

CROSS-DIFFUSION SYSTEMS WITH ENTROPY STRUCTURE*

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Abstract. Some results on cross-diffusion systems with entropy structure are reviewed. The focus is on local-in-time existence results for general systems with normally elliptic diffusion operators, due to Amann, and global-in-time existence theorems by Lepoutre, Moussa, and co-workers for cross-diffusion systems with an additional Laplace structure. The boundedness-by-entropy method allows for global bounded weak solutions to certain diffusion systems. Furthermore, a partial result on the uniqueness of weak solutions is recalled, and some open problems are presented.

Key words. Strongly coupled parabolic systems, local existence of solutions, global existence of solutions, gradient flow, duality method, boundedness-by-entropy method, nonlinear Aubin-Lions lemma, Kullback-Leibler entropy.

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1. Introduction. Multi-species systems from physics, biology, chemistry, etc. can be modeled by reaction-diffusion equations. When the gradient of the density of one species induces a flux of another species, cross diffusion occurs. Mathematically, this means that the diffusion matrix involves nonvanishing off-diagonal elements. In many applications, it turns out that the diffusion matrix is neither symmetric nor positive definite, which considerably complicates the mathematical analysis (see the examples in Section 2 and [25, Section 4.1]). In recent years, some progress has been made in this analysis by identifying a structural condition, namely a formal gradient-flow or entropy structure, allowing for a mathematical treatment. In this review, we report on selected results obtained from several researchers.

The cross-diffusion equations have the form

$$\partial_t u_i - \sum_{j=1}^n \operatorname{div}(A_{ij}(u)\nabla u_j) = f_i(u) \quad \text{in } \Omega, \quad t > 0, \quad i = 1, \dots, n, \quad (1.1)$$

where $u_i(x, t)$ is the density or concentration or volume fraction of the i th species of a multicomponent mixture, $u = (u_1, \dots, u_n)$, $A_{ij}(u)$ are the diffusion coefficients, $f_i(u)$ is the reaction term of the i th species, and $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with smooth boundary. We impose no-flux and initial conditions

$$\sum_{j=1}^n A_{ij}\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad t > 0, \quad u_i(0) = u_i^0 \quad \text{in } \Omega, \quad i = 1, \dots, n, \quad (1.2)$$

with the exterior normal unit vector ν on $\partial\Omega$, but Dirichlet or mixed Dirichlet-Neumann boundary conditions could be considered as well [20]. Setting $A(u) = (A_{ij}(u))$ and $f(u) = (f_1(u), \dots, f_n(u))$, we may write (1.1) more compactly as

$$\partial_t u - \operatorname{div}(A(u)\nabla u) = f(u) \quad \text{in } \Omega, \quad t > 0.$$

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In contrast to scalar parabolic equations, generally there do not exist maximum principles or a regularity theory for diffusion systems. For instance, there exist Hölder continuous solutions to certain parabolic systems that develop singularities in finite time [37]. Here, the situation is even worse: The diffusion matrix $A(u)$ is generally neither symmetric nor positive definite such that coercivity theory cannot be applied. Our approach is to assume a structure inspired from thermodynamics: We suppose that there exists a convex function $h : \mathbb{R}^n \rightarrow \mathbb{R}$, called an entropy density, such that the (possibly nonsymmetric) matrix product $h''(u)A(u)$ is positive semidefinite (in the sense $z^\top h''(u)A(u)z \geq 0$ for all $z \in \mathbb{R}^n$). Here, $h''(u)$ denotes the Hessian of h at the point u . We say that A has a *strict* entropy structure if $h''(u)A(u)$ is positive definite for all u . Then the entropy $\mathcal{H}[u] = \int_\Omega h(u)dx$ is a Lyapunov functional along solutions to (1.1)-(1.2) if $f(u) \cdot h'(u) \leq 0$ for all u :

$$\frac{d\mathcal{H}}{dt} = \int_\Omega \partial_t u \cdot h'(u)dx = - \int_\Omega \nabla u : h''(u)A(u)\nabla u dx + \int_\Omega f(u) \cdot h'(u)dx \leq 0, \quad (1.3)$$

where “:” denotes the Frobenius matrix product. If $h''(u)A(u)$ is positive definite, this yields gradient estimates needed for the global existence analysis.

