GLOBAL EXISTENCE OF WEAK SOLUTIONS AND WEAK–STRONG UNIQUENESS FOR NONISOTHERMAL MAXWELL–STEFAN SYSTEMS

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Abstract. The dynamics of multicomponent gas mixtures with vanishing barycentric velocity is described by Maxwell–Stefan equations with mass diffusion and heat conduction. The equations consist of the mass and energy balances, coupled to an algebraic system that relates the partial velocities and driving forces. The global existence of weak solutions to this system in a bounded domain with no-flux boundary conditions is proved by using the boundedness-by-entropy method. A priori estimates are obtained from the entropy inequality which originates from the consistent thermodynamic modeling. Furthermore, the weak–strong uniqueness property is shown by using the relative entropy method.

1. Introduction

The dynamics of multicomponent gaseous mixtures with vanishing barycentric velocity and constant temperature can be described by the Maxwell–Stefan equations [23, 26]. The existence of local-in-time smooth and global-in-time weak solutions to these systems has been proved in [2, 13, 16, 21]. The analysis of *nonisothermal* gas mixtures is, however, incomplete. The existence of local-in-time solutions was shown in [19], while [15] investigated a special nonisothermal case. In this paper, we prove the existence of global-in-time weak solutions and the weak–strong uniqueness property for a rather general nonisothermal Maxwell–Stefan system. The novelty of our approach is the consistent thermodynamic modeling.

1.1. Model equations.

The evolution of the mass densities $\rho_i(x,t)$ of the ith gas component and the temperature $\theta(x,t)$ of the mixture is described by the mass and energy balances

$$\partial_t\rho_i + \operatorname{div} J_i = 0, \quad \partial_t(\rho e) + \operatorname{div} J_e = 0, \quad i = 1,\ldots,n, \tag{1}$$

$$J_i = \rho_i u_i, \quad J_e = -\kappa(\theta)\nabla\theta + \sum_{j=1}^n (\rho_j e_j + p_j)u_j \quad \text{in } \Omega,\ t>0, \tag{2}$$

2000 *Mathematics Subject Classification.* 35K51, 76N10, 76R50, 80A17.

Key words and phrases. Gas mixture, Maxwell–Stefan equations, nonisothermal model, nonequilibrium thermodynamics, existence of weak solutions, weak–strong uniqueness.

The authors acknowledge partial support from the Austrian Science Fund (FWF), grants P33010 and F65. This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, ERC Advanced Grant no. 101018153.

where $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, J_i and J_e are the diffusion and energy fluxes, respectively, u_i are the diffusional velocities, $\rho = \sum_{i=1}^n \rho_i$ is the total mass density, p_i the partial pressure with the total pressure $p = \sum_{i=1}^n p_i$, $\rho_i e_i$ the partial internal energy $\rho_i e_i$ with the total energy $\rho e = \sum_{i=1}^n \rho_i e_i$, and $\kappa(\theta)$ is the heat conductivity. Equations (1)–(2) are supplemented with the boundary and initial conditions

$$J_i \cdot \nu = 0, \quad J_e \cdot \nu = \lambda(\theta - \theta_0) \quad \text{on } \partial\Omega,\ t > 0, \tag{3}$$

$$\rho_i(0) = \rho_i^0, \quad \theta(0) = \theta^0 \quad \text{in } \Omega,\ i = 1, \ldots, n, \tag{4}$$

where ν is the exterior unit normal vector to $\partial\Omega$, $\theta_0 > 0$ is the given background temperature, and $\lambda > 0$ is a relaxation constant. The boundary conditions mean that the gas components cannot leave the domain, while heat exchange through the boundary is possible and proportional to the difference between the gas and background temperatures. To close the model, we need to determine u_i, $\rho_i e_i$, and p_i.

The velocities u_i are computed from the constrained algebraic Maxwell–Stefan system

$$-\theta \sum_{j=1}^n b_{ij}\rho_i\rho_j(u_i - u_j) = d_i \quad \text{for } i = 1, \ldots, n, \quad \sum_{i=1}^n \rho_i u_i = 0, \tag{5}$$

where the constant coefficients $b_{ij} = b_{ji} > 0$ model the interaction between the ith and jth components. The driving force d_i is given by

$$d_i = \rho_i\theta\nabla\frac{\mu_i}{\theta} - \theta(\rho_i e_i + p_i)\nabla\frac{1}{\theta}, \quad i = 1, \ldots, n, \tag{6}$$

where μ_i is the chemical potential. The constraint

$$\nabla p = 0 \quad \text{in } \Omega,\ t > 0, \tag{7}$$

is needed in order for our system to be thermodynamically consistent. We refer to Section 2 for details.

The internal energies $\rho_i e_i$ and chemical potentials μ_i are determined from the Helmholtz free energy (see (16)), and the pressure is computed from the Gibbs–Duhem relation. As shown in Section 2, these quantities are explicitly given by

$$\begin{aligned} \mu_i &= \frac{\theta}{m_i}\log\frac{\rho_i}{m_i} - c_w\theta(\log\theta - 1), & \rho_i e_i &= c_w\rho_i\theta, \\ \rho_i\eta_i &= -\frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) + c_w\rho_i\log\theta, & p_i &= \frac{\rho_i\theta}{m_i}, \quad i = 1, \ldots, n, \end{aligned} \tag{8}$$

where $\rho_i\eta_i$ is the entropy density of the ith component and $c_w > 0$ is the heat capacity. Then the driving force d_i and energy flux J_e simplify to

$$d_i = \frac{\nabla(\rho_i\theta)}{m_i}, \quad J_e = -\kappa\nabla\theta + \theta\sum_{i=1}^n \frac{\rho_i u_i}{m_i}. \tag{9}$$

The matrix M associated to the algebraic system (5) is singular (since $\sum_{i=1}^n d_i = 0$) and thus not positive definite. However, we recall in Section 3.1 that it is positive definite on the subspace $L = \{\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n : \sqrt{\boldsymbol{\rho}} \cdot \boldsymbol{y} = 0\}$ (here, $\sqrt{\boldsymbol{\rho}}$ is the vector with components $\sqrt{\rho_i}$). Therefore, the Bott–Duffin inverse of M, denoted by $M^{BD} = M^{BD}(\boldsymbol{\rho})$, exists and is symmetric and positive definite on L. Moreover, we show in Section 3.3 below that the fluxes can be expressed as a linear combination of the entropy variables (or thermo-chemical potentials) $\boldsymbol{\mu}/\theta = (\mu_1/\theta, \ldots, \mu_n/\theta)$ and $-1/\theta$,

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -Q(\boldsymbol{\rho}, \theta) \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}, \quad \text{where } Q(\boldsymbol{\rho}, \theta) = \begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix}, \tag{10}$$

and $A = (A_{ij}) \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} = (B_i) \in \mathbb{R}^n$, $a > 0$ are given by

$$A_{ij}(\boldsymbol{\rho}) = M_{ij}^{BD}\sqrt{\rho_i\rho_j}, \quad B_i(\boldsymbol{\rho}, \theta) = \theta \sum_{j=1}^n \frac{A_{ij}}{m_j}, \quad a(\boldsymbol{\rho}, \theta) = \theta^2\bigg(\kappa + \sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j}\bigg). \tag{11}$$

Here, variables in bold font are n-dimensional vectors. The Onsager matrix Q turns out to be positive semidefinite (see (33)), which reveals the parabolic structure of equations (1)–(2).

1.2. **State of the art.** The isothermal Maxwell–Stefan equations can be derived from the multispecies Boltzmann equations in the diffusive approximation [6]. The high-friction limit in Euler (–Korteweg) equations reveals a formal gradient-flow form of the Maxwell–Stefan equations [17], leading to Fick–Onsager diffusion fluxes instead of (5). In fact, it is shown in [5] that the Fick–Onsager and generalized Maxwell–Stefan approaches are equivalent. A formal Chapman–Enskog expansion of the stationary nonisothermal model was given in [27]. Another nonisothermal Maxwell–Stefan system was derived in [1], but with a different energy flux than ours.

Maxwell–Stefan systems with nonvanishing barycentric velocities can be formulated in the framework of hyperbolic–parabolic systems, which allows one to perform a local-in-time existence analysis [13]. Global-in-time regular solutions around the constant equilibrium state were found to exist in [14]. An existence analysis for Maxwell-Stefan systems coupled to the Navier–Stokes equations for the barycentric velocity can be found in [8] for the incompressible case and in [4] for the compressible situation. For steady-state problems, we refer to, e.g., [7, 24].

When the barycentric velocity vanishes, the (isothermal) Maxwell–Stefan equations can be solved by generalized parabolic theory. The existence of local-in-time classical solutions was proved in [2], while the existence of global-in-time weak solutions with general initial data was shown in [21]. Concerning the nonisothermal equations, we refer to [15], where an existence analysis for global-in-time weak solutions was presented. However, this model has some modeling deficiencies explained below. Therefore, our first aim is to prove the global existence for a thermodynamically consistent nonisothermal model.

The uniqueness of strong solutions to the isothermal Maxwell–Stefan equations was shown in [2, 16, 19], but the uniqueness of weak solutions for general coefficients b_{ij} is still unsolved. A very special case (the coefficients b_{ij} have two degrees of freedom only) was investigated in [9]. It was shown in [18] that strong solutions are unique in the class of weak solutions, which is known as the weak–strong uniqueness property. Our second aim is to prove this property for the nonisothermal case.

Let us detail the main differences of our work compared to [15]:

(i) The most important difference is the lack of validity of the Onsager reciprocity relations in the model of [15]. The relations imply the symmetry of the coefficients of the Onsager matrix; see (10). The choice in [15] leads to a cancelation in the entropy inequality, thus simplifying the estimation. Our results do not rely on this simplification; see Remark 6 for further details.

(ii) The constraint (7) on the pressure is not taken into account in [15]. This condition is not necessary mathematically, but its lack creates an inconsistency with the assumption of vanishing barycentric velocity. Indeed, a difference in pressure induces a force difference, which can result in an acceleration according to Newton's second law, if there is no additional force to balance it.

(iii) According to Onsager's reciprocity relations, the Onsager matrix Q in (10) has to be positive semidefinite. We show that Q is in fact positive definite on the subspace $L = \{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} \cdot \sqrt{\boldsymbol{\rho}} = 0\}$. In [15], is is *assumed* that this subspace equals $\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{y} \cdot \mathbf{1} = 0\}$. This is not consistent with the thermodynamic modeling.

(iv) We consider different molar masses m_i, while they are assumed to be the same in [15]. When we assume equal molar masses, the cross-terms cancel, and we end up with the simple heat flux $J_e = -\kappa\nabla\theta$ (see (9) and the constraint in (5)), thus decoupling the equations.

1.3. **Main results.** We impose the following assumptions:

(A1) Domain: $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary, and $T > 0$. We set $\Omega_T = \Omega \times (0,T)$ and $\mathbb{R}_+ = [0,\infty)$.

(A2) Data: $\rho_i^0 \in L^\infty(\Omega)$ satisfies $\rho_i^0 \geq 0$ in Ω and $0 < \rho_* \leq \sum_{i=1}^n \rho_i^0 \leq \rho^*$ in Ω for some $\rho_*, \rho^* > 0$ and for all $i = 1,\ldots,n$; $\theta^0 \in L^\infty(\Omega)$ satisfies $\inf_\Omega \theta^0 > 0$.

(A3) Coefficients: $b_{ij} = b_{ji} > 0$ for all $i,j = 1,\ldots,n$.

(A4) Heat conductivity: $\kappa \in C^0(\mathbb{R}_+^n \times \mathbb{R}_+)$ satisfies $c_\kappa(1+\theta^2) \leq \kappa(\theta) \leq C_\kappa(1+\theta^2)$ for some $c_\kappa, C_\kappa > 0$ and all $(\boldsymbol{\rho},\theta) \in \mathbb{R}_+^n \times \mathbb{R}_+$.

The lower bound for the total mass density ρ is needed to derive uniform estimates for the temperature. The proof of Lemma 10 in [18] shows that $M_{ij}^{BD}(\boldsymbol{\rho})$ is bounded for all $\boldsymbol{\rho} \in \mathbb{R}_+^n$. The growth condition for the heat conductivity is used to derive higher integrability bounds for the temperature, which are needed to derive a uniform estimate for the discrete time derivative of the temperature. We may also assume reaction terms R_i in (1) with the properties that the total reaction rate $\sum_{i=1}^n R_i$ vanishes and the vector of reaction rates R_i is derived from a convex, nonnegative potential [11, Section 2.2].

The first main result is the existence of solutions.

Theorem 1 (Existence of weak solutions)**.** *Let Assumptions (A1)–(A4) hold. Then there exists a weak solution to* (1)–(8) *satisfying* $\rho_i > 0$, $\theta > 0$ *a.e. in* $\Omega_T = \Omega \times (0,T)$ *and*

$$\begin{aligned}
&\sqrt{\rho_i} \in L^\infty(\Omega_T) \cap C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \partial_t\rho_i \in L^2(0,T;H^1(\Omega)^*), \\
&\theta \in C_w^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \partial_t(\rho\theta) \in L^{16/11}(0,T;W^{1,16/11}(\Omega)^*), \\
&\theta^2,\, \log\theta \in L^2(0,T;H^1(\Omega)), \quad i = 1,\ldots,n,
\end{aligned}$$

the weak formulation

$$\int_0^T \langle \partial_t \rho_i, \phi_i\rangle_{H^1(\Omega)^*}\mathrm{d}t + \int_0^T\int_\Omega \sum_{i,j=}^n M_{ij}^{BD}\big(2\nabla\sqrt{\rho_j} + \rho_j\nabla\log\theta\big)\mathrm{d}x\mathrm{d}t = 0,$$

$$\int_0^T\int_\Omega \langle \partial_t(\rho\theta), \phi_0\rangle_{W^{1,16/5}(\Omega)^*}\mathrm{d}t + \int_0^T\int_\Omega\sum_{i,j=1}^n \frac{\theta M_{ij}^{BD}}{m_im_j}\sqrt{\rho_i}\big(2\nabla\sqrt{\rho_j} + \sqrt{\rho_j}\nabla\log\theta\big)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T\int_\Omega \kappa\nabla\theta\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t = \lambda\int_0^T\int_{\partial\Omega}(\theta_0-\theta)\phi_0\mathrm{d}s\mathrm{d}t$$

holds for all $\phi_1,\ldots,\phi_n \in L^2(0,T;H^1(\Omega))$ and $\phi_0\in L^{16/5}(0,T;W^{1,16/5}(\Omega)^*)$, and the initial conditions (4) *are satisfied in the sense $\rho_i(0)=\rho_i^0$ in $L^2(\Omega)$ and $\theta(0)=\theta^0$ weakly in $L^2(\Omega)$.*

The idea of the proof is to apply the boundedness-by-entropy method, which automatically yields $L^\infty(\Omega_T)$ bounds [20]. More precisely, we formulate system (1)–(2) in terms of the relative entropy variables $(\mu_i-\mu_n)/\theta$ for $i=1,\ldots,n-1$ and $\log\theta$. We show in Lemma 3 that this defines the mass densities and temperature uniquely as a function of $(w_1,\ldots,w_{n-1},w)$. We introduce the mathematical entropy density

$$h(\boldsymbol{\rho}',\theta) = \sum_{i=1}^n \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w\rho\log\theta,$$

where the nth partial mass density is computed from $\rho_n = \rho - \sum_{i=1}^{n-1}\rho_i$, i.e., h depends on $\boldsymbol{\rho}' = (\rho_1,\ldots,\rho_{n-1})$ and θ. Gradient estimates for $(\boldsymbol{\rho},\theta)$ are first derived from the entropy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h(\boldsymbol{\rho}',\theta)\mathrm{d}x + \int_\Omega\frac{\kappa}{\theta^2}|\nabla\theta|^2\mathrm{d}x + \sum_{i,j=1}^n\int_\Omega M_{ij}^{BD}\frac{d_i}{\theta\sqrt{\rho_i}}\frac{d_j}{\theta\sqrt{\rho_j}}\mathrm{d}x = 0,$$

which becomes an inequality for weak solutions. Second, as in [15], the energy balance equation (2) yields a bound for θ^2 in $L^2(0,T;H^1(\Omega))$. As mentioned before, the derivation of the entropy inequality differs from that one in [15], because the cross-term

$$I_5 = 2\int_\Omega\sum_{i=1}^{n-1}\frac{B_i}{\theta}\nabla\frac{\mu_i-\mu_n}{\theta}\cdot\nabla\log\theta\mathrm{d}x,$$

which cancels out in [15], needs to be controlled. (We recall definition (11) of B_i.) This is done by observing that the sum $I_4+I_5+I_8$ (see (40)) is nonnegative,

$$I_4+I_5+I_8 = \int_\Omega\sum_{i,j=1}^n A_{ij}\nabla\left(\frac{\mu_i}{\theta}+\frac{1}{m_i}\log\theta\right)\cdot\nabla\left(\frac{\mu_j}{\theta}+\frac{1}{m_j}\log\theta\right)\mathrm{d}x \geq 0,$$

as (A_{ij}) is positive semidefinite due to (33).

