1(B_M)$ and $L^p(B_M)\hookrightarrow L^{p/(p-1)}(B_M)$:

$$\begin{aligned} \sigma\int_0^t\bigg(\int_{B_M} f_i g(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^K\widehat{f}_i^+ f_i\mathrm{d}v\bigg)\mathrm{d}s &\le \int_0^t\|f_i\|_{L^1(B_M)}\|\widehat{f}_i\|_{L^p(B_M)}\|f_i\|_{L^{p/(p-1)}(B_M)}\mathrm{d}s \\ &\le \int_0^t\|f_i\|_{L^1(B_M)}^{p/(p-1)}\|f_i\|_{L^{p/(p-1)}(B_M)}^{p/(p-1)}\mathrm{d}s + C\int_0^t\|\widehat{f}_i\|_{L^p(B_M)}^p\mathrm{d}s \\ &\le C(M)\int_0^t\|f_i\|_{L^p(B_M)}^{2p/(p-1)}\mathrm{d}s + C\int_0^t\|\widehat{f}_i\|_{L^p(B_M)}^p\mathrm{d}s. \end{aligned}$$

Since $2p/(p-1)<p$ (because of $p>3$), the elementary inequality $z^{2p/(p-1)}\le C(\delta)+(\delta/2)z^p$ for $z\ge 0$ yields

$$\begin{aligned} &\sigma\int_0^t\bigg(\int_{B_M} f_i g(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^K\widehat{f}_i^+ f_i\mathrm{d}v\bigg)\mathrm{d}s \\ &\qquad\le \int_0^t\bigg(C(\delta) + \frac{\delta}{2}\|f_i\|_{L^p(B_M)}^p + C\|\widehat{f}_i\|_{L^p(B_M)}^p\bigg)\mathrm{d}s, \end{aligned}$$

and the second term on the right-hand side can be absorbed by the left-hand side of (22). We write $f_i\nabla f_i = \frac{1}{2}\nabla f_i^2$, use Young's inequality, and integrate by parts in the third term on the right-hand side of (22) (we denote the measure on ∂B_M with $d\Sigma_v$):

$$
\begin{aligned}
-\sigma\sum_{j=1}^n\int_0^t\int_{B_M} & c_{ji}^\varepsilon[\widehat{f}]\bigg(|\nabla f_i|^2+\frac{m_if_i}{T_{ji}^\varepsilon[\widehat{f}]}(v-u_{ji}^\varepsilon[\widehat{f}])\cdot\nabla f_i\bigg)\mathrm{d}v\mathrm{d}s \\
&\le -\frac{\sigma}{2}\sum_{i,j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[\widehat{f}]|\nabla f_i|^2\mathrm{d}v\mathrm{d}s+\frac{\sigma}{2}\sum_{i,j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[\widehat{f}]\frac{m_i^2|u_{ji}^\varepsilon[\widehat{f}]|^2}{T_{ji}^\varepsilon[\widehat{f}]^2}f_i^2\mathrm{d}v\mathrm{d}s \\
&\quad -\frac{\sigma}{2}\sum_{i,j=1}^s\int_0^t\int_{\partial B_M}c_{ji}^\varepsilon[\widehat{f}]\frac{m_i}{T_{ji}^\varepsilon[\widehat{f}]}|v|f_i^2\mathrm{d}\Sigma_v\mathrm{d}s+3\sum_{i,j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[\widehat{f}]\frac{m_i}{T_{ji}^\varepsilon[\widehat{f}]}f_i^2\mathrm{d}v\mathrm{d}s \\
&\le C(\varepsilon)\int_0^t\int_{B_M}f_i^2\mathrm{d}v\mathrm{d}s,
\end{aligned}
$$

using $v\cdot\nu=|v|$ and bounds (20) in the last step. Then (22) gives

$$
\int_{B_M}f_i(t)^2\mathrm{d}v+\delta\int_0^t\|f_i\|_{W^{1,p}(B_M)}^p\mathrm{d}s\le C(\delta)+C(\varepsilon)\int_0^t\int_{B_M}f_i^2\mathrm{d}v\mathrm{d}s+C\int_0^t\|\widehat{f}_i\|_{L^p(B_M)}^p\mathrm{d}s,
$$

and it follows from Gronwall's inequality that, for any $T>0$,

$$
\sup_{0<t<T}\|f_i\|_{L^2(B_M)}^2+\int_0^T\|f_i\|_{W^{1,p}(B_M)}^p\mathrm{d}t\le C\big(\delta,\varepsilon,T,\|f^0\|_{L^2(\mathbb{R}^3)}\big)\bigg(1+\int_0^T\|\widehat{f}_i\|_{L^p(B_M)}^p\mathrm{d}t\bigg). \tag{23}
$$

This estimate allows us to derive a bound for the time derivative,

$$
\|\partial_tf_i\|_{L^{p/(p-1)}(0,T;W^{1,p}(B_M)')}\le C(\delta,\varepsilon,T,f^0).
$$

Estimate (23) shows that the mapping $F:X\times[0,1]\to X$, $(\widehat{f},\sigma)\mapsto f$, is well defined. Moreover, the function $F(\cdot,0):X\to X$ is constant. Standard arguments show the continuity of F, and the compactness of F follows from the compact embedding $W^{1,p}(B_M)\hookrightarrow L^p(B_M)$, the bounds for f_i in $L^p(0,T;W^{1,p}(B_M))$ and $W^{1,p/(p-1)}(0,T;W^{1,p}(B_M)')$, and the Aubin–Lions lemma [12].

To apply the Leray–Schauder fixed-point theorem, we need to show that the set $\{f\in X:F(f,\sigma)=f\}$ of fixed points of $F(\cdot,\sigma)$ is bounded in X uniformly in $\sigma\in[0,1]$. To this end, we set $\widehat{f}=f$ in (21), use the test function f_i in its weak formulation, and estimate similarly as above:

$$
\begin{aligned}
\frac12\int_{B_M}&f_i^2(t)\mathrm{d}v-\frac12\int_{B_M}(f_i^0)^2\mathrm{d}v+\delta(1-\sigma)\int_0^t\int_{B_M}|f_i|^p\mathrm{d}v\mathrm{d}s+\delta\int_0^t\int_{B_M}|\nabla f_i|^p\mathrm{d}v\mathrm{d}s \\
&\le-\int_0^t\int_{B_M}\langle v\rangle^Kf_i^2\mathrm{d}v\mathrm{d}s+\int_0^t\bigg(\int_{B_M}f_ig(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^Kf_i^+\mathrm{d}v\bigg)\mathrm{d}s \\
&\quad-\varepsilon\int_0^t\int_{B_M}|\nabla f_i|^2\mathrm{d}v\mathrm{d}s+C(\varepsilon,M)\int_0^t\int_{B_M}f_i^2\mathrm{d}v\mathrm{d}s\le C(\varepsilon,M)\int_0^t\int_{B_M}f_i^2\mathrm{d}v\mathrm{d}s,
\end{aligned}
$$

where we used the inequality $(\int_{B_M} f_i\mathrm{d}v)^2 \le C(M)\int_{B_M} f_i^2\mathrm{d}v$. We deduce from Gronwall's inequality and the Poincaré–Wirtinger inequality that f_i is bounded in $L^p(0,T;W^{1,p}(B_M))$ uniformly in $\sigma\in[0,1]$. Therefore, we can apply the Leray–Schauder fixed-point theorem to infer the existence of a fixed point to (21) with $\sigma=1$, i.e. a solution $f_i\in L^p(0,T;L^p(B_M))$, $i=1,\ldots,s$, to (17).

3.2. **Limit** $M\to\infty$**.** Let $f_i^M := f_i$ be a weak solution to (17). We first derive some estimates uniform in M and then pass to the limit $M\to\infty$.

Lemma 5. *The solution f_i^M to* (17)*, constructed in the previous subsection, is nonnegative in $B_M\times(0,T)$, and the mass is controlled, $\|f_i^M(t)\|_{L^1(B_M)}\le\|f_i^0\|_{L^1(B_M)}$ for $t>0$.*

Proof. We use the test function $(f_i^M)^- = \min\{0,f_i^M\}$ in the weak formulation of (17), use $(f_i^0)^-=0$, and integrate by parts in the collision operator:

$$
\begin{aligned}
&\frac12\int_{B_M}(f_i^M)^-(t)^2\mathrm{d}v + \delta\int_0^t\int_{B_M}|\nabla(f_i^M)^-|^p\mathrm{d}v\mathrm{d}s \\
&= \sum_{j=1}^s\int_0^t\int_{B_M} c_{ji}[f^M]\bigg(-\frac12|\nabla(f_i^M)^-|^2 + \frac32\frac{m_i}{T_{ji}[f^M]}|(f_i^M)^-|^2\bigg)\mathrm{d}v\mathrm{d}s \\
&\quad + \frac12\sum_{i,j=1}^s\int_0^t\int_{B_M} c_{ji}^\varepsilon[f^M]\frac{m_i^2|u_{ji}^\varepsilon[f^M]|^2}{T_{ji}^\varepsilon[f^M]^2}|(f_i^M)^-|^2\mathrm{d}v\mathrm{d}s \\
&\quad - \frac12\sum_{i,j=1}^s\int_0^t\int_{\partial B_M} c_{ji}^\varepsilon[f^M]\frac{m_i}{T_{ji}^\varepsilon[f^M]}|v||(f_i^M)^-|^2\mathrm{d}\Sigma_v\mathrm{d}s \\
&\quad - \delta\int_0^t\int_{B_M}\langle v\rangle^K|(f_i^M)^-|^2\mathrm{d}v\mathrm{d}s \\
&\quad + \delta\int_0^t\bigg(\int_{B_M}(f_i^M)^-g(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^K(f_i^M)^+\mathrm{d}v\bigg)\mathrm{d}s \\
&\le C(\varepsilon)\int_0^t\int_{B_M}|(f_i^M)^-|^2\mathrm{d}v\mathrm{d}s,
\end{aligned}
$$

since the last term in the last but one step is nonpositive. We conclude from Gronwall's lemma that $(f_i^M)^-(t)=0$ and hence $f_i^M(t)\ge0$ in B_M for $t>0$. Next, we use the test function $\phi=1$ in the weak formulation of (17):

$$
\begin{aligned}
\int_{B_M} f_i^M(t)\mathrm{d}v &= \int_{B_M} f_i^0\mathrm{d}v - \delta\int_0^t\int_{B_M}\langle v\rangle^K f_i^M\mathrm{d}v\mathrm{d}s \\
&\quad + \delta\bigg(\int_0^t\int_{B_M} g(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^K f_i^M\mathrm{d}v\bigg)\mathrm{d}s \le \int_{B_M} f_i^0\mathrm{d}v,
\end{aligned}
$$

since $\int_{B_M} g(v)\mathrm{d}v\le\int_{\mathbb{R}^3} g(v)\mathrm{d}v=1$. This proves the mass control. $\square$

We show now some estimates uniform in M.