Introducing the entropy variables $w_i = \partial h / \partial u_i$ or $w = h'(u)$, we may write (1.1) equivalently as

$$\partial_t u(w) - \operatorname{div}(B(w)\nabla w) = f(u(w)), \quad B(w) := A(u(w))h''(u(w))^{-1}, \quad (1.4)$$

where $u(w) = (h')^{-1}(w)$ is interpreted as a function of $w = (w_1, \dots, w_n)$ and $h''(u)^{-1}$ is the inverse of the Hessian of h . By assumption, $B(w)$ is positive semidefinite, which indicates a (nonstandard) parabolic structure.

The entropy structure will be made more explicit for two examples in Section 2. In Sections 3 and 4, the local and global in time existence of solutions, respectively, will be reviewed. Furthermore, we comment in Section 5 on uniqueness results, and we close in Section 6 with some open problems.

2. Examples. We present two prototypic examples.

EXAMPLE 1 (Maxwell-Stefan equations). The dynamics of a fluid mixture of $n = 3$ components with volume fractions $u_1, u_2, u_3 = 1 - u_1 - u_2$ can be described by the Maxwell-Stefan equations [38], defined by (1.1) with

$$A(u) = \frac{1}{a(u)} \begin{pmatrix} d_2 + (d_0 - d_2)u_1 & (d_0 - d_1)u_1 \\ (d_0 - d_2)u_2 & d_1 + (d_0 - d_1)u_2 \end{pmatrix},$$

where $d_i > 0$ and $a(u) = d_1 d_2 (1 - u_1 - u_2) + d_0 (d_1 u_1 + d_2 u_2) > 0$. The model can be generalized to $n \geq 3$ components; see [4, 26]. For simplicity, we set $f \equiv 0$. Define the entropy density

$$h(u) = \sum_{i=1}^2 u_i (\log u_i - 1) + (1 - u_1 - u_2) (\log(1 - u_1 - u_2) - 1),$$

where $u = (u_1, u_2)$, and the entropy $\mathcal{H}[u] = \int_\Omega h(u)dx$. A formal computation shows that

$$\frac{d\mathcal{H}}{dt} + \int_\Omega \frac{1}{a(u)} \left(d_2 \frac{|\nabla u_1|^2}{u_1} + d_1 \frac{|\nabla u_2|^2}{u_2} + d_0 \frac{|\nabla(u_1 + u_2)|^2}{1 - u_1 - u_2} \right) dx = 0,$$

and in particular, $h''(u)A(u)$ is positive definite for $u_i > 0$. The entropy variables become $w_i = \partial h / \partial u_i = \log(u_i / (1 - u_1 - u_2))$ with inverse $u_i(w) = e^{w_i} / (1 + e^{w_1} + e^{w_2})$, which lies in the triangle $G = \{u \in \mathbb{R}^2 : u_1, u_2 > 0, 1 - u_1 - u_2 > 0\}$. This property makes sense since u_i are volume fractions and they are expected to be bounded. This property can be exploited in the existence analysis to obtain *bounded* solutions without using a maximum principle (which generally cannot be applied). \square

EXAMPLE 2 (Population model). The evolution of two interacting species may be modeled by equations (1.1) with the diffusion matrix

$$A(u) = \begin{pmatrix} a_{10} + a_{11}u_1 + a_{12}u_2 & a_{12}u_1 \\ a_{21}u_2 & a_{20} + a_{21}u_1 + a_{22}u_2 \end{pmatrix},$$

where $a_{ij} \geq 0$ [36]. We neglect the environmental potential and source terms, so $f \equiv 0$. The entropy is given by $\mathcal{H}[u] = \int_{\Omega} h(u) dx$, where $h(u) = a_{21}u_1(\log u_1 - 1) + a_{12}u_2(\log u_2 - 1)$. A formal computation shows that

$$\frac{d\mathcal{H}}{dt} + \int_{\Omega} \left\{ \left(\frac{a_{10}}{u_1} + a_{21}a_{11} \right) |\nabla u_1|^2 + \left(\frac{a_{20}}{u_2} + a_{12}a_{22} \right) |\nabla u_2|^2 + 4|\nabla \sqrt{u_1 u_2}|^2 \right\} dx = 0. \quad (2.1)$$

The entropy variables are $w_1 = a_{21} \log u_1$, $w_2 = a_{12} \log u_2$. Then the population densities are $u_1 = e^{w_1/a_{21}}$, $u_2 = e^{w_2/a_{12}} > 0$. An upper bound cannot be expected.