From a technical viewpoint, we approximate equations (1)–(2) by replacing the time derivative by the implicit Euler discretization to avoid issues with the time regularity and by adding a higher-order regularization to achieve $H^2(\Omega)$ and hence $L^\infty(\Omega)$ regularity for the entropy variables. The approximation is chosen in such a way that a discrete entropy inequality can be derived, yielding uniform estimates for both the compactness of the fixed-point operator (to obtain a solution to the approximate problem) and the de-regularization limit (to obtain a solution to the original problem).

Our second main result concerns the weak–strong uniqueness property.

Theorem 2 (Weak–strong uniqueness). *Let the assumptions of Theorem 1 hold, let $\lambda = 0$ in (3), let $(\boldsymbol{\rho}, \theta)$ be a weak solution and $(\bar{\boldsymbol{\rho}}, \bar\theta)$ be a strong solution to (1)–(8). We assume that there exist $m, M > 0$ such that*

$$0 < \rho_i \le \rho^*, \quad 0 < \theta \le M, \quad 0 < \bar\rho_i \le \rho^*, \quad 0 < m \le \bar\theta \le M \quad \text{in } \Omega_T.$$

Furthermore, we suppose that $\bar u_i$, $|\nabla \log \bar\theta| \in L^\infty(\Omega_T)$ for $i = 1, \ldots, n$ and that the thermal conductivity κ is Lipschitz continuous. If the initial data of $(\boldsymbol{\rho}, \theta)$ and $(\bar{\boldsymbol{\rho}}, \bar\theta)$ coincide then $\boldsymbol{\rho}(x,t) = \bar{\boldsymbol{\rho}}(x,t)$ and $\theta(x,t) = \bar\theta(x,t)$ for a.e. $x \in \Omega$ and all $t > 0$.

By a strong solution, we understand a solution that has sufficient regularity to satisfy the entropy equality stated in Lemma 14; see Section 5. Observe that we require the boundedness of the temperature θ, which is not proved in Theorem 1. The proof of Theorem 2 is based on the relative entropy, defined by

$$\begin{aligned} H(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar\theta) &= \int_\Omega \bigg(h(\boldsymbol{\rho}, \theta) - h(\bar{\boldsymbol{\rho}}, \bar\theta) - \sum_{i=1}^n \frac{\partial h}{\partial \rho_i}(\bar{\boldsymbol{\rho}}, \bar\theta)(\rho_i - \bar\rho_i) - \frac{\partial h}{\partial E}(\bar{\boldsymbol{\rho}}, \bar\theta)(E - \bar E) \bigg) \mathrm{d}x \\ &= \int_\Omega \bigg\{ \sum_{i=1}^n \frac{1}{m_i} \bigg(\rho_i \log \frac{\rho_i}{\bar\rho_i} - (\rho_i - \bar\rho_i) \bigg) - c_w \rho \bigg(\log \frac{\theta}{\bar\theta} - (\theta - \bar\theta) \bigg) \bigg\} \mathrm{d}x, \end{aligned} \tag{12}$$

where $E = c_w \rho \theta$ and $\bar E = c_w \rho \bar\theta$ are the internal energy densities. The idea is to compute the time derivative:

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t}(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar\theta) + c \int_\Omega \sum_{i=1}^n \rho_i |u_i - \bar u_i|^2 \mathrm{d}x + c \int_\Omega |\nabla(\log\theta - \log\bar\theta)|^2 \mathrm{d}x \\ \le C \int_\Omega \bigg(\sum_{i=1}^n (\rho_i - \bar\rho_i)^2 + (\theta - \bar\theta)^2 \bigg) \mathrm{d}x, \end{aligned}$$

where $c > 0$ is some constant and $C > 0$ depends on the $L^\infty(\Omega_T)$ norms of θ, $\bar u_i$, and $\nabla \log \bar\theta$, $i = 1, \ldots, n$. The difficulty is to estimate the expressions arising from the time derivative of the relative entropy in such a way that only $\bar u_i$ and $\bar\theta$ need to be bounded. Thanks to the positive lower bound for $\bar\theta$, we can bound the right-hand side in terms of the relative entropy,

$$\int_\Omega \bigg(\sum_{i=1}^n (\rho_i - \bar\rho_i)^2 + (\theta - \bar\theta)^2 \bigg) \mathrm{d}x \le \int_\Omega H(\boldsymbol{\rho}, \theta | \bar{\boldsymbol{\rho}}, \bar\theta) \mathrm{d}x.$$

Then Gronwall's lemma shows that $H((\boldsymbol{\rho}, \theta)(t) | (\bar{\boldsymbol{\rho}}, \bar\theta)(t)) = 0$ for $t > 0$ and hence $(\boldsymbol{\rho}, \theta)(t) = (\bar{\boldsymbol{\rho}}, \bar\theta)(t)$. Compared to [18], we include the temperature terms and combine them with the entropy variables w_i in such a way that the positive semidefiniteness of M^{BD} can be exploited.

The paper is organized as follows. We detail the thermodynamic modeling of equations (1)–(8) in Section 2. The inversion of the Maxwell–Stefan system (5), the definition of the (relative) entropy variables, and the formulations of the fluxes in terms of the relative entropy variables, as well as the corresponding weak formulation is presented in Section 3. Section 4 is concerned with the proof of Theorem 1, and Theorem 2 is proved in Section 5.

2. Modeling

We consider the following system of equations modeling the dynamics of a nonisothermal gas mixture of n components with mass diffusion and heat conduction:

$$\partial_t\rho_i + \operatorname{div}(\rho_i(v+u_i)) = 0, \quad i=1,\ldots,n, \tag{13}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v\otimes v) = \rho b - \nabla p, \tag{14}$$

$$\begin{aligned}\partial_t\Big(\rho e + \frac12\rho|v|^2\Big) &+ \operatorname{div}\Big(\Big(\rho e+\frac12\rho|v|^2\Big)v\Big) \\ &= \operatorname{div}(\kappa\nabla\theta) - \operatorname{div}\sum_{j=1}^n(\rho_ie_i+p_i)u_i - \operatorname{div}(pv) + \rho r + \rho b\cdot v + \sum_{i=1}^n\rho_ib_i\cdot u_i.\end{aligned} \tag{15}$$

Besides of the variables introduced in the introduction, v denotes the barycentric velocity of the mixture. The quantities ρ_ib_i are the body forces, where $\rho b = \sum_{i=1}^n\rho_ib_i$ is the total force exerted on the mixture, and ρr is the total heat supply due to radiation. The diffusional velocities u_i, the partial internal energy densities ρ_ie_i, and the partial pressures p_i are determined from the free energy; see below.

Equations (13)–(15) correspond to a so-called class-I model. They can be derived either via an entropy invariant model reduction [3] or in the high-friction limit [12] from a class-II model, in which each component has its own velocity v_i. Equations (13) are the partial mass balances, (14) is the momentum balance, and (15) the energy balance. As proved in [12], system (13)–(15) and (5) fits into the general theory of hyperbolic–parabolic composite-type systems introduced in [22] and further explored in [25].

As mentioned in the introduction, system (1)–(2) and (7) is supplemented by the constrained Maxwell–Stefan system (5) for the velocities u_i. These equations can be derived from a class-II model in the diffusion approximation [3, Section 14, (210)] or in the high-friction limit [12, Section 2, (2.50)] with the driving forces

$$d_i = -\frac{\rho_i}{\rho}\nabla p + \rho_i\theta\nabla\frac{\mu_i}{\theta} - \theta(\rho_ie_i+p_i)\nabla\frac1\theta + \rho_i(b-b_i),$$

where μ_i is the chemical potential of the ith component. Since the pressure is uniform in space, $\nabla p=0$, and we have neglected external forces, the driving force becomes (6). Then equations (1)–(2) and (7) are obtained by setting $v=0$ and $r=b_i=0$.

The internal energy densities ρ_ie_i, partial pressures p_i, and the chemical potential μ_i are determined from the Helmholtz free energy. We assume that the gas is a simple mixture, which implies that these quantities can be calculated from the partial free energy densities $\psi_i(\rho_i,\theta)$, $i=1,\ldots,n$. We have

$$\mu_i = \frac{\partial\psi_i}{\partial\rho_i}, \quad \rho_i\eta_i = -\frac{\partial\psi_i}{\partial\theta}, \quad \rho_i e_i = \psi_i + \theta\rho_i\eta_i, \quad p_i = \rho_i\mu_i - \psi_i,$$

where $\rho_i\eta_i$ is the entropy density of the ith component and the equation for p_i is called the Gibbs–Duhem relation. Defining the partial Helmholtz free energy as

$$\psi_i = \theta\frac{\rho_i}{m_i}\bigg(\log\frac{\rho_i}{m_i} - 1\bigg) - c_w\rho\theta(\log\theta - 1), \quad i=1,\ldots,n, \tag{16}$$

the thermodynamic quantities are given by (8). Moreover, the driving force d_i and enthalpy $h_i := \rho_i e_i + p_i$ read as

$$d_i = \frac{\nabla(\rho_i\theta)}{m_i}, \quad h_i = \bigg(c_w + \frac{1}{m_i}\bigg)\rho_i\theta, \quad i=1,\ldots,n. \tag{17}$$

This corresponds to equations (9).

3. Preparations

3.1. **Inversion of the Maxwell–Stefan system.** We discuss the inversion of the Maxwell–Stefan system (5) following [12] and [18, Section 2]. We write (5) equivalently as

$$-\theta\sqrt{\rho_i}\sum_{j=1}^n M_{ij}\sqrt{\rho_j}u_j = d_i, \quad i=1,\ldots,n, \tag{18}$$

where the matrix $M(\boldsymbol{\rho}) = (M_{ij}) \in \mathbb{R}^{n\times n}$ is given by

$$M_{ij} = \begin{cases} \sum_{k=1,\,k\neq i}^n b_{ik}\rho_k & \text{if } i=j, \\ -b_{ij}\sqrt{\rho_i\rho_j} & \text{if } i\neq j. \end{cases} \tag{19}$$

We wish to invert $M\boldsymbol{v} = \boldsymbol{w}$, where $v_i = \sqrt{\rho_i}u_i$ and $w_i = -d_i/(\theta\sqrt{\rho_i})$. Since (b_{ij}) is symmetric, $0 = (M\boldsymbol{v})_i = \sum_{i\neq j} b_{ij}\sqrt{\rho_j}(\sqrt{\rho_j}v_i - \sqrt{\rho_i}v_j)$ shows that the kernel of M consists of span$\{\sqrt{\boldsymbol{\rho}}\}$. Thus, we can invert M only on the subspace $L = \{\boldsymbol{y}\in\mathbb{R}^n : \sqrt{\boldsymbol{\rho}}\cdot\boldsymbol{y} = 0\}$. We define the projections P_L on L and $P_{L^\perp}$ on $L^\perp$ by

$$(P_L)_{ij} = \delta_{ij} - \rho^{-1}\sqrt{\rho_i\rho_j}, \quad (P_{L^\perp})_{ij} = \rho^{-1}\sqrt{\rho_i\rho_j} \quad \text{for } i,j=1,\ldots,n,$$

where δ_{ij} is the Kronecker symbol. The matrix $M = (M_{ij})$ is positive definite on L [18, Lemma 4]:

$$\boldsymbol{z}^T M\boldsymbol{z} \geq \mu_M|P_L\boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z}\in\mathbb{R}^n, \tag{20}$$

where $\mu_M = \min_{i\neq j} b_{ij} > 0$. Since the matrix $MP_L + P_{L^\perp}$ is invertible [18, Lemma 4], we can define the Bott–Duffin inverse of M with respect to L as $M^{BD} = P_L(MP_L + P_{L^\perp})^{-1}$. Hence, we can invert (18) by

$$\sqrt{\rho_i}u_i = -\sum_{j=1}^n M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}, \quad i=1,\ldots,n. \tag{21}$$

The matrix $M^{BD} = M^{BD}(\boldsymbol{\rho})$ is symmetric and positive definite on L [18, Lemma 4],

$$\boldsymbol{z}^T M^{BD} \boldsymbol{z} \geq \mu |P_L \boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{22}$$

where $\mu = (2\sum_{i\neq j}(b_{ij}+1))^{-1}$.