Lemma 6. *Let $0<\theta<1-3/p$. Then there exists a constant $C(\delta,\varepsilon)>0$ independent of M such that*

$$\sup_{0<t<T}\int_{B_M}\big(f_i^M(t)^2+\langle v\rangle^\theta f_i^M(t)\big)\mathrm{d}v+\int_0^T\int_{B_M}\langle v\rangle^{K+\theta}f_i^M\mathrm{d}v\mathrm{d}s \\ +\int_0^T\int_{B_M}\big(|\nabla f_i^M|^2+|\nabla f_i^M|^p\big)\mathrm{d}v\mathrm{d}s\le C(\delta,\varepsilon).$$

Proof. We use the test function f_i^M in the weak formulation of (17), use $\varepsilon\le c_{ji}[f^M]\le C(\varepsilon)$, and integrate by parts in the drift part of the collision operator:

$$\frac12\int_{B_M}f_i^M(t)^2\mathrm{d}v-\frac12\int_{B_M}(f_i^0)^2\mathrm{d}v+\delta\int_0^t\int_{B_M}\langle v\rangle^K(f_i^M)^2\mathrm{d}v\mathrm{d}s+\delta\int_0^t\int_{B_M}|\nabla f_i^M|^p\mathrm{d}v\mathrm{d}s \\ \le\delta\int_0^t\bigg(\int_{B_M}f_i^Mg(v)\mathrm{d}v\bigg)\bigg(\int_{B_M}\langle v\rangle^Kf_i^M\mathrm{d}v\bigg)\mathrm{d}s-\varepsilon\int_0^t\int_{B_M}|\nabla f_i^M|^2\mathrm{d}v\mathrm{d}s \\ +C(\varepsilon)\int_0^t\int_{B_M}(f_i^M)^2\mathrm{d}v\mathrm{d}s.$$

Because of the mass control from Lemma 5, $\int_{B_M}f_i^Mg(v)\mathrm{d}v\le\int_{B_M}f_i^M\mathrm{d}v\le C(f_i^0)$. Hence,

$$\frac12\int_{B_M}f_i^M(t)^2\mathrm{d}v+\delta\int_0^t\int_{B_M}\langle v\rangle^K(f_i^M)^2\mathrm{d}v\mathrm{d}s+\delta\int_0^t\int_{B_M}|\nabla f_i^M|^p\mathrm{d}v\mathrm{d}s \\ +\varepsilon\int_0^t\int_{B_M}|\nabla f_i^M|^2\mathrm{d}v\mathrm{d}s \\ \le C+C(f_i^0)\int_0^t\int_{B_M}\langle v\rangle^Kf_i^M\mathrm{d}v\mathrm{d}s+C(\varepsilon)\int_0^t\int_{B_M}(f_i^M)^2\mathrm{d}v\mathrm{d}s. \tag{24}$$

To control the second term on the right-hand side, we derive a bound for $\langle v\rangle^{K+\theta}f_i^M$ for some $\theta>0$. This is done by using the test function $\langle v\rangle^\theta$ in (17):

$$\int_{B_M}\langle v\rangle^\theta f_i^M(t)\mathrm{d}v-\int_{B_M}\langle v\rangle^\theta f_i^0\mathrm{d}v+\delta\int_0^t\int_{B_M}\langle v\rangle^{K+\theta}f_i^M\mathrm{d}v \\ \le C(g)\int_0^t\int_{B_M}\langle v\rangle^Kf_i^M\mathrm{d}v\mathrm{d}s+\delta C\int_0^t\int_{B_M}\langle v\rangle^{\theta-2}|\nabla f_i^M|^{p-2}|\nabla f_i^M\cdot v|\mathrm{d}v\mathrm{d}s \\ -\theta\sum_{j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[f^M]\langle v\rangle^{\theta-2}v\cdot\bigg(\nabla f_i^M+\frac{m_if_i^M}{T_{ji}[f^M]}(v-u_{ji}[f^M])\bigg)\mathrm{d}v\mathrm{d}s \\ =:I_1+I_2+I_3, \tag{25}$$

where $C(g)>0$ depends on the integral $\int_{B_M}\langle v\rangle^\theta g(v)\mathrm{d}v$ which is bounded uniformly in M. The first term is estimated according to

$$I_1\le\int_0^t\int_{B_M}\bigg(\frac{\delta}{4}\langle v\rangle^{K+\theta}+C(\delta,g,K)\bigg)f_i^M\mathrm{d}v\mathrm{d}s$$

$$\leq \frac{\delta}{4}\int_0^t\int_{B_M}\langle v\rangle^{K+\theta}f_i^M\mathrm{d}v\mathrm{d}s + C(\delta,g,K,f_i^0),$$

and the integral on the right-hand side can be absorbed by the left-hand side of (25). We use Young's inequality with exponents p and $p/(p-1)$ to find that

$$\begin{aligned} I_2 &\leq \delta C\int_0^t\int_{B_M}\langle v\rangle^{\theta-1}|\nabla f_i^\varepsilon|^{p-1}\mathrm{d}v\mathrm{d}s \\ &\leq \frac{\delta}{2}\int_0^t\int_{B_M}|\nabla f_i^M|^p\mathrm{d}v\mathrm{d}s + C\delta\int_0^t\int_{B_M}\langle v\rangle^{p(\theta-1)}\mathrm{d}v\mathrm{d}s. \end{aligned}$$

The integral over $\langle v\rangle^{p(\theta-1)}$ is bounded uniformly in M if $p(\theta-1)<-3$, which is equivalent to $\theta<1-3/p$. We integrate by parts in the first part of I_3:

$$\begin{aligned} -\sum_{j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[f^M]\langle v\rangle^{\theta-2}v\cdot\nabla f_i^M\mathrm{d}v\mathrm{d}s &= \sum_{j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[f^M]\operatorname{div}(\langle v\rangle^{\theta-2}v)f_i^M\mathrm{d}v\mathrm{d}s \\ &\quad - \sum_{j=1}^s\int_0^t\int_{\partial B_M}c_{ji}^\varepsilon[f^M]\langle v\rangle^{\theta-2}(v\cdot\nu)f_i^M\mathrm{d}v\mathrm{d}s, \end{aligned}$$

where ν is the exterior unit normal vector to ∂B_M. Since B_M is a ball around the origin, $\nu=v/|v|$ and hence $v\cdot\nu=|v|$, and we infer that the surface integral is nonpositive. Then, using $\langle v\rangle^{\theta-2}\leq 1$ and the mass control,

$$-\theta\sum_{j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[f^M]\langle v\rangle^{\theta-2}v\cdot\nabla f_i^M\mathrm{d}v\mathrm{d}s \leq C(\varepsilon)\sum_{j=1}^s\int_0^t\int_{B_M}\langle v\rangle^{\theta-2}f_i^M\mathrm{d}v\mathrm{d}s \leq C(\varepsilon,f_i^0).$$

The second part of I_3 is estimated according to

$$\begin{aligned} &\theta\sum_{j=1}^s\int_0^t\int_{B_M}c_{ji}^\varepsilon[f^M]\frac{m_i}{T_{ji}[f^M]}\langle v\rangle^{\theta-2}\big(|v|^2-v\cdot u_{ji}^\varepsilon[f^M]\big)f_i^M\mathrm{d}v\mathrm{d}s \\ &\leq C(\varepsilon)\int_0^t\int_{B_M}(\langle v\rangle^\theta+\langle v\rangle^{\theta-1})f_i^M\mathrm{d}v\mathrm{d}s \leq C(\delta,\varepsilon)+\frac{\delta}{4}\int_0^t\int_{B_M}\langle v\rangle^{K+\theta}f_i^M\mathrm{d}v\mathrm{d}s. \end{aligned}$$

Summarizing, we infer from (25) that

$$\int_{B_M}\langle v\rangle^\theta f_i^M(t)\mathrm{d}v + \frac{\delta}{2}\int_0^t\int_{B_M}\langle v\rangle^{K+\theta}f_i^M\mathrm{d}v \leq C(\delta,\varepsilon)+\frac{\delta}{2}\int_0^t\int_{B_M}|\nabla f_i^M|^p\mathrm{d}v\mathrm{d}s.$$

We add this inequality to (24) and use the inequality $\langle v\rangle^K\leq C(\delta)+(\delta/8)\langle v\rangle^{K+\theta}$ as well as the mass control:

$$\begin{aligned} \int_{B_M}\left(\frac{1}{2}f_i^M(t)^2+\langle v\rangle^\theta f_i^M(t)\right)\mathrm{d}v + \int_0^t\int_{B_M}\left(\frac{\delta}{2}\langle v\rangle^{K+\theta}f_i^M+\varepsilon|\nabla f_i^M|^2+\frac{\delta}{2}|\nabla f_i^M|^p\right)\mathrm{d}v\mathrm{d}s \\ \leq C(\delta,\varepsilon)+C(\varepsilon)\int_0^t\int_{B_M}(f_i^M)^2\mathrm{d}v\mathrm{d}s. \end{aligned}$$

We apply Gronwall's lemma and then take the supremum over $t \in (0,T)$ to finish the proof. $\square$

Lemma 6 gives uniform bounds for f_i^M in $L^\infty(0,T;L^2(B_M))$ and $L^p(0,T;W^{1,p}(B_M))$. Then, together with the bounds (20), we infer that $\partial_t f_i^M$ is bounded in $L^{p/(p-1)}(0,T;W^{-1,p}(B_M)')$ uniformly in M. The condition $p>3$ implies that the embedding $W^{1,p}(B_M) \hookrightarrow L^\infty(B_M)$ is compact. Then the Aubin–Lions lemma, together with a Cantor diagonal argument, yields the existence of a subsequence, which is not relabeled, such that, as $M\to\infty$,

$$f_i^M \to f_i \quad \text{strongly in } L^p(0,T;L^\infty(B)) \quad \text{for every ball } B\subset\mathbb{R}^3.$$

We claim that

$$f_i^M \to f_i \quad \text{strongly in } L^1(0,T;L^1(\mathbb{R}^3)).$$

Indeed, we know from Lemma 6 that $\int_B \langle v\rangle^\theta f_i^M(t)\mathrm{d}v \le C$ for all balls $B\subset\mathbb{R}^3$ uniformly in M and for $t\in(0,T)$. Then Fatou's lemma implies that

$$\int_{\mathbb{R}^3}\langle v\rangle^\theta f_i(t)\mathrm{d}v = \int_{\mathbb{R}^3}\liminf_{M\to\infty}\langle v\rangle^\theta f_i^M(t)1_{B_M}\mathrm{d}v \le \liminf_{M\to\infty}\int_{\mathbb{R}^3}\langle v\rangle^\theta f_i^M(t)1_{B_M}\mathrm{d}v \le C,$$

and this bound holds uniformly for $t\in(0,T)$. Set $f_i^M(t) := 0$ outside of B_M and let $R<M$. We write

