A degenerate fourth-order parabolic equation modeling Bose-Einstein condensation. Part I: Local existence of solutions

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Abstract

A degenerate fourth-order parabolic equation modeling condensation phenomena related to Bose-Einstein particles is analyzed. The model is a Fokker-Planck-type approximation of the Boltzmann-Nordheim equation, only keeping the leading order term. It maintains some of the main features of the kinetic model, namely mass and energy conservation and condensation at zero energy. The existence of a local-in-time nonnegative continuous weak solution is proven. If the solution is not global, it blows up with respect to the L^∞ norm in finite time. The proof is based on approximation arguments, interpolation inequalities in weighted Sobolev spaces, and suitable a priori estimates for a weighted gradient L^2 norm.

Key words: Degenerate parabolic equation, fourth-order parabolic equation, existence of weak solutions, Bose-Einstein condensation, weighted spaces.

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1 Introduction

The dynamics of weakly interacting quantum particles like bosons can be described by the homogeneous Boltzmann-Nordheim equation for the distribution function $f(x,t)$ depending on the energy $x \geq 0$ and time $t > 0$ [16],

$$f_t(x_1,t) = \frac{1}{\sqrt{x_1}} \int_D S(f_3 f_4(1+f_1)(1+f_2) - f_1 f_2(1+f_3)(1+f_4))dx_3 dx_4, \tag{1.1}$$

where $x_2 = x_3 + x_4 - x_1$, $D = \{x_3 + x_4 > x_1\}$, and the transition rate S in the energy space depends on $x_1, \ldots, x_4$. The main feature of this equation is the existence of finite-time blow-up solutions if

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the initial density is sufficiently dense, modeling the condensation process [11]. The post-nucleation self-similar solution was investigated in detail by Spohn [23]. Due to the high complexity of the Boltzmann-Nordheim equation, approximate Fokker-Planck-type equations modeling condensation phenomena related to Bose-Einstein particles were studied in the literature.

For instance, if the energy exchange of each collision is small, the Fokker-Planck approximation of the Nordheim equation in the non-relativistic regime leads to the so-called Kompaneets equation [21]. It was originally suggested to describe the evolution of a homogeneous plasma when radiation interacts with matter via Compton scattering. Escobedo et al. [10] showed that this equation develops singularites at zero energy.

Another Fokker-Planck model was studied by Kaniadakis and Quarati [19, 20], proposing a nonlinear correction to the linear drift term to account for the presence of quantum indistinguishable particles (bosons and fermions). The model was derived in [1] from a Boltzmann Bose-Einstein model in the crazing collision limit. Toscani [24] proved that the limit equation possesses global-in-time solutions if the initial mass is sufficiently small and the solutions blow up in finite time if the initial mass is large enough.

A Fokker-Planck-type equation, only containing the superlinear drift term, was analyzed recently by Carrillo et al. [8]. The existence of a unique measure-valued solution, which concentrates the mass at the origin, was proven. Moreover, all mass concentrates in the long-time limit $t \to \infty$.

All these Fokker-Planck equations are of first or second order. A *higher-order* Fokker-Planck approximation of the Boltzmann-Nordheim equation was motivated by Josserand et al. [16]. This model is the subject of this paper. Assuming that the main contribution to the collision operator on the right-hand side of (1.1) comes from the neighborhood of $x \approx x_1 \approx x_2 \approx x_3 \approx x_4$, the integrand of the collision operator can be expanded to second order, leading to the fourth-order parabolic equation

$$u_t = x^{-1/2}\Big(x^{13/2}(u^4(u^{-1})_{xx} - u^2(\log u)_{xx})\Big)_{xx}, \quad x \in (0,\infty),\ t > 0, \tag{1.2}$$

where $u(x,t)$ denotes the energy distribution. This approximation maintains some of the features of the original Boltzmann equation. Indeed, assuming no-flux-type boundary conditions at $x = 0$ and $x \to \infty$, this equation conserves the total mass $N = \int_0^\infty x^{1/2}udx$ and the kinetic energy $E = \int_0^\infty x^{3/2}udx$. Furthermore, the entropy $S = \int_0^\infty((1+u)\log(1+u) - u\log u)x^{1/2}dx$ is nondecreasing, and the equilibrium is reached at the Bose-Einstein distribution $u = (e^{(x-\mu)/T} - 1)^{-1}$, where μ and T are some parameters [16].

We expect that the local approximation (1.2) contains the relevant information on the finite-time collapse of the distribution function. For such a study, it is reasonable to keep only the leading-order cubic term in (1.2). Furthermore, we restrict ourselves to the finite energy interval $(0, L)$ for an arbitrarily large $L > 0$ to avoid some technicalities due to infinite domains. Because of the condensation at energy $x = 0$, we expect that the density essentially vanishes for large energies which makes Neumann-type boundary conditions at $x = L$ plausible.

More precisely, in this paper we shall subsequently consider the slightly generalized problem given by

$$\begin{cases} u_t = x^{-\beta}\Big(x^\alpha u^{n+2}(u^{-1})_{xx}\Big)_{xx}, & x \in \Omega,\ t>0,\\ x^\alpha u^{n+2}(u^{-1})_{xx} = \Big(x^\alpha u^{n+2}(u^{-1})_{xx}\Big)_{xx} = 0, & x=0,\ t>0,\\ u_x = u_{xxx} = 0, & x = L,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $n > 0$, and $\Omega = (0, L) \subset \mathbb{R}$, with a given nonnegative function u_0.
The boundary conditions at $x = 0$ correspond to those imposed in [16, Formulas (13)-(14)]. In the original equation, we have $\alpha = 13/2$, $\beta = 1/2$, and $n = 2$. The approximate equation in (1.3) still conserves mass and energy. Moreover, it admits the stationary solutions $u(x) = x^{-\sigma}$ with $\sigma \in \{0, 1, \frac{7}{6}, \frac{3}{2}\}$, containing the same Kolmogorov-Zkharov spectra as the full Boltzmann-Nordheim equation [16, Section 3.3]. This indicates that there is condensation at zero energy $x = 0$.

