THE DERRIDA-LEBOWITZ-SPEER-SPOHN EQUATION: EXISTENCE, NON-UNIQUENESS, AND DECAY RATES OF THE SOLUTIONS

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Abstract. The logarithmic fourth-order equation

$$\partial_t u + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{ij}^2 (u \partial_{ij}^2 \log u) = 0, \quad u(0,\cdot) = u_0,$$

called the Derrida-Lebowitz-Speer-Spohn equation, with periodic boundary conditions is analyzed. The global-in-time existence of weak nonnegative solutions in space dimensions $d \leq 3$ is shown. Furthermore, a family of entropy–entropy dissipation inequalities is derived in arbitrary space dimensions, and rates of the exponential decay of the weak solutions to the homogeneous steady state are estimated. The proofs are based on the algorithmic entropy construction method developed by the authors and on an exponential variable transformation. Finally, an example for non-uniqueness of the solution is provided.

AMS Classification. 35K30, 35B40, 35Q40, 35Q99.

Keywords. Logarithmic fourth-order equation, entropy–entropy dissipation method, existence of weak solutions, long-time behavior of solutions, decay rates, nonuniqueness of solutions.

1. Introduction

The logarithmic fourth-order equation

$$\partial_t u + \frac{1}{2}\partial_{ij}^2 (u \partial_{ij}^2 \log u) = 0, \quad u(0,\cdot) = u_0 \geq 0, \tag{1}$$

appears in various places in mathematical physics (notice that we employed the summation convention). It has been first derived by Derrida, Lebowitz, Speer, and Spohn [11, 12], and we shall therefore refer to (1) as the *DLSS equation*. Derrida et al. studied in [11, 12] interface fluctuations in a two-dimensional spin system, the so-called (time-discrete) Toom model. In a suitable scaling limit, a random variable u related to the deviation of the interface from a straight line satisfies the one-dimensional equation (1). The multi-dimensional DLSS equation appears in quantum semiconductor modeling as the zero-temperature,

The authors acknowledge partial support from the Deutsche Forschungsgemeinschaft (DFG), grants JU359/5 (Priority Program "Multi-scale problems") and JU359/7. This research is also part of the ESF program "Global and and geometrical aspects of nonlinear partial differential equations (GLOBAL)"..

zero-field limit of the quantum drift-diffusion model [19]. The variable u describes the electron density in a microelectronic device or in a quantum plasma. In both applications, the variable u is a nonnegative quantity.

In fact, to prove preservation of positivity or non-negativity of solutions constitutes the main analytical difficulty in rigorous studies of (1). There is generally no maximum principle available for fourth-order equations, which would allow to conclude from $u_0 \geq 0$ that also $u(t, \cdot) \geq 0$ at later times $t > 0$. Consequently, one has to rely on suitable regularization techniques and a priori estimates. The latter are difficult to obtain because of the highly nonlinear structure of the equation. We remark that similar difficulties appear in studies of the thin-film equation

$$\partial_t u + \operatorname{div}(u^\alpha \nabla \Delta u) = 0, \quad u(0, \cdot) = u_0 \geq 0.$$

For this equation it is well-known that preservation of positivity strongly depends on the parameter $\alpha > 0$. For a certain range of α's, there are solutions which are strictly positive initially, but which vanish at certain points after finite time [2].

In the present paper, we prove global-in-time existence of non-negative weak solutions to (1) on the d-dimensional torus $\mathbb{T}^d$, and we calculate rates for the exponential decay of the solutions to the homogeneous steady state. Moreover, we provide a family of initial data u_0 for which there exist at least two solutions. These results are new in the literature (also see below). Our method of proof is based on the entropy construction method recently developed in [17] to derive a priori estimates and an exponential transformation of variables to prove the nonnegativity of solutions.

The first mathematically rigorous treatment of (1) is due to [4]. There, *local-in-time* existence of classical solutions for strictly positive $W^{1,p}(\mathbb{T}^d)$ initial data with $p > d$ was proven. The existence result is obtained by means of classical semigroup theory applied to the equation

$$2\partial_t \sqrt{u} + \Delta^2 \sqrt{u} - \frac{(\Delta \sqrt{u})^2}{\sqrt{u}} = 0, \quad \sqrt{u}(0, x) = \sqrt{u_0(x)} > 0, \quad x \in \mathbb{T}^d,$$

which is equivalent to (1) as long as u remains bounded away from zero. Lacking suitable a priori estimates, existence was proven only locally in time (for $d > 1$), even for strictly positive solutions.

More information is available in dimension $d = 1$ because equation (1) is then well-posed in H^1. The *Fisher information*

$$F = \int_{\mathbb{T}} (\sqrt{u})_x^2 dx$$

is a Lyapunov functional, $dF/dt \leq 0$, which allows to relate global existence of solutions to strict positivity: if a classical solution breaks down at $t = t^*$, then the limit profile $\lim_{t \nearrow t^*} u(t, x)$ is still in H^1 but vanishes at some point $x \in \mathbb{T}$.

This observation has motivated the study on *nonnegative weak* solutions instead of positive classical solutions. In [18] global existence for the one-dimensional DLSS equation was shown in the class of functions with finite generalized entropy, $\widetilde{E}_0 = \int_{\mathbb{T}} (u - \log u) dx < \infty$, and with physically motivated boundary conditions. The key ingredient in the proof is the observation that $\widetilde{E}_0$ constitutes another Lyapunov functional for (1), providing nonnegativity of the solutions. The restriction to one spatial dimension is essential, since $\widetilde{E}_0$ is seemingly *not* a Lyapunov functional in dimensions $d > 1$.

In the following years, the one-dimensional DLSS equation with (mainly) periodic boundary conditions was extensively studied in the context of entropy–entropy production methods, and the exponentially fast decay of the solutions to the steady state has been proved [6, 7, 13, 15, 17, 20]. A numerical study of the long-time asymptotics for various boundary conditions can be found in [9].

Concerning the multi-dimensional problem, we remark that an independent investigation of (generalizations to) the DLSS equation has been just finished [14]. There, it is proven that (1) constitutes the gradient flow for the Fisher information with respect to the Wasserstein measure. The resulting existence theorem is more general than ours as it also holds in the non-physical dimensions $d \geq 4$, and on unbounded domains. Clearly, the treatment of the DLSS equation as a gradient flow promotes a deeper understanding of its nature. Our approach in the present note is complementary as it is very direct and much simpler (and also much shorter). Furthermore, we point out that the decay estimates derived by our methods are slightly sharper than those of [14], and we are able to present an example of non-uniqueness of solutions.

In the following we describe our results in more detail. The global existence result is based on the fact that the *physical entropy*

$$\widetilde{E}_1 = \int_{\mathbb{T}^d} u \log\Big(\frac{u}{\int u dx}\Big) dx \geq 0 \tag{2}$$

is a Lyapunov functional in any space dimension $d \geq 1$. In fact, multiplying (1) formally by $\log u$, integrating over $\mathbb{T}^d$ and integrating by parts leads to

$$\frac{d\widetilde{E}_1}{dt} + \frac{1}{2}\int_{\mathbb{T}^d} u\|\nabla^2 \log u\|^2 dx = 0,$$

where $\nabla^2 \log u$ is the Hessian of $\log u$ and $\|\cdot\|$ is the Euclidean norm. We call the above integral the *entropy production*. Since there is no lower bound for u available, this does not yield an H^2 bound on $\log u$. However, we are able to show that

$$\frac{1}{4}\int_{\mathbb{T}^d} u\|\nabla^2 \log u\|^2 dx \geq \kappa_1 \int_{\mathbb{T}^d} \|\nabla^2\sqrt{u}\|^2 dx, \quad \text{where } \kappa_1 = \frac{4d-1}{d(d+2)}, \tag{3}$$

leading to an H^2 bound for $\sqrt{u}$. This motivates to rewrite the nonlinearity in (1) in terms of $\sqrt{u}$, yielding the following equivalent form of the DLSS equation:

$$\partial_t u + \partial^2_{ij}\big(\sqrt{u}\partial^2_{ij}\sqrt{u} - \partial_i\sqrt{u}\partial_j\sqrt{u}\big) = 0, \quad x \in \mathbb{T}^d,\ t > 0, \tag{4}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{T}^d. \tag{5}$$

The other crucial idea, which eventually provides non-negativity of u, is an exponential variable transformation. To be more precise, for the fixed-point argument leading to the existence result, we also work in the original formulation

(1),

$$\partial_t u + \frac{1}{2}\partial_{ij}^2(u\partial_{ij}^2 y) = 0,$$

with the exponential variable $y = \log u$. In a suitable regularization regime, y is bounded in modulus, and hence $u = \exp(y)$ is strictly positive.

Our proof of the crucial inequality (3) is inspired by the algorithmic entropy construction method of [17]; there, the task of deriving inequalities like (3) is reformulated as a decision problem for polynomial systems. The latter can be solved (at least in principle) by computer algebra systems. The solution of the decision problem determines how to perform integration by parts in a way which leads to the desired inequality. By this method, the proof of (3) becomes quite short and elementary. Positivity of u is needed to make the computer-aided manipulations mathematically rigorous.

Our existence result reads as follows.