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s &= \int_0^T\int_{B_R}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s + \int_0^T\int_{\{R\le|v|\le M\}}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s \\ &\quad + \int_0^T\int_{\{|v|>M\}}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s =: J_1^M + J_2^M + J_3^M.\end{aligned}$$

Because of the strong convergence of (f_i^M) in B_R, we have $J_1^M\to 0$ as $M\to\infty$. We deduce from the uniform bound for $\langle v\rangle^\theta f_i^M$ in $L^1(\mathbb{R}^3)$ that

$$J_2^M \le \frac{1}{R^\theta}\int_0^T\int_{\{R\le|v|\le M\}}\langle v\rangle^\theta|f_i^M - f_i|\mathrm{d}v\mathrm{d}s \le \frac{C}{R^\theta}.$$

In a similar way, since $f_i^M = 0$ in $\{|v|>M\}$, we have

$$J_3^M \le \frac{1}{R^\theta}\int_0^T\int_{\{|v|>M\}}\langle v\rangle^\theta f_i\mathrm{d}v \le \frac{C}{R^\theta}.$$

We conclude that

$$\limsup_{M\to\infty}\int_0^T\int_{\mathbb{R}^3}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s \le \frac{C}{R^\theta} \quad \text{for all } R>0.$$

Since the left-hand side is independent of R, it follows that $\limsup_{M\to\infty}\int_0^T\int_{\mathbb{R}^3}|f_i^M - f_i|\mathrm{d}v\mathrm{d}s = 0$, proving the claim.

We also obtain, for a subsequence, the weak convergences

$$\begin{aligned}\nabla f_i^M &\rightharpoonup \nabla f_i \quad \text{weakly in } L^p(0,T;L^p(B)),\\ \partial_t f_i^M &\rightharpoonup \partial_t f_i \quad \text{weakly in } L^p(0,T;W^{1,p}(B)')\end{aligned}$$

as $M \to \infty$ for any ball $B \subset \mathbb{R}^3$. These convergences are sufficient to pass to the limit $M \to \infty$ in (17), and the limit $f_i^\varepsilon := f_i$ is a weak solution to

$$\begin{aligned}\partial_t f_i^\varepsilon &+ \delta\bigg(\langle v\rangle^K f_i^\varepsilon - g(v)\int_{\mathbb{R}^3}\langle v\rangle^K f_i^\varepsilon \mathrm{d}v\bigg) - \delta\,\mathrm{div}\,\big(|\nabla f_i^\varepsilon|^{p-2}\nabla f_i^\varepsilon\big) \\ &= \sum_{j=1}^s c_{ji}^\varepsilon[f^\varepsilon]\,\mathrm{div}\left(\nabla f_i^\varepsilon + \frac{m_i f_i^\varepsilon}{T_{ji}^\varepsilon[f^\varepsilon]}(v - u_{ji}^\varepsilon[f^\varepsilon])\right) \quad \text{in } \mathbb{R}^3,\ t>0,\end{aligned} \tag{26}$$

with the initial conditions (2).

3.3. **Limit $\varepsilon \to 0$.** Let f_i^ε be a weak solution to (2) and (26). An integration yields the conservation of mass:

$$\int_{\mathbb{R}^3} f_i^\varepsilon(t)\mathrm{d}v = n_i = \int_{\mathbb{R}^3} f_i^0 \mathrm{d}v > 0. \tag{27}$$

Strictly speaking, we cannot use the test function $\phi = 1$ in (26) and we need to work with a cutoff function ψ_R; we refer to Appendix B for details.

Lemma 7. *There exists a constant $C(\delta, T) > 0$ independent of ε such that*

$$\begin{aligned}\sup_{0<t<T}\sum_{i=1}^s \int_{\mathbb{R}^3}\big(f_i^\varepsilon(t)^2 + \langle v\rangle^\theta f_i^\varepsilon(t)\big)\mathrm{d}v + \sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ij}[f^\varepsilon]|\nabla f_i^\varepsilon|^2 \mathrm{d}v\mathrm{d}s \\ + \sum_{i=1}^s\int_0^T\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p \mathrm{d}v\mathrm{d}s + \sum_{i=1}^s\int_0^T\int_{\mathbb{R}^3}\big(\langle v\rangle^K (f_i^\varepsilon)^2 + \langle v\rangle^{K+\theta} f_i^\varepsilon\big)\mathrm{d}v\mathrm{d}s \le C(\delta, T).\end{aligned}$$

Proof. We split the proof in several steps.

Step 1: Test function $\langle v\rangle^\theta$. Let $0 < \theta < 1 - 3/p$. We use $\langle v\rangle^\theta$ as a test function in (26). Again, $\langle v\rangle^\theta$ cannot be used as a test function but we may use $\langle v\rangle^\theta \psi_R(v)$ for some cutoff function ψ_R; see Appendix B. Then, summing over $i = 1, \ldots, s$,

$$\begin{aligned}\sum_{i=1}^s &\int_{\mathbb{R}^3}\langle v\rangle^\theta f_i^\varepsilon(t)\mathrm{d}v - \sum_{i=1}^s\int_{\mathbb{R}^3}\langle v\rangle^\theta f_i^0 \mathrm{d}v + \delta\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s \\ &= \delta\int_0^t\bigg(\int_{\mathbb{R}^3}\langle v\rangle^\theta g(v)\mathrm{d}v\bigg)\bigg(\sum_{i=1}^s\int_{\mathbb{R}^3}\langle v\rangle^K f_i^\varepsilon \mathrm{d}v\bigg)\mathrm{d}s \\ &\quad - \delta\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^{p-2}\nabla f_i^\varepsilon\cdot\nabla\langle v\rangle^\theta \mathrm{d}v + \sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\nabla\langle v\rangle^\theta\cdot\nabla f_i^\varepsilon \mathrm{d}v\mathrm{d}s \\ &\quad - \sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\frac{m_i}{T_{ji}^\varepsilon[f^\varepsilon]}(v - u_{ji}^\varepsilon[f^\varepsilon])\cdot\nabla\langle v\rangle^\theta f_i^\varepsilon \mathrm{d}v\mathrm{d}s \\ &=: I_4 + \cdots + I_7.\end{aligned} \tag{28}$$

We estimate the right-hand side term by term. First, the integral over $\langle v\rangle^\theta g(v)$ is bounded. Using $\langle v\rangle^K \le (\delta/8)\langle v\rangle^{K+\theta} + C(\delta)$ and mass conservation (27), we can estimate

$$I_4 \le C(\delta) + \frac{\delta}{8}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s,$$

and the last integral can be absorbed by the left-hand side of (28). Because of $|\nabla\langle v\rangle^\theta| \le \theta\langle v\rangle^{\theta-1}$ and Young's inequality, the term I_5 becomes

$$\begin{aligned} I_5 &\le \delta C\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{\theta-1}|\nabla f_i^\varepsilon|^{p-1}\mathrm{d}v \\ &\le \frac{C}{p}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{p(\theta-1)}\mathrm{d}v\mathrm{d}s + \frac{p-1}{p}\delta^{p/(p-1)}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s \\ &\le C + \delta^{p/(p-1)}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s \le C + \frac{\delta}{2}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s, \end{aligned}$$

taking into account that the integral over $\langle v\rangle^{p(\theta-1)}$ is bounded since $p(\theta-1) < -3$ and choosing $\delta > 0$ sufficiently small such that $\delta^{p/(p-1)} \le \delta/2$. Integrating by parts in I_6 leads to

$$\begin{aligned} I_6 &= -\sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\Delta\langle v\rangle^\theta f_i\mathrm{d}v\mathrm{d}s \\ &\le C\sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\langle v\rangle^{\theta-2} f_i\mathrm{d}v\mathrm{d}s \le C\sum_{i,j=1}^s\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\mathrm{d}s, \end{aligned} \tag{29}$$

where we used $\langle v\rangle^{\theta-2} \le 1$ (note that $\theta < 1$) and mass conservation. It follows from Jensen's inequality, applied to the probability measure $(f_i/n_i)\mathrm{d}v$, that for $q \ge 0$ and $r \ge 1$,

$$\left(\int_{\mathbb{R}^3}\langle v\rangle^q\frac{f_i^\varepsilon}{n_i}\mathrm{d}v\right)^r \le \int_{\mathbb{R}^3}\langle v\rangle^{qr}\frac{f_i^\varepsilon}{n_i}\mathrm{d}v. \tag{30}$$

The final term I_7 becomes

$$\begin{aligned} I_7 &= -\theta\sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3}\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}m_i\langle v\rangle^{\theta-2}\big(|v|^2 - v\cdot u_{ji}^\varepsilon[f^\varepsilon]\big)f_i^\varepsilon\mathrm{d}v\mathrm{d}s \\ &\le C\sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\langle v\rangle^{\theta-1}\frac{|u_{ji}^\varepsilon[f^\varepsilon]|}{T_{ji}^\varepsilon[f^\varepsilon]}f_i^\varepsilon\mathrm{d}v\mathrm{d}s \\ &\le C\sum_{i,j=1}^s\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\frac{|u_{ji}^\varepsilon[f^\varepsilon]|}{T_{ji}^\varepsilon[f^\varepsilon]}\mathrm{d}s, \end{aligned}$$

where we used $\langle v\rangle^{\theta-1}\le 1$ and mass conservation. In view of definition (19) and Jensen's inequality (30), we have

$$|u_{ji}^\varepsilon[f^\varepsilon]|^K \le \max\big\{|u_i^\varepsilon[f^\varepsilon]|, |u_j^\varepsilon[f^\varepsilon]|\big\}^K \le \bigg(\sum_{i=1}^s \frac{1}{n_i}\int_{\mathbb{R}^3}\langle v\rangle \min\{f_i^\varepsilon, g(v)/\varepsilon\}\mathrm{d}v\bigg)^K \tag{31}$$
$$\le C\bigg(\sum_{i=1}^s \int_{\mathbb{R}^3}\langle v\rangle f_i^\varepsilon \mathrm{d}v\bigg)^K \le C\sum_{i=1}^s \int_{\mathbb{R}^3}\langle v\rangle^K f_i^\varepsilon \mathrm{d}v.$$

Thus, by Young's inequality and $\langle v\rangle^K \le C(\delta) + (\delta/8)\langle v\rangle^{K+\theta}$,

$$I_7 \le \sum_{i,j=1}^s \int_0^t |u_{ji}^\varepsilon[f^\varepsilon]|^K \mathrm{d}s + C\sum_{i,j=1}^s \int_0^t \bigg|\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}\bigg|^{K/(K-1)}\mathrm{d}s \tag{32}$$
$$\le C(\delta) + \frac{\delta}{8}\sum_{i=1}^s \int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s + C\sum_{i,j=1}^s \int_0^t \bigg|\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}\bigg|^{K/(K-1)}\mathrm{d}s.$$

Let us distinguish two cases, according to the value of γ.