From a mathematical point of view, significant challenges for the analysis stem from the fact that the parabolic equation in (1.3) degenerates both at $u = 0$ and at $x = 0$; accordingly, the literature does not yet provide any result for this equation, except for the heuristic study on self-similar solutions in [16]. It will turn out that this double degeneracy drastically distinguishes the solution behavior in (1.3) from that in related well-studied degenerate fourth-order parabolic equations such as the thin-film equation $u_t + (u^n u_{xxx})_x = 0$ [3, 2, 9]. Whereas e.g. the Neumann problem for the latter equation always possesses a globally defined continuous weak solution which remains bounded [5, 6], we shall see in the forthcoming paper [18] that the particular interplay of degeneracies in (1.3) can enforce solutions to blow up with respect to their spatial norm in $L^\infty(\Omega)$ within finite time. More generally, quite various types of higher-order diffusion equations such as e.g. the quantum diffusion or Derrida-Lebowitz-Speer-Spohn equation [14, 17], equations of epitaxial thin-film growth [25], or also some nonlinear sixth-order equations [7, 12, 22] have recently attracted considerable interest. To the best of our knowledge, however, such effects of spontaneous singularity formation, only due to a pure diffusion mechanism without any presence of external forces, have not been detected in any of these examples.

Against this background, the furthest conceivable outcome of any existence theory can only address *local* solvability. The goal of the present work is to establish an essentially optimal result in this direction, asserting local existence of a continuous weak solution u that conserves mass and that can be extended up to a maximal existence time $T_{max} \in (0, \infty]$ at which $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ must blow up whenever $T_{max} < \infty$.

Before we state our main result, we introduce some notation. We define for $\gamma \in \mathbb{R}$ the weighted Sobolev space

$$W^{1,2}_\gamma(\Omega) = \{v \in W^{1,2}_{\mathrm{loc}}(\Omega) : \|v\|^2_{L^2(\Omega)} + \|x^{\gamma/2} v_x\|_{L^2(\Omega)} < \infty\}$$

with norm $\|v\|_\gamma = (\|v\|^2_{L^2(\Omega)} + \|x^{\gamma/2} v_x\|_{L^2(\Omega)})^{1/2}$. We denote by χ_Q the characteristic function on the set $Q \subset \mathbb{R}^n$. The space $C^{4,1}(\bar{\Omega} \times (0, T))$ consists of all functions u such that u_{xxxx} and u_t exist and are continuous on $\bar{\Omega} \times (0, T)$. Furthermore, for any (not necessarily open) subset $Q \subset \mathbb{R}^n$, $C_0^\infty(Q)$ is the space of all functions such that $\mathrm{supp}(f) \subset Q$ is compact.

Definition 1.1 *Let n, α, $\beta \in \mathbb{R}$, and $T > 0$, and suppose that $u_0 \in C^0(\bar{\Omega})$ is nonnegative. Then by a* continuous weak solution *of (1.3) in $\Omega \times (0,T)$ we mean a nonnegative function $u \in C^0(\bar{\Omega} \times [0,T))$ with the properties $u \in C^{4,1}(((0,L] \times (0,T)) \cap \{u > 0\})$ as well as*

$$\chi_{\{u>0\}} x^\alpha u^n u_{xx} \in L^1_{loc}(\bar{\Omega} \times [0,T)) \qquad \text{and} \qquad \chi_{\{u>0\}} x^\alpha u^{n-1} u_x^2 \in L^1_{loc}(\bar{\Omega} \times [0,T)), \tag{1.4}$$

for which $u(\cdot,t)$ is differentiable with respect to x at $x = L$ for a.e. $t \in (0,T)$ with

$$u_x(L,t) = 0 \qquad \text{for a.e. } t \in (0,T), \tag{1.5}$$

and which satisfies the integral identity

$$-\int_0^T \int_\Omega x^\beta u \phi_t dx dt - \int_\Omega x^\beta u_0 \phi(\cdot,0) dx = \int_0^T \int_\Omega \chi_{\{u>0\}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \phi_{xx} dx dt \tag{1.6}$$

for all $\phi \in C_0^\infty(\bar{\Omega} \times [0,T))$ fulfilling $\phi_x(L,t) = 0$ for all $t \in (0,T)$.

Note that if u is a positive classical solution in the sense of this definition and $\alpha > 1$, then partial integration in (1.6) shows that u satisfies the boundary conditions in (1.3). Our main result reads as follows.

Theorem 1.1 (Local existence of solutions) *Let $n \in (n^*, 3)$, where $n^* = 1.5361\ldots$ is the unique positive root of the polynomial $n \mapsto n^3 + 5n^2 + 16n - 40$. Let $\alpha > 3$ and $\beta \in (-1, \alpha - 4)$. Then for any $\gamma \in (5 - \alpha + \beta, 1)$ and each nonnegative function $u_0 \in W^{1,2}_\gamma(\Omega)$, there exists $T_{max} \in (0,\infty]$ such that (1.3) possesses a continuous weak solution $u \in L^\infty_{loc}([0,T_{max}); W^{1,2}_\gamma(\Omega))$. Furthermore,*

$$\text{if } T_{max} < \infty \text{ then } \limsup_{t \to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \tag{1.7}$$

and the solution conserves the mass in the sense that

$$\int_\Omega x^\beta u(x,t) dx = \int_\Omega x^\beta u_0(x) dx \quad \text{for a.e. } t \in (0, T_{max}).$$

Note that the physical values $\alpha = \frac{13}{2}$, $\beta = \frac{1}{2}$, and $n = 2$ are admissible choices in the theorem.

A cornerstone in our analysis will consist in establishing an a priori estimate of the form

$$\begin{aligned}
\frac{d}{dt}\int_\Omega x^\gamma u_x^2 dx \quad &+ \quad c\int_\Omega x^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c\int_\Omega x^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx + c\int_\Omega x^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\
&+ \quad c\int_\Omega x^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c\int_\Omega x^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&\leq \quad C + C\Big(\int_\Omega x^\gamma u_x^2 dx\Big)^{\frac{n+2}{2}}
\end{aligned} \tag{1.8}$$

for appropriate $c > 0$ and $C > 0$, which can formally be derived from (1.3) under the restrictions for α, β, γ, and n made in Theorem 1.1. Upon integration, (1.8) will imply appropriate weighted integral estimates for u and its derivatives on small time intervals, inter alia the inequality

$$\int_\Omega x^\gamma u_x^2(x,t) dx \leq \tilde{C} \qquad \text{for all } t \in (0,T) \tag{1.9}$$