3.2. **Entropy variables.** The mathematical analysis becomes easier when formulating the system in terms of the so-called entropy variables. To this end, we introduce the mathematical entropy density

$$h = -\sum_{i=1}^n \rho_i \eta_i = \sum_{i=1}^n \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) - c_w \rho \log\theta, \tag{23}$$

which is the negative of the physical (total) entropy density (8). Summing the mass balances (1) over $i = 1, \ldots, n$ and using the constraint $\sum_{i=1}^n \rho_i u_i = 0$ from (5), we obtain $\partial_t \rho = 0$. Thus, the total density is determined by the initial total density, $\rho(x,t) = \sum_{i=1}^n \rho_i^0(x)$ for $x \in \Omega$, and is independent of time. This suggests to compute only the first $n-1$ mass densities, since the last one can be determined by $\rho_n = \rho - \sum_{i=1}^{n-1}\rho_i$. Then we interpret the entropy density h as a function of $(\boldsymbol{\rho}', \theta) := (\rho_1, \ldots, \rho_{n-1}, \theta)$:

$$h(\boldsymbol{\rho}', \theta) = \sum_{i=1}^{n-1} \frac{\rho_i}{m_i}\left(\log\frac{\rho_i}{m_i} - 1\right) + \frac{\rho_n}{m_n}\left(\log\frac{\rho_n}{m_n} - 1\right) - c_w\rho\log\theta$$

with the partial derivatives

$$\frac{\partial h}{\partial \rho_i} = \frac{1}{m_i}\log\frac{\rho_i}{m_i} - \frac{1}{m_n}\log\frac{\rho_n}{m_n}, \quad i = 1, \ldots, n-1, \quad \frac{\partial h}{\partial\theta} = -c_w\frac{\rho}{\theta}.$$

The Hessian matrix

$$\mathrm{D}^2 h = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0}^T & c_w\rho/\theta^2 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \text{where } R_{ij} = \frac{\delta_{ij}}{m_i\rho_i} + \frac{1}{m_n\rho_n},$$

is positive definite, showing that the entropy is convex.

According to thermodynamics [3], the entropy variables equal $(\mu_1/\theta, \ldots, \mu_n/\theta, -1/\theta)$. We set

$$q_i = \frac{\mu_i}{\theta} = \frac{1}{m_i}\log\frac{\rho_i}{m_i} - c_w(\log\theta - 1) \quad \text{for } i = 1, \ldots, n. \tag{24}$$

Since the nth partial density is determined by the densities $\rho_1, \ldots, \rho_{n-1}$, we prefer to work with the relative entropy variables

$$w_i = q_i - q_n = \frac{\mu_i - \mu_n}{\theta} = \frac{\partial h}{\partial \rho_i}, \quad i = 1, \ldots, n-1. \tag{25}$$

Setting additionally $w = \log\theta$, our new set of variables is $(w_1, \ldots, w_{n-1}, w)$. The following lemma states that the mapping $(\rho_1, \ldots, \rho_n, \theta) \mapsto (w_1, \ldots, w_{n-1}, w)$ is invertible.

Lemma 3. *Let $(w_1, \ldots, w_{n-1}, w) \in \mathbb{R}^n$ and $\rho > 0$ be given. Then there there exists a unique $(\rho_1, \ldots, \rho_n, \theta) \in \mathbb{R}_+^{n+1}$ with $\rho_i > 0$ for $i = 1, \ldots, n$ satisfying $\sum_{i=1}^n \rho_i = \rho$, $w_i = \partial h/\partial\rho_i$ for $i = 1, \ldots, n-1$, and $w = \log\theta$.*

Proof. The proof is similar to [8, Lemma 6] with some small changes. Given $w \in \mathbb{R}$, the temperature equals $\theta = \exp(w) > 0$. The function

$$f(s) = \sum_{i=1}^{n-1} m_i e^{m_i w_i} \bigg(\frac{\rho - s}{m_n}\bigg)^{m_i/m_n} \quad \text{for } s \in [0, \rho],$$

is strictly decreasing and $0 = f(\rho) < f(s) < f(0)$ for $s \in (0, \rho)$. By continuity, there exists a unique fixed point $s_0 \in (0, \rho)$. Then $\rho_i := m_i \exp(m_i w_i)((\rho - s_0)/m_n)^{m_i/m_n}$ for $i = 1, \ldots, n$ satisfies $\rho_i > 0$ and $\sum_{i=1}^{n-1} \rho_i = f(s_0) = s_0 < \rho$. Consequently, $\rho_n := \rho - \sum_{i=1}^{n-1} \rho_i = \rho - s_0 > 0$ and $\rho_i/m_i = \exp(m_i w_i)(\rho_n/m_n)^{m_i/m_n}$ is equivalent to

$$w_i = \frac{1}{m_i} \log \frac{\rho_i}{m_i} - \frac{1}{m_n} \log \frac{\rho_n}{m_n} = \frac{\partial h}{\partial \rho_i}$$

for $i = 1, \ldots, n-1$, which finishes the proof. □

3.3. **Formulation of the fluxes and parabolicity.** We can compute the fluxes as a linear combination of $\nabla(w_1, \ldots, w_{n-1}, w)$ or $\nabla(q_1, \ldots, q_n, -1/\theta)$.

Lemma 4. *It holds for $i = 1, \ldots, n$ that*

$$J_i = -\sum_{j=1}^{n-1} A_{ij} \nabla w_j - \frac{B_i}{\theta} \nabla w = -\sum_{j=1}^{n} A_{ij} \nabla q_j - B_i \nabla \bigg(-\frac{1}{\theta}\bigg), \tag{26}$$

$$\begin{aligned} J_e &= -\kappa\theta\nabla w - \sum_{j=1}^{n-1} B_j \nabla w_j - \theta \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j} \nabla w \\ &= -\sum_{j=1}^{n} B_j \nabla q_j - \theta^2 \bigg(\kappa + \sum_{i,j=1}^{n} \frac{A_{ij}}{m_i m_j}\bigg) \nabla \bigg(-\frac{1}{\theta}\bigg), \end{aligned} \tag{27}$$

where the coefficients

$$A_{ij} = M_{ij}^{BD} \sqrt{\rho_i \rho_j}, \quad B_i = \theta \sum_{j=1}^{n} A_{ij} \bigg(c_w + \frac{1}{m_j}\bigg) = \theta \sum_{j=1}^{n} \frac{A_{ij}}{m_j} \tag{28}$$

for $i, j = 1, \ldots, n$ depend on $(\boldsymbol{\rho}, \theta)$ and satisfy the relations

$$\sum_{i=1}^{n} A_{ij} = \sum_{j=1}^{n} A_{ij} = \sum_{i=1}^{n} B_i = 0. \tag{29}$$

Proof. We wish to express the driving force $d_j = \nabla(\rho_j \theta)/m_j$ from (9) in terms of $\nabla q_j = \nabla \log \rho_j/m_j - c_w \nabla \log \theta$. A computation, using $w = \log \theta$, yields

$$d_j = \rho_j \theta \nabla q_j + \rho_j \theta \bigg(c_w + \frac{1}{m_j}\bigg) \nabla w. \tag{30}$$

Therefore, by (21), for $i = 1, \ldots, n$,

$$J_i = \rho_i u_i = -\sqrt{\rho_i}\sum_{j=}^n M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}} = -\sum_{j=1}^n M_{ij}^{BD}\sqrt{\rho_i\rho_j}\bigg\{\nabla q_j + \bigg(c_w + \frac{1}{m_j}\bigg)\nabla w\bigg\}$$
$$= -\sum_{j=1}^n A_{ij}\nabla q_j - \sum_{j=1}^n A_{ij}\bigg(c_w + \frac{1}{m_j}\bigg)\nabla\log\theta = -\sum_{j=1}^n A_{ij}\nabla q_j - \frac{B_i}{\theta}\nabla\log\theta.$$

This shows the second relation in (26). The first relation then follows from (29) (which is proved below), since, using $q_j = w_j + q_n$ for $j = 1, \ldots, n-1$ (see (25)),

$$\sum_{j=1}^n A_{ij}\nabla q_j = \sum_{j=1}^{n-1} A_{ij}(\nabla w_j + \nabla q_n) + A_{in}\nabla q_n = \sum_{j=1}^{n-1} A_{ij}\nabla w_j. \tag{31}$$

Next, we compute the energy flux defined in (2). We use (17), (21), and (30):

$$\begin{aligned} J_e &= -\kappa\theta\nabla w + \sum_{i=1}^n \sqrt{\rho_i}\theta\bigg(c_w + \frac{1}{m_i}\bigg)\sqrt{\rho_i}u_i \\ &= -\kappa\theta\nabla w - \theta\sum_{i,j=1}^n \sqrt{\rho_i}\bigg(c_w + \frac{1}{m_i}\bigg)M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}} \\ &= -\kappa\theta\nabla w - \theta\sum_{i,j=1}^n \bigg(c_w + \frac{1}{m_i}\bigg)M_{ij}^{BD}\sqrt{\rho_i\rho_j}\bigg\{\nabla q_j + \bigg(c_w + \frac{1}{m_j}\bigg)\nabla w\bigg\} \\ &= -\kappa\theta\nabla w - \sum_{j=1}^n B_j\nabla q_j - \theta\sum_{i,j=1}^n A_{ij}\bigg(c_w + \frac{1}{m_i}\bigg)\bigg(c_w + \frac{1}{m_j}\bigg)\nabla w \\ &= -\kappa\theta\nabla w - \sum_{j=1}^n B_j\nabla q_j - \theta\sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j}\nabla w, \end{aligned}$$

where the last equation follows from (29). Moreover, because of

$$\sum_{j=1}^n B_j\nabla q_j = \sum_{j=1}^{n-1} B_j\nabla(w_j + q_n) + B_n\nabla q_n = \sum_{j=1}^{n-1} B_j\nabla w_j, \tag{32}$$

we have proved (27).

It remains to verify (29). We recall the property $P_L(MP_L + P_{L^\perp})^{-1}P_{L^\perp} = 0$ from [28, Lemma 2], which implies that $M^{BD}P_{L^\perp} = 0$. Hence, $L^\perp \subset \ker M^{BD}$ and since $L^\perp = \operatorname{span}\{\sqrt{\boldsymbol{\rho}}\}$, we conclude that $\sum_{j=1}^n M_{ij}^{BD}\sqrt{\rho_j} = 0$. This shows that, by the definition of A_{ij},

$$\sum_{j=1}^n A_{ij} = \sqrt{\rho_i}\sum_{j=1}^n M_{ij}^{BD}\sqrt{\rho_j} = 0.$$

The symmetry of (A_{ij}) immediately gives $\sum_{i=1}^n A_{ij} = 0$. Finally, by the definition of B_i,

$$\sum_{i=1}^n B_i = \theta \sum_{i,j=1}^n M_{ij}^{BD}\sqrt{\rho_i\rho_j}\left(c_w + \frac{1}{m_j}\right) = \theta\sum_{j=1}^n\left(c_w + \frac{1}{m_j}\right)\sum_{i=1}^n A_{ij} = 0.$$

This finishes the proof. □

The previous proof shows that we can formulate the diffusion fluxes in different ways.

Corollary 5. *It holds for $i = 1, \dots, n$ that*

$$J_i = \rho_i u_i = -\sum_{j=1}^n A_{ij}\nabla\left(q_j + \frac{w}{m_j}\right) = -\sqrt{\rho_i}\sum_{j=1}^n M_{ij}^{BD}\frac{d_j}{\theta\sqrt{\rho_j}}.$$

We claim that the Onsager matrix $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ in (10) is positive semidefinite. Let $a = \theta(\kappa + \sum_{i,j=1}^n A_{ij}/(m_i m_j))$. We compute for $\xi \in \mathbb{R}^{n+1}$:

$$\begin{aligned} \xi^T Q\xi &= \sum_{i,j=1}^n A_{ij}\xi_i\xi_j + 2\sum_{i=1}^n B_i\xi_i\xi_{n+1} + a\xi_{n+1}^n \\ &= \sum_{i,j=1}^n A_{ij}\xi_i\xi_j + 2\theta\sum_{i,j=1}^n \frac{A_{ij}}{m_j}\xi_i\xi_{n+1} + \theta^2\left(\kappa + \sum_{i,j=1}^n\frac{A_{ij}}{m_i m_j}\right)\xi_{n+1}^2 \\ &= \sum_{i,j=1}^n A_{ij}\left(\xi_i + \frac{\theta\xi_{n+1}}{m_i}\right)\left(\xi_j + \frac{\theta\xi_{n+1}}{m_j}\right) + \kappa\theta^2\xi_{n+1}^2 \geq 0, \end{aligned} \tag{33}$$

where the nonnegativity follows from the positive semidefiniteness (22) of M^{BD}. This reveals the parabolicity of our system in terms of the entropy variables.

3.4. **Weak formulation.** The previous subsection shows that we can write our system as the mass and energy balances (1)–(2) with the fluxes (26)–(27). The weak formulation in the relative entropy variables (25) reads as

$$\int_0^T\langle\partial_t\rho_i,\phi_i\rangle\mathrm{d}t + \int_0^T\int_\Omega\left(\sum_{j=1}^{n-1}A_{ij}\nabla w_j + e^{-w}B_i\nabla w\right)\cdot\nabla\phi_i\mathrm{d}x\mathrm{d}t = 0, \tag{34}$$

$$\begin{aligned} &\int_0^T\langle\partial_t E,\phi_0\rangle\mathrm{d}t + \int_0^T\int_\Omega e^w\left(\kappa + \sum_{i,j=1}^n\frac{A_{ij}}{m_i m_j}\right)\nabla w\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T\int_\Omega\sum_{j=1}^{n-1}B_j\nabla w_j\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}t = \lambda\int_0^T\int_{\partial\Omega}(\theta_0-\theta)\phi_0\mathrm{d}s\mathrm{d}t \end{aligned} \tag{35}$$

for test functions $\phi_1, \dots, \phi_n \in L^2(0,T;H^1(\Omega))$ and $\phi_0 \in L^\infty(0,T;W^{1,\infty}(\Omega))$. According to (8), the energy is given by $E = c_w\rho\theta$. Moreover, ρ_i, A_{ij}, B_i, and E are interpreted as functions of $(w_1, \dots, w_{n-1}, w)$.

4. Proof of Theorem 1

The proof follows the lines of [15, Section 3], which is based on the boundedness-by-entropy method [20], but some details are different. We approximate equations (34)–(35) by replacing the time derivative by the implicit Euler scheme and adding a higher-order regularization in w_i. The existence of solutions to the approximate system is shown by means of the Leray–Schauder fixed-point theorem, where the compactness of the fixed-point operator is obtained by the approximate entropy inequality. This inequality yields estimates uniform in the regularization parameters, allowing for the de-regularization limit via the Aubin–Lions compactness lemma.