Case 1: $\gamma \ge 0$. We distinguish the subcases $\gamma\ge 2$ and $0\le\gamma<2$. First, let $\gamma\ge 2$. Jensen's inequality (30) leads to

$$c_{ji}^\varepsilon[f^\varepsilon] \le \varepsilon + C|T_j^{\varepsilon,\uparrow}|^{\gamma/2} \le 1 + C\bigg(\int_{\mathbb{R}^3}\langle v\rangle^2 f_i^\varepsilon \mathrm{d}v\bigg)^{\gamma/2} \le 1 + C\int_{\mathbb{R}^3}\langle v\rangle^\gamma f_i^\varepsilon \mathrm{d}v.$$

If $0\le\gamma<2$, we apply Young's inequality:

$$c_{ji}^\varepsilon[f^\varepsilon] \le \varepsilon + C|T_j^{\varepsilon,\uparrow}|^{\gamma/2} \le 1 + C\bigg(\int_{\mathbb{R}^3}\langle v\rangle^2 f_i^\varepsilon \mathrm{d}v\bigg)^{\gamma/2} \le 1 + C\int_{\mathbb{R}^3}\langle v\rangle^2 f_i^\varepsilon \mathrm{d}v.$$

Summarizing, we obtain for all $\gamma\ge 0$:

$$c_{ji}^\varepsilon[f^\varepsilon] \le 1 + C\int_{\mathbb{R}^3}\langle v\rangle^{\max\{\gamma,2\}} f_i^\varepsilon \mathrm{d}v. \tag{33}$$

Consequently, if we choose K sufficiently large, (29) yields

$$I_6 \le C + C\sum_{i=1}^s \int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{\max\{\gamma,2\}} f_i^\varepsilon \mathrm{d}v \le C(\delta) + \frac{\delta}{8}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v.$$

To estimate the last term in (32), we bound $T_{ji}^\varepsilon[f^\varepsilon]$ from below. For this, we choose an arbitrary $\lambda>0$ and set $u_i^\varepsilon = u_i^\varepsilon[f^\varepsilon]$:

$$T_i^{\varepsilon,\downarrow}[f^\varepsilon] \ge C\int_{\mathbb{R}^3} f_i^\varepsilon |v-u_i^\varepsilon|^2 \mathrm{d}v \ge C\int_{\{|v-u_i^\varepsilon|>\lambda\}} f_i^\varepsilon |v-u_i^\varepsilon|^2\mathrm{d}v \tag{34}$$
$$\ge C\lambda^2 \int_{\{|v-u_i^\varepsilon|>\lambda\}} f_i^\varepsilon \mathrm{d}v = C\lambda^2\bigg(n_i - \int_{\{|v-u_i^\varepsilon|\le\lambda\}} f_i^\varepsilon\mathrm{d}v\bigg).$$

Applying the Cauchy–Schwarz inequality to the last integral, we have

$$T_i^{\varepsilon,\downarrow}[f^\varepsilon] \geq C\lambda^2\bigg\{n_i - \|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}\bigg(\int_{\{|v-u_i^\varepsilon|\leq\lambda\}}\mathrm{d}v\bigg)^{1/2}\bigg\} \geq C\lambda^2\big(n_i - C\lambda^{3/2}\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}\big),$$

since the integral over any ball in $\mathbb{R}^3$ with radius λ is of the order λ^3. We obtain with the choice $\lambda = C_0 n_i^{2/3}\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}^{-2/3}$ for some $C_0 > 0$:

$$T_i^{\varepsilon,\downarrow}[f^\varepsilon] \geq CC_0^2(1 - CC_0^{3/2})n_i^{7/3}\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}^{-4/3}$$

and therefore, choosing $C_0 > 0$ sufficiently small,

$$T_{ji}^\varepsilon[f^\varepsilon] \geq \min\big\{T_i^{\varepsilon,\downarrow}[f^\varepsilon], T_j^{\varepsilon,\downarrow}[f^\varepsilon]\big\} \geq C\bigg(\sum_{k=1}^s\|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\bigg)^{-2/3}. \tag{35}$$

We continue with the estimate of the last term in (32). We infer from Young's inequality with exponents $3(K-1)/(2K)$ and $3(K-1)/(K-3)$ as well as estimate (33) and Jensen's inequality (30) that

$$\begin{aligned}\sum_{i,j=1}^s\bigg|\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}\bigg|^{K/(K-1)} &\leq \sum_{i,j=1}^s T_{ji}^\varepsilon[f^\varepsilon]^{-3/2} + C\sum_{i,j=1}^s c_{ji}^\varepsilon[f^\varepsilon]^{3K/(K-3)} \\ &\leq C + C\sum_{k=1}^s\|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + C\sum_{i=1}^s\int_{\mathbb{R}^3}\langle v\rangle^{3K\max\{\gamma,2\}/(K-3)}f_i^\varepsilon\mathrm{d}v.\end{aligned}$$

For sufficiently large $K > 0$, we have $3K\max\{\gamma,2\}/(K-3) < K+\theta$. Hence,

$$\sum_{i,j=1}^s\int_0^t\bigg|\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}\bigg|^{K/(K-1)}\mathrm{d}s \leq C(\delta) + C\sum_{i=1}^s\int_0^t\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\mathrm{d}s + \frac{\delta}{8}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v.$$

We infer from (32) that

$$I_7 \leq C(\delta) + \frac{\delta}{4}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v + C\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}(f_i^\varepsilon)^2\mathrm{d}v\mathrm{d}s.$$

Case 2: $\gamma < 0$. It follows from (35) that

$$c_{ji}^\varepsilon[f^\varepsilon] \leq \varepsilon + C|T_j^{\varepsilon,\downarrow}|^{\gamma/2} \leq 1 + C\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}^{-2\gamma/3} \leq 1 + C\bigg(\sum_{k=1}^s\|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\bigg)^{-\gamma/3}. \tag{36}$$

Therefore, estimates (29), (32) lead to

$$\begin{aligned}I_6 + I_7 \leq{}& C\sum_{i,j=1}^s\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\mathrm{d}s + C(\delta) + \frac{\delta}{8}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v\mathrm{d}s \\ &+ C\sum_{i,j=1}^s\int_0^t\bigg|\frac{c_{ji}^\varepsilon[f^\varepsilon]}{T_{ji}^\varepsilon[f^\varepsilon]}\bigg|^{K/(K-1)}\mathrm{d}s\end{aligned}$$

$$\leq C(\delta) + \frac{\delta}{8}\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s$$
$$+ C\sum_{i,j=1}^s \int_0^t \bigg(\sum_{k=1}^s \|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\bigg)^{K(2-\gamma)/(3(K-1))} \mathrm{d}s.$$

The Gagliardo–Nirenberg inequality

$$\|f_k\|_{L^2(\mathbb{R}^3)} \leq C\|f_k\|_{L^1(\mathbb{R}^3)}^{1-\xi}\|\nabla f_k\|_{L^p(\mathbb{R}^3)}^{\xi}, \quad \text{where } \xi = \frac{3p}{2(4p-3)},$$

and mass conservation imply that

$$I_6 + I_7 \leq C(\delta) + \frac{\delta}{8}\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s + \frac{\delta}{2}\int_0^t \int_{\mathbb{R}^3} |\nabla f_i^\varepsilon|^p \mathrm{d}v\mathrm{d}s,$$

as long as $2\xi(2-\gamma)/3p$ or equivalently $p > (5-\gamma)/4$.

In both cases, summarizing the estimates for $I_4, \ldots, I_7$, we conclude from (28) that

$$\sum_{i=1}^s \int_{\mathbb{R}^3} \langle v\rangle^{\theta} f_i^\varepsilon(t)\mathrm{d}v + \frac{\delta}{2}\int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s$$
$$\leq C(\delta) + \frac{\delta}{2}\int_0^t \int_{\mathbb{R}^3} |\nabla f_i^\varepsilon|^p \mathrm{d}v\mathrm{d}s + C\sum_{i=1}^n \int_0^t \int_{\mathbb{R}^3} (f_i^\varepsilon)^2 \mathrm{d}v\mathrm{d}s. \tag{37}$$

We still need to control the integrals on the right-hand side of (37), which is done in the next step.

Step 2: Test function f_i^ε. We use the test function f_i^ε in (26) and sum over $i = 1, \ldots, s$:

$$\frac{1}{2}\sum_{i=1}^s \int_{\mathbb{R}^3} f_i^\varepsilon(t)^2 \mathrm{d}v - \frac{1}{2}\sum_{i=1}^s \int_{\mathbb{R}^3} (f_i^0)^2 \mathrm{d}v + \delta\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K (f_i^\varepsilon)^2 \mathrm{d}v\mathrm{d}s \tag{38}$$
$$+ \delta\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla f_i^\varepsilon|^p \mathrm{d}v\mathrm{d}s + \sum_{i,j=1}^s \int_0^t \int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]|\nabla f_i^\varepsilon|^2 \mathrm{d}v\mathrm{d}s$$
$$= \delta\sum_{i=1}^s \int_0^t \bigg(\int_{\mathbb{R}^3} f_i^\varepsilon g(v)\mathrm{d}v\bigg)\bigg(\int_{\mathbb{R}^3} \langle v\rangle^K f_i^\varepsilon \mathrm{d}v\bigg)\mathrm{d}s$$
$$- \frac{1}{2}\sum_{i,j=1}^s \int_0^t \int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\frac{m_i}{T_{ji}^\varepsilon[f^\varepsilon]}(v - u_{ji}^\varepsilon[f^\varepsilon]) \cdot \nabla (f_i^\varepsilon)^2 \mathrm{d}v\mathrm{d}s$$
$$=: I_8 + I_9.$$

We use mass conservation to infer that $\int_{\mathbb{R}^3} f_i^\varepsilon g(v)\mathrm{d}v \leq \int_{\mathbb{R}^3} f_i^\varepsilon \mathrm{d}v \leq C$ and hence,

$$I_8 \leq \delta C\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K f_i^\varepsilon \mathrm{d}v \leq C + \frac{\delta}{8}\sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+\theta} f_i^\varepsilon \mathrm{d}v\mathrm{d}s,$$

and the last integral can be absorbed by the left-hand side of (38). By integration by parts and the lower bound (35), we have

$$
\begin{aligned}
I_9 &= \frac{1}{2}\sum_{i,j=1}^{s}\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]\frac{m_i}{T_{ji}^\varepsilon[f^\varepsilon]}\operatorname{div}(v-u_{ji}^\varepsilon[f^\varepsilon])(f_i^\varepsilon)^2\mathrm{d}v\mathrm{d}s \\
&= \frac{3}{2}\sum_{i,j=1}^{s}\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\frac{m_i}{T_{ji}^\varepsilon[f^\varepsilon]}\|f_i^\varepsilon\|_{L^2(\mathbb{R}^3)}^2\mathrm{d}s \le C\sum_{i,j,k=1}^{s}\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^{10/3}\mathrm{d}s.
\end{aligned} \tag{39}
$$