for some $\tilde{C} > 0$ and appropriately small $T > 0$.
A rigorous variant of (1.8) is shown in Lemmas 4.1 and 4.2. In view of the degeneracies in (1.3), our analysis will rely on a suitable regularization. To achieve this, we shall replace $x^{-\beta}$ and x^{α} by $(x+\varepsilon)^{-\beta}$ and $g_\varepsilon(x)$, respectively, where $\varepsilon > 0$, g_ε is positive in Ω, and $g_{\varepsilon,x}$ vanishes on the boundary. The latter condition ensures that the approximate flux $J = -g_\varepsilon(x)(-u^n u_{xx} + 2u^{n-1}u_x^2)$ vanishes on the boundary as well. We emphasize that unlike typical approaches in related equations such as the thin-film equation, our regularized problems are still degenerate at $u = 0$. To circumvent obstacles stemming from this, we shall first consider stricly positive initial data only; however, this will require additional efforts in ruling out that the local-in-time approximate solutions thereby obtained do not approach this critical level $u = 0$ within finite time (see Lemma 5.2).
The limit process $\varepsilon \to 0$ will then be carried out on the basis of a spatio-temporal Hölder estimate for the approximate solutions, which thanks to the fact that $\gamma < 1$ can be derived from (1.9) along with the adaptation of a well-known argument from parabolic theory, which turns this into an appropriate Hölder estimate with respect to time (Lemma 6.1).

The paper is organized as follows. In Section 2, we introduce the family of approximate problems. Interpolation inequalities in weighted spaces, which are needed for the existence analysis, are shown in Section 3. The proof of the a priori estimates (Lemmas 4.1 and 4.2) is the subject of Section 4. Then Section 5 is concerned with the local existence for the approximate problems and the absence of dead core formation. A Hölder estimate for the approximate solutions is derived in Section 6. Finally, the proof of Theorem 1.1 is presented in Section 7.

2 A family of approximate problems

We formulate a family of approximate problems in which the singularity at $x = 0$ is removed but the boundary conditions in (1.3) hold at $x = L$ and $x = 0$. To this end, we let $\varepsilon_0 = \min\{1, \sqrt{L/2}\}$, and for $\varepsilon \in (0, \varepsilon_0)$, we choose $\zeta_\varepsilon \in C_0^\infty(\Omega)$ satisfying $0 \le \zeta_\varepsilon \le 1$ and $\zeta_\varepsilon(y) = 1$ for $y \in (\varepsilon^2, L - \varepsilon^2)$. Furthermore, we set

$$z_\varepsilon(x) = \varepsilon + \int_0^x \zeta_\varepsilon(y)dy, \quad x \in [0, L].$$

Then the function z_ε belongs to $C^\infty([0, L])$, $z_\varepsilon(x) \ge \varepsilon$ for all $x \in [0, L]$, and it satisfies homogeneous Neumann boundary conditions, $z_{\varepsilon,x}(0) = z_{\varepsilon,x}(L) = 0$. Then $g_\varepsilon := z_\varepsilon^\alpha$ belongs to $C^\infty([0, L])$ and satisfies $g_\varepsilon \ge \varepsilon^\alpha$ on $[0, L]$ and $g_{\varepsilon,x}(0) = g_{\varepsilon,x}(L) = 0$. Further pointwise estimates for g_ε are summarized in the following lemma.

Lemma 2.1 (Properties of g_ε) *Let $\alpha > 0$. Then the following properties hold:*
(i) There exists a positive decreasing function $\Lambda : [0, \varepsilon_0) \to (0, 1)$ such that $\inf_{(0,\varepsilon_0)} \Lambda > 0$, $\Lambda(0) = 1$, and for all $\varepsilon \in (0, \varepsilon_0)$,

$$\Lambda(\varepsilon)(x + \varepsilon)^\alpha \le g_\varepsilon(x) \le (x + \varepsilon)^\alpha, \quad x \in [0, L].$$

(ii) There exists $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$0 \le g_{\varepsilon x}(x) \le c(x + \varepsilon)^{\alpha - 1}, \quad x \in [0, L].$$

(iii) There exists $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{g_{\varepsilon x}(x)^2}{g_\varepsilon(x)} \le c(x+\varepsilon)^{\alpha-2}, \quad x \in [0, L].$$

PROOF. (i) Since $\zeta_\varepsilon \le 1$, we have $z_\varepsilon(x) \le \varepsilon + x$ for $x \in [0, L]$. This yields the second inequality, $g_\varepsilon(x) = z_\varepsilon(x)^\alpha \le (x+\varepsilon)^\alpha$. To prove the first one, we divide $[0, L]$ into three subintervals. First, for $x \in [0, \varepsilon^2]$ the property $z_\varepsilon(x) \ge \varepsilon$ yields

$$\frac{z_\varepsilon(x)}{x+\varepsilon} \ge \frac{\varepsilon}{x+\varepsilon} \ge \frac{\varepsilon}{\varepsilon^2+\varepsilon} = \frac{1}{1+\varepsilon}.$$

Next, if $x \in (\varepsilon^2, L-\varepsilon^2)$ then $\zeta_\varepsilon(x) = 1$, whence using that $z_\varepsilon(\varepsilon^2) \ge \varepsilon$, we obtain

$$\frac{z_\varepsilon(x)}{x+\varepsilon} = \frac{1}{x+\varepsilon}\left(z_\varepsilon(x) + \int_{\varepsilon^2}^x \zeta_\varepsilon(y)dy\right) \ge \frac{\varepsilon + (x-\varepsilon^2)}{x+\varepsilon} \ge 1 - \frac{\varepsilon^2}{\varepsilon^2+\varepsilon} = \frac{1}{1+\varepsilon}.$$

We finally consider the case $x \in [L-\varepsilon^2, L]$, in which because of the nonnegativity of z_ε and the fact that $\zeta_\varepsilon = 1$ on $[\varepsilon^2, L-\varepsilon^2]$, we infer that

$$\frac{z_\varepsilon(x)}{x+\varepsilon} \ge \frac{z_\varepsilon(L-\varepsilon^2)}{x+\varepsilon} = \frac{\varepsilon + (L-2\varepsilon^2)}{x+\varepsilon} \ge \frac{L+\varepsilon-2\varepsilon^2}{L+\varepsilon} = 1 - \frac{2\varepsilon^2}{L+\varepsilon}.$$

The claim hence follows by defining $\Lambda(\varepsilon) = \min\{1/(1+\varepsilon), 1-2\varepsilon^2/(L+\varepsilon)\}$.