Let $\varepsilon \in (0,1)$, $N \in \mathbb{N}$, and $\tau = T/N$. We set $w_0 = \log\theta_0$ and $\boldsymbol{w} = (w_1,\ldots,w_{n-1},w)$. Let $\bar{\boldsymbol{w}} = (\bar w_1,\ldots,\bar w_{n-1},\bar w) \in L^\infty(\Omega;\mathbb{R}^n)$ be given. We define for test functions $\phi_i \in H^2(\Omega)$, $i=0,\ldots,n-1$, the approximate scheme

$$0 = \frac{1}{\tau}\int_\Omega \big(\rho_i(\boldsymbol{w}) - \rho_i(\bar{\boldsymbol{w}})\big)\phi_i \mathrm{d}x + \int_\Omega \bigg(\sum_{j=1}^{n-1} A_{ij}\nabla w_j + e^{-w}B_i\nabla w\bigg)\cdot\nabla\phi_i \mathrm{d}x + \varepsilon\int_\Omega \big(\mathrm{D}^2 w_i : \mathrm{D}^2\phi_i + w_i\phi_i\big)\mathrm{d}x, \tag{36}$$

$$\begin{aligned} 0 = {} & \frac{1}{\tau}\int_\Omega \big(E(\boldsymbol{w}) - E(\bar{\boldsymbol{w}})\big)\phi_0 \mathrm{d}x + \int_\Omega e^w\bigg(\kappa(e^w) + \sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j}\bigg)\nabla w\cdot\nabla\phi_0 \mathrm{d}x \\ & + \int_\Omega \sum_{i=1}^{n-1} B_i\nabla w_i\cdot\nabla\phi_0\mathrm{d}x - \lambda\int_{\partial\Omega}(e^{w_0} - e^w)\phi_0\mathrm{d}s \\ & + \varepsilon\int_\Omega (e^{w_0}+e^w)(w-w_0)\phi_0\mathrm{d}x + \varepsilon\int_\Omega e^w\big(\mathrm{D}^2 w : \mathrm{D}^2\phi_0 + |\nabla w|^2\nabla w\cdot\nabla\phi_0\big)\mathrm{d}x, \end{aligned} \tag{37}$$

where $\mathrm{D}^2 w_i$ is the Hesse matrix of w_i, the double point ":" denotes the Frobenius matrix product, we recall that $E(\boldsymbol{w}) = c_w\rho\theta$, and A_{ij} and B_i are interpreted as functions of $\boldsymbol{w}$. The higher-order regularization yields solutions $w_i, w \in H^2(\Omega)$, and the $W^{1,4}(\Omega)$ regularization allows us to estimate the higher-order terms when using the test function $e^{-w_0} - e^{-w}$ (see the estimate of I_{11} below). The lower-order regularization $(e^{w_0} - e^w)(w-w_0)$ provides an ε-dependent $L^2(\Omega)$ bound for w.

4.1. Solution of the linearized approximate problem.

Let $\boldsymbol{w}^* \in W^{1,4}(\Omega;\,\mathbb{R}^n)$ and $\sigma \in [0,1]$. We want to find a solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to the linear problem

$$a(\boldsymbol{w},\boldsymbol{\phi}) = \sigma F(\boldsymbol{\phi}) \quad \text{for } \boldsymbol{\phi} = (\phi_1,\ldots,\phi_{n-1},\phi_0) \in H^2(\Omega;\mathbb{R}^n), \tag{38}$$

where

$$\begin{aligned} a(\boldsymbol{w},\boldsymbol{\phi}) = {} & \int_\Omega \kappa(e^{w^*})e^{w^*}\nabla w\cdot\nabla\phi_0\mathrm{d}x + \varepsilon\int_\Omega\sum_{i=1}^{n-1}\big(\mathrm{D}^2 w_i : \mathrm{D}^2\phi_i + w_i\phi_i\big)\mathrm{d}x \\ & + \varepsilon\int_\Omega (e^{w_0} + e^{w^*})w\phi_0\mathrm{d}x + \varepsilon\int_\Omega e^{w^*}\big(\mathrm{D}^2 w : \mathrm{D}^2\phi_0 + |\nabla w^*|^2\nabla w\cdot\nabla\phi_0\big)\mathrm{d}x, \end{aligned}$$

$$
\begin{aligned}
F(\boldsymbol{\phi}) = &-\int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w}^*)\nabla w_j^* \cdot \nabla\phi_i \mathrm{d}x - \int_\Omega e^{w^*} \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w}^*)}{m_i m_j}\nabla w^* \cdot \nabla\phi_0 \mathrm{d}x \\
&- \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*)e^{-w^*}\nabla w^* \cdot \nabla\phi_i \mathrm{d}x - \int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w}^*)\nabla w_i^* \cdot \nabla\phi_0 \mathrm{d}x \\
&- \frac{1}{\tau}\int_\Omega \sum_{i=1}^{n-1}(\rho_i^* - \bar\rho_i)\phi_i \mathrm{d}x - \frac{1}{\tau}\int_\Omega (E^* - \bar E)\phi_0 \mathrm{d}x + \lambda\int_{\partial\Omega}(e^{w_0} - e^{w^*})\phi_0 \mathrm{d}s \\
&+ \varepsilon\int_\Omega (e^{w_0} + e^{w^*})w_0\phi_0 \mathrm{d}x,
\end{aligned}
$$

where we abbreviated $\rho_i^* = \rho_i(\boldsymbol{w}^*)$, $\bar\rho_i = \rho_i(\bar{\boldsymbol{w}})$, $E^* = c_w\rho e^{w^*}$, and $\bar E = c_w\rho e^{\bar w}$. The bilinear form a is clearly coercive on $H^2(\Omega;\mathbb{R}^n)$, and both a and F are continuous on this space. By the Lax–Milgram lemma, there exists a unique solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to (38).

4.2. **Solution of the approximate problem.** The solution $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ to (38) defines the fixed-point operator $S : W^{1,4}(\Omega;\mathbb{R}^n) \times [0,1] \to W^{1,4}(\Omega;\mathbb{R}^n)$, $S(\boldsymbol{w}^*,\sigma) = \boldsymbol{w}$. The operator is continuous, compact (because of the compact embedding $H^2(\Omega;\mathbb{R}^n) \hookrightarrow W^{1,4}(\Omega;\mathbb{R}^n)$), and it satisfies $S(\boldsymbol{w}^*,0) = 0$ for all $\boldsymbol{w}^* \in W^{1,4}(\Omega;\mathbb{R}^n)$. It remains to find a uniform bound for all fixed points of $S(\cdot,\sigma)$. Let $\boldsymbol{w} \in H^2(\Omega;\mathbb{R}^n)$ be such a fixed point. Then $\boldsymbol{w}$ solves (38) with $\boldsymbol{w}^* = \boldsymbol{w}$. We choose the test functions $\phi_i = w_i$ for $i = 1,\ldots,n-1$ and $\phi_0 = e^{-w_0} - e^{-w}$ in (38):

$$
\begin{aligned}
0 = &\frac{\sigma}{\tau}\int_\Omega \sum_{i=1}^{n-1}(\rho_i - \bar\rho_i)w_i \mathrm{d}x + \frac{\sigma}{\tau}\int_\Omega (E - \bar E)(-e^{-w})\mathrm{d}x + \frac{\sigma}{\tau}\int_\Omega (E - \bar E)e^{-w_0}\mathrm{d}x \\
&+ \sigma\int_\Omega \sum_{i,j=1}^{n-1} A_{ij}(\boldsymbol{w})\nabla w_i \cdot \nabla w_j \mathrm{d}x + 2\sigma\int_\Omega \sum_{i=1}^{n-1} B_i(\boldsymbol{w})e^{-w}\nabla w_i \cdot \nabla w \mathrm{d}x \\
&+ \int_\Omega \kappa(e^w)|\nabla w|^2 \mathrm{d}x + \varepsilon\int_\Omega \sum_{i=1}^{n-1}\big(|\mathrm{D}^2 w_i|^2 + w_i^2\big)\mathrm{d}x + \sigma\int_\Omega \sum_{i,j=1}^{n} \frac{A_{ij}(\boldsymbol{w})}{m_i m_j}|\nabla w|^2 \mathrm{d}x \\
&- \sigma\lambda\int_{\partial\Omega}(e^{w_0} - e^{w})(e^{-w_0} - e^{-w})\mathrm{d}s + \varepsilon\int_\Omega (e^{w_0} + e^{w})(e^{-w_0} - e^{-w})(w - \sigma w_0)\mathrm{d}x \\
&+ \varepsilon\int_\Omega \big(|\mathrm{D}^2 w|^2 - \mathrm{D}w : (\nabla w \otimes \nabla w) + |\nabla w|^4\big)\mathrm{d}x =: I_1 + \cdots + I_{11}.
\end{aligned} \tag{39}
$$

We estimate the terms $I_1,\ldots,I_{11}$ step by step. First, by the convexity of the entropy and arguing similarly as in [15, Section 3, Step 2],

$$
\begin{aligned}
I_1 + I_2 &= \frac{\sigma}{\tau}\int_\Omega \sum_{i=1}^{n-1}\bigg((\rho_i - \bar\rho_i)\frac{\partial h}{\partial \rho_i} + (\theta - \bar\theta)\frac{\partial h}{\partial\theta}\bigg)\mathrm{d}x \\
&\ge \frac{\sigma}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta) - h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta)\big)\mathrm{d}x,
\end{aligned}
$$

where we have set $\theta = e^w$ and $\bar\theta = e^{\bar w}$. Definition (25) of w_i, definition (28) of B_i, and the relations

$$\sum_{j=1}^{n-1} A_{ij}(\boldsymbol{w})\nabla w_j = \sum_{j=1}^{n} A_{ij}(\boldsymbol{w})\nabla q_j, \quad \sum_{i=1}^{n-1} B_i(\boldsymbol{w})\nabla w_i = \sum_{j=1}^{n} B_i(\boldsymbol{w})\nabla q_i$$

from (31)–(32) allow us to rewrite the sum $I_4 + I_5 + I_8$ as

$$I_4 + I_5 + I_8 = \sigma \int_\Omega \sum_{i,j=1}^{n} A_{ij}(\boldsymbol{w})\nabla\Big(q_i + \frac{w}{m_i}\Big)\cdot\nabla\Big(q_j + \frac{w}{m_j}\Big)\mathrm{d}x. \tag{40}$$

This expression is nonnegative because of the positive semidefiniteness of $A_{ij} = M_{ij}^{BD}\sqrt{\rho_i\rho_j}$; see (22). Furthermore, since $\sinh(z)/z \geq 1$ for $z \in \mathbb{R}$, $z \neq 0$,

$$\begin{aligned}
I_9 &= \sigma\lambda \int_{\partial\Omega} e^{-w-w_0}(e^w - e^{w_0})^2\mathrm{d}x \geq 0,\\
I_{10} &= 2\varepsilon \int_\Omega \sinh(w-w_0)(w-\sigma w_0)\mathrm{d}x = 2\varepsilon\int_\Omega (w-w_0)(w-\sigma w_0)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x\\
&= \varepsilon\int_\Omega w^2\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x + \varepsilon\int_\Omega \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x\\
&\geq \varepsilon\int_\Omega w^2\mathrm{d}x + \varepsilon\int_\Omega \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0}\mathrm{d}x.
\end{aligned}$$

We claim that there exists $m = m(w_0,\sigma) > 0$ such that for all $w \in \mathbb{R}$,

$$g(w) = \big(w^2 - 2(1+\sigma)ww_0 + 2\sigma w_0^2\big)\frac{\sinh(w-w_0)}{w-w_0} \geq -m,$$

where $w_0 \in \mathbb{R}$ and $\sigma \in (0,1]$ are given. Indeed, this follows from $g(w) \to \infty$ as $|w| \to \infty$ and $g((1+\sigma)w_0) < 0$ (unless $w_0 = 0$). We conclude that

$$I_{10} \geq \varepsilon\int_\Omega w^2\mathrm{d}x - \varepsilon m.$$

Finally, we can estimate

$$I_{11} = \frac{\varepsilon}{2}\int_\Omega \big(|\mathrm{D}^2 w|^2 + |\mathrm{D}^2 w - \nabla w\otimes\nabla w|^2 + |\nabla w|^4\big)\mathrm{d}x \geq \frac{\varepsilon}{2}\int_\Omega\big(|\mathrm{D}^2 w|^2 + |\nabla w|^4\big)\mathrm{d}x.$$

Summarizing these estimates, we find that

$$\begin{aligned}
&\frac{\sigma}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta) + Ee^{-w_0}\big)\mathrm{d}x + \varepsilon C\big(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\big)\\
&\quad + \int_\Omega \kappa(e^w)|\nabla w|^2\mathrm{d}x \leq \frac{\sigma}{\tau}\int_\Omega\big(h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) + \bar E e^{-w_0}\big)\mathrm{d}x + \varepsilon m.
\end{aligned} \tag{41}$$

The right-hand side is bounded since $\bar{\boldsymbol{w}} \in L^\infty(\Omega;\mathbb{R}^n)$ by assumption, implying that $(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) \in L^\infty(\Omega;\mathbb{R}^n)$. The first term on the left-hand side is bounded from below since, by definition (23) of h and $Ee^{-w_0} = c_w\rho\theta/\theta_0$,

$$h(\rho_1,\ldots,\rho_{n-1},\theta) + Ee^{-w_0} = \sum_{i=1}^n \frac{\rho_i}{m_i}\bigg(\log\frac{\rho_i}{m_i} - 1\bigg) - c_w\rho\bigg(\log\theta - \frac{\theta}{\theta_0}\bigg).$$

Thus, we obtain a uniform bound for $\boldsymbol{w}$ in $H^2(\Omega;\mathbb{R}^n)$ and consequently also in $W^{1,4}(\Omega;\mathbb{R}^n)$. We can apply the Leray–Schauder fixed-point theorem to conclude the existence of a fixed point of $S(\cdot,1)$. This, in turn, shows that $\boldsymbol{w}$ is a weak solution to the approximate problem (36)–(37).

Remark 6 (Treatment of the cross-terms)**.** In the paper [15], the fluxes are given by

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} M & -\boldsymbol{G} \\ \boldsymbol{G}^T & \kappa\theta^2 \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},$$

where $M = M(\boldsymbol{\rho},\theta) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{G} = \boldsymbol{G}(\boldsymbol{\rho},\theta) \in \mathbb{R}^n$. A multiplication of this equation by $\nabla(\boldsymbol{\mu}/\theta, -1/\theta)$ shows that the cross-terms cancel out,

$$-\nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^n M_{ij}\nabla\frac{\mu_i}{\theta}\cdot\nabla\frac{\mu_j}{\theta} + \kappa|\nabla\log\theta|^2 \ge 0,$$

since M is assumed to be positive semidefinite in [15]. In the present work, we have

$$\begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = -\begin{pmatrix} A & \boldsymbol{B} \\ \boldsymbol{B}^T & a \end{pmatrix} \nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix},$$

and the cross-terms do not cancel. This is compensated by the sum $\sum_{i,j=1}^n A_{ij}/(m_i m_j)$. Indeed, a computation shows that (also see (40))

$$-\nabla \begin{pmatrix} \boldsymbol{\mu}/\theta \\ -1/\theta \end{pmatrix}^T : \begin{pmatrix} \boldsymbol{J} \\ J_e \end{pmatrix} = \sum_{i,j=1}^n A_{ij}\nabla\bigg(q_i + \frac{w}{m_i}\bigg)\cdot\nabla\bigg(q_j + \frac{w}{m_j}\bigg) + \kappa|\nabla\log\theta|^2 \ge 0,$$

since A is positive semidefinite because of (33). □

4.3. **Discrete entropy inequality.** We derive some estimates from (39) with $\sigma = 1$, which are uniform in (ε,τ), by exploiting the sum $I_4 + I_5 + I_8$, which we have neglected in (41). Taking into account that the estimate of I_{10} becomes for $\sigma = 1$

$$I_{10} = 2\varepsilon\int_\Omega \sinh(w-w_0)(w-w_0)\mathrm{d}x \ge 2\varepsilon\int_\Omega (w-w_0)^2\mathrm{d}x \ge 0,$$

we obtain the discrete entropy inequality

$$\begin{aligned} \frac{\sigma}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta) + Ee^{-w_0}\big)\mathrm{d}x + \varepsilon C\big(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\big) \\ + \int_\Omega \kappa(e^w)|\nabla w|^2\mathrm{d}x + \int_\Omega \sum_{i,j=1}^n A_{ij}\nabla\bigg(q_i + \frac{w}{m_i}\bigg)\cdot\nabla\bigg(q_j + \frac{w}{m_j}\bigg)\mathrm{d}x \\ \le \frac{\sigma}{\tau}\int_\Omega \big(h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) + \bar{E}e^{-w_0}\big)\mathrm{d}x. \end{aligned} \tag{42}$$

Lemma 7. *It holds that*

$$\int_\Omega \sum_{i,j=1}^n A_{ij}\nabla\Big(q_i+\frac{w}{m_i}\Big)\cdot\nabla\Big(q_j+\frac{w}{m_j}\Big)\mathrm{d}x \ge \int_\Omega\sum_{i=1}^n \frac{\mu}{m_i^2}|2\nabla\sqrt{\rho_i}+\sqrt{\rho_i}\nabla w|^2\mathrm{d}x, \tag{43}$$

where $\mu>0$ is defined in (22).