Let $\gamma \ge 0$. Then the Gagliardo–Nirenberg inequality with $\zeta = 3p/(8p-6)\in(0,1)$ and mass conservation lead to

$$
I_9 \le C\sum_{i,j,k=1}^{s}\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\|\nabla f_k^\varepsilon\|_{L^p(\mathbb{R}^3)}^{10\zeta/3}\|f_k^\varepsilon\|_{L^1(\mathbb{R}^3)}^{10(1-\zeta)/3}\mathrm{d}s \le C\sum_{i,j,k=1}^{s}\int_0^t c_{ji}^\varepsilon[f^\varepsilon]\|\nabla f_k^\varepsilon\|_{L^p(\mathbb{R}^3)}^{5p/(4p-3)}\mathrm{d}s.
$$

Then, using Young's inequality, estimate (33) for $c_{ji}^\varepsilon[f^\varepsilon]$, and Jensen's inequality (30),

$$
\begin{aligned}
I_9 &\le \frac{\delta}{8}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s + C(\delta)\sum_{i,j=1}^{s}|c_{ji}^\varepsilon[f^\varepsilon]|^{(4p-3)/(4p-8)} \\
&\le C + \frac{\delta}{8}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s + C(\delta)\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{(2+\gamma)(4p-3)/(4p-8)}f_i^\varepsilon\mathrm{d}v \\
&\le C(\delta) + \frac{\delta}{8}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s + \frac{\delta}{8}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v,
\end{aligned}
$$

if we choose $K+\theta > (2+\gamma)(4p-3)/(4p-8)$.

If $\gamma < 0$, estimates (36) and (39) imply that

$$
I_9 \le C\sum_{k=1}^{s}\int_0^t \|f_k^\varepsilon\|_{L^2(\mathbb{R}^3)}^{(10-2\gamma)/3}\mathrm{d}s,
$$

and Gagliardo–Nirenberg and Young's inequalities allow us to bound I_9 similarly as above as

$$
I_9 \le C(\delta) + \delta\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s,
$$

as long as $p > 2-\gamma/4$.

In both cases, we insert the estimates for I_8 and I_9 into (38) to obtain

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{s}\int_{\mathbb{R}^3} f_i^\varepsilon(t)^2\mathrm{d}v + \frac{\delta}{2}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^K(f_i^\varepsilon)^2\mathrm{d}v\mathrm{d}s + \frac{\delta}{4}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s \\
&\quad + \sum_{i,j=1}^{s}\int_0^t\int_{\mathbb{R}^3} c_{ji}^\varepsilon[f^\varepsilon]|\nabla f_i^\varepsilon|^2\mathrm{d}v\mathrm{d}s \le C(\delta) + \frac{\delta}{4}\sum_{i=1}^{s}\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v\mathrm{d}s.
\end{aligned}
$$

Step 3: End of the proof. We add the previous inequality to (37),

$$
\begin{aligned}
&\sum_{i=1}^s\int_{\mathbb{R}^3}\big(f_i^\varepsilon(t)^2+\langle v\rangle^\theta f_i^\varepsilon(t)\big)\mathrm{d}v+\frac{\delta}{2}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^K(f_i^\varepsilon)^2\mathrm{d}v\mathrm{d}s \\
&\quad+\frac{\delta}{4}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v\mathrm{d}s+\sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3}c_{ji}^\varepsilon[f^\varepsilon]|\nabla f_i^\varepsilon|^2\mathrm{d}v\mathrm{d}s \\
&\quad+\frac{\delta}{4}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v\mathrm{d}s\le C(\delta)+C\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}(f_i^\varepsilon)^2\mathrm{d}v\mathrm{d}s.
\end{aligned}
$$

Then Gronwall's lemma concludes the proof. □

Lemma 8. *There exists a constant $C(\delta,T)>0$ independent of ε and a number $r>1$ such that*

$$\|\partial_t f_i^\varepsilon\|_{L^r(0,T;W^{-1,p}(\mathbb{R}^3))}\le C(\delta,T).$$

Proof. The estimate for $\langle v\rangle^{K+\theta}f_i^\varepsilon$ in Lemma 7 and bounds (33), (36) show that $c_{ji}^\varepsilon[f^\varepsilon]$ is uniformly bounded in $L^{(K+\theta)/(2+\gamma)}(0,T)$ (or better), while $T_{ji}^\varepsilon[f^\varepsilon]^{-1}$ is uniformly bounded in $L^\infty(0,T)$ because of the lower bound (35) and the estimate for f_i^ε in $L^\infty(0,T;L^2(\mathbb{R}^3))$. Furthermore, we conclude from (31) that $|u_{ji}^\varepsilon[f^\varepsilon]|^{K+\theta}\le C\sum_{i=1}^s\langle v\rangle^{K+\theta}f_i^\varepsilon\mathrm{d}v$ (using the Jensen inequality (30)) is uniformly bounded in $L^1(0,T)$. This shows that $c_{ji}[f^\varepsilon]T_{ji}[f^\varepsilon]^{-1}u_{ji}[f^\varepsilon]$ is uniformly bounded in $L^{(K+\theta)/(3+\gamma)}(0,T)$. Furthermore, by Young's inequality and Lemma 7,

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}^3}(\langle v\rangle^K f_i^\varepsilon)^{(K+2\theta)/(K+\theta)}\mathrm{d}v\mathrm{d}s&=\int_0^T\int_{\mathbb{R}^3}\big(\langle v\rangle^{K+\theta}f_i^\varepsilon\big)^{K/(K+\theta)}\big(\langle v\rangle^K(f_i^\varepsilon)^2\big)^{\theta/(K+\theta)}\mathrm{d}v\mathrm{d}s \\
&\le C\int_0^T\|\langle v\rangle^{K+\theta}f_i^\varepsilon\|_{L^1(\mathbb{R}^3)}\mathrm{d}s+C\int_0^T\|\langle v\rangle^K(f_i^\varepsilon)^2\|_{L^1(\mathbb{R}^3)}\mathrm{d}s\le C.
\end{aligned}
$$

Together with the uniform bounds for f_i^ε from Lemma 7, this yields a uniform bound for $\partial_t f_i^\varepsilon$ in $L^r(0,T;W^{-1,p}(\mathbb{R}^3))$ for some $r>1$, finishing the proof. □

The bounds of Lemmas 7 and 8 and the compact embedding $W^{1,p}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3;\langle v\rangle^K\mathrm{d}v)\hookrightarrow L^2(\mathbb{R}^3)$ (see Lemma 13 in Appendix A) allow us to apply the Aubin–Lions lemma to conclude the existence of a subsequence (not relabeled) such that, as $\varepsilon\to0$,

$$f_i^\varepsilon\to f_i\quad\text{strongly in }L^2(0,T;L^2(\mathbb{R}^3)).$$

Furthermore, we obtain weak convergences for ∇f_i^ε and $\partial_t f_i^\varepsilon$ in suitable spaces. At this point, it is straightforward to pass to the limit $\varepsilon\to0$ in (26) to infer that $f_i^\delta:=f_i$ is a weak solution to

$$
\begin{aligned}
\partial_t f_i^\delta&+\delta\bigg(\langle v\rangle^K f_i^\delta-g(v)\int_{\mathbb{R}^3}\langle v\rangle^K f_i^\delta\mathrm{d}v\bigg)-\delta\operatorname{div}\big(|\nabla f_i^\delta|^{p-2}\nabla f_i^\delta\big) \\
&=\sum_{j=1}^s c_{ji}[f^\delta]\operatorname{div}\bigg(\nabla f_i^\delta+\frac{m_if_i^\delta}{T_{ji}[f^\delta]}(v-u_{ji}[f^\delta])\bigg)\quad\text{in }\mathbb{R}^3,\ t>0.
\end{aligned}\tag{40}
$$

We observe that the collision operator on the right-hand side is identical to that one in (1) and in particular, it conserves mass, momentum, and energy; see Lemma 2.

3.4. **Limit** $\delta \to 0$**.** Let f_i^δ be the solution to (2) and (40), constructed in the previous subsection. To perform the limit $\delta \to 0$, we derive some estimates uniform in δ. First, we note that mass conservation still holds, i.e. $\|f_i^\delta\|_{L^1(\mathbb{R}^3)} = n_i$ for $i = 1, \ldots, s$.

Lemma 9. *There exists a constant $C > 0$ independent of δ (but depending on the initial data) such that*

$$\sup_{0<t<T} \sum_{i=1}^s \int_{\mathbb{R}^3} \big(f_i^\delta(t)\log f_i^\delta(t) + f_i^\delta(t)|v|^2\big)\mathrm{d}v \le C,$$

$$\sum_{i,j=1}^s \int_0^T \int_{\mathbb{R}^3} c_{ji}[f^\delta] f_i^\delta \left|\nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]}\right|^2 \mathrm{d}v\mathrm{d}s \le C,$$

$$\delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla (f_i^\delta)^{(p-1)/p}|^p \mathrm{d}v\mathrm{d}s + \delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+2} f_i^\delta \mathrm{d}v\mathrm{d}s \le C.$$

Proof. We split the proof in several parts.