(ii) As $0 \le z_{\varepsilon,x} \le 1$, we have $g_{\varepsilon,x} = \alpha z_\varepsilon^{\alpha-1} z_{\varepsilon,x} \le \alpha z_\varepsilon^{\alpha-1}$ in $[0, L]$. Thus, (i) implies (ii).

(iii) This follows directly from (i) and (ii). □

With the above choices of $\varepsilon_0 > 0$ and g_ε, we proceed to regularize the original problem appropriately. The idea is to replace in the first equation in (1.3), rewritten in the form $u_t = x^{-\beta}(-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1}u_x^2)_{xx}$, the coefficients $x^{-\beta}$ and x^α by $(x+\varepsilon)^{-\beta}$ and $g_\varepsilon(x)$, respectively. Accordingly, for $\varepsilon \in (0, \varepsilon_0)$, we shall consider the approximate problem

$$\begin{cases} u_t = \frac{1}{(x+\varepsilon)^\beta} \cdot \Big\{ - g_\varepsilon(x)u^n u_{xx} + 2g_\varepsilon(x)u^{n-1}u_{xx} \Big\}_{xx}, & x \in \Omega,\ t > 0, \\ u_x = u_{xxx} = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.10}$$

The boundary behavior of g_ε guarantees that the flux

$$J(x,t) = -g_\varepsilon(x)u^n u_{xx} + 2g_\varepsilon(x)u^{n-1}u_x^2 \tag{2.11}$$

vanishes on $\partial\Omega = \{0, L\}$. This results upon expanding J_x according to

$$\begin{aligned} J_x = & -g_\varepsilon(x)u^n u_{xxx} + (4-n)g_\varepsilon(x)u^{n-1}u_x u_{xx} + 2(n-1)g_\varepsilon(x)u^{n-2}u_x^3 \\ & -g_{\varepsilon,x}(x)u^n u_{xx} + 2g_{\varepsilon,x}(x)u^{n-1}u_x^2, \end{aligned} \tag{2.12}$$

and evaluating this expression on $\partial\Omega$:

Lemma 2.2 (Boundary flux vanishes) *Let $n > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $T > 0$, and $\varepsilon \in (0, \varepsilon_0)$, and let $u \in C^{4,1}(\overline{\Omega} \times (0,T))$ be a positive classical solution of (2.10). Then $J_x(x,t) = 0$ for all $x \in \partial\Omega$ and $t \in (0,T)$, where J is defined in (2.11).*

PROOF. The statement is a consequence of (2.12) and the identities $u_x = u_{xxx} = g_{\varepsilon,x} = 0$ on $\partial\Omega$. $\Box$

The above choice of boundary conditions ensures that the total mass is preserved.

Lemma 2.3 (Conservation of total mass) *Under the assumptions of Lemma 2.2, we have*

$$\frac{d}{dt}\int_\Omega (x+\varepsilon)^\beta u(x,t)dx = 0 \qquad \text{for all } t \in (0,T).$$

PROOF. The claim immediately results by integrating (2.10) over Ω and using that $J_x = 0$ on $\partial\Omega$. $\Box$

3 Some interpolation inequalities

As a preparation for our subsequent analysis, let us collect some interpolation inequalities in weighted spaces. The first of these reads as follows.

Lemma 3.1 *Let $n \in \mathbb{R} \setminus \{-1,1\}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$. Then for any $\eta > 0$, one can find $C(\eta) > 0$ such that for all positive functions $u \in C^2(\bar\Omega)$ satisfying $u_x = 0$ on $\partial\Omega$, we have*

$$\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx + \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \\
\le \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
+ C(\eta) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx
\end{aligned} \tag{3.1}$$

for all $\varepsilon > 0$.

The proof of Lemma 3.1 will be achieved in a series of steps to be presented separately in Lemmas 3.2-3.4. We first estimate the last integral on the left-hand side of (3.1) by a sum involving a small portion of the first term in (3.1).

Lemma 3.2 *Let $n \in \mathbb{R} \setminus \{-1\}$ and α, β, and γ be arbitrary real numbers. Then for all $\eta > 0$, there exists $C(\eta) > 0$ such that whenever $\varepsilon > 0$ and $u \in C^2(\bar\Omega)$ is positive with $u_x = 0$ on $\partial\Omega$, the inequality*

$$\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \le \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + C(\eta) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \tag{3.2}$$

holds.

PROOF. Using $u_x = 0$ on $\partial\Omega$, we may integrate by parts and use Young's inequality to find that

$$\begin{aligned}
\Gamma &:= \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \\
&= -\frac{1}{n+1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^{n+1} u_{xx} dx - \frac{\alpha-\beta+\gamma-4}{n+1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-5} u^{n+1} u_x dx \\
&\le \frac{\eta}{2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + \frac{1}{2(n+1)^2\eta} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \\
&\quad + \frac{1}{2}\Gamma + \frac{(\alpha-\beta+\gamma-4)^2}{2(n+1)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx.
\end{aligned}$$

Rearranging yields (3.2). $\square$

Using the above preparation, we can control the first term in (3.1) as desired:

Lemma 3.3 *Let $n \in \mathbb{R} \setminus \{-1\}$ and $\alpha,\ \beta,\ \gamma \in \mathbb{R}$. Then for all $\eta > 0$, one can find $C(\eta) > 0$ with the property that any positive function $u \in C^3(\bar{\Omega})$ with $u_x = 0$ on $\partial\Omega$ satisfies*

$$\begin{aligned}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx \quad &\le \quad \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + C(\eta) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \end{aligned} \tag{3.3}$$

for each $\varepsilon > 0$.

PROOF. Since $u_x = 0$ on $\partial\Omega$, an integration by parts shows that

$$\begin{aligned}\Gamma \quad := \quad & \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx = -\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_x u_{xxx} dx \\ & - n \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx - (\alpha-\beta+\gamma-2) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^n u_x u_{xx} dx, \end{aligned} \tag{3.4}$$

where by Young's inequality we find that

$$-\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_x u_{xxx} dx \le \frac{\eta}{2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c_1 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx$$

and

$$\begin{aligned} -n \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx \quad &\le \quad \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + c_2 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \end{aligned}$$

as well as

$$\begin{aligned} -(\alpha-\beta+\gamma-2) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^n u_x u_{xx} dx \quad &\le \quad \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + c_3 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \end{aligned}$$

with $c_1 := \frac{1}{2\eta}, c_2 := \frac{n^2}{\eta}$ and $c_3 := \frac{(\alpha-\beta+\gamma-2)^2}{\eta}$. Since Lemma 3.2 provides $c_4 > 0$ such that

$$(c_1 + c_2) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \le \frac{1}{2}\Gamma + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx,$$

(3.4) thereby proves (3.3). $\square$

Now the latter allows us to also estimate the second term in (3.1) in the claimed manner.