We deduce from Assumption (A4) that $\kappa(e^w)|\nabla w|^2\ge c_\kappa|\nabla w|^2$, and in view of (42), this quantity is bounded in $L^2(\Omega)$. Therefore, Lemma 7 yields a gradient bound for $\sqrt{\rho_i}$ in $L^2(\Omega)$, since

$$4|\nabla\sqrt{\rho_i}|^2\le|2\nabla\sqrt{\rho_i}+\sqrt{\rho_i}\nabla w|^2+\rho_i|\nabla w|^2.$$

Proof of Lemma 7. It follows from (24) and (29) that

$$\sum_{i,j=1}^n A_{ij}\nabla q_i=\sum_{i,j=1}^n A_{ij}\frac{\nabla\log\rho_i}{m_i}-c_w\sum_{i,j=1}^n A_{ij}\nabla w=\sum_{i,j=1}^n A_{ij}\frac{\nabla\rho_i}{m_i\rho_i}$$

and therefore, in view of the definition $A_{ij}=M_{ij}^{BD}\sqrt{\rho_i\rho_j}$ and the positive definiteness (22) on the subspace L,

$$\begin{aligned}\sum_{i,j=1}^n A_{ij}\nabla\Big(q_i+\frac{w}{m_i}\Big)\cdot\nabla\Big(q_j+\frac{w}{m_j}\Big)&=\sum_{i,j=1}^n A_{ij}\Big(\frac{\nabla\rho_i}{m_i\rho_i}+\frac{\nabla w}{m_i}\Big)\cdot\Big(\frac{\nabla\rho_j}{m_j\rho_j}+\frac{\nabla w}{m_j}\Big)\\&=\sum_{i,j=1}^n M_{ij}^{BD}\frac{1}{m_i}\Big(\frac{\nabla\rho_i}{\sqrt{\rho_i}}+\sqrt{\rho_i}\nabla w\Big)\cdot\frac{1}{m_j}\Big(\frac{\nabla\rho_j}{\sqrt{\rho_j}}+\sqrt{\rho_j}\nabla w\Big)\\&\ge\mu\bigg|P_L\Big(\frac{1}{m_i}\Big(\frac{\nabla\rho_i}{\sqrt{\rho_i}}+\sqrt{\rho_i}\nabla w\Big)\Big)_{i=1}^n\bigg|^2.\end{aligned}$$

We insert the definition of the projection matrix P_L:

$$\begin{aligned}\bigg[P_L\Big(\frac{1}{m_j}\Big(\frac{\nabla\rho_j}{\sqrt{\rho_j}}+\sqrt{\rho_j}\nabla w\Big)\Big)_{j=1}^n\bigg]_i&=\sum_{j=1}^n\Big(\delta_{ij}-\frac{\sqrt{\rho_i\rho_j}}{\rho}\Big)\frac{1}{m_j}\Big(\frac{\nabla\rho_j}{\sqrt{\rho_j}}+\sqrt{\rho_j}\nabla w\Big)\\&=\frac{1}{m_i}\Big(\frac{\nabla\rho_i}{\sqrt{\rho_i}}+\sqrt{\rho_i}\nabla w\Big)-\frac{\sqrt{\rho_i}}{\rho}\sum_{j=1}^n\frac{1}{m_j}(\nabla\rho_j+\rho_j\nabla w)=\frac{1}{m_i}\Big(\frac{\nabla\rho_i}{\sqrt{\rho_i}}+\sqrt{\rho_i}\nabla w\Big).\end{aligned}$$

The last step follows from the pressure constraint (7). Indeed, by (8),

$$\sum_{j=1}^n\frac{1}{m_j}(\nabla\rho_j+\rho_j\nabla w)=\frac{1}{\theta}\sum_{j=1}^n\frac{\nabla(\rho_j\theta)}{m_j}=\frac{1}{\theta}\nabla p=0. \tag{44}$$

We have shown that

$$\sum_{i,j=1}^n A_{ij}\nabla\Big(q_i+\frac{w}{m_i}\Big)\cdot\nabla\Big(q_j+\frac{w}{m_j}\Big)\ge\sum_{i=1}^n\frac{\mu}{m_i^2}\big|2\nabla\sqrt{\rho_i}+\sqrt{\rho_i}\nabla w\big|^2,$$

which equals (43) after integration over Ω. ∎

Remark 8. We observe that the sum (44) vanishes even without requiring the constraint (7). Indeed, by (17),

$$\sum_{j=1}^n \frac{1}{m_j}(\nabla\rho_j + \rho_j\nabla w) = \frac{1}{\theta}\sum_{j=1}^n \frac{1}{m_j}\nabla(\rho_j\theta) = \frac{1}{\theta}\sum_{j=1}^n d_j = 0.$$

The fact that $\sum_{j=1}^n d_j$ vanishes is a necessary condition for the invertibility of the linear system (18). □

In view of Lemma 7 and the lower bound $\kappa \ge c_\kappa(1+\theta^2)$, we conclude from (42) the following discrete entropy inequality.

Lemma 9 (Discrete entropy inequality)**.** *It holds that*

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \big(h(\rho_1,\ldots,\rho_{n-1},\theta) + Ee^{-w_0}\big)\mathrm{d}x + \varepsilon C\big(\|\boldsymbol{w}\|_{H^2(\Omega)}^2 + \|\nabla w\|_{L^4(\Omega)}^4\big) \\
&\quad + \int_\Omega \big(|\nabla w|^2 + |\nabla\theta|^2\big)\mathrm{d}x + \int_\Omega \sum_{i=1}^n \frac{\mu}{m_i^2}\big|2\nabla\sqrt{\rho_i} + \sqrt{\rho_i}\nabla w\big|^2\mathrm{d}x \\
&\le \frac{1}{\tau}\int_\Omega \big(h(\bar\rho_1,\ldots,\bar\rho_{n-1},\bar\theta) + \bar{E}e^{-w_0}\big)\mathrm{d}x.
\end{aligned}$$

Finally, we derive an estimate for the temperature.

Lemma 10. *There exists a constant $C>0$, only depending on λ, Ω, $\partial\Omega$, and θ^0 such that*

$$\frac{c_w}{2\tau}\int_\Omega \rho\theta^2\mathrm{d}x + \frac{c_\kappa}{2}\int_\Omega (1+\theta^2)|\nabla\theta|^2\mathrm{d}x \le C + C\int_\Omega \sum_{i=1}^n |\nabla\sqrt{\rho_i}|^2\mathrm{d}x + \frac{c_w}{2\tau}\int_\Omega \rho\bar\theta^2\mathrm{d}x.$$

Proof. We use θ as a test function in the approximate energy equation (37). Observing that $\nabla w_i = \nabla\rho_i/(m_i\rho_i) - \nabla\rho_n/(m_n\rho_n)$ by (25) and $\sum_{i=1}^n B_i\nabla w_i = \sum_{i=1}^n B_i(m_i\rho_i)^{-1}\nabla\rho_i$ by (29), we find that

$$\begin{aligned}
0 &= \frac{c_w}{\tau}\int_\Omega \rho(\theta-\bar\theta)\mathrm{d}x + \int_\Omega \kappa(\theta)|\nabla\theta|^2\mathrm{d}x + \int_\Omega \sum_{i,j=1}^n \frac{A_{ij}}{m_im_j}|\nabla\theta|^2\mathrm{d}x \\
&\int_\Omega \sum_{i=1}^n \frac{B_i}{m_i\rho_i}\nabla\rho_i\cdot\nabla\theta\mathrm{d}x - \lambda\int_{\partial\Omega}(\theta_0-\theta)\theta\mathrm{d}s + \varepsilon\int_\Omega (\theta_0+\theta)(\log\theta - \log\theta_0)\theta\mathrm{d}x \\
&+ \varepsilon\int_\Omega \left(|\mathrm{D}^2\theta|^2 - \frac{1}{\theta}\mathrm{D}^2\theta : (\nabla\theta\otimes\nabla\theta) + \frac{|\nabla\theta|^4}{\theta^2}\right)\mathrm{d}x = J_1 + \cdots + J_7.
\end{aligned}$$

We deduce from Young's inequality and Assumption (A4) on κ that

$$J_1 \ge \frac{c_w}{2\tau}\int_\Omega \rho(\theta^2 - \bar\theta^2)\mathrm{d}x, \quad J_2 \ge c_\kappa\int_\Omega (1+\theta^2)|\nabla\theta|^2\mathrm{d}x.$$

Furthermore, $J_3 \geq 0$. Definition (28) of B_i and A_{ij} as well as the bound $\rho_j \leq \rho^*$ show that

$$
\begin{aligned}
J_4 &= \theta \sum_{i,j=1}^n \frac{A_{ij}}{m_i m_j \rho_i} \nabla\rho_i \cdot \nabla\theta \mathrm{d}x = \theta \sum_{i,j=1}^n \frac{M_{ij}^{BD}}{m_i m_j} \frac{\sqrt{\rho_j}}{\sqrt{\rho_i}} \nabla\rho_i \cdot \nabla\theta \mathrm{d}x \\
&\geq -\frac{c_\kappa}{2} \int_\Omega \theta^2 |\nabla\theta|^2 \mathrm{d}x - C \int_\Omega \sum_{i=1}^n |\nabla\sqrt{\rho_i}|^2 \mathrm{d}x.
\end{aligned}
$$

The integrals J_5 are J_6 are bounded from below since

$$
J_5 \geq -\frac{\lambda}{4} \int_{\partial\Omega} \theta_0^2 \mathrm{d}s \geq -C(\lambda, \partial\Omega, \theta_0),
$$

and the dominant term in J_6 is $\theta^2 \log\theta$, which is bounded from below by a negative constant. Finally, J_7 is nonnegative:

$$
J_7 = \frac{\varepsilon}{2} \int_\Omega \left(|\mathrm{D}^2\theta|^2 + \frac{|\nabla\theta|^4}{\theta^2} + \left| \mathrm{D}^2\theta - \frac{1}{\theta} \nabla\theta \otimes \nabla\theta \right|^2 \right) \mathrm{d}x \geq 0.
$$

Collecting these estimates finishes the proof. $\square$

4.4. **Uniform estimates.** Let $(w_1^k, \ldots, w_{n-1}^k, w^k)$ be a solution to the approximate scheme (36)–(37) with $(w_1^{k-1}, \ldots, w_{n-1}^{k-1}, w^{k-1}) = (\bar{w}_1, \ldots, \bar{w}_{n-1}, \bar{w})$. We set $\theta^k = \exp(w^k)$ and $\rho_i^k = \rho_i(w^k)$ determined from Lemma 3. Furthermore, we set $E^k = c_w \rho \theta^k$, recalling that $\rho = \sum_{i=1}^n \rho_i^0$. We introduce the piecewise constant in time functions

$$
\begin{aligned}
&\rho_i^{(\tau)}(x,t) = \rho_i^k(x), \quad q_i^{(\tau)} = \frac{1}{m_i} \log \frac{\rho_i^k}{m_i} - c_w(\log\theta^k - 1) \quad \text{for } i = 1, \ldots, n, \\
&\theta^{(\tau)}(x,t) = \theta^k(x), \quad E^{(\tau)}(x,t) = E^k(x), \quad w_i^{(\tau)}(x,t) = w_i^k(x) \quad \text{for } i = 1, \ldots, n-1,
\end{aligned}
$$

where $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$, and $k = 1, \ldots, N$. At time $t = 0$, we set $\rho_i^{(\tau)}(0) = \rho_i^0$ and $\theta^{(\tau)}(0) = \theta^0$. Furthermore, we introduce the shift operator $(\sigma_\tau \rho_i^{(\tau)})(x,t) = \rho_i^{k-1}(x)$ if $t \in ((k-1)\tau, k\tau]$. Then $(\boldsymbol{\rho}^{(\tau)}, \theta^{(\tau)})$ solves

$$
\begin{aligned}
0 &= \frac{1}{\tau} \int_0^T \int_\Omega (\rho_i^{(\tau)} - \sigma_\tau \rho_i^{(\tau)}) \phi_i \mathrm{d}x\mathrm{d}t + \varepsilon \int_0^T \int_\Omega \big(\mathrm{D}^2 w_i^{(\tau)} : \mathrm{D}^2\phi_i + w_i^{(\tau)} \phi_i \big) \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \left(\sum_{j=1}^{n-1} A_{ij}(\boldsymbol{w}^{(\tau)}) \nabla w_j^{(\tau)} + e^{-w^{(\tau)}} B_i(\boldsymbol{w}^{(\tau)}) \nabla w^{(\tau)} \right) \cdot \nabla\phi_i \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
0 &= \frac{1}{\tau} \int_0^T \int_\Omega (E^{(\tau)} - \sigma_\tau E^{(\tau)}) \phi_0 \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \kappa(\theta^{(\tau)}) \nabla\theta^{(\tau)} \cdot \nabla\phi_0 \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^{n-1} B_j(\boldsymbol{w}^{(\tau)}) \nabla w_i^{(\tau)} \cdot \nabla\phi_0 \mathrm{d}x - \lambda \int_0^T \int_{\partial\Omega} (\theta_0 - \theta^{(\tau)}) \phi_0 \mathrm{d}s\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \sum_{i,j=1}^n \frac{A_{ij}(\boldsymbol{w}^{(\tau)})}{m_i m_j} \nabla\theta^{(\tau)} \cdot \nabla\phi_0 \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{46}
$$

$$+\varepsilon\int_0^T\int_\Omega(\theta_0+\theta^{(\tau)})(\log\theta^{(\tau)}-\log\theta_0)\phi_0\mathrm{d}x\mathrm{d}t$$
$$+\varepsilon\int_0^T\int_\Omega\theta^{(\tau)}\big(\mathrm{D}^2\log\theta^{(\tau)}:\mathrm{D}^2\phi_0+|\nabla\log\theta^{(\tau)}|^2\nabla\log\theta^{(\tau)}\cdot\nabla\phi_0\Big)\mathrm{d}x\mathrm{d}t.$$