The model can be generalized to $n \geq 2$ species with diffusion coefficients

$$A_{ij}(u) = \delta_{ij} \left(a_{i0} + \sum_{k=1}^n a_{ik}u_k \right) + a_{ij}u_i, \quad i, j = 1, \dots, n. \quad (2.2)$$

The entropy structure is more delicate than in the two-species case. Indeed, assume that there exist numbers $\pi_i > 0$ such that the equations

$$\pi_i a_{ij} = \pi_j a_{ji}, \quad i, j = 1, \dots, n, \quad (2.3)$$

are satisfied. Then $h(u) = \sum_{i=1}^n \pi_i u_i (\log u_i - 1)$ is an entropy density, i.e. $d\mathcal{H}/dt \leq 0$ [8]. Equations (2.3) are recognized as the detailed-balance condition for the Markov chain with transition rates a_{ij} , and $\pi = (\pi_1, \dots, \pi_n)$ is the corresponding invariant measure [25, Section 5.1]. \square

3. Local existence of classical solutions. A very general result on the local-in-time existence of classical solutions to diffusion systems was proved by Amann (see [2, Section 1] or [3, Theorem 14.1]). A special version reads as follows.

THEOREM 3.1 (Amann [2]). *Let $G \subset \mathbb{R}^n$ be open, $A_{ij}, f_i \in C^\infty(G)$, all eigenvalues of $A(u)$ have positive real parts for all $u \in G$, and $u^0 \in V := \{v \in W^{1,p}(\Omega; \mathbb{R}^n) : v(\bar{\Omega}) \subset G\}$, where $p > d$. Then there exists a unique maximal solution u to (1.1)-(1.2) satisfying $u \in C^0([0, T^*]; V) \cap C^\infty(\bar{\Omega} \times (0, T^*); \mathbb{R}^n)$, where $0 < T^* \leq \infty$.*

An elliptic operator $u \mapsto \operatorname{div}(A(u)\nabla u)$ with the property that all eigenvalues of $A(u)$ have positive real parts is called *normally elliptic*. We claim that any cross-diffusion system with strict entropy structure is normally elliptic.

LEMMA 3.2 (Eigenvalues of A). *Let $A \in \mathbb{R}^{n \times n}$. We assume that there exists a symmetric, positive definite matrix $H \in \mathbb{R}^{n \times n}$ such that HA is positive definite. Then every eigenvalue of A has a positive real part.*

In the context of cross-diffusion systems, H stands for the Hessian $h''(u)$.

Proof. Let $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}$ be an eigenvalue of A with eigenvector $u = v + iw$, where $v, w \in \mathbb{R}^n$ with $v \neq 0$ or $w \neq 0$. It follows from $Au = \lambda u$

that $Av = \xi v - \eta w$, $Aw = \eta v + \xi w$. We multiply both equations by $v^\top H$, $w^\top H$, respectively:

$$0 < v^\top HAv = \xi v^\top Hv - \eta v^\top Hw, \quad 0 < w^\top HAw = \eta w^\top Hv + \xi w^\top Hw.$$

Since H is symmetric, we have $v^\top Hw = w^\top Hv$. Therefore, adding both identities,

$$0 < v^\top HAv + w^\top HAw = \xi(v^\top Hv + w^\top Hw).$$

We infer from the positive definiteness of H that $\xi > 0$, proving the claim. \square

4. Global existence of weak solutions. The classical solution of Amann can be continued for all time under some assumptions [3, Theorem 15.3].

THEOREM 4.1 (Amann [3]). *Let u be the classical maximal solution to (1.1)-(1.2) on $[0, T^*)$. Assume that $u|_{[0, T]}$ is bounded away from ∂G for each $T > 0$ and that there exists $\alpha > 0$ such that $\|u(t)\|_{C^{0, \alpha}} \leq C(T)$ for all $0 \leq t \leq T < \infty$, $t < T^*$. Then $T^* = \infty$.*

Unfortunately, it is not easy to derive a uniform bound in the Hölder norm. A possibility is to show that the gradient $\nabla u_i(t)$ satisfies some higher integrability, namely $L^p(\Omega)$ for $p > d$, since $W^{1, p}(\Omega)$ embeds continuously into $C^{0, \alpha}(\Omega)$ for $\alpha = 1 - p/d > 0$. Estimates in the $W^{1, p}$ norm with $p > d$ for a particular system were derived in, e.g., [23, 29].