Lemma 3.4 *Let $n \in \mathbb{R} \setminus \{-1, 1\}$ and $\alpha,\ \beta,\ \gamma \in \mathbb{R}$. Then for all $\eta > 0$, we can pick $C(\eta) > 0$ such that if $u \in C^3(\bar{\Omega})$ is positive and satisfies $u_x = 0$ on $\partial\Omega$, then for all $\varepsilon > 0$, we have*

$$\begin{aligned}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \quad &\le \quad \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + C(\eta) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx. \qquad (3.5)\end{aligned}$$

PROOF. Once more integrating by parts and using Young's inequality, we see that

$$\begin{aligned}\Gamma \quad &:= \quad \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\ &= \quad -\frac{3}{n-1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx - \frac{\alpha-\beta+\gamma-2}{n-1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^{n-1} u_x^3 dx \\ &\le \quad \frac{1}{4}\Gamma + c_1 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + \frac{1}{4}\Gamma + c_2 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx\end{aligned}$$

with $c_1 := \frac{9}{(n-1)^2}$ and $c_2 := \frac{(\alpha-\beta+\gamma-4)^2}{(n-1)^2}$. Thus,

$$\Gamma \le 2c_1 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + 2c_2 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx,$$

whence invoking Lemma 3.3, we readily arrive at (3.5). □

PROOF of Lemma 3.1. We only need to combine Lemmas 3.2, 3.3, and 3.4. □

The following inequality is closely related to those used in the context of the thin-film equation $u_t + (u^n u_{xxx})_x = 0$ [4].

Lemma 3.5 *Let $n \in \mathbb{R} \setminus \{3\}$ and $\alpha,\ \beta,\ \gamma \in \mathbb{R}$. Then for all $\eta \in (0,1)$ and any positive $u \in C^2(\bar{\Omega})$ fulfilling $u_x = 0$ on $\partial\Omega$, the inequality*

$$\begin{aligned}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \quad &\le \quad \frac{25}{(1-\eta)(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + \frac{(\alpha-\beta+\gamma)^2}{\eta(1-\eta)(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \qquad (3.6)\end{aligned}$$

is valid for all $\varepsilon > 0$.

PROOF. We integrate by parts using $u_x = 0$ on $\partial\Omega$ and apply Young's inequality to obtain the estimate

$$\begin{aligned}\Gamma \quad &:= \quad \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\ &= \quad -\frac{5}{n-3} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-3} u_x^4 u_{xx} dx - \frac{\alpha-\beta+\gamma}{n-3} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-1} u^{n-3} u_x^5 dx \\ &\le \quad \frac{1}{2}\Gamma + \frac{1}{2} \cdot \frac{25}{(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + \frac{\eta}{2}\Gamma + \frac{1}{2\eta} \cdot \frac{(\alpha-\beta+\gamma)^2}{(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx,\end{aligned}$$

which can readily be checked to be equivalent to (3.6). $\Box$

The following two lemmas are concerned with estimates on the Hölder and L^∞ norms of functions in $W^{1,2}_{loc}(\Omega)$.

Lemma 3.6 *Let $\gamma \in (-\infty, 1)$. Then there exists $c(\gamma) > 0$ such that for any $\varepsilon \in [0,1)$ and any $u \in W^{1,2}_{loc}(\Omega)$,*

$$|u(x_2) - u(x_1)| \le c \left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \right)^{\frac{1}{2}} |x_2 - x_1|^\theta \quad \textit{for all } x_1, x_2 \in \Omega,$$

where $\theta := \min\{\frac{1}{2}, \frac{1-\gamma}{2}\}$.

Proof. Let $0 < x_1 < x_2 < L$ and suppose first that $\gamma \in [0,1)$. Then by the Cauchy-Schwarz inequality,

$$|u(x_2) - u(x_1)| = \left| \int_{x_1}^{x_2} u_x(x) dx \right| \le \left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} (x+\varepsilon)^{-\gamma} dx \right)^{\frac{1}{2}}.$$

Employing the Hölder continuity of $x \mapsto x^{1-\gamma}$, we obtain

$$\int_{x_1}^{x_2} (x+\varepsilon)^{-\gamma} dx = \frac{1}{1-\gamma} \left((x_2+\varepsilon)^{1-\gamma} - (x_1+\varepsilon)^{1-\gamma} \right) \le \frac{c_1}{1-\gamma} |x_2 - x_1|^{1-\gamma}.$$

The result thus follows with $c = (c_1/(1-\gamma))^{\frac{1}{2}}$.
If $\gamma \in (-\infty, 0)$, we replace γ by $-\gamma$ in the above arguments and use the Lipschitz continuity of $x \mapsto x^{1+|\gamma|}$. $\Box$

Lemma 3.7 *Let $\gamma \in (-\infty, 1)$ and $\beta \in \mathbb{R}$. Then there exists $c = c(\beta, L) > 0$ such that for all $\varepsilon \in [0,1)$ and any $u \in W^{1,2}_{loc}(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \le c \left(\int_\Omega (x+\varepsilon)^\beta |u| dx + \left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \right)^{1/2} \right).$$

Proof. Assuming that $B = \int_\Omega (x+\varepsilon)^\beta |u| dx$ is finite, we see that there exists $x_0 \in (\frac{L}{2}, L)$ such that $(x_0+\varepsilon)^\beta |u(x_0)| \le \frac{2B}{L}$, for otherwise the inequality $B \ge \int_{\frac{L}{2}}^{L} (x+\varepsilon)^\beta |u| dx > \frac{L}{2} \cdot \frac{2B}{L} = B$ gives a contradiction. Since $\frac{L}{2} \le x_0 + \varepsilon \le L+1$, we infer that

$$|u(x_0)| \le c_1 \int_\Omega (x+\varepsilon)^\beta |u| dx,$$

where $c_1 = \frac{2}{L} \cdot \max\{(\frac{L}{2})^{-\beta}, (L+1)^{-\beta}\}$. The conclusion thus follows from Lemma 3.6. $\Box$

4 A differential inequality for $\int_\Omega x^\gamma u_x^2$

A key role in our analysis will be played by the following a priori estimate for the functional $y(t) := \int_\Omega (x+\varepsilon)^\beta u_x^2 dx$ in terms of a weighted norm of u in $L^{n+2}(\Omega)$. In Lemma 4.2 below, we shall turn this into an autonomous differential equation for $y(t)$, which will be essential for our local existence proof.