Theorem 1. *Let $T > 0$ and $d \leq 3$. Furthermore, let u_0 be a nonnegative measurable function on $\mathbb{T}^d$ with finite physical entropy $E_1(u_0) = \int_T (u_0(\log u_0 - 1) + 1)dx < +\infty$. Then there exists a weak solution u to (4)-(5) satisfying*

$$u(t,\cdot) \geq 0 \text{ a.e.}, \quad u \in W^{1,1}(0,T; H^{-2}(\mathbb{T}^d)), \quad \sqrt{u} \in L^2(0,T; H^2(\mathbb{T}^d)),$$

and for all $z \in L^\infty(0,T; H^2(\mathbb{T}^d))$,

$$\int_0^T \langle \partial_t u, z\rangle_{H^{-2},H^2} dt + \int_0^T \int_{\mathbb{T}^d} \big(\sqrt{u}\partial_{ij}^2\sqrt{u} - \partial_i\sqrt{u}\partial_j\sqrt{u}\big)\partial_{ij}^2 z dx = 0.$$

The theorem is valid in the physically relevant dimensions $d \leq 3$. This restriction is related to the lack of certain Sobolev-embeddings in higher dimensions $d \geq 4$. Most prominently, the fixed-point argument exploits the continuous embedding $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ to conclude absolute boundedness of y and hence strict positivity of $u = \exp(y)$. We have chosen periodic boundary conditions in order to avoid boundary integrals. For the treatment of nonhomogeneous boundary conditions of Dirichlet-Neumann-type in one space dimension, we refer to [15].

Our second result concerns the long-time behavior of weak solutions to the homogeneous steady state u_∞, and generally the systematic investigation of Lyapunov functionals. More specifically, we determine a range of parameters $\gamma > 0$, for which the *entropies*

$$\widetilde{E}_\gamma = \frac{1}{\gamma(\gamma-1)}\int_{\mathbb{T}^d} \big(u(t,\cdot)^\gamma - u_\infty^\gamma\big)dx$$

monotonically decay to zero. (Recall that $\widetilde{E}_1$ is the physical entropy from (2).) Starting from the results of [4], Lyapunov functionals of this (and more general) type have been investigated for $d = 1$. Here, we extend the entropy construction method developed in [17] to the multi-dimensional case.

To prove entropy decay, we multiply (4) formally by $v^{2(\gamma-1)}/(\gamma-1)$, where $v = \sqrt{u}$, integrate over the torus and integrate by parts. This leads to

$$\frac{d\widetilde{E}_\gamma}{dt} + \frac{1}{\gamma-1}\int_{\mathbb{T}^d} v^2\partial_{ij}^2(\log v)\partial_{ij}^2(v^{2(\gamma-1)})dx = 0.$$

Next, we need to relate the entropy production to the entropy itself. For this, inequality (3) in generalized. We show, by using the method of [17], that if $0 < \gamma < 2(d+1)/(d+2)$ then

$$\frac{1}{2(\gamma-1)}\int_{\mathbb{T}^d} v^2\partial^2_{ij}(\log v)\partial^2_{ij}(v^{2(\gamma-1)})dx \geq \kappa_\gamma \int_{\mathbb{T}^d}(\Delta v^\gamma)^2 dx, \tag{6}$$

where

$$\kappa_\gamma = \frac{-(d+2)^2\gamma^2 + 2(d+1)(d+2)\gamma - (d-1)^2}{\gamma^2(-(d+2)^2\gamma^2 + 2(d+1)(d+2)\gamma)}.$$

The constant κ_γ is positive if and only if $(\sqrt{d}-1)^2/(d+2) < \gamma < (\sqrt{d}+1)^2/(d+2)$. By a Beckner-type inequality, we can relate the integral of Δv^γ to the entropy itself, giving $d\widetilde{E}_\gamma/dt + c\widetilde{E}_\gamma \leq 0$ for some $c > 0$. A similar strategy works for the physical entropy, $\gamma = 1$. Eventually, Gronwall's lemma yields the following exponential decay estimates:

Theorem 2. *Assume that u is either a positive classical solution to* (4)-(5)*, or the weak solution from Theorem 1. Let* $u_\infty \equiv \mathrm{meas}(\mathbb{T}^d)^{-1}\int_{\mathbb{T}^d} u_0 dx > 0$*. Then the entropies decays exponentially fast,*

$$\widetilde{E}_\gamma(u(t,\cdot)) \leq \widetilde{E}_\gamma(u_0)\exp(-16\pi^4\gamma^2\kappa_\gamma t) \quad \textit{for} \quad 1 \leq \gamma < \frac{(\sqrt{d}+1)^2}{d+2},$$

and the solution itself decays exponentially in the L^1 *norm,*

$$\|u(t,\cdot) - u_\infty\|_{L^1(\mathbb{T}^d)} \leq (2\widetilde{E}_1(u_0))^{1/2}\exp(-8\pi^4\kappa_1 t).$$

In order to make the above inequalities rigorous, we consider a regularized semi-discrete version of (4) for which we obtain positive H^2 solutions. Since the fourth-order differential operator in (1) is not strictly elliptic in $y = \log u$ ($u = 0$ may be possible), we add the regularization $-\varepsilon(\Delta^2 y + y)$ for $\varepsilon > 0$ to the right-hand side of (1). Unfortunately, this regularization destroys the dissipative structure of the DLSS equation, and we cannot prove anymore the entropy–entropy production inequality for $\gamma \neq 1$. To cure this problem, we need to add the expression

$$\varepsilon \mathrm{div}(|\nabla \log \max\{v,\mu\}|^2 \nabla y) \quad \text{for some } \mu > 0.$$

The third main result of this paper concerns the nonuniqueness of solutions. We show that, for a family of particular initial data, there exist at least two solutions to (4)-(5) in the class of nonnegative functions in $L^1(0,T;H^2(\mathbb{T}^d))$ with finite physical entropy E_1. Recall that uniqueness holds in the class of positive smooth functions [4]. Here, the initial data are chosen in such a way that they vanish on a set of measure zero, and that they represent classical solutions to the stationary and hence, to the transient equation. On the other hand, our existence result provides a solution which converges to the homogeneous positive steady state u_∞. Therefore, this solution is not equal to the first one. This observation may give a criterium how to choose the physically relevant solution: it should dissipate the physical entropy.

Throughout this paper, we make the following simplification. Due to the scaling invariance of (5) with respect to $x \to \xi x$, $t \to \xi^4 t$, and $u \to \eta u$ for ξ, $\eta > 0$, we may assume that the torus $\mathbb{T}^d$ is normalized, $\mathbb{T}^d \cong [0,1]^d$. We further assume that the initial datum has unit mass, $\int_{\mathbb{T}^d} u_0 dx = 1$; notice that the DLSS equation is mass preserving.

The paper is organized as follows. In section 2 we show some inequalities needed for the analysis of the DLSS equation. In particular we prove (3) and (6). Theorem 2 is proved in section 3 for smooth positive solutions. Then the existence of solutions is shown in section 4. Section 5 is devoted to the proof of Theorem 2 for weak solutions. Finally, in section 6 the non-uniqueness result is presented.

2. Some inequalities

We collect some inequalities which are needed in the following sections. We start with a lower bound on the Euclidean norm of a matrix. Let $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ be a matrix and $a \in \mathbb{R}^d$ be a vector. We define the Euclidean norm of A and a, respectively, by $\|A\|^2 = \sum_{i,j} a_{ij}^2$ and $\|a\|^2 = \sum_j a_j^2$. Furthermore, $\operatorname{tr} A = \sum_j a_{jj}$ is the trace of A and

$$A : (a)^2 = \sum_{i,j=1}^d a_{ij} a_i a_j.$$

Lemma 3. *Let $A \in \mathbb{R}^{d\times d}$ be a real symmetric matrix and let $a \in \mathbb{R}^d$ be a non-zero vector. Then*

$$\|A\|^2 \geq \frac{1}{d}(\operatorname{tr} A)^2 + \frac{d}{d-1}\left(\frac{A : (a)^2}{\|a\|^2} - \frac{\operatorname{tr} A}{d}\right)^2. \tag{7}$$

Proof. Since A is real and symmetric, one can assume (by the spectral theorem) without loss of generality that A is a diagonal matrix, $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$; recall that norms and traces are invariant under orthogonal transformations. Furthermore, one can also assume, by homogeneity of (7), that $a = (a_1, \ldots, a_d)^\top$ is a unit vector, $\sum_j a_j^2 = 1$. It is elementary to conclude, by Cauchy's inequality, that $\sum_j a_j^4 \geq 1/d$ with equality if and only if $a_1^2 = \cdots = a_d^2 = 1/d$. In the latter case, (7) reduces to $\sum_j \lambda_j^2 \geq 1/d \cdot (\sum_j \lambda_j)^2$, which is true, again by Cauchy's inequality. From now on, we assume that

$$\sum_{j=1}^d a_j^2 > \frac{1}{d}. \tag{8}$$

We prove (7) by determining the minimal value of

$$F(\lambda) = \frac{1}{2}\|A\|^2 = \frac{1}{2}\sum_{j=1}^d \lambda_j^2, \quad \lambda = (\lambda_1, \ldots, \lambda_d)^\top,$$

subject to the constraints

$$t = \frac{1}{d}\operatorname{tr} A = \frac{1}{d}\sum_{j=1}^d \lambda_j \quad \text{and} \quad s = A : (a)^2 = \sum_{j=1}^d \lambda_j a_j^2$$

for given values of s, $t \in \mathbb{R}$. A constrained extremum of F is attained at a point $\lambda = \bar\lambda$ if and only if, for some α, $\beta \in \mathbb{R}$,

$$(\bar\lambda_1, \ldots, \bar\lambda_d) = \nabla_\lambda F = \alpha \nabla_\lambda t + \beta \nabla_\lambda s = (\alpha/d)(1, \ldots, 1) + \beta(a_1^2, \ldots, a_d^2),$$

so that $\bar\lambda_j = \alpha/d + \beta a_j^2$. The Lagrange multipliers α, $\beta \in \mathbb{R}$ are determined by the constraints, i.e. as the solution to the linear system

$$t = \frac{1}{d} \sum_{j=1}^d \bar\lambda_j = \frac{\alpha}{d} + \frac{\beta}{d}, \tag{9}$$

$$s = \sum_{j=1}^d \bar\lambda_j a_j^2 = \frac{\alpha}{d} + \Big(\sum_{j=1}^d a_j^4 \Big) \beta. \tag{10}$$