Another approach is to find weak solutions using the entropy method as outlined in the introduction. The key elements of the existence proof are the definition of an approximate problem and a compactness argument. We are aware of two approaches in the literature. In both approaches, the time derivative is replaced by the implicit Euler discretization. This avoids issues with the (low) time regularity. To define the change of unknowns $u(w)$, we need bounded approximate solutions w . The first approach regularizes the equations by adding a weak form of $\varepsilon((-\Delta)^s w + w)$. Since $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ for $s > d/2$, this yields bounded weak solutions. The second approach formulates the implicit Euler scheme as a fixed-point equation involving the solution operator $(M - \Delta)^{-1}$ for sufficiently large $M > 0$. This allows one to exploit the regularization property of the solution operator $(M - \Delta)^{-1} : L^p(\Omega) \rightarrow W^{2, p}(\Omega)$, and the continuous embedding $W^{2, p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > d$ yields bounded solutions. We detail both approaches in the following subsections.

4.1. Boundedness-by-entropy method. This method does not only give the global existence of solutions but it also yields L^∞ bounds. It was first used in [5] and made systematic in [24]. The first key assumption is that the derivative $h' : G \rightarrow \mathbb{R}^n$ is invertible, where $G \subset \mathbb{R}^n$ is a bounded set. Then $u(w(x, t)) = (h')^{-1}(w(x, t)) \in G$ yields lower and upper bounds for the densities u_i ; see Example 1. The second key assumption is the positive definiteness of $h''(u)A(u)$. Applications indicate that this property does not hold uniformly in u . Therefore, we impose a weaker condition.

(H1) $h \in C^2(G; [0, \infty))$ is convex with invertible derivative $h' : G \rightarrow \mathbb{R}^n$.

(H2) $G \subset (0, 1)^n$ and for $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$ and $u = (u_1, \dots, u_n) \in G$,

$$z^\top h''(u)A(u)z \geq \kappa \sum_{i=1}^n u_i^{2m-2} z_i^2, \quad \text{where } m \geq \frac{1}{2}, \kappa > 0. \quad (4.1)$$

(H3) $A = (A_{ij}) \in C^0(G; \mathbb{R}^{n \times n})$ and $|A_{ij}(u)| \leq C_A |u_j|^a$ for all $u \in G$, $i, j = 1, \dots, n$, where $C_A, a > 0$.

(H4) $f \in C^0(G; \mathbb{R}^n)$ and $\exists C_f > 0: \forall u \in G: f(u) \cdot h'(u) \leq C_f(1 + h(u))$.

Hypothesis (4.1) is satisfied with $m = \frac{1}{2}$ in Examples 1 and 2 if $a_{10} > 0$, $a_{20} > 0$ and $m = 1$ in Example 2 if $a_{11} > 0$, $a_{22} > 0$. The following theorem is proved in [24, Theorem 2]; also see [25, Section 4.4].

THEOREM 4.2 (Global existence [24]). *Let (H1)-(H4) hold and let $u^0 \in L^1(\Omega; \mathbb{R}^n)$ be such that $u^0(\Omega) \subset \overline{G}$. Then there exists a bounded weak solution u to (1.1)-(1.2) satisfying $u(\Omega, t) \subset \overline{G}$ for all $t > 0$ and $u \in L^2_{\text{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^n))$, $\partial_t u \in L^2_{\text{loc}}(0, \infty; H^1(\Omega; \mathbb{R}^n)')$, for all $T > 0$ and $\phi \in L^2(0, T; H^1(\Omega; \mathbb{R}^n))$,*

$$\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_{\Omega} \nabla \phi : A(u) \nabla u dx dt = \int_0^T \int_{\Omega} f(u) \cdot \phi dx dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $H^1(\Omega)'$, and $u(0) = u^0$ holds in $H^1(\Omega; \mathbb{R}^n)'$.

The idea of the proof is to solve first for given u^{k-1} the regularized problem

$$\begin{aligned} & \frac{1}{\tau} \int_{\Omega} (u(w^k) - u(w^{k-1})) \cdot \phi dx + \int_{\Omega} \nabla \phi : B(w^k) \nabla w^k dx \\ & + \int_{\Omega} \left(\sum_{|\alpha|=s} D^{\alpha} w^k \cdot D^{\alpha} \phi + w^k \cdot \phi \right) dx = \int_{\Omega} f(u(w^k)) \cdot \phi dx \end{aligned} \quad (4.2)$$

for $\phi \in H^s(\Omega; \mathbb{R}^n)$, where $s > d/2$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n = s$ is a multiindex, $D^{\alpha} = \partial^s / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$ is a partial derivative of order m , $u(w) := (h')^{-1}(w)$, and w^k is an approximation of $w(\cdot, k\tau)$ with the time step $\tau > 0$. This problem is solved by the Leray-Schauder theorem. Uniform estimates are derived from a discrete version of the entropy-production identity (1.3) and Hypothesis (H2).