Lemma 4.1 (A priori estimate in terms of a weighted L^{n+2} norm) *Let $n_\star = 1.5361\ldots$ be the unique positive root of $n \mapsto P(n) := n^3 + 5n^2 + 16n - 40$, and let $n \in (n_\star, 3)$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$. Then there exist $\varepsilon_\star \in (0, \varepsilon_0)$, $c > 0$, and $K > 0$ such that if for some $T > 0$ and $\varepsilon \in (0, \varepsilon_\star)$, $u \in C^{4,1}(\bar\Omega \times (0,T))$ is a positive classical solution to (2.10), then*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx &+ c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&\le K\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \qquad \text{for all } t \in (0,T). \qquad (4.1)
\end{aligned}$$

Proof. With the notation (2.11), we can write the first equation in (2.10) as $u_t = (x+\varepsilon)^{-\beta} J_{xx}$. Since $u_x = J_x = 0$ on $\partial\Omega$ by Lemma 2.2, an integration by parts gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx &= -\int_\Omega ((x+\varepsilon)^\gamma u_x)_x u_t dx = -\int_\Omega ((x+\varepsilon)^\gamma u_x)_x (x+\varepsilon)^{-\beta} J_{xx} dx \\
&= \int_\Omega \Big[(x+\varepsilon)^{-\beta+\gamma} u_{xx} + \gamma (x+\varepsilon)^{-\beta+\gamma-1} u_x\Big]_x J_x dx
\end{aligned}$$

for all $t \in (0,T)$. Computing

$$\begin{aligned}
&\Big[(x+\varepsilon)^{-\beta+\gamma} u_{xx} + \gamma(x+\varepsilon)^{-\beta+\gamma-1} u_x\Big]_x \\
&\quad = (x+\varepsilon)^{-\beta+\gamma} u_{xxx} + (2\gamma-\beta)(x+\varepsilon)^{-\beta+\gamma-1} u_{xx} + \gamma(\gamma-\beta-1)(x+\varepsilon)^{-\beta+\gamma-2} u_x
\end{aligned}$$

and expanding J_x by means of (2.12), we thus obtain the identity

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx = &-\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx \\
&+ (4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-1} u_x u_{xx} u_{xxx} dx \\
&+ 2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^3 u_{xxx} dx \\
&- \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^n u_{xx} u_{xxx} dx \\
&+ 2\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-1} u_x^2 u_{xxx} dx \\
&- (2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^n u_{xx} u_{xxx} dx
\end{aligned}$$

$$
\begin{aligned}
& + (2\gamma-\beta)(4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-1} u_x u_{xx}^2 dx \\
& + 2(2\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
& - (2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_{\varepsilon x}(x) u^{n} u_{xx}^2 dx \\
& + 2(2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_{\varepsilon x}(x) u^{n-1} u_x^2 u_{xx} dx \\
& - \gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n} u_x u_{xxx} dx \\
& + \gamma(\gamma-\beta-1)(4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-1} u_x^2 u_{xx} dx \\
& + 2\gamma(\gamma-\beta-1)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\
& - \gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_{\varepsilon x}(x) u^{n} u_x u_{xx} dx \\
& + 2\gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_{\varepsilon x}(x) u^{n-1} u_x^3 dx \\
=: \quad & I_1 + \cdots + I_{15} \qquad \text{for all } t \in (0,T).
\end{aligned}
\tag{4.2}
$$

Our goal is to adequately apply the interpolation inequalities in Lemma 3.1 and Lemma 3.5 and to identify those integrals which absorb the $O(\eta)$ contributions in (3.1) and (3.6) such that finally only a possibly large multiple of the integral over $(x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}$ remains.
To achieve this, we observe that the integral I_1 is nonpositive and thus can be used to absorb positive contributions. Apart from this, the only absorptive contribution to be used in the sequel will result from I_3, which we therefore rearrange first: Namely, by two further integrations by parts, once more relying on the fact that $u_x = 0$ on $\partial\Omega$, this term can be rewritten according to

$$
\begin{aligned}
I_3 \quad = \quad & -6(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
& - 2(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-3} u_x^4 u_{xx} dx \\
& - 2(\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
& - 2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-2} u_x^3 u_{xx} dx \\
= \quad & -6(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
& - \frac{2}{5}(n-1)(n-2)(3-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx \\
& + \frac{2}{5}(\gamma-\beta)(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-3} u_x^5 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{5}(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-3} u_x^5 dx \\
&-2(\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
&-2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-2} u_x^3 u_{xx} dx \\
=:\ & I_{31}+\cdots+I_{36} \qquad \text{for all } t\in(0,T).
\end{aligned}
\tag{4.3}
$$

In order to specify our choice of $\varepsilon_\star$, let us note that, according to our restriction on n and with P as specified in the formulation of the lemma, we have $P(n)>0$, which implies that when $n<2$,

$$
\begin{aligned}
& 4(3-n)\Big\{6(n-1)-\frac{(4-n)^2}{4}-\frac{10(n-1)(2-n)}{3-n}\Big\} \\
=\ & 24(3-n)(n-1)-(3-n)(4-n)^2-40(n-1)(2-n) \\
=\ & -24n^2+96n-72-3n^2+24n-48+n^3-8n^2+16n+40n^2-120n+80 \\
=\ & P(n)>0.
\end{aligned}
$$