By assumption (8), the extremal point is unique and consequently the global minimum point of the convex functional F under the constraints. The minimal value is given by

$$F(\bar\lambda) = \frac{1}{2} \sum_{j=1}^d \bar\lambda_j^2 = \frac{\alpha^2}{2d} + \frac{\alpha\beta}{d} + \frac{\beta^2}{2} \sum_{j=1}^d a_j^4.$$

Take the square of (9) to simplify this expression to

$$F(\bar\lambda) = \frac{dt^2}{2} + \frac{\beta^2}{2} \Big(\sum_{j=1}^d a_j^4 - \frac{1}{d} \Big).$$

Now calculate β from (9) and (10),

$$\beta = (s - t) \Big(\sum_{j=1}^d a_j^4 - \frac{1}{d} \Big)^{-1}$$

and insert this in the above expression for $F(\bar\lambda)$ to find

$$F(\bar\lambda) = \frac{dt^2}{2} + \frac{1}{2}(s-t)^2 \Big(\sum_{j=1}^d a_j^4 - \frac{1}{d} \Big)^{-1}.$$

In conclusion,

$$\|A\|^2 \geq 2F(\bar\lambda) = \frac{1}{d}\big(\operatorname{tr} A\big)^2 + \Big(A : (a)^2 - \frac{1}{d} \operatorname{tr} A \Big)^2 \Big(\sum_{j=1}^d a_j^4 - \frac{1}{d} \Big)^{-1},$$

and the final formula (7) follows since obviously $\sum_j a_j^4 \leq (\sum_j a_j^2)^2 = 1$. □

The main result of this section is the following inequality.

Lemma 4. *Let $v \in H^2(\mathbb{T}^d) \cap W^{1,4}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ in dimension $d \geq 2$. Assume that v is strictly positive. Then, for any $0 < \gamma < 2(d+1)/(d+2)$,*

$$\frac{1}{2(\gamma - 1)} \int_{\mathbb{T}^d} v^2 \partial_{ij}^2 (\log v) \partial_{ij}^2 (v^{2(\gamma-1)}) dx \geq \kappa_\gamma \int_{\mathbb{T}^d} (\Delta v^\gamma)^2 dx, \tag{11}$$

if $\gamma \neq 1$, *or*

$$\int_{\mathbb{T}^d} v^2 \partial_{ij}^2 (\log v)^2 dx \geq \kappa_1 \int_{\mathbb{T}^d} (\Delta v)^2 dx, \tag{12}$$

if $\gamma = 1$, *respectively, where*

$$\kappa_\gamma = \frac{p(\gamma)}{\gamma^2 (p(\gamma) - p(0))} \quad \text{and} \quad p(\gamma) = -\gamma^2 + \frac{2(d+1)}{d+2}\gamma - \Big(\frac{d-1}{d+2}\Big)^2. \tag{13}$$

The function $\nabla^2 v$ denotes the Hessian of v. By Sobolev embedding, it is sufficient to assume $v \in H^2(\mathbb{T}^d)$ in space dimensions $d \leq 3$. The condition $0 < \gamma < 2(d+1)/(d+2)$ ensures that $p(\gamma) > p(0)$ such that κ_γ is well defined; if the stronger condition $(\sqrt{d}-1)^2/(d+2) < \gamma < (\sqrt{d}+1)^2/(d+2)$ holds, then $\kappa_\gamma > 0$. Finally, we remark that the method of [17] directly applies to the one-dimensional situation, yielding (11) and (12), respectively, for $0 \leq \gamma \leq \frac{3}{2}$, with $\kappa_\gamma = \min(\gamma, 12 - 8\gamma)/\gamma^3$.

Proof. In order to simplify the computations, we introduce the functions θ, λ and μ, respectively, by (recall that $v > 0$)

$$\theta = \frac{|\nabla v|}{v}, \quad \lambda = \frac{1}{d}\frac{\Delta v}{v}, \quad (\lambda + \mu)\theta^2 = \frac{1}{v^3}\nabla^2 v : (\nabla v)^2,$$

and $\rho \geq 0$ by

$$\|\nabla^2 v\|^2 = \Big(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2\Big)v^2.$$

We need to show that ρ is well defined. But this is clear since

$$\|\nabla^2 v\|^2 \geq \Big(d\lambda^2 + \frac{d}{d-1}\mu^2\Big)v^2$$

follows directly from (7) after taking $A = \nabla^2 v$ and $a = \nabla v$.

We compute the left-hand side of (11),

$$\begin{aligned} J &= \frac{1}{2(\gamma-1)} \int_{\mathbb{T}^d} v^2 \partial_{ij}^2 (\log v) \partial_{ij}^2 (v^{2(\gamma-1)}) dx \\ &= \frac{1}{2(\gamma-1)} \int_{\mathbb{T}^d} (v\partial_{ij}^2 v - \partial_i v \partial_j v) \partial_{ij}^2 (v^{2(\gamma-1)}) dx \\ &= \int_{\mathbb{T}^d} (v\partial_{ij}^2 v - \partial_i v \partial_j v) v^{2(\gamma-2)} \big(v\partial_{ij}^2 v + (2\gamma-3)\partial_i v \partial_j v\big) dx \\ &= \int_{\mathbb{T}^d} v^{2\gamma} \Big(\frac{\|\nabla^2 v\|^2}{v^2} - 2(2-\gamma)\frac{\nabla^2 v}{v^2} : \Big(\frac{\nabla v}{v}\Big)^2 + (3-2\gamma)\frac{|\nabla v|^4}{v^4}\Big) dx, \end{aligned}$$

and express it in terms of the functions θ, λ, μ, and ρ defined above,

$$J = \int_{\mathbb{T}^d} v^{2\gamma} \Big(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2 - 2(2-\gamma)(\lambda+\mu)\theta^2 + (3-2\gamma)\theta^4\Big) dx.$$

This integral is compared to

$$\begin{aligned} K &= \frac{1}{\gamma^2} \int_{\mathbb{T}^d} (\Delta v^\gamma)^2 dx = \int_{\mathbb{T}^d} v^{2(\gamma-2)} \big(v\Delta v + (\gamma-1)|\nabla v|^2\big)^2 dx \\ &= \int_{\mathbb{T}^d} v^{2\gamma} \big(d\lambda + (\gamma-1)\theta^2\big)^2 dx. \end{aligned}$$

More precisely, we shall determine a constant $c_0 > 0$ independent of v such that $J - c_0 K \geq 0$ for all (positive) functions v. Our strategy is an adaption of the method developed in [17]. We formally perform integration by parts in the expression $J - c_0 K$ by adding a linear combination of certain "dummy" integrals – which are actually zero and hence do not change the value of $J - c_0 K$. The coefficients in the linear combination are determined in such a way that makes the resulting integrand pointwise non-negative. The latter is a decision problem from real algebraic geometry, and it is solved with computer aid.

We shall rely on the following two "dummy" integral expressions:

$$J_1 = \int_{\mathbb{T}^d} \operatorname{div}\big(v^{2\gamma-2}(\nabla^2 v - \Delta v \mathbb{I}) \cdot \nabla v\big) dx,$$
$$J_2 = \int_{\mathbb{T}^d} \operatorname{div}\big(v^{2\gamma-3} |\nabla v|^2 \nabla v\big) dx,$$

where $\mathbb{I}$ is the unit matrix in $\mathbb{R}^{d\times d}$. Clearly, in view of the periodic boundary conditions, $J_1 = J_2 = 0$. The goal is to find constants c_0, c_1, and c_2 such that $J - c_0 K = J - c_0 K + c_1 J_1 + c_2 J_2 \geq 0$; moreover, c_0 should be as large as possible. Since, with the above notations,

$$\begin{aligned} J_1 &= \int_{\mathbb{T}^d} v^{2(\gamma-2)}\big(v^2(\|\nabla^2 v\|^2 - (\Delta v)^2) + 2(\gamma-1)v(\nabla^2 v - \Delta v\mathbb{I}) : (\nabla v)^2\big) dx \\ &= \int_{\mathbb{T}^d} v^{2\gamma}\Big(- d(d-1)\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2 \\ &\qquad + 2(\gamma-1)(-(d-1)\lambda\theta^2 + \mu\theta^2)\Big) dx \end{aligned}$$

and

$$\begin{aligned} J_2 &= \int_{\mathbb{T}^d} v^{2(\gamma-2)}\big((2\nabla^2 v + \Delta v \mathbb{I}) : (\nabla v)^2 + (2\gamma-3)|\nabla v|^4\big) dx \\ &= \int_{\mathbb{T}^d} v^{2\gamma}\big((d+2)\lambda\theta^2 + 2\mu\theta^2 + (2\gamma-3)\theta^4\big) dx, \end{aligned}$$

we obtain

$$\begin{aligned} J - c_0 K + c_1 J_1 + c_2 J_2 &= \int_{\mathbb{T}^d} v^{2\gamma}\big\{ d\lambda^2\big[1 - dc_0 - (d-1)c_1\big] \\ &+ \lambda\theta^2\big[2(\gamma-1)(1 - dc_0 - (d-1)c_1) + (d+2)c_2 - 2\big] + Q(\theta,\mu,\rho)\big\} dx, \end{aligned} \tag{14}$$

where Q is a polynomial in θ, μ, and ρ with coefficients depending on c_0, c_1, and c_2 but not on λ. We choose to eliminate λ from the above integrand by defining c_1 and c_2 appropriately. The linear system

$$\begin{aligned} 1 - dc_0 - (d-1)c_1 &= 0, \\ 2(\gamma-1)(1 - dc_0 - (d-1)c_1) + (d+2)c_2 - 2 &= 0 \end{aligned}$$

has the solution $c_1 = (1 - dc_0)/(d-1)$ and $c_2 = 2/(d+2)$. With this choice, the polynomial Q in (14) reads as

$$Q(\theta,\mu,\rho) = \frac{1}{(d-1)^2(d+2)}\big(b_1\mu^2 + 2b_2\mu\theta^2 + b_3\theta^4 + b_4\rho^2\big),$$

where

$$\begin{aligned} b_1 &= d^2(d+2)(1-c_0),\\ b_2 &= d(d-1)\big((d+2)(\gamma+c_0(1-\gamma))-2d-1\big),\\ b_3 &= (d-1)^2\big(d(3-2\gamma)-c_0(d+2)(\gamma-1)^2\big),\\ b_4 &= d(d+2)(d-1)(1-c_0). \end{aligned}$$