Let $u^{(\tau)}(x, t) = u(w^k(x))$ for $x \in \Omega$ and $t \in ((k-1)\tau, k\tau]$, $k = 1, \dots, N$, be piecewise constant functions in time. If $t = 0$, we set $u^{(\tau)}(\cdot, 0) = u^0$. We also need the time shift operator $(\sigma_{\tau} u^{(\tau)})(\cdot, t) = u(w^{k-1})$ for $t \in ((k-1)\tau, k\tau]$. It follows from the boundedness and the discrete entropy-production inequality that [25, Section 4.4]

$$\|u^{(\tau)}\|_{L^{\infty}(0, T; L^1(\Omega))} \leq C, \quad (4.3)$$

$$\tau^{-1} \|u^{(\tau)} - \sigma_{\tau} u^{(\tau)}\|_{L^2(0, T; H^s(\Omega)')} + \|(u^{(\tau)})^m\|_{L^2(0, T; H^1(\Omega))} \leq C, \quad (4.4)$$

where $C > 0$ is independent of ε and τ . (In fact, we have even a bound for $(u^{(\tau)})$ in $L^{\infty}(0, T; L^{\infty}(\Omega))$.) If $m = 1$, we deduce relative compactness for $(u^{(\tau)})$ in $L^2(Q_T)$ (where $Q_T = \Omega \times (0, T)$) from the discrete Aubin-Lions lemma in the version of [15]. When $m \neq 1$, we need the nonlinear version of [8, 11, 39].

LEMMA 4.3 (Nonlinear Aubin-Lions). *Let $T > 0$, $m > 0$, and let $(u^{(\tau)})$ be a family of nonnegative functions that are piecewise constant in time with uniform time step $\tau > 0$. Assume that there exists $C > 0$ such that (4.4) holds for all $\tau > 0$.*

- *Let $m > 1$ and let $(u^{(\tau)})$ be bounded in $L^{\infty}(Q_T)$. Then $(u^{(\tau)})$ is relatively compact in $L^p(Q_T)$ for any $p < \infty$ [39, Lemma 9].*
- *Let $1/2 \leq m \leq 1$. Then $(u^{(\tau)})$ is relatively compact in $L^{2m}(0, T; L^{pm}(\Omega))$, where $p \geq 1/m$ and $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous [11, Theorem 3].*
- *Let $\max\{0, 1/2 - 1/d\} < m < 1/2$ and let (4.3) hold. Then $(u^{(\tau)})$ is relatively compact in $L^1(0, T; L^{d/(d-1)}(\Omega))$ [8, Theorem 22].*

Another version of the nonlinear Aubin-Lions lemma is shown in [31].

Theorem 4.2 can be directly applied to the Maxwell-Stefan equations from Example 1 yielding the global existence of bounded weak solutions.

4.2. Cross-diffusion system with Laplace structure. Theorem 4.2 can be only applied to situations in which the densities are bounded (volume fractions). However, the method of proof can be adapted to cases, in which the domain G is not bounded. The main difference is that we cannot work in $L^\infty(\Omega)$ anymore but only in $L^p(\Omega)$ for suitable $p < \infty$. The precise value of p depends on m in Hypothesis (H2), and a global existence result can be proved under certain growth conditions on $A_{ij}(u)$ and $f_i(u)$. As an example, consider the population model from Example 2 for $n \geq 2$ species. The following theorem was proved in [8].

THEOREM 4.4 (Population model, linear A_{ij} [8]). *Let $u_i^0 \geq 0$ be such that $\int_\Omega h(u^0)dx < \infty$ and let the detailed-balance condition (2.3) and $a_{ii} > 0$ hold. Then there exists a weak solution $u = (u_1, \dots, u_n)$ to (1.1)-(1.2) with diffusion matrix (2.2) satisfying $u_i \geq 0$ in Ω , $t > 0$, and $u_i \in L^2_{\text{loc}}(0, \infty; H^1(\Omega))$, $\partial_t u_i \in L^q_{\text{loc}}(0, T; W^{1,q}(\Omega)')$, where $q = 2d + 2$ and $q' = (2d + 2)/(2d + 1)$.*

We have assumed that there is self-diffusion $a_{ii} > 0$, yielding an L^2 estimate for ∇u_i , which is stronger than the L^2 estimate for ∇u_i^m with $m < 1$. An existence result with vanishing self-diffusion $a_{ii} = 0$ was shown in [7] for the two-species model. Here, we only have an L^2 bound for $\nabla \sqrt{u_i}$. The lack of regularity for ∇u_i can be compensated by exploiting the gradient estimate for $\nabla \sqrt{u_1 u_2}$ in (2.1) and an $L^2 \log L^2$ estimate coming from the Lotka-Volterra reaction terms.