Step 1: Test function $\log f_i^\delta$*.* We use the test function $\log f_i^\delta$ in (40). Again, strictly speaking, this test function cannot be used since we cannot exclude that $f_i^\delta = 0$. We show in Appendix B how this argument can be made rigorous. We obtain from formulation (7) and property (16)

$$\begin{aligned}
&\sum_{i=1}^s \int_{\mathbb{R}^3} f_i^\delta(t)\log f_i^\delta(t)\mathrm{d}v - \sum_{i=1}^s \int_{\mathbb{R}^3} f_i^0 \log f_i^0 \mathrm{d}v + \delta c_p \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla (f_i^\delta)^{(p-1)/p}|^p \mathrm{d}v\mathrm{d}s \\
&\quad + \sum_{i,j=1}^s \int_0^t \int_{\mathbb{R}^3} c_{ji}[f^\delta] f_i^\delta \left|\nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]}\right|^2 \mathrm{d}v\mathrm{d}s \\
&\le -\delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K f_i^\delta \log f_i^\delta \mathrm{d}v\mathrm{d}s \\
&\quad + \delta \sum_{i=1}^s \int_0^t \bigg(\int_{\mathbb{R}^3} \log f_i^\delta g(v)\mathrm{d}v\bigg)\bigg(\int_{\mathbb{R}^3} \langle v\rangle^K f_i^\delta \mathrm{d}v\bigg)\mathrm{d}s =: I_{10} + I_{11}.
\end{aligned} \tag{41}$$

By mass conservation,

$$\int_{\mathbb{R}^3} \log f_i^\delta g(v)\mathrm{d}v \le \int_{\{f_i^\delta \ge 1\}} \log f_i^\delta g(v)\mathrm{d}v \le C \int_{\{f_i^\delta \ge 1\}} (1 + f_i^\delta) g(v)\mathrm{d}v \le C,$$

and consequently,

$$I_{11} \le C\delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K f_i^\delta \mathrm{d}v\mathrm{d}s \le C\delta + \frac{\delta}{32} \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+2} f_i^\delta \mathrm{d}v\mathrm{d}s.$$

The term I_{10} can be written as

$$I_{10} \le \delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K f_i^\delta \bigg(\log \frac{1}{f_i^\delta} \bigg)^+ \mathrm{d}v\mathrm{d}s,$$

recalling that $z^+ = \max\{0,z\}$. We choose $0 < \alpha < 1/(K+2)$ and use the inequality $\log z \le z^\alpha/\alpha$ for $z = 1/f_i^\delta > 1$ as well as Young's inequality to estimate

$$\begin{aligned} \langle v\rangle^K f_i^\delta \bigg(\log \frac{1}{f_i^\delta} \bigg)^+ &= \langle v\rangle^K 1_{\{f_i^\delta<1\}} f_i^\delta \log \frac{1}{f_i^\delta} \le \frac{1}{\alpha} \langle v\rangle^K (f_i^\delta)^{1-\alpha} \\ &= \alpha^{-1} \langle v\rangle^{-1} \big(\langle v\rangle^{K+1} (f_i^\delta)^{1-\alpha} \big) \le \alpha^{-1/\alpha} \langle v\rangle^{-1/\alpha} + \langle v\rangle^{(K+1)/(1-\alpha)} f_i^\delta. \end{aligned}$$

It follows from $K > 1$ that $-1/\alpha < -(K+2) < -3$ and hence, the integral over $\langle v\rangle^{-1/\alpha}$ is finite. This yields, since $(K+1)/(1-\alpha) < K+2$,

$$I_{10} \le C\delta + \delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{(K+1)/(1-\alpha)} f_i^\delta \mathrm{d}v\mathrm{d}s \le C\delta + \frac{\delta}{32} \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+2} f_i^\delta \mathrm{d}v\mathrm{d}s.$$

We insert the estimate for I_{10} and I_{11} into (41) to find that

$$\begin{aligned} &\sum_{i=1}^s \int_{\mathbb{R}^3} f_i^\delta(t) \log f_i^\delta(t) \mathrm{d}v + \delta c_p \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla (f_i^\delta)^{(p-1)/p}|^p \mathrm{d}v\mathrm{d}s \\ &\quad + \sum_{i,j=1}^s \int_0^t \int_{\mathbb{R}^3} c_{ji}[f^\delta] f_i^\delta \bigg| \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \bigg|^2 \mathrm{d}v\mathrm{d}s \\ &\le C + \frac{\delta}{16} \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+2} f_i^\delta \mathrm{d}v\mathrm{d}s. \end{aligned} \tag{42}$$

We need to estimate the right-hand side.

Step 2: Test function $|v|^2$. We use the test function $|v|^2$ (more precisely a suitable cutoff function, see Appendix B) in (40). Since the collision operator conserves the energy (see Lemma 2), the corresponding integral vanishes, and we end up with

$$\begin{aligned} &\sum_{i=1}^s \int_{\mathbb{R}^3} f_i^\delta(t) |v|^2 \mathrm{d}v - \sum_{i=1}^s \int_{\mathbb{R}^3} f_i^0 |v|^2 \mathrm{d}v + \delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^K |v|^2 f_i^\delta \mathrm{d}v\mathrm{d}s \\ &\quad = \sum_{i=1}^s \int_0^t \bigg(\int_{\mathbb{R}^3} |v|^2 g(v) \mathrm{d}v \bigg) \bigg(\int_{\mathbb{R}^3} \langle v\rangle^K f_i^\delta \mathrm{d}v \bigg) \mathrm{d}s \\ &\qquad - 2\delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla f_i^\delta|^{p-2} \nabla f_i^\delta \cdot v \mathrm{d}v\mathrm{d}s \\ &\quad \le C(\delta) + \frac{\delta}{8} \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} \langle v\rangle^{K+2} f_i^\delta \mathrm{d}v\mathrm{d}s + 2\delta \sum_{i=1}^s \int_0^t \int_{\mathbb{R}^3} |\nabla f_i^\delta|^{p-1} |v| \mathrm{d}v\mathrm{d}s. \end{aligned}$$

Since $\langle v\rangle^K|v|^2 = \langle v\rangle^{K+2} - \langle v\rangle^K \geq \frac12\langle v\rangle^{K+2} - C$, the last term on the left-hand side is bounded from below by

$$\begin{aligned}\delta\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^K|v|^2 f_i^\delta \mathrm{d}v\mathrm{d}s &\geq \frac{\delta}{2}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v\mathrm{d}s - C\delta\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}f_i^\delta\mathrm{d}v\mathrm{d}s \\ &\geq \frac{\delta}{2}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v\mathrm{d}s - C\delta,\end{aligned}$$

where we used again mass conservation in the last step. Therefore,

$$\sum_{i=1}^s\int_{\mathbb{R}^3}f_i^\delta(t)|v|^2\mathrm{d}v + \frac{3\delta}{8}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v\mathrm{d}s \leq C + 2\delta\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}|\nabla f_i^\delta|^{p-1}|v|\mathrm{d}v\mathrm{d}s. \tag{43}$$

We estimate the term on the right-hand side of (43). Let $q > 1$. We apply Young's inequality twice with exponents $(p, p/(p-1))$ and $(q, q/(q-1))$:

$$\begin{aligned}2\delta\int_{\mathbb{R}^3}&|\nabla f_i^\delta|^{p-1}|v|\mathrm{d}v \leq C\delta\int_{\mathbb{R}^3}\big(|v||f_i^\delta|^{(p-1)/p}\big)|\nabla(f_i^\delta)^{(p-1)/p}|^{p-1}\mathrm{d}s \\ &\leq C\delta\int_{\mathbb{R}^3}|v|^p|f_i^\delta|^{p-1}\mathrm{d}v + \frac{\delta}{4}c_p\int_{\mathbb{R}^3}|\nabla(f_i^\delta)^{(p-1)/p}|^p\mathrm{d}v \qquad (44)\\ &\leq \delta\int_{\mathbb{R}^3}\bigg(\frac{q-1}{q}\big(C|v|^p(f_i^\delta)^{1-1/q}\big)^{q/(q-1)} + \frac1q(f_i^\delta)^{(p-2+1/q)q}\bigg)\mathrm{d}v \\ &\quad + \frac{\delta}{4}c_p\int_{\mathbb{R}^3}|\nabla(f_i^\delta)^{(p-1)/p}|^p\mathrm{d}vv \\ &\leq C\delta\int_{R^3}|v|^{pq/(q-1)}f_i^\delta\mathrm{d}v + \frac{\delta}{q}\int_{\mathbb{R}^3}(f_i^\delta)^{1+q(p-2)}\mathrm{d}v + \frac{\delta}{4}c_p\int_{\mathbb{R}^3}|\nabla(f_i^\delta)^{(p-1)/p}|^p\mathrm{d}v,\end{aligned}$$

where $c_p > 0$ is as in (42). We deduce from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}&\|\psi\|_{L^r(\mathbb{R}^3)} \leq C\|\nabla\psi\|_{L^p(\mathbb{R}^3)}^\theta\|\psi\|_{L^{p/(p-1)}(\mathbb{R}^3)}^{1-\theta}, \quad \text{where} \\ &r = \frac{p}{p-1}(1+q(p-2)), \quad \theta = \frac{3q(p-1)(p-2)}{2(2p-3)(1+q(p-2))},\end{aligned}$$

applied to $\psi = (f_i^\delta)^{(p-1)/p}$, that

$$\begin{aligned}\frac{\delta}{q}\int_{\mathbb{R}^3}(f_i^\delta)^{1+q(p-2)}\mathrm{d}v &= \frac{\delta}{q}\|(f_i^\delta)^{(p-1)/p}\|_{L^r(\mathbb{R}^3)}^r \leq C\delta\|\nabla(f_i^\delta)^{(p-1)/p}\|_{L^p(\mathbb{R}^3)}^{r\theta}\|f_i^\delta\|_{L^1(\mathbb{R}^3)}^{(p-1)(1-\theta)/p} \\ &\leq C\delta\|\nabla(f_i^\delta)^{(p-1)/p}\|_{L^p(\mathbb{R}^3)}^{r\theta} \leq \frac{\delta}{4}c_p\|\nabla(f_i^\delta)^{(p-1)/p}\|_{L^p(\mathbb{R}^3)}^p + C\delta,\end{aligned}$$

where we used mass conservation in the last but one step and the fact $r\theta < p$ as well as Young's inequality in the last step. Choosing $q = 4/3$, the first term on the right-hand side of (44) is estimated according to

$$C\delta\int_{R^3}|v|^{pq/(q-1)}f_i^\delta\mathrm{d}v = C\delta\int_{R^3}|v|^{4p}f_i^\delta\mathrm{d}v \leq \frac{\delta}{4}\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v + C\delta,$$

if we choose $K > 4p-2$ so that $4p < K+2$. We conclude from (44) that

$$2\delta\int_{\mathbb{R}^3}|\nabla f_i^\delta|^{p-1}|v|\mathrm{d}v \le C\delta + \frac{\delta}{4}\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v + \frac{\delta}{2}c_p\|\nabla(f_i^\delta)^{(p-1)/p}\|_{L^p(\mathbb{R}^3)}^p$$

and then from (43) that

$$\begin{aligned}\sum_{i=1}^s\int_{\mathbb{R}^3}f_i^\delta|v|^2\mathrm{d}v + \frac{\delta}{8}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v\mathrm{d}s& \\ \le C + \frac{\delta}{2}c_p\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}|\nabla(f_i^\delta)^{(p-1)/p}|^p\mathrm{d}v\mathrm{d}s&.\end{aligned}$$

Step 3: End of the proof. We add the previous inequality to (42):

$$\begin{aligned}&\sum_{i=1}^s\int_{\mathbb{R}^3}\big(f_i^\delta(t)\log f_i^\delta(t) + f_i^\delta(t)|v|^2\big)\mathrm{d}v + \frac{\delta}{2}c_p\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}|\nabla(f_i^\delta)^{(p-1)/p}|^p\mathrm{d}v\mathrm{d}s \\ &\quad + \frac{\delta}{16}\sum_{i=1}^s\int_0^t\int_{\mathbb{R}^3}\langle v\rangle^{K+2}f_i^\delta\mathrm{d}v\mathrm{d}s + \sum_{i,j=1}^s\int_0^t\int_{\mathbb{R}^3}c_{ji}[f^\delta]f_i^\delta\left|\nabla\log\frac{f_i^\delta}{M_{ij}[f^\delta]}\right|^2\mathrm{d}v\mathrm{d}s \le C.\end{aligned}$$

This concludes the proof. □

The energy bound in Lemma 9 shows that the temperature $T_i[f^\delta]$, defined in (3), is bounded from above uniformly in δ and $(0,T)$. This implies that $c_{ji}[f^\delta]$, defined in (4), is bounded from above uniformly in δ and $(0,T)$ when $\gamma \ge 0$. We claim that the temperature $T_{ji}[f^\delta]$ is also uniformly bounded from below, which implies that $c_{ji}[f^\delta]$ is bounded from above uniformly in δ and $(0,T)$ also when $\gamma < 0$.