The discrete entropy inequality in Lemma 9 and the temperature estimates in Lemma 10 yield, after summation over $k=1,\ldots,N$,

$$\begin{aligned}\sup_{0<t<T}\int_\Omega\bigg(&h(\rho_1^{(\tau)}(t),\ldots,\rho_{n-1}^{(\tau)}(t),\theta^{(\tau)}(t))+\frac{c_w}{\theta_0}\rho\theta^{(\tau)}(t)\bigg)\mathrm{d}x\\&+\int_0^T\int_\Omega\big(|\nabla\log\theta^{(\tau)}|^2+|\nabla\theta^{(\tau)}|^2\big)\mathrm{d}x\mathrm{d}t\\&+\varepsilon C\int_0^T\big(\|\boldsymbol{w}^{(\tau)}\|_{H^2(\Omega)}^2+\|\nabla w^{(\tau)}\|_{L^4(\Omega)}^4\big)\mathrm{d}t\\&+\int_0^T\int_\Omega\sum_{i=1}^n\frac{\mu}{m_i^2}\big|2\nabla(\rho_i^{(\tau)})^{1/2}+(\rho_i^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)}\big|^2\mathrm{d}x\mathrm{d}t\\\le\int_\Omega&\big(h(\rho_1^0,\ldots,\rho_{n-1}^0,\theta^0)+c_w\rho\theta^0\big)\mathrm{d}x,\end{aligned}\tag{47}$$

$$\begin{aligned}c_w\sup_{0<t<T}\int_\Omega\rho(\theta^{(\tau)})^2\mathrm{d}x&+c_\kappa\int_0^T\int_\Omega(1+(\theta^{(\tau)})^2)|\nabla\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}t\\\le C(T)+C\int_0^T&\int_\Omega\sum_{i=1}^n|\nabla(\rho_i^{(\tau)})^{1/2}|^2\mathrm{d}x\mathrm{d}t+\frac{c_w}{2}\int_\Omega\rho(\theta^0)^2\mathrm{d}x.\end{aligned}\tag{48}$$

Lemma 11. *There exists $C>0$ not depending on (ε,τ) such that*

$$\|\boldsymbol{\rho}^{(\tau)}\|_{L^\infty(\Omega_T)}+\|\theta^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}\le C,\tag{49}$$
$$\|\log\theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}+\|\theta^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}\le C,\tag{50}$$
$$\varepsilon^{1/2}\|\boldsymbol{w}^{(\tau)}\|_{L^2(0,T;H^2(\Omega))}+\varepsilon^{1/4}\|\nabla w^{(\tau)}\|_{L^4(\Omega_T)}\le C,\tag{51}$$

Proof. Estimates (49) and (51) are an immediate consequence of (47) and $\rho\ge\rho_*>0$. Bound (47) also shows that $\sup_{(0,T)}\int_\Omega(-\log\theta^{(\tau)}+\theta^{(\tau)})\mathrm{d}x$ is uniformly bounded from above. Thus, $\log\theta^{(\tau)}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Then the uniform bounds for $\nabla\log\theta^{(\tau)}$ and $\nabla\theta^{(\tau)}$ as well as the Poincaré–Wirtinger inequality yield bounds for $\log\theta^{(\tau)}$ and $\theta^{(\tau)}$ in $L^2(\Omega_T)$, proving (50). □

Lemma 12. *There exists $C>0$ not depending on (ε,τ) such that for $i=1,\ldots,n$,*

$$\|(\rho_i^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\Omega))}+\|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}\le C,\tag{52}$$
$$\|\theta^{(\tau)}\|_{L^\infty(0,T;L^2(\Omega))}+\|(\theta^{(\tau)})^2\|_{L^2(0,T;H^1(\Omega))}+\|\theta^{(\tau)}\|_{L^{16/3}(\Omega_T)}\le C.\tag{53}$$

Proof. We infer from (47) that

$$\int_0^T\int_\Omega|\nabla(\rho_i^{(\tau)})^{1/2}|^2\mathrm{d}x\mathrm{d}t \le C\int_0^T\int_\Omega\big|2\nabla(\rho_i^{(\tau)})^{1/2}|^2+(\rho_i^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)}\big|^2\mathrm{d}x\mathrm{d}t \\ + C\int_0^T\int_\Omega|\nabla\log\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}t \le C,$$

and the $L^\infty(\Omega_T)$ bound (49) gives for $i=1,\ldots,n$,

$$\|\rho_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} \le 2\|\rho_i^{(\tau)}\|^{1/2}_{L^\infty(\Omega_T)}\|\nabla(\rho_i^{(\tau)})^{1/2}\|_{L^2(\Omega_T)}+\|\rho_i^{(\tau)}\|_{L^2(\Omega_T)} \le C.$$

Therefore, the right-hand side of (48) is uniformly bounded, which proves the first two estimates in (53). The remaining one is a consequence of the Gagliardo–Nirenberg inequality with $\eta=3/4$:

$$\|(\theta^{(\tau)})^2\|^{8/3}_{L^{8/3}(\Omega_T)} \le C\int_0^T\|(\theta^{(\tau)})^2\|^{8\eta/3}_{H^1(\Omega)}\|(\theta^{(\tau)})^2\|^{8(1-\eta)/3}_{L^1(\Omega)}\mathrm{d}t \\ \le \|\theta^{(\tau)}\|^{4/3}_{L^\infty(0,T;L^2(\Omega))}\int_0^T\|(\theta^{(\tau)})^2\|^2_{H^1(\Omega)}\mathrm{d}t \le C.$$

This finishes the proof. □

The following lemma can be proved as in [15, Lemma 9].

Lemma 13. *There exists $C>0$ not depending on (ε,τ) such that*

$$\|\rho_i^{(\tau)}-\sigma_\tau\rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)}+\|\theta^{(\tau)}-\sigma_\tau\theta^{(\tau)}\|_{L^{16/15}(0,T;W^{2,16}(\Omega)^*)} \le C\tau. \tag{54}$$

4.5. **The limit** (ε,τ)**.** The bounds (50), (52), and (54) allow us to apply the Aubin–Lions lemma in the version of [10]. There exist subsequences, which are not relabeled, such that as $(\varepsilon,\tau)\to 0$,

$$\rho_i^{(\tau)}\to\rho_i, \quad \theta^{(\tau)}\to\theta \quad \text{strongly in } L^2(\Omega_T),\ i=1,\ldots,n-1.$$

The convergence also holds for $i=n$ since $\rho_n^{(\tau)}=1-\sum_{i=1}^{n-1}\rho_i^{(\tau)}$. Thanks to the $L^\infty(\Omega_T)$ bound for $\rho_i^{(\tau)}$ and the $L^{16/3}(\Omega_T)$ bound for $\theta^{(\tau)}$, we have

$$\rho_i^{(\tau)}\to\rho_i \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r<\infty, \\ \theta^{(\tau)}\to\theta \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r<16/3.$$

We claim that $\rho_i>0$ and $\theta>0$ a.e. in Ω_T. The positivity of ρ_i is proved as in [15, p. 16]. The strong convergence of $(\theta^{(\tau)})$ implies a.e. convergence and in particular $\log\theta^{(\tau)}\to Z$ a.e. Thus, $\theta^{(\tau)}\to\exp(Z)$ a.e. We conclude that $\theta=\exp(Z)>0$ a.e. in Ω_T.

It follows that $\log\theta\in L^2(\Omega_T)$ and estimate (50) yields

$$\nabla\log\theta^{(\tau)}\rightharpoonup\nabla\log\theta \quad \text{weakly in } L^2(\Omega_T). \tag{55}$$

Furthermore, in view of (50), (52), and (54), up to subsequences,

$$\rho_i^{(\tau)}\rightharpoonup\rho_i, \quad \theta^{(\tau)}\rightharpoonup\theta \quad \text{weakly in } L^2(0,T;H^1(\Omega)),$$

$$\begin{aligned}
&\tau^{-1}(\rho_i^{(\tau)} - \sigma_\tau\rho_i^{(\tau)}) \rightharpoonup \partial_t\rho_i \quad \text{weakly in } L^2(0,T;H^2(\Omega)^*), \\
&\tau^{-1}(\theta^{(\tau)} - \sigma_\tau\theta^{(\tau)}) \rightharpoonup \partial_t\rho_i \quad \text{weakly in } L^{16/15}(0,T;W^{2,16}(\Omega)^*),
\end{aligned}$$

and the bounds (51) show that

$$\varepsilon\log\theta^{(\tau)} \to 0, \quad \varepsilon w_i^{(\tau)} \to 0 \quad \text{strongly in } L^2(0,T;H^2(\Omega)).$$

The embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ is compact, giving $\theta^{(\tau)} \to \theta$ strongly in $L^2(0,T;L^2(\partial\Omega))$.

These convergences are sufficient to pass to the limit $(\varepsilon,\tau) \to 0$ in (45)–(46), showing that $(\boldsymbol{\rho},\theta)$ solves the weak formulation (34)–(35). We only detail the limits in the terms $A_{ij}^{(\tau)} = A_{ij}(\boldsymbol{w}^{(\tau)})$ and $B_i^{(\tau)} = B_i(\boldsymbol{w}^{(\tau)})$. We know that $\nabla(\rho_i^{(\tau)})^{1/2} \rightharpoonup \nabla\rho_i^{1/2}$ weakly in $L^2(\Omega_T)$ and

$$\frac{A_{ij}^{(\tau)}}{m_j(\rho_j^{(\tau)})^{1/2}} = M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{(\rho_i^{(\tau)})^{1/2}}{m_j} \to M_{ij}^{BD}(\boldsymbol{\rho})\frac{\rho_i^{1/2}}{m_j} = \frac{A_{ij}}{m_j\rho_j^{1/2}}$$

strongly in $L^\gamma(\Omega_T)$ for all $\gamma < \infty$. Using (31) and (24), this implies that

$$\begin{aligned}
\sum_{j=1}^{n-1} A_{ij}^{(\tau)}\nabla w_j^{(\tau)} &= \sum_{j=1}^{n}\frac{A_{ij}^{(\tau)}}{m_j}\nabla\log\frac{\rho_i^{(\tau)}}{m_j} = 2\sum_{j=1}^{n} M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{(\rho_i^{(\tau)})^{1/2}}{m_j}\nabla(\rho_j^{(\tau)})^{1/2} \\
&\rightharpoonup 2\sum_{j=1}^{n} M_{ij}^{BD}\frac{\rho_i^{1/2}}{m_j}\nabla\rho_j^{1/2} \quad \text{weakly in } L^s(\Omega_T),\ s < 2.
\end{aligned}$$

Since the sequence is bounded in $L^2(\Omega_T)$, this convergence also holds in this space. Similarly,

$$\begin{aligned}
B_i^{(\tau)}e^{-w^{(\tau)}}\nabla w^{(\tau)} &= \sum_{j=1}^{n}\frac{A_{ij}^{(\tau)}}{m_j}\nabla\log\theta^{(\tau)} \rightharpoonup \sum_{j=1}^{n}\frac{A_{ij}}{m_j}\nabla\log\theta \quad \text{weakly in } L^2(\Omega_T), \\
A_{ij}^{(\tau)}\nabla\theta^{(\tau)} &= M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})(\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2}\nabla\theta^{(\tau)} \rightharpoonup A_{ij}\nabla\theta \quad \text{weakly in } L^2(\Omega_T),
\end{aligned}$$

and using $\theta^{(\tau)} \to \theta$ strongly in $L^r(\Omega_T)$ for $r < 16/3$,

$$\sum_{i=1}^{n-1} B_i^{(\tau)}\nabla w_i^{(\tau)} = 2\sum_{i,j=1}^{n}\frac{M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})}{m_im_j}\theta^{(\tau)}(\rho_j^{(\tau)})^{1/2}\nabla(\rho_i^{(\tau)})^{1/2} \rightharpoonup 2\sum_{i=1}^{n}\frac{B_i}{m_i\rho_i^{1/2}}\nabla\rho_i^{1/2}$$

weakly in $L^s(\Omega_T)$ for $s < 16/11$, and since the right-hand side lies in $L^{16/11}(\Omega_T)$, this convergence also holds in $L^{16/11}(\Omega_T)$.

Next, we claim that $\rho_i(0)$ and $\theta(0)$ satisfy the initial data. The time derivative of the linear interpolant

$$\widetilde{\rho}_i^{(\tau)}(t) = \rho_i^k - \frac{k\tau - t}{\tau}(\rho_i^k - \rho_i^{k-1}) \quad \text{for } (k-1)\tau < t < k\tau$$

is bounded since, because of (54),

$$\|\partial_t\widetilde{\rho}_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le \tau^{-1}\|\rho_i^{(\tau)} - \sigma_\tau\rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)} \le C.$$

Thus, $\widetilde{\rho}_i^{(\tau)}$ is uniformly bounded in $H^1(0,T;H^2(\Omega)^*)\hookrightarrow C^0([0,T];H^2(\Omega)^*)$ and we conclude for a subsequence that $\rho_i^0=\widetilde{\rho}_i^{(\tau)}(0)\rightharpoonup r_i$ weakly in $H^2(\Omega)^*$ for some $r_i\in H^2(\Omega)^*$. It follows that $r_i=\rho_i^0$. As $\widetilde{\rho}_i^{(\tau)}$ and $\rho_i^{(\tau)}$ converge to the same limit,

$$\|\widetilde{\rho}_i^{(\tau)}-\rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)}\le\|\rho_i^{(\tau)}-\sigma_\tau\rho_i^{(\tau)}\|_{L^2(0,T;H^2(\Omega)^*)}\le C\tau\to 0,$$

this shows that $\rho_i^0=r_i=\rho_i(0)$ in $H^2(\Omega)^*$. In an analogous way, we verify that $\theta(0)=\theta^0$ in $W^{2,16}(\Omega)^*$.

The initial data are satisfied in better spaces. Indeed, going back to (34)–(35), the regularity of ρ_i implies that $\partial_t\rho_i\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;H^1(\Omega)^*)\hookrightarrow C^0([0,T];L^2(\Omega))$ and thus $\rho_i(0)=\rho_i^0$ in the sense of $L^2(\Omega)$. The temperature satisfies $\theta\in L^\infty(0,T;L^2(\Omega))\cap C^0([0,T];W^{2,16}(\Omega)^*)$, which gives $\theta\in C^0_w([0,T];L^2(\Omega))$. Consequently, $\theta(0)=\theta^0$ weakly in $L^2(\Omega)$. Moreover, we deduce from $|\kappa\nabla\theta|\le C_\kappa(|\nabla\theta|+\theta|\nabla\theta^2|)\in L^{16/11}(\Omega_T)$ that $\partial_t\theta\in L^{16/11}(0,T;W^{1,16/11}(\Omega)^*)$. This completes the proof.