If $c_0 \leq 1$, then $b_4 \geq 0$. We wish to choose $c_0 \leq 1$ in such a way that the remaining sum $b_1\mu^2 + 2b_2\mu\theta^2 + b_3\theta^4$ is nonnegative as well, for any μ and θ. This is the case if (i) $b_1 > 0$ and (ii) $b_1 b_3 - b_2^2 \geq 0$. Condition (ii) is equivalent to

$$\begin{aligned} 0 &\leq (1-c_0)(d+2)\big(2(d+1)\gamma-(d+2)\gamma^2\big)-(d-1)^2\\ &= (1-c_0)(d+2)^2(p(\gamma)-p(0))-(d-1)^2, \end{aligned}$$

which is further equivalent to (recall that $p(\gamma) > p(0)$ on the considered range of γ's)

$$c_0 \leq \frac{p(\gamma)}{p(\gamma)-p(0)}.$$

The best choice for c_0 is obviously to make it equal to the right-hand side. As $p(0) < 0$, one has in particular that $c_0 < 1$, so condition (i) is satisfied as well. Thus we have found constants c_0, c_1, and c_2 for which the expression $J - c_0K + c_1J_1 + c_2J_2$ is nonnegative. With $\kappa_\gamma = c_0/\gamma^2$, Lemma 4 is proven. □

Remark 5. Elimination of λ from the integrand in (14) is clearly not the only strategy to initiate the polynomial reduction process. However, from numerical studies of the multivariate polynomial, there is strong evidence that this strategy leads to the optimal values for c_0, at least for γ close to one.

As a consequence of Lemma 4 for $v = \sqrt{u}$ and $\gamma = 1$, we obtain the inequality (3) which connects the entropy production of (4) to the smoothness of its solution.

Lemma 6. *For all $d \geq 1$ and all strictly positive functions u such that $\sqrt{u} \in H^2(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^d)$ it holds*

$$\frac{1}{4}\int_{\mathbb{T}^d} u\|\nabla^2 \log u\|^2 dx \geq \kappa_1 \int_{\mathbb{T}^d} \|\nabla^2\sqrt{u}\|^2 dx, \quad \textit{where } \kappa_1 = \frac{4d-1}{d(d+2)}.$$

We also need the following generalized convex Sobolev inequalities.

Lemma 7. *Let $f \in H^2(\mathbb{T}^d)$ be nonnegative. Then, for $1 < p \leq 2$,*

$$\frac{p}{p-1}\Big(\int_{\mathbb{T}^d} f^2 dx - \Big(\int_{\mathbb{T}^d} f^{2/p} dx\Big)^p\Big) \leq \frac{1}{8\pi^4}\int_{\mathbb{T}^d} (\Delta f)^2 dx. \tag{15}$$

Furthermore,

$$\int_{\mathbb{T}^d} f^2 \log\big(f^2/\|f\|_{L^2}^2\big) dx \leq \frac{1}{8\pi^4}\int_{\mathbb{T}^d} (\Delta f)^2 dx. \tag{16}$$

Inequality (16) represents the limit of (15) as $p \searrow 1$. Unfortunately, (15) does seemingly *not* generalize for parameters $0 < p < 1$ in dimensions $d > 1$. The reason is that the functional on the left-hand side is convex in f only if $1 \le p \le 2$, see the discussion in [5]. This limits our decay estimates to entropies E_γ with $\gamma \ge 1$.

Proof. We only prove inequality (15) as (16) follows in a completely analogous manner. The estimate is a consequence of the Beckner-type inequality and the Poincaré inequality. For the one-dimensional torus $\mathbb{T}$, the former reads as [3]

$$\frac{p}{p-1}\Big(\int_{\mathbb{T}} f^2 dx_j - \Big(\int_{\mathbb{T}} f^{2/p} dx_j\Big)^p\Big) \le \frac{1}{2\pi^2}\int_{\mathbb{T}} |\partial_j f|^2 dx_j$$

(see, e.g., [13] for an easy proof). In several space dimensions, we obtain the same result since the above inequality *tensorizes.* Indeed, by employing the relation

$$\int_{\mathbb{T}^d} f^2 dx - \Big(\int_{\mathbb{T}^d} f^{2/p} dx\Big)^p \le \sum_{j=1}^d \int_{\mathbb{T}^d}\Big(\int_0^1 f^2 dx_j - \Big(\int_0^1 f^{2/p} dx_j\Big)^p\Big) dx$$

from Proposition 4.1 in [22], it follows that

$$\begin{aligned}\frac{p}{p-1}\Big(\int_{\mathbb{T}^d} f^2 dx - \Big(\int_{\mathbb{T}^d} f^{2/p} dx\Big)^p\Big) &\le \frac{1}{2\pi^2}\sum_{j=1}^d \int_{\mathbb{T}^d}\Big(\int_0^1 (\partial_j f)^2 dx_j\Big) dx \\ &= \frac{1}{2\pi^2}\int_{\mathbb{T}^d} |\nabla f|^2 dx.\end{aligned}$$

Now, Poincaré's inequality for multi-periodic functions with zero mean,

$$\int_{\mathbb{T}^d} |\nabla f|^2 dx \le \frac{1}{4\pi^2}\int_{\mathbb{T}^d} \|\nabla^2 f\|^2 dx = \frac{1}{4\pi^2}\int_{\mathbb{T}^d} (\Delta f)^2 dx,$$

gives the assertion. □

In section 4 we frequently refer to the Gagliardo-Nirenberg inequalities which we recall for convenience [16].

Lemma 8. *Let m, $k \in \mathbb{N}_0$ with $0 \le k \le m$, $0 \le \theta < 1$, and $1 \le p, q, r \le \infty$. If both*

$$k - \frac{d}{p} \le \theta\Big(m - \frac{d}{q}\Big) + (1-\theta)\Big(-\frac{d}{r}\Big) \quad \text{and} \quad \frac{1}{p} \le \frac{\theta}{q} + \frac{1-\theta}{r},$$

then any function $f \in W^{m,q}(\mathbb{T}^d) \cap L^r(\mathbb{T}^d)$ belongs to $W^{k,p}(\mathbb{T}^d)$, and there exists a constant $C > 0$ independent of f such that

$$\|f\|_{W^{k,p}} \le C\|f\|_{W^{m,q}}^\theta \|f\|_{L^r}^{1-\theta}. \tag{17}$$

If additionally $k \ge 1$, we conclude from (17) that

$$\|\nabla^k f\|_{L^p} \le C\|\nabla^m f\|_{L^q}^\theta \|f\|_{L^r}^{1-\theta} \tag{18}$$

by means of the Poincaré inequality.

3. Decay rates for smooth positive solutions

We show Theorem 2 first for smooth positive solutions by using Lemmas 4 and 7. The proof for weak solutions is based on estimates for the semi-discrete, regularized problem and is therefore presented later in section 5. Both proofs are identical in their structure, but the proof for smooth solutions is stripped of the technicalities that are introduced by the regularization process.

The essential tool to derive the a priori estimates are the so-called *relative entropies*

$$E_\gamma(u_1|u_2) = \int_{\mathbb{T}^d} \phi_\gamma\Big(\frac{u_1}{u_2}\Big)u_2 dx, \quad \gamma \notin \{0,1\},$$

where u_1 and u_2 are nonnegative functions on $\mathbb{T}^d$ with unit mean value, and ϕ_γ is given by

$$\phi_\gamma(s) = \frac{1}{\gamma(\gamma-1)}\big(s^\gamma - \gamma s + \gamma - 1\big), \quad s \geq 0. \tag{19}$$

The natural continuation for $\gamma = 1$ is $\phi_1(s) = s(\log s - 1) + 1$; the functional E_1 corresponds to the physical entropy. The functions ϕ_γ are nonnegative and convex and attain their minimal value at $s = 1$. Consequently, E_γ is nonnegative (possibly $+\infty$) and vanishes if and only if $u_1 = u_2$.

To obtain the a priori estimates (and the decay rates), we consider entropies of solutions $u_1 = u$ relative to the spatial homogeneous steady state $u_2 \equiv 1$:

$$E_\gamma(u(t,\cdot)) = \frac{1}{\gamma(\gamma-1)}\Big(\int_{\mathbb{T}^d} u(t,x)^\gamma dx - 1\Big), \quad \gamma \geq 1. \tag{20}$$

We obtain the following entropy–entropy production estimate.

Proposition 9. *Assume that u is a smooth positive solution to* (4)-(5) *in dimensions $d \geq 2$. Then*

$$\frac{dE_\gamma}{dt} + 2\kappa_\gamma \int_{\mathbb{T}^d} (\Delta u^{\gamma/2})^2 dx \leq 0, \quad \textit{for} \quad 0 < \gamma < \frac{2(d+1)}{d+2}, \tag{21}$$

where $\kappa_\gamma > 0$ is defined in (13).