Since in the case $n\in[2,3)$ we clearly have

$$
6(n-1)-\frac{(4-n)^2}{4}\ge 6(2-1)-\frac{(4-2)^2}{4}=5>0,
$$

this entails that for any choice of $n\in(n_\star,3)$,

$$
6(n-1)-\frac{(4-n)^2}{4}-\frac{10(n-1)(2-n)_+}{3-n}>0.
$$

Consequently, with $\Lambda(\varepsilon)$ as in Lemma 2.1, we can pick $\varepsilon_\star\in(0,\varepsilon_0)$ such that with $\Lambda_\star:=\Lambda(\varepsilon_\star)$, we have

$$
\Big\{6(n-1)-\frac{(4-n)^2}{4}\Big\}\Lambda_\star-\frac{10(n-1)(2-n)_+}{3-n}>0,
$$

and thereupon fix a number $\mu\in(0,1)$ sufficiently close to 1 and $\eta>0$ suitably small such that still

$$
\Big\{6(n-1)-\frac{(4-n)^2}{4\mu}\Big\}\Lambda_\star-\frac{10(n-1)(2-n)_+}{3-n}-\Big\{\Lambda_\star+\frac{50}{(3-n)^2}+1\Big\}\eta>0, \tag{4.4}
$$

and such that moreover

$$
(1-\mu-\eta)\Lambda_\star-\eta>0. \tag{4.5}
$$

Upon these choices, we first use Young's inequality to estimate I_2 according to

$$
I_2\le \mu\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx+\frac{(4-n)^2}{4\mu}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx. \tag{4.6}
$$

Next, recalling Lemma 2.1, we obtain

$$
\begin{aligned}
I_4\ &\le\ \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx+c_1\int_\Omega (x+\varepsilon)^{-\beta+\gamma}\frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)}\cdot u^n u_{xx}^2 dx \\
&\le\ \frac{\eta}{2}|I_1|+c_2\Gamma_1,
\end{aligned}
\tag{4.7}
$$

where

$$\Gamma_1 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx \tag{4.8}$$

and c_1 and c_2, as all numbers $c_3, c_4, \dots$ appearing below, denote positive constants depending on n, α, β, and γ, but neither on $\varepsilon \in (0, \varepsilon_\star)$ nor on the solution u.
Similarly, we find $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned} I_5 &\leq \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_3 \int_\Omega (x+\varepsilon)^{-\beta+\gamma} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n-2} u_x^4 dx \\ &\leq \frac{\eta}{4} |I_1| + c_4 \Gamma_2 \end{aligned} \tag{4.9}$$

with

$$\Gamma_2 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx, \tag{4.10}$$

and then $c_5 > 0$ and $c_6 > 0$ satisfying

$$\begin{aligned} I_6 &\leq \frac{\eta}{8} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_5 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_{xx}^2 dx \\ &\leq \frac{\eta}{8} |I_1| + c_5 \Gamma_1 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} I_7 &\leq \frac{\eta}{2} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_6 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_{xx}^2 dx \\ &\leq \frac{\eta}{2} \tilde{I}_{31} + c_6 \Gamma_1 \end{aligned} \tag{4.12}$$

where

$$\tilde{I}_{31} := \frac{I_{31}}{-6(n-1)} = \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx. \tag{4.13}$$

In much the same manner, we derive the inequalities

$$\begin{aligned} I_8 &\leq \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_7 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\ &\leq \frac{\eta}{4} \tilde{I}_{31} + c_7 \Gamma_2 \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} I_{10} &\leq \frac{\eta}{8} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_8 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^n u_x^2 dx \\ &\leq \frac{\eta}{8} \tilde{I}_{31} + c_9 \Gamma_3 \end{aligned} \tag{4.15}$$

for some positive c_7, c_8 and c_9 and

$$\Gamma_3 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx, \tag{4.16}$$

as well as

$$
\begin{aligned}
I_{11} &\leq \frac{\eta}{16}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_{10}\int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} g_\varepsilon(x) u^n u_x^2 dx \\
&\leq \frac{\eta}{16}|I_1| + c_{10}\Gamma_3
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
I_{12} &\leq \frac{\eta}{16}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_{11}\int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} g_\varepsilon(x) u^n u_x^2 dx \\
&\leq \frac{\eta}{16}\tilde{I}_{31} + c_{11}\Gamma_3
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
I_{14} &\leq \frac{\eta}{32}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_{12}\int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n+2} dx \\
&\leq \frac{\eta}{32}\tilde{I}_{31} + c_{13}\Gamma_4,
\end{aligned}
\tag{4.19}
$$

where

$$
\Gamma_4 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \tag{4.20}
$$

and c_{10}, c_{11}, c_{12}, and c_{13} are positive constants.
As for the remaining terms on the right of (4.2), we again apply Lemma 2.1 to find $c_{14} > 0$ and $c_{15} > 0$ such that

$$
I_9 \leq c_{14}\Gamma_1 \tag{4.21}
$$

and

$$
I_{13} \leq c_{15} J_2, \tag{4.22}
$$

whereas Young's inequality provides $c_{16} > 0$ fulfilling

$$
\begin{aligned}
I_{15} &\leq c_{16}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^{n-1} |u_x|^3 dx \\
&\leq c_{16}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx + c_{16}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \\
&= c_{16}\Gamma_2 + c_{16}\Gamma_4.
\end{aligned}
\tag{4.23}
$$

Finally, abbreviating

$$
\tilde{I}_{32} := \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx, \tag{4.24}
$$

using Young's inequality we obtain constants $c_{17}, ..., c_{21}$ such that

$$
\begin{aligned}
I_{33} &\leq \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{17}\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\
&\leq \frac{\eta}{2}\tilde{I}_{32} + c_{17}\Gamma_2
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
I_{34} &\leq \frac{\eta}{4}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{18}\int_\Omega (x+\varepsilon)^{-\beta+\gamma}\frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)}\cdot u^{n-2}u_x^4 dx \\
&\leq \frac{\eta}{4}\tilde{I}_{32} + c_{18}\Gamma_2
\end{aligned}
\tag{4.26}
$$

as well as

$$
\begin{aligned}
I_{35} &\leq \frac{\eta}{8}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{19}\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_{xx}^2 dx \\
&\leq \frac{\eta}{8}\tilde{I}_{32} + c_{19}\Gamma_1
\end{aligned}
\tag{4.27}
$$

and

$$
\begin{aligned}
I_{36} &\leq \frac{\eta}{16}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{20}\int_\Omega (x+\varepsilon)^{-\beta+\gamma}\frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)}\cdot u^n u_{xx}^2 dx \\
&\leq \frac{\eta}{16}\tilde{I}_{32} + c_{21}\Gamma_1.
\end{aligned}
\tag{4.28}
$$