5. Proof of Theorem 2

Let $(\boldsymbol{\rho},\theta)$ be a weak solution and $(\bar{\boldsymbol{\rho}},\bar\theta)$ be a strong solution to (1)–(8). We introduce the entropy

$$H(\boldsymbol{\rho}(t),\theta(t))=\int_\Omega\bigg(\sum_{i=1}^n\frac{\rho_i}{m_i}\bigg(\log\frac{\rho_i}{m_i}-1\bigg)-c_w\rho\log\theta\bigg)\mathrm{d}x.$$

Lemma 14 (Entropy equality for strong solutions)**.** *Let $(\bar{\boldsymbol{\rho}},\bar\theta)$ be a strong solution to* (1)–(8) *(in the sense mentioned after Theorem 2) with $\lambda=0$. Then*

$$H(\bar{\boldsymbol{\rho}}(t),\bar\theta(t))+\int_0^t\int_\Omega\frac{\kappa(\bar\theta)}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x\mathrm{d}s+\frac12\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j|\bar u_i-\bar u_j|^2\mathrm{d}x\mathrm{d}s=H(\bar{\boldsymbol{\rho}}(0),\bar\theta(0)).$$

Proof. We use (1) and (2) and integrate by parts to obtain

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t}&=\int_\Omega\bigg(\sum_{i=1}^n\frac{\partial_t\bar\rho_i}{m_i}\log\frac{\bar\rho_i}{m_i}-\frac{c_w}{\rho}\partial_t(\rho\bar\theta)\bigg)\mathrm{d}x\\
&=\int_\Omega\bigg\{\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\nabla\log\frac{\bar\rho_i}{m_i}+\frac{\nabla\bar\theta}{\bar\theta^2}\bigg(-\bar\kappa\nabla\bar\theta+\bar\theta\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\bigg)\bigg\}dx\\
&=-\int_\Omega\frac{\bar\kappa}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x+\int_\Omega\sum_{i=1}^n\frac{\bar u_i}{m_i}\cdot(\nabla\bar\rho_i+\bar\rho_i\nabla\log\bar\theta)\mathrm{d}x\\
&=-\int_\Omega\frac{\bar\kappa}{\bar\theta^2}|\nabla\bar\theta|^2\mathrm{d}x+\int_\Omega\sum_{i=1}^n\frac{1}{\bar\theta}\bar u_i\cdot\bar d_i\mathrm{d}x,\end{aligned}$$

where $\bar\kappa = \kappa(\bar\theta)$ and we used (17) in the last step. By the algebraic system (5) and the symmetry of (b_{ij}),

$$\sum_{i=1}^n \frac{1}{\bar\theta}\bar u_i \cdot \bar d_i = -\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j(\bar u_i - \bar u_j)\cdot \bar u_i = -\frac{1}{2}\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j|\bar u_i - \bar u_j|^2. \tag{56}$$

This shows the claim. □

Lemma 15 (Entropy inequality for weak solutions)**.** *Let* $(\bar{\boldsymbol\rho},\bar\theta)$ *be a weak solution to* (1)–(8) *with* $\lambda = 0$*. Then*

$$H(\boldsymbol\rho(t),\theta(t)) + \int_0^t\int_\Omega \frac{\kappa}{\theta^2}|\nabla\theta|^2 \mathrm{d}x\mathrm{d}s + \frac12\int_0^t\int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i-u_j|^2\mathrm{d}x\mathrm{d}s \le H(\boldsymbol\rho^0,\theta^0).$$

Proof. Let $(\boldsymbol\rho^k,\theta^k)$ for $k=1,\ldots,N$ be a solution to the approximate problem (36)–(37), constructed in Section 4.2. According to (42), this solution satisfies

$$\begin{aligned} H(\boldsymbol\rho^k,\theta^k) &+ \tau\int_\Omega \kappa(\theta^k)|\nabla\log\theta^k|^2\mathrm{d}x \\ &+ \tau\int_\Omega\sum_{i,j=1}^n A_{ij}^k\nabla\left(q_i^k + \frac{w^k}{m_i}\right)\cdot\nabla\left(q_j^k+\frac{w^k}{m_j}\right)\mathrm{d}x \le H(\boldsymbol\rho^{k-1},\theta^{k-1}), \end{aligned}$$

where the superindex k denotes the kth time step. By Corollary 5 as well as relations (21) and (56),

$$\begin{aligned} \sum_{i,j=1}^n A_{ij}^k\nabla\left(q_i^k + \frac{w^k}{m_i}\right)\cdot\nabla\left(q_j^k+\frac{w^k}{m_j}\right) &= \sum_{i,j=1}^n (M_{ij}^{BD})^k\frac{d_i^k}{\theta^k(\rho_i^k)^{1/2}}\cdot\frac{d_j^k}{\theta^k(\rho_j^k)^{1/2}} \\ &= -\sum_{i=1}^n\frac{1}{\theta^k}d_i^k\cdot u_i^k = \frac12\sum_{i,j=1}^n b_{ij}\rho_i^k\rho_j^k|u_i^k-u_j^k|^2. \end{aligned}$$

Therefore,

$$H(\boldsymbol\rho^k,\theta^k) + \tau\int_\Omega\kappa(\theta^k)|\nabla\log\theta^k|^2\mathrm{d}x + \frac{\tau}{2}\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^k\rho_j^k|u_i^k-u_j^k|^2\mathrm{d}x \le H(\boldsymbol\rho^{k-1},\theta^{k-1}).$$

We sum over $k=1,\ldots,j$ with $t\in((j-1)\tau,j\tau]$ and use the notation of Section 4.4:

$$\begin{aligned} H(\boldsymbol\rho^{(\tau)}(t),\theta^{(\tau)}(t)) &+ \int_0^t\int_\Omega\kappa(\theta^{(\tau)})|\nabla\log\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}s \\ &+ \frac12\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)}-u_j^{(\tau)}|^2\mathrm{d}x\mathrm{d}s \le H(\boldsymbol\rho^0,\theta^0) \end{aligned} \tag{57}$$

for a.e. $t\in(0,T)$.

It remains to pass to the limit $(\varepsilon,\tau)\to 0$ in (57). We deduce from the strong convergence of $(\boldsymbol{\rho}^{(\tau)})$ and $(\theta^{(\tau)})$ that

$$H(\boldsymbol{\rho}(t),\theta(t)) \le \liminf_{(\varepsilon,\tau)\to 0} H(\boldsymbol{\rho}^{(\tau)}(t),\theta^{(\tau)}(t)).$$

We deduce from the strong convergence $\rho_i^{(\tau)}\to\rho_i$ in $L^q(\Omega_T)$ for any $q<\infty$ and the boundedness of M_{ij}^{BD} that $M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\to M_{ij}^{BD}(\boldsymbol{\rho})$ strongly in any $L^q(\Omega_T)$. In view of the weak convergences $\nabla\log\theta^{(\tau)}\rightharpoonup\nabla\log\theta$ from (55) and $\nabla(\rho_i^{(\tau)})^{1/2}\rightharpoonup\nabla\rho_i^{1/2}$ from (52) weakly in $L^2(\Omega_T)$, we have

$$2\nabla(\rho_i^{(\tau)})^{1/2}+\rho_i^{(\tau)}\nabla\log\theta^{(\tau)}\rightharpoonup 2\nabla\rho_i^{1/2}+\rho_i\nabla\log\theta \quad \text{weakly in } L^2(\Omega_T).$$

Hence, using (21),

$$\begin{aligned}(\rho_i^{(\tau)})^{1/2}u_i^{(\tau)} &= \sum_{j=1}^n M_{ij}^{BD}(\boldsymbol{\rho}^{(\tau)})\frac{1}{m_j}\big(2\nabla(\rho_i^{(\tau)})^{1/2}+\rho_i^{(\tau)}\nabla\log\theta^{(\tau)}\big)\\ &\rightharpoonup \sum_{j=1}^n M_{ij}^{BD}(\boldsymbol{\rho})\frac{1}{m_j}\big(2\nabla\rho_i^{1/2}+\rho_i\nabla\log\theta\big)=\rho_i^{1/2}u_i.\end{aligned}$$

weakly in $L^2(\Omega_T)$, where the last identity is the definition of u_i. Then, taking into account the boundedness of $\rho_i^{(\tau)}$ in $L^\infty(\Omega_T)$, for any $i,j=1,\ldots,n$,

$$(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2}u_i^{(\tau)}\rightharpoonup(b_{ij}\rho_i\rho_j)^{1/2}u_i \quad \text{weakly in } L^2(\Omega_T).$$

As the $L^2(\Omega_T)$ norm is weakly lower semicontinuous,

$$\begin{aligned}\int_0^T\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i-u_j|^2\mathrm{d}x\mathrm{d}s &\le \liminf_{(\varepsilon,\tau)\to 0}\int_0^T\int_\Omega\sum_{i,j=1}^n\big|(b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)})^{1/2}(u_i^{(\tau)}-u_j^{(\tau)})\big|^2\mathrm{d}x\mathrm{d}s\\ &=\liminf_{(\varepsilon,\tau)\to 0}\int_0^T\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i^{(\tau)}\rho_j^{(\tau)}|u_i^{(\tau)}-u_j^{(\tau)}|^2\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Finally, $\kappa(\theta^{(\tau)})^{1/2}\nabla\log\theta^{(\tau)}\to\kappa(\theta)^{1/2}\nabla\log\theta$ weakly in $L^1(\Omega_T)$ and, because of the uniform bounds, also in $L^2(\Omega_T)$. Hence,

$$\int_0^t\int_\Omega\frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2\mathrm{d}x\mathrm{d}s\le\liminf_{(\varepsilon,\tau)\to 0}\int_0^t\int_\Omega\frac{\kappa(\theta^{(\tau)})}{(\theta^{(\tau)})^2}|\nabla\theta^{(\tau)}|^2\mathrm{d}x\mathrm{d}s.$$

Thus, applying the limit inferior $(\varepsilon,\tau)\to 0$ to both sides of (57) yields the result. □

Lemma 16 (Relative entropy inequality)**.** *Let the assumptions of Theorem 2 hold and let* $\rho_i(0)=\bar\rho_i(0)$ *for* $i=1,\ldots,n$ *and* $\theta(0)=\bar\theta(0)$. *Then*

$$(58)\quad H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t))+\frac{\mu_M}{2}\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2\mathrm{d}x\mathrm{d}s$$

$$+ \frac{c_\kappa}{2}\int_0^t\int_\Omega |\nabla(\log\theta - \log\bar\theta)|^2 \mathrm{d}x\mathrm{d}s \le C\int_0^t\int_\Omega\bigg(\sum_{j=1}^n(\rho_j-\bar\rho_j)^2 + (\theta-\bar\theta)^2\bigg)\mathrm{d}x\mathrm{d}s,$$

where the relative entropy $H(\boldsymbol{\rho},\theta|\bar{\boldsymbol{\rho}},\bar\theta)$ *is defined in* (12).

Proof. We use the test functions $\phi_i = m_i^{-1}\log(\bar\rho_i/m_i) - c_w\log\bar\theta$ and $\phi_0 = -1/\bar\theta$ in the weak formulations satisfied by $\rho_i - \bar\rho_i$ and $\rho(\theta-\bar\theta)$, respectively,

$$\int_\Omega(\rho_i-\bar\rho_i)(t)\phi_i(t)\mathrm{d}x = \int_0^t\int_\Omega(\rho_i-\bar\rho_i)\partial_t\phi_i\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega(\rho_iu_i - \bar\rho_i\bar u_i)\cdot\nabla\phi_i\mathrm{d}x\mathrm{d}s,$$

$$\begin{aligned}\int_\Omega c_w\rho(\theta-\bar\theta)(t)\phi_0(t)\mathrm{d}x &= \int_0^t\int_\Omega c_w\rho(\theta-\bar\theta)\partial_t\phi_0\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega(\kappa\nabla\theta - \bar\kappa\nabla\bar\theta)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega\sum_{j=1}^n(h_ju_j - \bar h_j\bar u_j)\cdot\nabla\phi_0\mathrm{d}x\mathrm{d}s,\end{aligned}$$

where $h_j = (c_w + 1/m_j)\rho_j\theta$, $\bar h_j = (c_w+1/m_j)\bar\rho_j\bar\theta$, and $\kappa = \kappa(\theta)$, $\bar\kappa = \kappa(\bar\theta)$. Strictly speaking, we cannot use ϕ_i as a test function since $\log\bar\rho_i$ and $1/\bar\theta$ may be not integrable. However, we can use a density argument similarly as in the proof of [18, Lemma 8]. Then, summing over $i=1,\ldots,n$,

$$\begin{aligned}&\int_\Omega\bigg\{\sum_{i=1}^n(\rho_i-\bar\rho_i)(t)\bigg(\frac{1}{m_i}\log\frac{\bar\rho_i}{m_i} - c_w\log\bar\theta\bigg)(t) - c_w\rho\frac{\theta-\bar\theta}{\bar\theta}(t)\bigg\}\mathrm{d}x \\ &\quad = \int_0^t\int_\Omega\bigg\{\sum_{i=1}^n\bigg((\rho_i-\bar\rho_i)\frac{\partial_t\bar\rho_i}{m_i\bar\rho_i} + (\rho_iu_i - \bar\rho_i\bar u_i)\cdot\frac{\nabla\bar\rho_i}{m_i\bar\rho_i}\bigg) + c_w\rho(\theta-\bar\theta)\partial_t\bigg(-\frac{1}{\bar\theta}\bigg)\bigg\}\mathrm{d}x\mathrm{d}s \\ &\qquad - \int_0^t\int_\Omega(\kappa\nabla\theta - \bar\kappa\nabla\bar\theta)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega\sum_{j=1}^n(h_ju_j - \bar h_j\bar u_j)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

We subtract this identity and the entropy equality from Lemma 14 for $(\bar{\boldsymbol{\rho}},\bar\theta)$ from the entropy inequality for $(\boldsymbol{\rho},\theta)$ obtained in Lemma 15 and insert equations (1)–(2) to replace the time derivatives $\partial_t\bar\rho_i$ and $\partial_t(-1/\bar\theta)$. A computation shows that

$$H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t)) \le K_1\cdots + K_5, \quad \text{where} \tag{59}$$

$$\begin{aligned}K_1 &= -\int_0^t\int_\Omega\big(\kappa|\nabla\log\theta|^2 - \bar\kappa|\nabla\log\bar\theta|^2\big)\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega\bar\kappa\nabla\bar\theta\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2} - \frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^t\int_\Omega(\kappa\nabla\theta - \bar\kappa\nabla\bar\theta)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s, \\ K_2 &= -\int_0^t\int_\Omega\sum_{i=1}^n\frac{\bar\rho_i\bar u_i}{m_i}\cdot\nabla\bigg(\frac{\rho_i}{\bar\rho_i}\bigg)\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega\sum_{i=1}^n\frac{\nabla\bar\rho_i}{m_i\bar\rho_i}\cdot(\rho_iu_i - \bar\rho_i\bar u_i)\mathrm{d}x\mathrm{d}s, \\ K_3 &= -\int_0^t\int_\Omega\sum_{i=1}^n\bar h_i\bar u_i\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2} - \frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s - \int_0^t\int_\Omega\sum_{i=1}^n(h_iu_i - \bar h_i\bar u_i)\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s,\end{aligned}$$

$$K_4 = -\frac{1}{2}\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j|u_i-u_j|^2\mathrm{d}x\mathrm{d}s,$$

$$K_5 = \frac{1}{2}\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j\big|\bar u_i-\bar u_j\big|^2\mathrm{d}x\mathrm{d}s.$$