Proof. For convenience, we work with the function $v = \sqrt{u}$ instead of u. If $\gamma \neq 1$, we integrate the DLSS equation (4) against the test function $v^{2(\gamma-1)}/2(\gamma-1)$. Then we obtain for the time derivative

$$\frac{1}{2(\gamma-1)}\int_{\mathbb{T}^d} \partial_t(v^2)v^{2(\gamma-1)}dx = \frac{1}{2\gamma(\gamma-1)}\int_{\mathbb{T}^d} \partial_t(v^{2\gamma})dx = \frac{1}{2}\frac{dE_\gamma}{dt}.$$

In combination with Lemma 4,

$$\frac{1}{2}\frac{dE_\gamma}{dt} = -\frac{1}{2(\gamma-1)}\int_{\mathbb{T}^d} v^2\partial^2_{ij}(\log v)\partial^2_{ij}(v^{2(\gamma-1)})dx \leq -\kappa_\gamma \int_{\mathbb{T}^d}(\Delta v^\gamma)^2 dx.$$

If $\gamma = 1$, we use the test function $\log v$ instead. □

Remark 10. As pointed out before, the coefficient function κ_γ follows a different law in dimension $d = 1$, see the remarks after Lemma 4. In conclusion, the entropy production is estimated as

$$\frac{dE_\gamma}{dt} + \frac{2\mu_\gamma}{\gamma^2}\int_{\mathbb{T}} |(u^{\gamma/2})_{xx}|^2 dx \leq 0,$$

where

$$\mu_\gamma = \begin{cases} 1 & \text{for } 0 < \gamma < 4/3 \\ 12/\gamma - 8 & \text{for } 4/3 < \gamma < 3/2. \end{cases}$$

Estimates for the limiting case $\gamma = 0$ are also available (see Theorem 3 in [7]).

The proof of Theorem 2 is immediate: applying Lemma 7 to $f = u^{\gamma/2}$ with $p = \gamma$, and taking into account that u has unit mass, we obtain from Proposition 9

$$\frac{dE_\gamma}{dt} + (2\pi)^4\gamma^2\kappa_\gamma E_\gamma \le 0.$$

Gronwall's lemma shows the entropy decay. The decay in the L^1 norm is a straight-forward consequence of the Csiszár-Kullback inequality [10, 21].

The values of κ_γ as a function of γ are plotted in Figure 1 (left). For $\gamma \ge 1$, these correspond to exponential decay rates of the respective entropy E_γ. (There is no immediate interpretation of κ_γ for $0 < \gamma < 1$.) The right figure shows the decay rate $8\pi^4\kappa_1$ in the L^1 norm for $\gamma = 1$ as a function of the dimension d. This rate is given by

$$8\pi^4\kappa_1 = 8\pi^4\frac{4d-1}{d(d+2)};$$

this is slightly better than the rate obtained in [14], which amounts to $24\pi^4/(d+2)$ for equation (1).

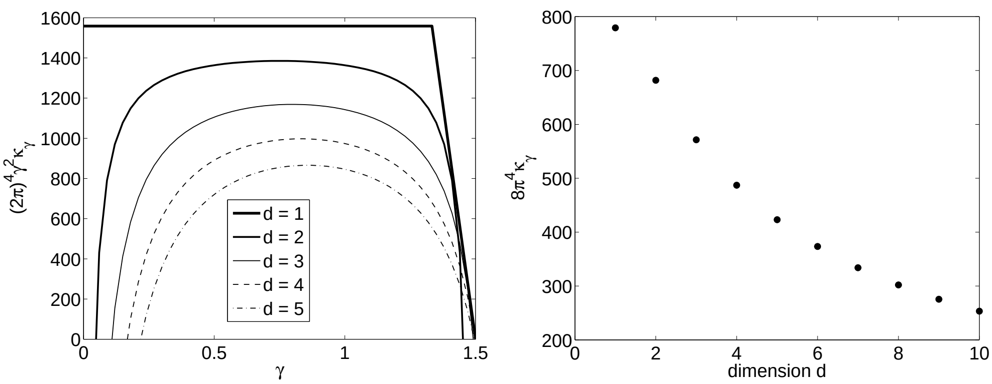

Figure 1. Decay rates for the entropy E_γ (left) and in the L^1 norm for $\gamma = 1$ (right) depending on the dimension d.

4. Existence of solutions

In this section we prove Theorem 1. The proof is divided into a series of lemmas. We continue to use $v = \sqrt{u}$ for easier notation.

4.1. **Existence of a time-discrete solution.** Let $T > 0$ be a terminal time and $\tau > 0$ a time step. Let w be a given function. We wish to find a solution $v \in H^2(\mathbb{T}^d)$ to the semi-discrete equation

$$\frac{1}{\tau}(v^2 - w^2) = -\partial_{ij}^2\big(v\partial_{ij}^2 v - \partial_i v\partial_j v\big). \tag{22}$$

Lemma 11. *Let $d \leq 3$. Assume that w is a nonnegative measurable function on $\mathbb{T}^d$ with finite entropy $E_1(w^2) < +\infty$ and unit mass $\int_{\mathbb{T}^d} w^2 dx = 1$. Then there exists a nonnegative weak solution $v \in H^2(\mathbb{T}^d)$ to* (22)*. Furthermore, v^2 has unit mass, the physical entropy is dissipated in the sense*

$$E_1(v^2) + 2\tau\kappa_1 \int_{\mathbb{T}^d} \|\nabla^2 v\|^2 dx \leq E_1(w^2), \tag{23}$$

and the entropies $E_\gamma(v^2)$ and $E_\gamma(w^2)$ are related by

$$(1 + 16\pi^4 \tau\gamma^2\kappa_\gamma) E_\gamma(v^2) \leq E_\gamma(w^2), \tag{24}$$

where $1 \leq \gamma < (\sqrt{d}+1)^2/(d+2)$ and κ_γ is defined in (13)*.*

Proof. Step 1: definition of the regularized problem. The solution to (22) is obtained as the limit of solutions to a regularized problem. For this, recall that (4) can be written as

$$\partial_t(v^2) = -\frac{1}{2}\partial_{ij}^2(v^2\partial_{ij}^2 y) \quad \text{with } y = \log(v^2).$$

We regularize (22) in the above formulation by adding a strongly elliptic operator in y:

$$\frac{1}{\tau}(v^2 - w^2) = -\frac{1}{2}\partial_{ij}^2(v^2\partial_{ij}^2 y) - \varepsilon(\Delta^2 y + y) + \varepsilon \operatorname{div}\big(|\nabla \log[v]_\mu|^2 \nabla y\big), \tag{25}$$

where ε, $\mu > 0$ are regularization parameters and $[v]_\mu = \max\{v, \mu\}$. The fourth-order operator $\varepsilon(\Delta^2 y + y)$ guarantees coercivity of the above right-hand side with respect to y. The nonlinear second-order operator allows to derive the a priori estimates for the general entropy E_γ.

Step 2: solution of the regularized problem. In order to solve (25) we employ the Leray-Schauder fixed-point theorem (see Theorem B.5 in [24]). Let $\sigma \in [0,1]$ and $\bar{v} \in W^{1,4}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, and introduce for y, $z \in H^2(\mathbb{T}^d)$,

$$\begin{aligned} a(y,z) &= \frac{1}{2}\int_{\mathbb{T}^d} \bar{v}^2 \partial_{ij}^2 y \partial_{ij}^2 z dx + \varepsilon \int_{\mathbb{T}^d} (\Delta y \Delta z + yz + |\nabla \log[\bar{v}]_\mu|^2 \nabla y \cdot \nabla z) dx, \\ f(z) &= \frac{\sigma}{\tau}\langle \bar{v}^2 - w^2, z\rangle_{H^{-2},H^2}. \end{aligned}$$

Since $\bar{v} \in W^{1,4}(\mathbb{T}^d)$, also $\log[\bar{v}]_\mu \in W^{1,4}(\mathbb{T}^d)$, hence $|\nabla \log[\bar{v}]_\mu|^2 \nabla y \cdot \nabla z$ is integrable. The bilinear form a is continuous and coercive since, by the Gagliardo-Nirenberg inequality (18),

$$a(y,y) \geq \varepsilon \int_{\mathbb{T}^d} \big((\Delta y)^2 + y^2\big) dx \geq C\varepsilon \|y\|_{H^2}^2. \tag{26}$$

Moreover, w^2 has finite physical entropy, so $w^2 \in L^1(\mathbb{T}^d) \hookrightarrow H^{-2}(\mathbb{T}^d)$ in space dimensions $d \leq 3$, yielding continuity of the linear form f. Consequently, Lax-Milgram's lemma provides the existence of a unique solution to

$$-a(y,z) = f(z) \quad \text{for all } z \in H^2(\mathbb{T}^d).$$

Define the fixed-point operator $S : W^{1,4}(\mathbb{T}^d) \times [0,1] \to W^{1,4}(\mathbb{T}^d)$ by $S(\bar{v},\sigma) := v = e^{y/2}$. Since $y \in H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, we have indeed that $v \in H^2(\mathbb{T}^d) \hookrightarrow W^{1,4}(\mathbb{T}^d)$.

We shall now verify the hypotheses of the Leray-Schauder theorem; the latter provides a solution v of $S(v,1)=v$. The operator S is constant at $\sigma=0$, $S(\bar v,0)=1$. By standard results for elliptic equations, S is continuous and compact since the embedding $H^2(\mathbb{T}^d)\hookrightarrow W^{1,4}(\mathbb{T}^d)$ is compact. It remains to show a uniform bound for all fixed points of $S(\cdot,\sigma)$. This bound is obtained from the production of the physical entropy and Lemma 6.