Lemma 10. *There exists a constant $c > 0$, only depending on the initial entropy (and in particular independent of δ), such that*

$$\inf_{0<t<T}T_{ji}[f^\delta(t)] \ge c > 0.$$

Proof. Define $\Phi(x) = \mu(1+x)\log(1+x) - \mu x$ for $x \ge 0$, where $\mu > 0$. Then $\Phi^*(y) = \mu e^{y/\mu} - y - \mu$ for $y \ge 0$ is its convex conjugate, and the Fenchel–Young inequality $xy \le \Phi(x) + \Phi^*(y)$ holds. We infer from the lower bound (34) and the Fenchel–Young inequality with $x = f_i^\delta$ and $y = 1$ that

$$\begin{aligned}T_i[f^\delta] &\ge C\lambda^2\bigg(n_i - \int_{\{|v-u_i|\le\lambda\}}f_i^\delta\mathrm{d}v\bigg) \\ &\ge C\lambda^2\bigg(n_i - \mu\int_{\mathbb{R}^3}(1+f_i^\delta)\log(1+f_i^\delta)\mathrm{d}v - \frac{4}{3}\pi\mu e^{1/\mu}\lambda^3\bigg) \\ &\ge C\lambda^2\bigg(n_i - \mu C_0 - \frac{4}{3}\pi\mu e^{1/\mu}\lambda^3\bigg),\end{aligned}$$

since the volume of the ball in $\mathbb{R}^3$ with radius λ equals $4\pi\lambda^3/3$, and C_0 depends on the initial data via the first estimate in Lemma 9. Then, choosing $\mu = 1/\log(C_0\lambda^{-3})$, a computation reveals that

$$T_i[f^\delta] \geq C\lambda^2\bigg(n_i - \frac{C_1}{\log(C_0\lambda^{-3})}\bigg), \quad \text{where } C_1 = C_0\bigg(1+\frac{4}{3}\pi\bigg).$$

It follows from the choice $\lambda = [C_0\exp(-2C_1/n_i)]^{1/3}$ that $T_i[f^\delta] \geq c > 0$ for $c = C\lambda^2 n_i/2$, and this inequality is uniform in $(0,T)$. It can be seen from (34) that C is proportional to $1/n_i$ such that the constant c only depends on the initial entropy and energy via C_0. Consequently, $T_{ji}[f^\delta] \geq \min\{T_i[f^\delta], T_j[f^\delta]\} \geq c > 0$. ☐

Remark 11. Observe that the uniform positive bound on $T_{ji}[f^\delta]$ yields a uniform bound for $c_{ji}[f^\delta]$ in $L^\infty(0,T)$ even in the case $\gamma < 0$ so that $c_{ji}[f^\delta]$ is uniformly bounded in $L^\infty(0,T)$ for any $\gamma \in \mathbb{R}$. We can also conclude a uniform positive bound for $c_{ji}[f^\delta]$ for every $\gamma \in \mathbb{R}$. ☐

Lemma 12. *There exists a constant $C > 0$ independent of δ such that*

$$\inf_{[0,T]} c_{ji}[f^\delta] \geq C^{-1}, \quad \sup_{[0,T]} c_{ji}[f^\delta] \leq C, \quad \|\nabla f_i^\delta\|_{L^2(0,T;L^1(\mathbb{R}^3))} \leq C.$$

Proof. The bounds for $c_{ji}[f^\delta]$ follow from definitions (3) and (4) as well as Lemmas 9 and 10. By the second estimate in Lemma 9 and the fact that $f_i^\delta|\nabla\log M_{ii}[f^\delta]|^2$ (which is bounded by the energy) is uniformly bounded in $L^\infty(0,T;L^1(\mathbb{R}^3))$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} c_{ji}[f^\delta]|\nabla(f_i^\delta)^{1/2}|^2\mathrm{d}v\mathrm{d}s &= \frac{1}{4}\int_0^T\int_{\mathbb{R}^3} c_{ji}[f^\delta]f_i^\delta|\nabla\log f_i^\delta|^2\mathrm{d}v\mathrm{d}s \\ &\leq \frac{1}{2}\int_0^T\int_{\mathbb{R}^3} c_{ji}[f^\delta]\bigg(f_i^\delta\bigg|\nabla\log\frac{f_i^\delta}{M_{ij}}[f^\delta]\bigg|^2 + f_i^\delta|\nabla\log M_{ij}[f^\delta]|^2\bigg)\mathrm{d}v\mathrm{d}s \leq C.\end{aligned}$$

Consequently, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_0^T\bigg(\int_{\mathbb{R}^3} c_{ji}[f^\delta]|\nabla f_i^\delta|\mathrm{d}v\bigg)^2\mathrm{d}s &= 4\int_0^T c_{ji}[f^\delta]^2\bigg(\int_{\mathbb{R}^3}(f_i^\delta)^{1/2}|\nabla(f_i^\delta)^{1/2}|\mathrm{d}v\bigg)^2\mathrm{d}s \\ &\leq 4\int_0^T c_{ji}[f^\delta]^2\|(f_i^\delta)^{1/2}\|^2_{L^2(\mathbb{R}^3)}\|\nabla(f_i^\delta)^{1/2}\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}s \\ &\leq 4\sup_{0<t<T}\|f_i^\delta(t)\|_{L^1(\mathbb{R}^3)}\int_0^T\int_{\mathbb{R}^3} c_{ji}[f^\delta]^2|\nabla(f_i^\delta)^{1/2}|^2\mathrm{d}v\mathrm{d}s \leq C.\end{aligned}$$

The lemma follows from the uniform lower bound for $c_{ji}[f^\delta]$. ☐

We claim that $\partial_t f_i^\delta$ is uniformly bounded in $L^r(0,T;W^{-1,1}(\mathbb{R}^3))$ for some $r > 1$. Indeed, by Lemma 9 and Jensen's inequality (30), $\delta\langle v\rangle^K f_i^\delta$ is uniformly bounded in $L^{(K+2)/K}(0,T;L^1(\mathbb{R}^3))$ and $f_i^\delta(v-u_{ji}[f^\delta])$ is uniformly bounded in $L^\infty(0,T;L^1(\mathbb{R}^3))$. Lemma 9 also shows that $\delta|\nabla f_i^\delta|^{p-2}\nabla f_i^\delta$ is uniformly bounded in $L^{p/(p-1)}(0,T;L^{p/(p-1)}(\mathbb{R}^3))$ and by Lemma 12, $c_{ji}[f^\delta]\nabla f_i^\delta$ is uniformly bounded in $L^2(0,T;L^1(\mathbb{R}^3))$. This shows the claim with $r = \min\{(K+2)/K, p/(p-1), 2\}$.

Since the embedding $W^{1,1}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3; (1+|v|^2)\mathrm{d}v) \hookrightarrow L^1(\mathbb{R}^3)$ is compact (the proof is similar to that one of Lemma 13), we can apply the Aubin–Lions lemma to conclude the existence of a subsequence (not relabeled) such that

$$f_i^\delta \to f_i \quad \text{strongly in } L^2(0,T;L^1(\mathbb{R}^3)).$$

Furthermore, for a subsequence,

$$\partial_t f_i^\delta \rightharpoonup \partial_t f_i \quad \text{weakly in } L^r(0,T;W^{-1,1}(\mathbb{R}^3)),$$

and $\delta \operatorname{div}(|\nabla f_i^\delta|^{p-2}\nabla f_i^\delta) \to 0$ strongly in $L^p(0,T;W^{-1,p}(\mathbb{R}^3))$.

Next, we claim that

$$\delta\langle v\rangle^K f_i^\delta \to 0 \quad \text{strongly in } L^1(0,T;L^1(\mathbb{R}^3)).$$

Indeed, the strong convergence of f_i^δ and the uniform bound for $\langle v\rangle^{K+2} f_i^\delta$ show that, for any $R>0$,

$$\begin{aligned}
\limsup_{\delta\to 0}\int_0^T\int_{\mathbb{R}^3}\delta\langle v\rangle^K f_i^\delta \mathrm{d}v\mathrm{d}s &= \limsup_{\delta\to 0}\int_0^T\bigg(\delta\int_{\{|v|\le R\}}\langle v\rangle^K f_i^\delta\mathrm{d}v + \delta\int_{\{|v|>R\}}\langle v\rangle^K f_i^\delta\mathrm{d}v\bigg)\mathrm{d}s \\
&= \limsup_{\delta\to 0}\int_0^T\int_{\{|v|>R\}}\langle v\rangle^K f_i^\delta\mathrm{d}v\mathrm{d}s \le \frac{1}{R^2}\limsup_{\delta\to 0}\int_0^T\int_{\mathbb{R}^3}\langle v\rangle^{K+2} f_i^\delta\mathrm{d}v\mathrm{d}s \le \frac{C}{R^2}.
\end{aligned}$$

This yields $\limsup_{\delta\to 0}\int_0^T\int_{\mathbb{R}^3}\langle v\rangle^K f_i^\delta \mathrm{d}v\mathrm{d}s = 0$, proving the claim.