The detailed-balance condition can be replaced by a “weak cross-diffusion” assumption which is automatically satisfied if (A_{ij}) is symmetric; see [8, Formula (12)].

Another generalization concerns *nonlinear* diffusion coefficients

$$A_{ij}(u) = \delta_{ij} \left(a_{i0} + \sum_{k=1}^n a_{ik} u_k^{s_k} \right) + s_j a_{ij} u_i u_j^{s_j-1}, \quad i, j = 1, \dots, n, \quad (4.5)$$

for $s_i \geq 0$. The corresponding cross-diffusion system can be analyzed by the method of the previous subsection. However, improved results can be obtained by exploiting the Laplace structure, meaning that (1.1) with (4.5) writes as

$$\partial_t u_i - \Delta(u_i p_i(u)) = f_i(u), \quad \text{where } p_i(u) = \alpha_{i0} + \sum_{j=1}^n \alpha_{ij} u_j^{s_j}, \quad (4.6)$$

and $\alpha_{ij} = a_{ij}$ for $i \neq j$ and $\alpha_{ii} = (s_i + 1)a_{ii}$. Let $a_{ii} > 0$ and $s_i \leq 2$. Then, by the entropy-production inequality, $\nabla u_i^{s_i/2}$ is bounded in $L^2(Q_T)$, and the Gagliardo-Nirenberg inequality with $q = 2 + 4/(ds_i)$ and $\theta = ds_i/(2 + ds_i)$ shows that

$$\begin{aligned} \|u_i^{s_i/2}\|_{L^q(Q_T)}^q &= \int_0^T \|u_i^{s_i/2}\|_{L^q(\Omega)}^q dt \leq \int_0^T \|u_i^{s_i/2}\|_{H^1(\Omega)}^{q\theta} \|u_i^{s_i/2}\|_{L^{2/s_i}(\Omega)}^{q(1-\theta)} dt \\ &\leq \|u_i\|_{L^\infty(0,T;L^1(\Omega))}^{qs_i(1-\theta)/2} \int_0^T \|u_i^{s_i/2}\|_{H^1(\Omega)}^2 dt \leq C. \end{aligned}$$

We deduce that u_i is bounded in $L^{s_i+2/d}(Q_T)$. Using the duality method of Pierre [35], an improved regularity result can be derived. Indeed, set $\bar{u} = \sum_{i=1}^n u_i$ and $\mu = \sum_{i=1}^n u_i p_i(u)/\bar{u}$. If $f_i(u)$ grows at most linearly in u_i , we find that \bar{u} solves $\partial_t \bar{u} - \Delta(\mu \bar{u}) \leq C\bar{u}$ for some constant $C > 0$ depending on f_i . Then (see, e.g., [30, Lemma 1.2] or the review [34])

$$\int_0^T \int_\Omega \mu \bar{u}^2 dx dt \leq C(T, u^0). \quad (4.7)$$

We infer that $u_i^2 p_i(u)$ is uniformly bounded in $L^1(Q_T)$, giving a bound for u_i in $L^{s_i+2}(Q_T)$. For $d > 1$, this bound is better than the bound in $L^{s_i+2/d}(Q_T)$ derived above. The improved regularity is a key element in proving the global existence of solutions [30, Theorem 1.10] (also see the precursor versions in [12, 13]). We define the entropy density $h(u) = \sum_{i=1}^n h_i(u_i)$, where

$$h_i(u_i) = \begin{cases} (u_i^{s_i} - s_i u_i + s_i - 1)/(s_i - 1) & \text{if } s_i \neq 1, \\ u_i(\log u_i - 1) + 1 & \text{if } s_i = 1. \end{cases}$$

THEOREM 4.5 (Population model, nonlinear A_{ij} [30]). *Assume that $s_i > 0$, $s_i s_j \leq 1$ for $i \neq j$, let the detailed-balance condition (2.3) hold, and $f_i(u) = b_{i0} - \sum_{j=1}^n b_{ij} u_j^{\alpha_{ij}}$ for $b_{ij} \geq 0$ and $\alpha_{ij} < 1$. Finally, let $u_i^0 \in L^1(\Omega) \cap H^1(\Omega)'$, $\int_{\Omega} h_i(u_i^0) < \infty$. Then there exists an integrable solution $u_i \geq 0$ to (4.6) and (1.2) such that for all smooth test functions ϕ satisfying $\nabla \phi_i \cdot \nu = 0$ on $\partial\Omega$,*

$$\begin{aligned} - \int_0^\infty \int_{\Omega} u \cdot \partial_t \phi dx dt - \int_0^\infty \int_{\Omega} \sum_{i=1}^n u_i p_i(u) \Delta \phi_i dx dt \\ = \int_0^\infty \int_{\Omega} f(u) \cdot \phi dx dt + \int_{\Omega} u^0(x) \cdot \phi(x, 0) dx. \end{aligned}$$