Let $v\in H^2(\mathbb{T}^d)$ be a fixed point of $S(\cdot,\sigma)$ for some $\sigma\in[0,1]$. Then v is a solution to (25) with σ/τ instead of $1/\tau$, and with $v=e^{y/2}>0$, $y\in H^2(\mathbb{T}^d)$. Since $\phi(s)=s(\log s-1)+1$ is convex, $\phi(s_1)-\phi(s_2)\le\phi'(s_1)(s_1-s_2)$ for all s_1, $s_2\ge 0$. Hence,

$$\begin{aligned}\frac{\sigma}{2\tau}(E_1(v^2)-E_1(w^2)) &= \frac{\sigma}{2\tau}\int_{\mathbb{T}^d}(\phi(v^2)-\phi(w^2))dx \\ &\le \frac{\sigma}{2\tau}\int_{\mathbb{T}^d}(v^2-w^2)\log(v^2)dx=-a(y,y) \\ &\le -\frac14\int_{\mathbb{T}^d}v^2\|\nabla^2\log(v^2)\|^2dx-\varepsilon\int_{\mathbb{T}^d}((\Delta y)^2+y^2)dx.\end{aligned}\tag{27}$$

The estimate of Lemma 6 shows that

$$\frac{\sigma}{\tau}(E_1(v^2)-E_1(w^2))+2\kappa_1\int_{\mathbb{T}^d}\|\nabla^2 v\|^2dx\le 0.$$

As a consequence,

$$E_1(v^2)\le E_1(w^2)\quad\text{and}\quad \|\nabla^2 v\|_{L^2}^2\le\frac{1}{2\tau\kappa_1}E_1(w^2).$$

In particular, $\nabla^2 v$ is uniformly bounded in $L^2(\mathbb{T}^d)$. Together with the elementary inequality $s\le\phi(s)+(e-1)$ for all $s\ge 0$, we obtain

$$\|v\|_{L^2}^2\le\int_{\mathbb{T}^d}(\phi(v^2)+e-1)dx=E_1(v^2)+e-1.$$

This means that v is uniformly bounded in $L^2(\mathbb{T}^d)$. Then the Gagliardo-Nirenberg inequality gives the desired uniform bound for v:

$$\|v\|_{H^2}^2\le C(\|\nabla^2 v\|_{L^2}^2+\|v\|_{L^2}^2)\le\Big(1+\frac{C}{2\tau\kappa_1}\Big)E_1(w^2)+2C.\tag{28}$$

The Leray-Schauder fixed-point theorem provides a solution v to $S(v,1)=v$, which we denote by v_ε. Obviously, v_ε satisfies (25).

Step 3: lower bound for v_ε. By construction of v_ε, there exists $y_\varepsilon\in H^2(\mathbb{T}^d)$ such that $v_\varepsilon=e^{y_\varepsilon/2}$. Going back to (27), we see that

$$\frac{1}{2\tau}(E_1(v_\varepsilon^2)-E_1(w^2))\le-\varepsilon\int_{\mathbb{T}^d}((\Delta y_\varepsilon)^2+y_\varepsilon^2)dx\le-\varepsilon C\|y_\varepsilon\|_{H^2}^2,$$

using the Gagliardo-Nirenberg inequality. Hence,

$$\|y_\varepsilon\|_{H^2}\le\Big(\frac{E_1(w^2)}{2\varepsilon\tau C}\Big)^{1/2}\le c\varepsilon^{-1/2},\tag{29}$$

where $c>0$ is here and in the following a generic constant independent of ε. In combination with the embedding $H^2(\mathbb{T}^d)\hookrightarrow L^\infty(\mathbb{T}^d)$, this gives $\|y_\varepsilon\|_{L^\infty}\le$

$c\varepsilon^{-1/2}$. Consequently, v_ε is strictly positive:

$$v_\varepsilon = \exp\Big(\frac{y_\varepsilon}{2}\Big) \geq \exp\Big(-\frac{c}{2\varepsilon^{1/2}}\Big) = \mu(\varepsilon) > 0.$$

Thus, with $\mu := \mu(\varepsilon)$, it holds $[v_\varepsilon]_\mu = v_\varepsilon$, and the respective fixed point $v_\varepsilon \in H^2(\mathbb{T}^d)$ satisfies

$$\begin{aligned} \frac{1}{\tau}(v_\varepsilon^2 - w^2) = &-\partial_{ij}^2(v_\varepsilon\partial_{ij}^2 v_\varepsilon - \partial_i v_\varepsilon \partial_j v_\varepsilon) \\ &- \varepsilon(\Delta^2 \log v_\varepsilon + \log v_\varepsilon) + \varepsilon \mathrm{div}(|\nabla \log v_\varepsilon|^2 \nabla \log v_\varepsilon). \end{aligned} \tag{30}$$

Step 4: the limit $\varepsilon \to 0$. The estimate (28) shows that the sequence (v_ε) is bounded in $H^2(\mathbb{T}^d)$. Thus, for a subsequence which is not relabeled, $v_\varepsilon \rightharpoonup v$ weakly in $H^2(\mathbb{T}^d)$ and $v_\varepsilon \to v$ strongly in $W^{1,4}(\mathbb{T}^d)$ and $L^\infty(\mathbb{T}^d)$ as $\varepsilon \to 0$ for some $v \in H^2(\mathbb{T}^d)$. For the first expression on the right-hand side in (30), we thus obtain

$$v_\varepsilon \partial_{ij}^2 v_\varepsilon - \partial_i v_\varepsilon \partial_j v_\varepsilon \rightharpoonup v\partial_{ij}^2 v - \partial_i v \partial_j v \quad \text{weakly in } L^2(\mathbb{T}^d).$$

In order to prove that v is indeed as solution to (22), we verify that the expressions involving the factor ε vanish as $\varepsilon \to 0$. From the refined coercivity estimate

$$a(y_\varepsilon, y_\varepsilon) \geq \varepsilon(c\|y_\varepsilon\|_{H^2}^2 + \|\nabla y_\varepsilon\|_{L^4}^4),$$

we learn that

$$\|\nabla y_\varepsilon\|_{L^4} \leq c\varepsilon^{-1/4}.$$

In combination with (29), this gives

$$\begin{aligned} &\big|\big\langle \varepsilon\big(\Delta^2 \log v_\varepsilon + \log v_\varepsilon - \mathrm{div}(|\nabla \log v_\varepsilon|^2 \nabla \log v_\varepsilon)\big), z\big\rangle_{H^{-2},H^2}\big| \\ &\quad \leq \varepsilon\big(\|\log v_\varepsilon\|_{H^2}\|z\|_{H^2} + \|\log v_\varepsilon\|_{L^2}\|z\|_{L^2} + \|\nabla \log v_\varepsilon\|_{L^4}^3 \|z\|_{W^{1,4}}\big) \\ &\quad \leq c(\varepsilon^{1/2} + \varepsilon^{1/4})\|z\|_{H^2} \end{aligned}$$

for any test function $z \in H^2(\mathbb{T}^d)$. Therefore,

$$\varepsilon\big(\Delta^2 \log v_\varepsilon + \log v_\varepsilon - \mathrm{div}(|\nabla \log v_\varepsilon|^2 \nabla \log v_\varepsilon)\big) \rightharpoonup 0 \quad \text{weakly in } H^{-2}(\mathbb{T}^d),$$

so v satisfies (22).

Step 5: verification of (23) *and* (24). Conservation of mass follows from the weak formulation of (22) by using $z \equiv 1$ as a test function. From (27) and Lemma 6 it follows that

$$E_1(v_\varepsilon^2) + 2\tau\kappa_1 \int_{\mathbb{T}^d} \|\nabla^2 v_\varepsilon\|^2 dx \leq E_1(w^2).$$

In the limit $\varepsilon \to 0$, this inequality gives (23) since (a subsequence of) v_ε converges weakly to v in $H^2(\mathbb{T}^d)$ and the L^2-norm of the Hessian of v_ε constitutes a weakly lower semicontinuous functional on $H^2(\mathbb{T}^d)$.

Next, we prove (24). Recall that the solutions v_ε of the regularized equation (25) are strictly positive and bounded in modulus. Hence $\log v_\varepsilon$ and v_ε^p, for arbitrary exponents $p \in \mathbb{R}$, are well-defined functions in $H^2(\mathbb{T}^d)$. Using the test function $\phi_\gamma'(v_\varepsilon)/2 = (v_\varepsilon^{2(\gamma-1)} - 1)/2(\gamma-1)$ in (30) gives (see (19) for the definition of ϕ_γ),

$$\begin{aligned}
\frac{1}{2\tau}(E_\gamma(v_\varepsilon^2) - E_\gamma(w^2)) &= \frac{1}{2\tau}\int_{\mathbb{T}^d}\big(\phi_\gamma(v_\varepsilon^2) - \phi_\gamma(w^2)\big)dx \\
&\le \frac{1}{2\tau}\int_{\mathbb{T}^d}\phi_\gamma'(v_\varepsilon^2)(v_\varepsilon^2 - w^2)dx = \frac{1}{2(\gamma-1)\tau}\int_{\mathbb{T}^d}(v_\varepsilon^2 - w^2)\partial_{ij}^2(v_\varepsilon^{2(\gamma-1)})dx \\
&= -\frac{1}{2(\gamma-1)}\int_{\mathbb{T}^d}(v_\varepsilon\partial_{ij}^2 v_\varepsilon - \partial_i v_\varepsilon\partial_j v_\varepsilon)\partial_{ij}^2(v_\varepsilon^{2(\gamma-1)})dx \\
&\quad - \frac{\varepsilon}{\gamma-1}\int_{\mathbb{T}^d}\big(\Delta(v_\varepsilon^{2(\gamma-1)})\Delta(\log v_\varepsilon) + |\nabla\log v_\varepsilon|^2\nabla(\log v_\varepsilon)\cdot\nabla(v_\varepsilon^{2(\gamma-1)})\big)dx \\
&\quad - \frac{\varepsilon}{2(\gamma-1)}\int_{\mathbb{T}^d}v_\varepsilon^{\gamma-1}\log v_\varepsilon dx \\
&= A_1 - \varepsilon A_2 - \varepsilon A_3.
\end{aligned}$$