The convergence $u_{ji}[f^\delta] \to u_{ji}[f]$ strongly in $L^q(0,T)$ for any $q<\infty$ follows from the uniform $L^\infty(0,T)$ bound of the energy $\sum_{i=1}^s\int_{\mathbb{R}^3} f_i^\delta|v|^2\mathrm{d}v$. To show the convergence of the temperature $T_{ji}[f^\delta]$, we need a uniform bound for a higher-order moment $\sum_{i=1}^s\int_{\mathbb{R}^3} f_i^\delta|v|^m\mathrm{d}v$ for some $m>2$. This is done in a similar way as in Step 2 of Lemma 9, where we used the test function $|v|^2$ in (40), but here we choose the test function $|v|^m$ with $m>2$. In this case, the collision operator gives a nonzero contribution, but our previous estimates show that it is bounded, since $u_{ji}[f^\delta]$ is uniformly bounded and $c_{ji}[f^\delta]$ and $T_{ji}[f^\delta]^{-1}$ are uniformly bounded from above. This yields the existence of a constant $C>0$ such that

$$\sup_{0<t<T}\sum_{i=1}^s\int_{\mathbb{R}^3}\langle v\rangle^m f_i^\delta(t)\mathrm{d}v \le C \quad \text{for some } m>2.$$

It follows from this bound that $T_{ji}[f^\delta] \to T_{ji}[f]$ strongly in $L^q(0,T)$ for every $q<\infty$. Now, we can pass to the limit $\delta\to 0$ in (40), showing that the limit function f_i is a weak solution to (1)–(6).

Appendix A. A compactness result

Lemma 13. *The space $W^{1,p}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3;(1+|v|^2)\mathrm{d}v)$ with $p>3$ is compactly embedded into $L^2(\mathbb{R}^3)$ and in $L^\infty(\mathbb{R}^3)$.*

Proof. The proof is inspired from [2, Lemma 1]. Let (f_n) be bounded in $V := W^{1,p}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3;(1+|v|^2)\mathrm{d}v)$. It follows from the continuous embedding $W^{1,p}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ that there exists a subsequence, which is not relabeled, such that $f_n \rightharpoonup f$ weakly in $L^\infty(\mathbb{R}^3)$ as $n\to\infty$. Let $B_M\subset\mathbb{R}^3$ be the ball around the origin with radius $M>0$. Then, in view of the compact embedding $W^{1,p}(B_M) \hookrightarrow L^\infty(B_M)$, up to a subsequence, $f_n \to f$ strongly in $L^\infty(B_M)$. Thanks to a Cantor diagonal argument, the subsequence (f_n) can be chosen independent of M. By the uniform bound in V and Fatou's lemma, we have $f \in V$. Next, for sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}\|f_n - f\|_{L^2(\mathbb{R}^3)} &= \int_{B_M} |f_n - f|^2 \mathrm{d}v + \int_{\mathbb{R}^3\setminus B_M} |f_n - f|^2 \mathrm{d}v \\ &\le \frac{\varepsilon}{2} + \frac{1}{M^2}\int_{\mathbb{R}^3} (1+|v|^2)|f_n - f|^2 \mathrm{d}v \le \varepsilon,\end{aligned}$$

if we choose also $M > 0$ sufficiently large. Hence, $f_n \to f$ strongly in $L^2(\mathbb{R}^3)$. We use the Gagliardo–Nirenberg inequality with $\beta = 3p/(5p-6) \in (0,1)$:

$$\|f_n - f\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla(f_n - f)\|^\beta_{L^p(\mathbb{R}^3)}\|f_n - f\|^{1-\beta}_{L^2(\mathbb{R}^3)} \le C\|f_n - f\|^{1-\beta}_{L^2(\mathbb{R}^3)} \to 0$$

as $n \to \infty$. This concludes the proof. $\square$

Appendix B. Rigorous test functions

We have used $\langle v\rangle^\theta$ for $\theta \ge 0$ and $\log f_i^\delta$ as test functions in the corresponding weak formulations, which is not rigorous. To make the computations rigorous, we need to approximate. First, we introduce the cutoff functions

$$\psi_R(x) = \psi_1\left(\frac{x}{R}\right), \quad \psi_1(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ \frac{1}{2}(1+\cos(\pi(|x|-1))) & \text{if } 1 \le |x| \le 2, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and use $\langle v\rangle^\theta \psi_R$ as a test function in (26) (we take $\theta = 0$ to verify the mass control). This leads to additional terms depending on ψ_R and $\nabla\psi_R$. We focus our attention to the most delicate one and use Hölder's inequality with exponents $p/(p-1)$ and p as well as $|\nabla\psi_R(v)| \le C/R$ in $\{R < |v| < 2R\}$ and $|\nabla\psi_R| = 0$ else:

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla f_i^\varepsilon|^{p-1}|\nabla\psi_R|\langle v\rangle^\theta \mathrm{d}v &\le \frac{\delta}{4}\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p \mathrm{d}v + C(\delta)\int_{\mathbb{R}^3}|\nabla\psi_R|^p\langle v\rangle^{p\theta}\mathrm{d}v \\ &\le \frac{\delta}{4}\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p \mathrm{d}v + \frac{C(\delta)}{R^p}\int_{\{|v|<2R\}}\langle v\rangle^{p\theta}\mathrm{d}v \le \frac{\delta}{4}\int_{\mathbb{R}^3}|\nabla f_i^\varepsilon|^p\mathrm{d}v + C(\delta)R^{-p+p\theta+3},\end{aligned}$$

and the last term vanishes as $R \to \infty$ since we have chosen $0 < \theta < 1 - 3/p$.

Second, we use the test function $\log(f_i^\delta + \eta) - \log\eta$ for $0 < \eta < 1$ in (40). For this, we observe that, by (16),

$$\begin{aligned}&\sum_{i,j=1}^s \int_{\mathbb{R}^3} c_{ij}[f^\delta] f_i^\delta \nabla\log\frac{f_i^\delta}{M_{ii}[f^\delta]}\cdot\nabla\log(f_i^\delta+\eta)\mathrm{d}v \\ &\quad = \sum_{i,j=1}^s\int_{\mathbb{R}^3} c_{ij}[f^\delta]f_i^\delta\left(1 - \frac{\eta}{f_i^\delta+\eta}\right)\nabla\log\frac{f_i^\delta}{M_{ij}[f^\delta]}\cdot\nabla\log f_i^\delta \mathrm{d}v\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} c_{ij}[f^\delta] f_i^\delta \left| \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \right|^2 \mathrm{d}v \\
&\quad - \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} c_{ij}[f^\delta] \frac{\eta}{f_i^\delta + \eta} \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \cdot \nabla f_i^\delta \mathrm{d}v \\
&= \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} c_{ij}[f^\delta] f_i^\delta \left(1 - \frac{\eta}{f_i^\delta + \eta} \right) \left| \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \right|^2 \mathrm{d}v \\
&\quad - \eta \sum_{i,j=1}^{s} \int_{\mathbb{R}^3} c_{ij}[f^\delta] \frac{f_i^\delta}{f_i^\delta + \eta} \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \cdot \nabla \log M_{ij}[f_i^\delta] \mathrm{d}v.
\end{aligned}
$$

Then we obtain from (40), putting all terms of order η to the right-hand side,

$$
\begin{aligned}
&\sum_{i=1}^{s} \int_{\mathbb{R}^3} \big((f_i^\delta(t) + \eta) \log(f_i^\delta(t) + \eta) - \eta \log \eta \big) \mathrm{d}v \\
&\qquad + \delta \sum_{i=1}^{s} \int_0^t \int_{\mathbb{R}^3} \langle v \rangle^K f_i^\delta \log(f_i^\delta(t) + \eta) \mathrm{d}v \mathrm{d}s \\
&\qquad + \delta c_p \sum_{i=1}^{s} \int_0^t \int_{\mathbb{R}^3} |\nabla (f_i^\delta + \eta)^{(p-1)/p}|^p \mathrm{d}v \mathrm{d}s \\
&\qquad + \sum_{i,j=1}^{s} \int_0^t \int_{\mathbb{R}^3} c_{ij}[f^\delta] f_i^\delta \left(1 - \frac{\eta}{f_i^\delta + \eta} \right) \left| \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \right|^2 \mathrm{d}v \mathrm{d}s \\
&\quad = \sum_{i=1}^{s} \int_{\mathbb{R}^3} \big((f_i^0 + \eta) \log(f_i^0 + \eta) - \eta \log \eta \big) \mathrm{d}v \\
&\qquad + \delta \sum_{i=1}^{s} \int_0^t \left(\int_{\mathbb{R}^3} \langle v \rangle^K f_i^\delta \mathrm{d}v \right) \left(\int_{\mathbb{R}^3} g(v) \log(f_i^\delta + \eta) \mathrm{d}v \right) \mathrm{d}v \\
&\qquad + \eta \sum_{i,j=1}^{s} \int_0^t \int_{\mathbb{R}^3} c_{ij}[f^\delta] \frac{f_i^\delta}{f_i^\delta + \eta} \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \nabla \log M_{ij}[f_i^\delta] \mathrm{d}v.
\end{aligned} \tag{45}
$$

The second term on the right-hand side can be bounded because of mass conservation. The last integral can be controlled by

$$
\begin{aligned}
&\eta \sum_{i,j=1}^{s} \int_0^t \int_{\mathbb{R}^3} c_{ij}[f^\delta] \frac{f_i^\delta}{f_i^\delta + \eta} \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \nabla \log M_{ij}[f_i^\delta] \mathrm{d}v \\
&\qquad \le \frac{1}{2} \sum_{i,j=1}^{s} \int_0^t \int_{\mathbb{R}^3} c_{ij}[f^\delta] f_i^\delta \left| \nabla \log \frac{f_i^\delta}{M_{ij}[f^\delta]} \right|^2 \mathrm{d}v
\end{aligned}
$$

$$+\frac{1}{2}\sum_{i,j=1}^{s}\int_0^t\int_{\mathbb{R}^3}c_{ij}[f^\delta]f_i^\delta\bigg|\frac{\eta}{f_i^\delta+\eta}\bigg|^2|\nabla\log M_{ij}[f^\delta]|^2\mathrm{d}v.$$

The first term on the right-hand side is absorbed by the left-hand side of (45). The function

$$G_\eta(v)=f_i^\delta(v)\bigg|\frac{\eta}{f_i^\delta(v)+\eta}\bigg|^2|\nabla\log M_{ij}[f^\delta](v)|^2$$

is uniformly bounded by $0\le G_\eta\le f_i^\delta|\nabla\log M_{ij}[f^\delta]|\in L^1(0,T;L^1(\mathbb{R}^3))$, and converges to zero a.e. in $\mathbb{R}^3\times(0,T)$. Therefore, by dominated convergence, $G_\eta\to 0$ strongly in $L^1(0,T;L^1(\mathbb{R}^3))$. Fatou's lemma allows us to perform the limit $\eta\to 0$ in (45). Then, proceeding as in Step 1 of the proof of Lemma 9, we derive the entropy inequality.

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Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA

Email address: hujw@uw.edu

Institute of Analysis and Scientific Computing, Technische Universität Wien, Wiedner Hauptstrasse 8–10, 1040 Wien, Austria

Email address: juengel@tuwien.ac.at

Institut für Angewandte Analysis, Universität Ulm, Helmholtzstrasse 18, 89081 Ulm, Germany

Email address: nicola.zamponi@uni-ulm.de