It is an open problem to show the same result for arbitrary $s_i > 0$. The key idea of the proof is to formulate the implicit Euler scheme

$$\tau^{-1}(u_i^k - u_i^{k-1}) = \Delta F_i(u^k) + f_i(u^k), \quad \text{where } F_i(u^k) = u_i^k p_i(u^k),$$

as the fixed-point equation

$$u^k = F^{-1}\left((M - \Delta)^{-1}(u^{k-1} - u^k + MF(u^k))\right),$$

where $F = (F_1, \dots, F_n)$ and $M > 0$ is a sufficiently large number. In fact, if M is large and $u_i^{k-1} > 0$, we can show that $v := u_i^{k-1} - u_i^k + MF_i(u^k) > 0$, and by the maximum principle, $(M - \Delta)^{-1}v > 0$. Then, if F is a homeomorphism on $[0, \infty)^n$, $u_i^k > 0$, which yields positivity. Moreover, elliptic regularity theory implies that for $v \in L^p(\Omega)$ with $p > d/2$, we have $(M - \Delta)^{-1}v \in W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. This shows that u_i^k is bounded in L^∞ and it defines a fixed-point operator on $L^\infty(\Omega; \mathbb{R}^n)$.

The main assumption is that F is a homeomorphism. Under this assumption, Theorem 4.5 can be considerably generalized; see [30, Theorem 1.7] for details.

5. Uniqueness of weak solutions. The uniqueness of weak solutions to diffusion systems is a delicate topic. One of the first uniqueness results was shown in [1], assuming that the elliptic operator is linear and the time derivative of u_i is integrable. The latter hypothesis was relaxed in [32] allowing for finite-energy solutions but to scalar equations only. The uniqueness of solutions was shown in [33] for a cross-diffusion system with a strictly positive definite diffusion matrix. For cross-diffusion systems with entropy structure (and not necessarily positive definite $A(u)$), there are much less papers. The first result was for a special two-species population model [27], later extended to a volume-filling system [39], and generalized in [9] for a class of cross-diffusion systems. In this section, we report on the result of [9].

We allow for cross-diffusion systems involving drift terms,

$$\partial_t u_i = \operatorname{div} \sum_{j=1}^n (A_{ij}(u) \nabla u_j + B_{ij}(u) \nabla \phi), \quad i = 1, \dots, n, \quad (5.1)$$

where ϕ is a potential solving the Poisson equation

$$-\Delta\phi = u_0 - f(x) \quad \text{in } \Omega, \quad u_0 := \sum_{i=1}^n a_i u_i, \quad (5.2)$$

$a_i \geq 0$ are some constants, and $f(x)$ is a given background density. The equations are complemented by (1.2) and $\nabla\phi \cdot \nu = 0$ on $\partial\Omega$, $t > 0$. For consistency, we need to impose the condition $\int_{\Omega} \sum_{i=1}^n a_i u_i^0 dx = \int_{\Omega} f(x) dx$.

The uniqueness proof only works for a special class of coefficients, namely

$$A_{ij}(u) = p(u_0)\delta_{ij} + a_j u_i q(u_0), \quad B_{ij}(u) = r(u_0)u_i \delta_{ij}, \quad i, j = 1, \dots, n, \quad (5.3)$$

for some functions p , q , and r . The main result is as follows.

THEOREM 5.1 (Uniqueness of bounded weak solutions [9]). *Let $u^0 \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Let (u, ϕ) be a weak solution to (5.1)-(5.3), (1.2) such that $u_0(\Omega, t) \subset [0, L]$ for some $L > 0$. Assume that there exists $M > 0$ such that for all $s \in [0, L]$,*

$$p(s) \geq 0, \quad p(s) + q(s)s \geq 0, \quad (5.4)$$

$$r(s)s \in C^1([0, L]), \quad \frac{(r(s) + r'(s)s)^2}{p(s) + q(s)s} \leq M. \quad (5.5)$$

Then (u, ϕ) is unique in the class of solutions satisfying $\int_{\Omega} \phi dx = 0$, $\nabla\phi \in L^\infty(0, T; L^\infty(\Omega))$, and $u_i \in L^2(0, T; H^1(\Omega))$, $\partial_t u_i \in L^2(0, T; H^1(\Omega)')$ for $i = 1, \dots, n$. In the case $r \equiv 0$, the boundedness of u_0 is not needed, provided that $\sqrt{p(u_0)}\nabla u_i$, $\sqrt{|q(u_0)|}\nabla u_i \in L^2(\Omega \times (0, T))$.