The term K_1 can be rewritten as

$$\begin{aligned} K_1 &= -\int_0^t\int_\Omega\frac{1}{\bar\theta}(\kappa\bar\theta-\bar\kappa\theta)\nabla(\log\theta-\log\bar\theta)\cdot\nabla\log\bar\theta\mathrm{d}x\mathrm{d}s \\ &\quad-\int_0^t\int_\Omega\kappa\big|\nabla(\log\theta-\log\bar\theta)\big|^2\mathrm{d}x\mathrm{d}s \\ &\quad+\int_0^t\int_\Omega\frac{\theta-\bar\theta}{\bar\theta}\nabla\log\bar\theta\cdot(\kappa\nabla\log\theta-\bar\kappa\nabla\log\bar\theta)\mathrm{d}x\mathrm{d}s =: K_{11}+K_{12}+K_{13}. \end{aligned}$$

The algebraic system (5) with $d_i=\nabla(\rho_i\theta)/m_i$ can be formulated as

$$-m_i\sum_{j=1}^n b_{ij}\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j)-\bar\rho_i\nabla\log\bar\theta=\nabla\bar\rho_i.$$

This allows us to rewrite K_2:

$$\begin{aligned} K_2 &= \int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\rho_j(u_i-u_j)\cdot\bar u_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\bar\rho_j(\bar u_i-\bar u_j)\cdot\bar u_i\mathrm{d}x\mathrm{d}s \\ &\quad+\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\rho_i\bar\rho_j(\bar u_i-\bar u_j)\cdot u_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i,j=1}^n b_{ij}\bar\rho_i\bar\rho_j(\bar u_i-\bar u_j)\cdot\bar u_i\mathrm{d}x\mathrm{d}s \\ &\quad+\int_0^t\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\rho_i\nabla\log\theta\cdot\bar u_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\rho_i\nabla\log\bar\theta\cdot\bar u_i\mathrm{d}x\mathrm{d}s \\ &\quad+\int_0^t\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\rho_i\nabla\log\bar\theta\cdot u_i\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\bar\rho_i\nabla\log\bar\theta\cdot\bar u_i\mathrm{d}x\mathrm{d}s \\ &=: K_{21}+\cdots+K_{28}. \end{aligned}$$

Furthermore, it follows from $h_i=(c_w+1/m_i)\rho_i\theta$ and $\sum_{i=1}^n\rho_iu_i=\sum_{i=1}^n\bar\rho_i\bar u_i=0$ that

$$\begin{aligned} K_3 &= -\int_0^t\int_\Omega\sum_{i=1}^n\bar h_i\bar u_i\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2}\bigg)\mathrm{d}x\mathrm{d}s-\int_0^t\int_\Omega\sum_{i=1}^n h_iu_i\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\ &= -\int_0^t\int_\Omega\sum_{i=1}^n\bigg(c_w+\frac{1}{m_i}\bigg)\bar\theta\bar\rho_i\bar u_i\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2}\bigg)\mathrm{d}x\mathrm{d}s \\ &\quad-\int_0^t\int_\Omega\sum_{i=1}^n\bigg(c_w+\frac{1}{m_i}\bigg)\theta\rho_iu_i\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^t \int_\Omega \sum_{i=1}^n \frac{\bar\rho_i\bar\theta}{m_i}\bar u_i\cdot\nabla\bigg(\frac{\theta}{\bar\theta^2}\bigg)\mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega \sum_{i=1}^n \frac{\rho_i\theta}{m_i}u_i\cdot\nabla\bigg(-\frac{1}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\
&= -\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar\rho_i\bar u_i}{m_i\bar\theta}\cdot\nabla\theta\mathrm{d}x\mathrm{d}s + 2\int_0^T \int_\Omega \sum_{i=1}^n \frac{\bar\rho_i\bar u_i\theta}{m_i\bar\theta^2}\cdot\nabla\bar\theta\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^T \int_\Omega \sum_{i=1}^n \frac{\rho_i u_i\theta}{m_i\bar\theta^2}\cdot\nabla\bar\theta\mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

We reformulate K_4 as

$$
\begin{aligned}
K_4 &= -\frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j|\bar u_i-\bar u_j|^2\mathrm{d}x\mathrm{d}s \\
&\quad - \int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j(u_i-u_j)\cdot(\bar u_i-\bar u_j)\mathrm{d}x\mathrm{d}s =: K_{41}+K_{42}+K_{43}.
\end{aligned}
$$

A long but straightforward computation shows that

$$
\begin{aligned}
&K_{21}+K_{22}+K_{23}+K_{24}+K_{42}+K_{43}+K_5 \\
&\quad = -\int_0^T \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i(\rho_j-\bar\rho_j)(u_i-\bar u_i)\cdot(\bar u_i-\bar u_j)\mathrm{d}x\mathrm{d}s =: L_1
\end{aligned}
$$

and

$$
\begin{aligned}
K_{25}+K_{26}+K_{27}+K_{28}+K_3 &= \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}(\rho_i-\bar\rho_i)(\nabla\log\theta-\nabla\log\bar\theta)\cdot\bar u_i\mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\bar\rho_i\bar u_i\cdot(\nabla\log\theta-\nabla\log\bar\theta)\bigg(1-\frac{\theta}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}\rho_i(u_i-\bar u_i)\cdot\nabla\log\bar\theta\bigg(1-\frac{\theta}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \sum_{i=1}^n \frac{1}{m_i}(\rho_i-\bar\rho_i)\bar u_i\cdot\nabla\log\bar\theta\bigg(1-\frac{\theta}{\bar\theta}\bigg)\mathrm{d}x\mathrm{d}s \\
&=: L_2+L_3+L_4+L_5.
\end{aligned}
$$

Inserting these expressions into (59), putting K_{12} on the left-hand side, and rearranging the terms, we find that

$$
H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t)) + \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2\mathrm{d}x\mathrm{d}s \tag{60}
$$

$$+\int_0^t\int_\Omega \kappa|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s \le K_{11}+K_{13}+L_1+\cdots+L_5.$$

The second term on the left-hand side can be bounded from below. Indeed, it follows from the symmetry of (b_{ij}), definition (19) of M_{ij}, and the positive definiteness (20) of M on L that

$$\begin{aligned}\frac12\sum_{i,j=1}^n & b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2 \\ &= \sum_{i=1}^n\bigg(\sum_{j=1,\,j\neq i}^n b_{ij}\rho_j\bigg)\rho_i|u_i-\bar u_i|^2 - \sum_{i,j=1,\,i\neq j}^n b_{ij}\rho_i\rho_j(u_i-\bar u_i)\cdot(u_j-\bar u_j) \\ &= \sum_{i,j=1}^n M_{ij}\sqrt{\rho_i}(u_i-\bar u_i)\cdot\sqrt{\rho_j}(u_j-\bar u_j)\ge \mu_M|P_L\boldsymbol{Y}|^2,\end{aligned}$$

where $Y_j=\sqrt{\rho_j}(u_j-\bar u_j)$. The norm of the projection is computed according to

$$\begin{aligned}|P_L\boldsymbol{Y}|^2 &= |\boldsymbol{Y}|^2-|P_{L^\perp}\boldsymbol{Y}|^2 = \sum_{i=1}^n\rho_i|u_i-\bar u_i|^2 - \sum_{i=1}^n\frac{\rho_i}{\rho^2}\bigg|\sum_{j=1}^n\rho_j(u_j-\bar u_j)\bigg|^2 \\ &= \sum_{i=1}^n\rho_i|u_i-\bar u_i|^2-\frac1\rho\bigg|\sum_{j=1}^n(\rho_j-\bar\rho_j)\bar u_j\bigg|^2 \ge \sum_{i=1}^n\rho_i|u_i-\bar u_i|^2 - C_1\sum_{j=1}^n(\rho_j-\bar\rho_j)^2,\end{aligned}$$

where we used $\sum_{i=1}^n\rho_iu_i=0$ in the third equality, and $C_1>0$ depends on ρ_* and the $L^\infty(\Omega_T)$ norms of $\bar u_j$, $j=1,\ldots,n$. Consequently,

$$\begin{aligned}\frac12\int_0^t\int_\Omega & \sum_{i,j=1}^n b_{ij}\rho_i\rho_j\big|(u_i-\bar u_i)-(u_j-\bar u_j)\big|^2\mathrm{d}x\mathrm{d}s \\ &\ge \mu_M\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2\mathrm{d}x\mathrm{d}s - C_2\int_0^t\int_\Omega\sum_{j=1}^n(\rho_j-\bar\rho_j)^2\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{61}$$

We turn to the estimation of the terms on the right-hand side of (60). By the Lipschitz continuity of κ and Young's inequality, K_{11} is estimated as

$$\begin{aligned}K_{11} &= -\int_0^t\int_\Omega\frac{1}{\bar\theta}\Big(\kappa(\bar\theta-\theta)+(\kappa-\bar\kappa)\theta\Big)\nabla\bar\theta\cdot\nabla(\log\theta-\log\bar\theta)\mathrm{d}x\mathrm{d}s \\ &\le \frac{c_\kappa}{8}\int_0^t\int_\Omega|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s + C_3\int_0^t\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x\mathrm{d}s,\end{aligned}$$

and $C_3>0$ depends on c_κ (see Assumption (A4)), and the $L^\infty(\Omega_T)$ norms of θ and $\nabla\log\bar\theta$. A similar estimate shows that

$$K_{13} = -\int_0^t\int_\Omega\frac{\theta-\bar\theta}{\bar\theta}\big(\kappa\nabla(\log\theta-\log\bar\theta)+(\kappa-\bar\kappa)\nabla\log\bar\theta\big)\cdot\nabla\log\bar\theta\mathrm{d}x\mathrm{d}s$$

$$\leq \frac{c_\kappa}{8}\int_0^t\int_\Omega|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s+C_4\int_0^t\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x\mathrm{d}s,$$

$$L_2\leq \frac{c_\kappa}{8}\int_0^t\int_\Omega|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s+C_5\int_0^t\int_\Omega\sum_{i=1}^n(\rho_i-\bar\rho_i)^2\mathrm{d}x\mathrm{d}s,$$

$$L_3\leq \frac{c_\kappa}{8}\int_0^t\int_\Omega|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s+C_6\int_0^t\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x\mathrm{d}s,$$

observing that C_4 depends on c_κ, δ and the $L^\infty(\Omega_T)$ norms of θ, $\nabla\log\bar\theta$, and $\bar u_i$, C_5 depends on the $L^\infty(\Omega_T)$ norms of $\bar u_i$, and C_6 depends on c_κ, ρ^*, δ, and the $L^\infty(\Omega_T)$ norms of $\bar u_i$ $(i=1,\ldots,n)$. Moreover, by Young's inequality again,

$$L_1\leq \frac{\mu_M}{4}\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2\mathrm{d}x\mathrm{d}s+C_7\int_0^t\int_\Omega\sum_{i=1}^n(\rho_i-\bar\rho_i)^2\mathrm{d}x\mathrm{d}s,$$

$$L_4\leq \frac{\mu_M}{4}\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2\mathrm{d}x\mathrm{d}s+C_8\int_0^t\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x\mathrm{d}s,$$

where C_7 depends on ρ^*, μ_M, and the $L^\infty(\Omega_T)$ norms of $\bar u_i$ $(i=1,\ldots,n)$, while C_8 depends on δ, ρ^*, and the $L^\infty(\Omega_T)$ norm of $\nabla\log\bar\theta$. Finally,

$$L_5\leq C_9\int_0^T\int_\Omega\sum_{i=1}^n(\rho_i-\bar\rho_i)^2\mathrm{d}x\mathrm{d}s+C_{10}\int_0^T\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x\mathrm{d}s,$$

where $C_9>0$ depends on the $L^\infty(\Omega_T)$ norms of $\bar u_i$ $(i=1,\ldots,n)$, and C_{10} depends on δ and the $L^\infty(\Omega_T)$ norm of $\nabla\log\bar\theta$.

Summarizing the previous estimations, we infer from (60), (61), and the lower bound for κ (see Assumption (A4)) the conclusion. $\square$

It remains to estimate the right-hand side of (58) in terms of the relative entropy. For this, we observe that, by [18, Lemma 16],

$$\int_\Omega\sum_{i=1}^n\frac{1}{m_i}\bigg(\rho_i\log\frac{\rho_i}{\bar\rho_i}-(\rho_i-\bar\rho_i)\bigg)\mathrm{d}x\geq C\int_\Omega\sum_{i=1}^n(\rho_i-\bar\rho_i)^2\mathrm{d}x.$$

Furthermore, for all functions $f\in C^1(\mathbb{R})$ with $f'(1)=0$,

$$\begin{aligned}f(s)-f(1)&=(s-1)\int_0^1f'(\sigma(s-1)+1)\mathrm{d}\sigma=(s-1)\int_0^1f'(\tau(s-1)+1)\big|_{\tau=0}^{\sigma}\mathrm{d}\sigma\\&=(s-1)^2\int_0^1\int_0^\sigma f''(\tau(s-1)+1)\mathrm{d}\tau\mathrm{d}\sigma.\end{aligned}$$

This yields, choosing $f(s)=-\log s+s-1$ and $s=\theta/\bar\theta$,

$$\int_\Omega c_w\rho\bigg(-\log\frac{\theta}{\bar\theta}+\frac{1}{\bar\theta}(\theta-\bar\theta)\bigg)\mathrm{d}x\geq\int_\Omega c_w\rho\frac{(\theta-\bar\theta)^2}{\max\{\theta,\bar\theta\}^2}\mathrm{d}x\geq C\int_\Omega(\theta-\bar\theta)^2\mathrm{d}x,$$

where $C>0$ depends on the lower bound for $\bar\theta$ in Ω_T. By definition of the relative entropy, we conclude from Lemma 16 that

$$H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t)) + \frac{\mu_M}{2}\int_0^t\int_\Omega\sum_{i=1}^n\rho_i|u_i-\bar u_i|^2\mathrm{d}x\mathrm{d}s$$
$$+\frac{c_\kappa}{2}\int_0^t\int_\Omega|\nabla(\log\theta-\log\bar\theta)|^2\mathrm{d}x\mathrm{d}s \le C\int_0^t H(\boldsymbol{\rho},\theta|\bar{\boldsymbol{\rho}},\bar\theta)\mathrm{d}s.$$

Gronwall's lemma shows that $H((\boldsymbol{\rho},\theta)(t)|(\bar{\boldsymbol{\rho}},\bar\theta)(t))=0$ and hence $\boldsymbol{\rho}(t)=\bar{\boldsymbol{\rho}}(t)$ and $\theta(t)=\bar\theta(t)=0$ in Ω for $t>0$. This finishes the proof.

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