Now, by Lemma 4,

$$A_1 \le -\kappa_\gamma\int_{\mathbb{T}^d}(\Delta u^{\gamma/2})^2 dx.$$

Furthermore, by Lemma 7, applied to $f = u^{\gamma/2}$ and $p = \gamma$, and since u has unit mass, we obtain

$$\frac{\gamma}{\gamma-1}\Big(\int_{\mathbb{T}^d}u^\gamma dx - 1\Big) \le \frac{1}{8\pi^4}\int_{\mathbb{T}^d}(\Delta u^{\gamma/2})^2 dx,$$

so finally,

$$A_1 \le -\frac{8\pi^4\gamma\kappa_\gamma}{\gamma-1}\Big(\int_{\mathbb{T}^d}u^\gamma dx - 1\Big) = -8\pi^4\gamma^2\kappa_\gamma E_\gamma(v_\varepsilon^2).$$

Now, we show that A_2 and A_3 are bounded from below, uniformly in $\varepsilon > 0$. This is clear for A_3 since $\gamma > 1$. The remaining integral can be written as

$$\begin{aligned}
A_2 &= \int_{\mathbb{T}^d}v_\varepsilon^{2(\gamma-1)}\Big(\Big(\frac{\Delta v_\varepsilon}{v_\varepsilon}\Big)^2 - 2(2-\gamma)\frac{\Delta v_\varepsilon}{v_\varepsilon}\Big|\frac{\nabla v_\varepsilon}{v_\varepsilon}\Big|^2 + 2(2-\gamma)\Big|\frac{\nabla v_\varepsilon}{v_\varepsilon}\Big|^4\Big)dx \\
&= \int_{\mathbb{T}^d}v_\varepsilon^{2(\gamma-1)}\Big(\Big(\frac{\Delta v_\varepsilon}{v_\varepsilon} - (2-\gamma)\Big|\frac{\nabla v_\varepsilon}{v_\varepsilon}\Big|^2\Big)^2 + \gamma(2-\gamma)\Big|\frac{\nabla v_\varepsilon}{v_\varepsilon}\Big|^4\Big)dx \\
&\ge 0
\end{aligned}$$

since $\gamma < (\sqrt{d}+1)^2/(d+2) \le 3/2$. These estimates give

$$\frac{1}{\tau}(E_1(v_\varepsilon^2) - E_1(w^2)) \le -16\pi^4\gamma^2\kappa_\gamma E_\gamma(v_\varepsilon^2).$$

We pass to the limit $\varepsilon \to 0$ in this inequality. As $v_\varepsilon \to v$ strongly in $L^\infty(\mathbb{T}^d)$, integration and limit commute and we conclude

$$\frac{1}{\tau}(E_1(v^2) - E_1(w^2)) \le -16\pi^4\gamma^2\kappa_\gamma E_\gamma(v^2)$$

from which (24) follows. This finishes the proof. □

4.2. **A priori estimates.** Let an arbitrary terminal time $T > 0$ be fixed in the following. Define the step function $v^{(\tau)} : [0, T) \to L^2(\mathbb{T}^d)$ recursively as follows. Let $v_0 = \sqrt{u_0}$, and for given $k \in \mathbb{N}$, let $v_k \in H^2(\mathbb{T}^d)$ be the non-negative solution (according to Lemma 11) to (22) with $w = v_{k-1}$. Now define $v^{(\tau)}(t) := v_k$ for $(k-1)\tau < t \le k\tau$. Then $v^{(\tau)}$ satisfies

$$\frac{1}{\tau}\big((v^{(\tau)})^2 - (\sigma_\tau v^{(\tau)})^2\big) = -\partial^2_{ij}\big(v^{(\tau)}\partial^2_{ij}v^{(\tau)} - \partial_i v^{(\tau)}\partial_j v^{(\tau)}\big), \tag{31}$$

where σ_τ denotes the shift operator $(\sigma_\tau v^{(\tau)})(t) = v^{(\tau)}(t-\tau)$ for $\tau \le t < T$. In order to pass to the continuum limit $\tau \to 0$ in (31), we need the following a priori estimate.

Lemma 12. *The function $v^{(\tau)}$ satisfies*

$$\|(v^{(\tau)})^2\|_{L^{11/10}(0,T;H^2(\mathbb{T}^d))} + \tau^{-1}\|(v^{(\tau)})^2 - (\sigma_\tau v^{(\tau)})^2\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))} \le c, \tag{32}$$

where the constant $c > 0$ is independent of τ.

Proof. From Lemma 11 we know that

$$\|v^{(\tau)}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} = \|u_0\|^{1/2}_{L^1(\mathbb{T}^d)} = 1, \quad \|\nabla^2 v^{(\tau)}\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le c.$$

In order to derive (32), we employ the Gagliardo-Nirenberg and Hölder inequalities. The former inequality shows that

$$\begin{aligned}
\|v^{(\tau)}\|^{8/3}_{L^{8/3}(0,T;L^\infty(\mathbb{T}^d))} &\le C\int_0^T \|v^{(\tau)}(t,\cdot)\|^{8\theta/3}_{H^2}\|v^{(\tau)}(t,\cdot)\|^{8(1-\theta)/3}_{L^2}dt \\
&\le C\|v^{(\tau)}\|^{8(1-\theta)/3}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\int_0^T \|v^{(\tau)}(t,\cdot)\|^{8\theta/3}_{H^2}dt,
\end{aligned} \tag{33}$$

where $\theta = d/4$. Since $8\theta/3 = 2d/3 \le 2$ in dimensions $d \le 3$, the right-hand side is uniformly bounded. Applying Hölder's inequality with respect to t, for $p = 9/5$ and $p' = 9/4$, we infer

$$\begin{aligned}
\|v^{(\tau)}\partial^2_{ij}v^{(\tau)}\|^{11/10}_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} &\le C\int_0^T \|v^{(\tau)}(t,\cdot)\|^{11/10}_{H^2}\|v^{(\tau)}(t,\cdot)\|^{11/10}_{L^\infty}dt \\
&\le C\|v^{(\tau)}\|^{11/10}_{L^{11p/10}(0,T;H^2(\mathbb{T}^d))}\|v^{(\tau)}\|^{11/10}_{L^{11p'/10}(0,T;L^\infty(\mathbb{T}^d))}.
\end{aligned} \tag{34}$$

Since $11p/10 = 99/50 \le 2$ and $11p'/10 = 99/40 \le 8/3$, the right-hand side is uniformly bounded in view of the boundedness of $v^{(\tau)}$ in $L^{8/3}(0,T;L^\infty(\mathbb{T}^d))$. On the other hand, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\|v^{(\tau)}\|^{16/7}_{L^{16/7}(0,T;W^{1,4}(\mathbb{T}^d))} &\le C\int_0^T \|v^{(\tau)}(t,\cdot)\|^{16\theta/7}_{H^2}\|v^{(\tau)}(t,\cdot)\|^{16(1-\theta)/7}_{L^2}dt \\
&\le C\|v^{(\tau)}\|^{16(1-\theta)/7}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\|v^{(\tau)}\|^{16\theta/7}_{L^{16\theta/7}(0,T;H^2(\mathbb{T}^d))},
\end{aligned} \tag{35}$$

where $\theta = (d+4)/8$. As $16\theta/7 = 2(d+4)/7 \leq 2$ in dimensions $d \leq 3$, $v^{(\tau)}$ is uniformly bounded in $L^{16/7}(0,T;W^{1,4}(\mathbb{T}^d))$. As a straightforward conclusion,

$$\begin{aligned}\|\partial_i v^{(\tau)}\partial_j v^{(\tau)}\|^{11/10}_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} &\leq \int_0^T \|\nabla v^{(\tau)}(t,\cdot)\|^{22/10}_{L^4}dt \\ &\leq \|v^{(\tau)}\|^{22/10}_{L^{22/10}(0,T;W^{1,4}(\mathbb{T}^d))} \leq c, \end{aligned} \tag{36}$$

since $22/10 < 16/7$. Estimates (34) and (36) together yield

$$\begin{aligned}&\|\nabla^2(v^{(\tau)})^2\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} \\ &\qquad\leq 2\sum_{i,j=1}^d \|v^{(\tau)}\partial_{ij}v^{(\tau)} + \partial_i v^{(\tau)}\partial_j v^{(\tau)}\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} \leq c.\end{aligned}$$

Moreover, by (33) and (35), since $22/10 < 8/3$ and $22/10 < 16/7$,

$$\|\nabla(v^{(\tau)})^2\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} \leq 2\|v^{(\tau)}\|_{L^{22/10}(0,T;L^4(\mathbb{T}^d))}\|\nabla v^{(\tau)}\|_{L^{22/10}(0,T;L^4(\mathbb{T}^d))}.$$

The right-hand side is bounded by the considerations above. This estimates the first term in (32). To obtain a uniform bound on the second term in (32), we combine again (34) and (36):

$$\begin{aligned}&\frac{1}{\tau}\|(v^{(\tau)})^2 - (\sigma_\tau v^{(\tau)})^2\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))} \\ &\quad\leq \sum_{i,j=1}^d \Big(\|v^{(\tau)}\partial^2_{ij}v^{(\tau)}\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} + \|\partial_i v^{(\tau)}\partial_j v^{(\tau)}\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))}\Big) \leq c.\end{aligned}$$

□

4.3. **The limit $\tau \to 0$.** The a priori estimates of the previous subsection are sufficient to pass to the limit $\tau \to 0$.