The proof is based on the H^{-1} method and the entropy method of Gajewski [19]. First, we show the uniqueness of $u_0 = \sum_{i=1}^n a_i u_i$, solving

$$\partial_t u_0 = \operatorname{div}(\nabla Q(u_0) + R(u_0)\nabla\phi),$$

where $Q(s) = \int_0^s (p(z) + q(z)z) dz$ and $R(s) = r(s)s$. Since Q is nondecreasing, the use of the H^{-1} technique seems to be natural. Given two solutions (u, ϕ) and (v, ψ) , the idea is to use the test function χ that solves the dual problem $-\Delta\chi = u_0 - v_0$ in Ω , $\nabla\chi \cdot \nu = 0$ on $\partial\Omega$ and to show that $\frac{d}{dt} \|\nabla\chi\|_{L^2(\Omega)}^2 \leq C \|\nabla\chi\|_{L^2(\Omega)}^2$, using the monotonicity of Q . This implies that $u_0 = v_0$ and $\phi = \psi$. Second, we differentiate (a regularized version of) the semimetric

$$d(u, v) = \sum_{i=1}^n \int_{\Omega} \left(h(u_i) + h(v_i) - 2h\left(\frac{u_i + v_i}{2}\right) \right) dx,$$

where $h(s) = s(\log s - 1) + 1$. Computing the time derivative of $d(u(t), v(t))$, it turns out that the drift terms cancel and we end up with $\frac{d}{dt} d(u, v) \leq 0$ implying that $u = v$.

Gajewski's semimetric is related to the relative entropy or Kullback-Leibler entropy $\mathcal{H}[u|v] = \mathcal{H}[u] - \mathcal{H}[v] - \mathcal{H}'[v] \cdot \mathcal{H}(u - v)$ used in statistics [28]. In fact, the proof of Theorem 5.1 can be performed as well with the symmetrized relative entropy $d_0(u, v) = \mathcal{H}[u|v] + \mathcal{H}[v|u]$. Both distances $d(u, v)$ and $d_0(u, v)$ behave like $|u - v|^2$ for "small" $|u - v|$, but they lead to different expressions when computed explicitly. The Kullback-Leibler entropy was also employed to derive explicit exponential convergence rates to equilibrium [6] and to prove weak-strong uniqueness results for (diagonal) reaction-diffusion systems [18].

6. Open problems.

We mention some open questions.

- *Reaction terms:* Hypothesis (H4) excludes reaction terms which grow superlinearly. The global existence of solutions to cross-diffusion systems with, for instance, quadratic reactions is an open problem. One approach could be to consider renormalized instead of weak solutions, as done in [17] for (diagonal) reaction-diffusion systems. This is currently under development [9]. Another idea is to exploit the entropy techniques devised for reaction-diffusion systems [16].
- *n-species population model:* It is an open problem to find global solutions to the population model with diffusion matrix (2.2) and $n \geq 3$ without detailed balance or “weak cross-diffusion”. Numerical experiments indicate that standard choices like the Boltzmann entropy, relative entropy, etc. are not Lyapunov functionals. So, the problem to find a priori estimates is open.
- *Uniqueness of solutions:* The uniqueness result presented in Theorem 5.1 is rather particular. One may ask whether weak-strong uniqueness of solutions can be shown like in [18] for diagonal diffusion systems. In fact, uniqueness of weak solutions is known to be delicate even for drift-diffusion equations; see, e.g., [14].
- *Regularity theory:* The duality method yields global L^p regularity results for cross-diffusion systems with Laplace structure (see (4.7)). Another approach is to apply maximal L^p regularity theory as done in [21] for Maxwell-Stefan systems, at least for local solutions. The (open) question is to what extent this theory can be applied to general systems with entropy structure?
- *Entropies:* Given a cross-diffusion system, a major open question is how an entropy structure can be detected. In thermodynamics, often the entropy (more precisely: free energy) and entropy production are given and the system of partial differential equations follows from these quantities. Furthermore, it is an open question how large is the class of cross-diffusion systems with entropy structure. Are there diffusion systems with normally elliptic operator, which have no entropy structure?

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