Lemma 13. *There exists some nonnegative function $u \in W^{1,1}(0,T;H^{-2}(\mathbb{T}^d))$ with $\sqrt{u} \in L^2(0,T;H^2(\mathbb{T}^d))$ such that, for a subsequence of $(v^{(\tau)})$, which is not relabeled, as $\tau \to 0$,*

$$\begin{aligned}\frac{1}{\tau}\Big((v^{(\tau)})^2 - \sigma_\tau(v^{(\tau)})^2\Big) &\rightharpoonup \partial_t u \quad \text{weakly in } L^{11/10}(0,T;H^{-2}(\mathbb{T}^d)),\\ v^{(\tau)}\partial^2_{ij}v^{(\tau)} &\rightharpoonup \sqrt{u}\partial^2_{ij}\sqrt{u} \quad \text{weakly in } L^1(0,T;L^2(\mathbb{T}^d)),\\ \partial_i v^{(\tau)}\partial_j v^{(\tau)} &\rightharpoonup \partial_i\sqrt{u}\partial_j\sqrt{u} \quad \text{weakly in } L^1(0,T;L^2(\mathbb{T}^d)).\end{aligned}$$

Moreover, u is a weak solution to (4)-(5).

Proof. Estimate (32) allows to apply the Aubin lemma [23], showing that, up to a subsequence, $(v^{(\tau)})^2 \to u$ in $L^{11/10}(0,T;L^\infty(\mathbb{T}^d))$ as $\tau \to 0$ for some limit function u. Here, we have used that $H^2(\mathbb{T}^d)$ embeddes compactly into $L^\infty(\mathbb{T}^d)$ in dimensions $d \leq 3$. In particular, $(v^{(\tau)})$ converges pointwise a.e. Since obviously, $(v^{(\tau)})^2$ is nonnegative, so is u, and we can define $\sqrt{u} \in L^{22/10}(0,T;L^\infty(\mathbb{T}^d))$; note that $v^{(\tau)}$ converges strongly to $\sqrt{u}$ in this space.

Now, the first claim follows directly from (32) and the construction of $v^{(\tau)}$. Estimate (32) further yields weak convergence of $v^{(\tau)}$ in $L^2(0,T;H^2(\mathbb{T}^d))$. The weak limit necessarily coincides with $\sqrt{u}$, the strong limit from above.

By Hölder's inequality,

$$\begin{aligned}\|v^{(\tau)} - \sqrt{u}\|^2_{L^2(0,T;L^\infty(\mathbb{T}^d)} &\le \|(v^{(\tau)} - \sqrt{u})^2\|_{L^{11/10}(0,T;L^\infty(\mathbb{T}^d))} \cdot T^{1/11} \\ &\le \|(v^{(\tau)})^2 - u\|_{L^{11/10}(0,T;L^\infty(\mathbb{T}^d))} \cdot T^{1/11}.\end{aligned} \tag{37}$$

In the last step, we have used that $(a-b)^2 \le |a^2 - b^2|$ for arbitrary nonnegative $a, b \in \mathbb{R}$. Now, by the Gagliardo-Nirenberg and Hölder inequalities,

$$\begin{aligned}&\|\nabla(v^{(\tau)} - \sqrt{u})\|^2_{L^2(0,T;L^4(\mathbb{T}^d))} \\ &\qquad \le C\|v^{(\tau)} - \sqrt{u}\|_{L^2(0,T;H^2(\mathbb{T}^d))}\|v^{(\tau)} - \sqrt{u}\|_{L^2(0,T;L^\infty(\mathbb{T}^d))}.\end{aligned}$$

The first term in the product is bounded (cf. estimate (23)); the second term converges to zero by (37) above. Thus $v^{(\tau)} \to \sqrt{u}$ strongly in $L^2(0,T;W^{1,4}(\mathbb{T}^d))$ and

$$\partial_i v^{(\tau)} \partial_j v^{(\tau)} \rightharpoonup \partial_i \sqrt{u} \partial_j \sqrt{u} \quad \text{weakly in } L^1(0,T;L^2(\mathbb{T}^d)).$$

The remaining limit follows from (37) and weak convergence of $v^{(\tau)}$ to $\sqrt{u}$ in $L^2(0,T;H^2(\mathbb{T}^d))$. Finally, since $L^2(0,T;H^2(\mathbb{T}^d)) \hookrightarrow L^2(0,T;L^\infty(\mathbb{T}^d))$, one verifies that $u = \sqrt{u} \cdot \sqrt{u} \in L^1(0,T;H^2(\mathbb{T}^d))$ by the Hölder and Gagliardo-Nirenberg estimates. □

5. Decay rates for nonnegative weak solutions

We prove Theorem 2 for the solutions constructed in the previous section.

First, we show $\kappa_\gamma > 0$ for $1 \le \gamma < (\sqrt{d}+1)^2/(d+2)$. Indeed, by definition, $\kappa_\gamma > 0$ if $p(\gamma) > 0$, with the quadratic polynomial $p(\gamma)$ given in (13). But $p(\gamma) > 0$ if and only if $\gamma_- < \gamma < \gamma_+$ where $\gamma_\pm$ are the two roots of p. Now, a computation yields $\gamma_\pm = (\sqrt{d} \pm 1)^2/(d+2)$, and it is immediately seen that $\gamma_- < 1 < \gamma_+$.

Next, set $t_n = n\tau$ for $n = 0, \ldots, M$. From (24) we know that

$$E_\gamma\big(v^{(\tau)}(t_{n+1},\cdot)^2\big) - E_\gamma\big(v^{(\tau)}(t_n,\cdot)^2\big) \le -(2\pi)^4\tau\gamma^2\kappa_\gamma E_\gamma\big(v^{(\tau)}(t_{n+1},\cdot)^2\big).$$

Summation over $n = 0, \ldots, M-1$ gives

$$\begin{aligned}E_\gamma\big(v^{(\tau)}(t_M,\cdot)^2\big) - E_\gamma(u_0) &\le -(2\pi)^4\tau\gamma^2\kappa_\gamma \sum_{j=1}^M E_\gamma\big(v^{(\tau)}(t_j,\cdot)^2\big) \\ &\le -(2\pi)^4\tau\gamma^2\kappa_\gamma \int_\tau^{t_M} E_\gamma\big(v^{(\tau)}(s,\cdot)^2\big)ds.\end{aligned}$$

Keep t fixed; perform the limits $\tau \to 0$ and $M \to \infty$ such that $t_M = M\tau \to t$. Since $v^{(\tau)} \to \sqrt{u}$ strongly in $L^2(0,T;L^\infty(\mathbb{T}^d))$ as $\tau \to 0$,

$$E_\gamma\big(u(t,\cdot)\big) \le E_\gamma(u_0) - (2\pi)^4\tau\gamma^2\kappa_\gamma \int_0^t E_\gamma\big(u(s,\cdot)\big)ds.$$

Gronwall's lemma leads to the desired decay estimate. Decay in the L^1-norm follows immediately from the Csiszár-Kullback inequality. This finishes the proof of Theorem 2.

6. Non-uniqueness of solutions

In dimensions $d \leq 3$, we provide a family of initial conditions for which the DLSS equation (4)-(5) has at least two solutions in the class $L^1(0, T; H^2(\mathbb{T}^d))$ for all $T > 0$. Namely, for arbitrary integers $n_1, \ldots, n_d$, let

$$\hat{u}(t, x) = \cos^2(n_1 \pi x_1) \cdots \cos^2(n_d \pi x_d), \quad x = (x_1, \ldots, x_n)^\top \in \mathbb{T}^d.$$

This function is C^∞ smooth, time-independent, spatially multi-periodic, and has finite physical entropy, $\int_{\mathbb{T}^d} (\hat{u}(\log \hat{u} - 1) + 1) dx < +\infty$. Moreover, a simple calculation shows that the distribution

$$\partial^2_{ij} \big(\sqrt{\hat{u}} \partial^2_{ij} \sqrt{\hat{u}} - \partial_i \sqrt{\hat{u}} \partial_j \sqrt{\hat{u}} \big)$$

is identically zero. In other words, $\hat{u}$ is a weak solution of the stationary, and hence also of the transient equation. This time-independent function is clearly not physical: it does not converge to the homogeneous steady state, and it does not dissipate the physical (or any other) entropy.

On the other hand, Theorems 1 and 2 provide the existence of a weak solution $u(t, \cdot)$ to (4) with initial datum $u_0(x) = \hat{u}(0, x)$ which converges to the constant steady state as $t \to \infty$. Thus, $u \neq \hat{u}$. Hence, we have found two weak solutions to (4)-(5) in the class of nonnegative functions in $L^1(0, T; H^2(\mathbb{T}^d))$.

Moreover, the above observation makes clear that one cannot expect strict positivity of weak solutions for $t > 0$ if the initial conditions attain zero somewhere. On the other hand, numerical experiments (see, e.g., [9]) lead to the conjecture that for strictly positive initial data, the solutions are also strictly positive.

We remark that the stationary solution $\hat{u}$ does *not* have the regularity stated in the conclusions of Theorem 1: observe that $\sqrt{\hat{u}} \notin L^2(0, T; H^2(\mathbb{T}^d))$. Whether the condition $\sqrt{u} \in L^2(0, T; H^2(\mathbb{T}^d))$ is sufficient to obtain entropy-dissipative solutions (or perhaps even uniqueness and positivity) remains an open question.

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