In light of (4.6)-(4.28), (4.2) and (4.3) thus yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \leq\ & -(1-\mu-\eta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx \\
& - \Big\{6(n-1) - \frac{(4-n)^4}{4\mu} - \eta\Big\}\cdot \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
& + \Big\{-\frac{2}{5}(n-1)(n-2)(3-n) + \eta\Big\}\cdot \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx \\
& + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
& + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx
\end{aligned}
\tag{4.29}
$$

for all $t \in (0,T)$ with some $c_{22} > 0$, where we have used that $\sum_{j=1}^N \frac{\eta}{2^j} < \eta$ for all $N \in \mathbb{N}$.
Now by means of Lemma 3.5 we can find $c_{23} > 0$ such that

$$
\begin{aligned}
&\Big\{-\frac{2}{5}(n-1)(n-2)(3-n) + \eta\Big\}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx \\
&\quad\leq \Big\{\frac{2}{5}(n-1)(2-n)_+(3-n) + \eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\
&\quad\leq \frac{25}{(1-\eta)(3-n)^2}\Big\{\frac{2}{5}(n-1)(2-n)_+(3-n) + \eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\qquad + c_{23}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&\quad\leq \frac{25}{(3-n)^2}\Big\{\frac{2}{5}(n-1)(2-n)_+(3-n) + 2\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\qquad + c_{23}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx.
\end{aligned}
$$

The last inequality follows from the fact that for $A := \frac{2}{5}(n-1)(2-n)_+(3-n) < 1$ $(0 < n < 3)$, we have $(A+\eta)/(1-\eta) \le A + 2\eta$ if $0 < \eta < \frac{1}{2}(1-A)$. Then applying Lemma 3.1, we obtain $c_{24} > 0$ satisfying

$$\begin{aligned}
c_{22} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx \quad &+ \quad (c_{22}+c_{23}) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&+ c_{22} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \\
&\le \quad \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ c_{24} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx. \qquad (4.30)
\end{aligned}$$

Therefore, (4.29) shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \le -(1-\mu-\eta) \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx \\
&- \Big\{ 6(n-1) - \frac{(4-n)^2}{4\mu} - \eta \Big\} \cdot \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx \\
&+ \Big\{ \frac{25}{(3-n)^2} \Big[\frac{2}{5}(n-1)(2-n)_+(3-n) + 2\eta \Big] + \eta \Big\} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ (c_{22}+c_{24}) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \qquad \text{for all } t \in (0,T).
\end{aligned}$$

Since clearly $1-\mu-\eta$ and $6(n-1) - \frac{(4-n)^2}{4\mu} - \eta$ are both positive thanks to (4.5) and (4.4), we may now use the lower estimate for g_ε established in Lemma 2.1 to infer that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \quad &\le \quad -\Big\{ (1-\mu-\eta)\Lambda_\star - \eta \Big\} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx \\
&- \Big\{ \Big[6(n-1) - \frac{(4-n)^2}{4\mu} - \eta \Big] \Lambda_\star - \frac{10(n-1)(2-n)_+}{3-n} - \frac{50\eta}{(3-n)^2} - \eta \Big\} \\
&\quad \times \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ (c_{22}+c_{24}) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \qquad \text{for all } t \in (0,T),
\end{aligned}$$

because $\varepsilon < \varepsilon_\star$ and hence $\Lambda(\varepsilon) \ge \Lambda_\star$ by the monotonicity of Λ asserted by Lemma 2.1. According to (4.5) and (4.4), after another application of (4.30), this entails (4.1). □

Under additional assumptions on the parameters α, β, and γ, we are able to derive a priori estimates for small times only depending on the initial data. More precisely, if the parameter γ is chosen large enough, then the weight in the integral on the right-hand side of (4.1) is sufficiently regular, whence from the above we can deduce a bound for $\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx$ for all sufficiently small $t > 0$. Since we plan to finally achieve a boundedness property for u itself with respect to the norm in $L^\infty(\Omega)$, we require that $\gamma < 1$. This explains the restriction on β in the following lemma.

Lemma 4.2 (A priori estimate for small times) *Let $n_* = 1.5361\dots$ and $\varepsilon_\star \in (0,1)$ be as in Lemma 4.1, let $\alpha > 0$, $\beta \in (-1, \alpha-4)$, and*

$$\gamma \in (5-\alpha+\beta, 1). \tag{4.31}$$

Then one can find $c > 0$ such that for all $A > 0$ and $B > 0$, there exists $T_0(A,B) \in (0,1)$ with the following property: If for some $\varepsilon \in (0,\varepsilon_\star)$ and $T \in (0, T_0(A,B))$, $u \in C^{4,1}(\bar\Omega \times [0,T))$ is positive and solves (2.10) in $\Omega \times (0,T)$ with

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,0)dx \le A \qquad \textit{and} \qquad \int_\Omega (x+\varepsilon)^\beta u(x,0)dx \le B, \tag{4.32}$$

then

$$\begin{aligned}
\sup_{t\in(0,T)} \int_\Omega (x+\varepsilon)^\gamma u(x,t)dx &\le c\int_0^T \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dxdt \qquad (4.33)\\
&+ c\int_0^T \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dxdt + c\int_0^T \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dxdt\\
&+ c\int_0^T \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dxdt + c\int_0^T \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dxdt\\
&\le A+1. \qquad (4.34)
\end{aligned}$$

In particular, in that case there exists $C(A,B) > 0$ such that the flux J, defined in (2.11), satisfies

$$\int_0^T \int_\Omega (x+\varepsilon)^{-\alpha-\beta+\gamma} J_x^2 dxdt \le C(A,B). \tag{4.35}$$

Proof. Let us first note that our hypothesis $\beta \in (-1, \alpha-4)$ entails the inequality $5-\alpha+\beta < 1$, whence the assumption $\gamma \in (5-\alpha+\beta, 1)$ indeed is meaningful. Then with $T_0(A,B) \in (0,1)$ to be fixed below, we assume that $T \in (0, T_0(A,B))$ and that u has the properties listed above. Thus, for each $t \in (0,T)$, by (4.32) and Lemma 2.3, we have $\int_\Omega (x+\varepsilon)^\beta u(x,t)dx \le B$, so that Lemma 3.7 says that

$$u(x,t) \le c_1 B + c_1 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac12} \qquad \text{for all } x \in \Omega \tag{4.36}$$

with some $c_1 > 0$, where we have used that $\gamma < 1$. Consequently, thanks to (4.31), the integral on the right-hand side of (4.1) can be estimated according to

$$\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2}(x,t)dx &\le 2^{n+2}(c_1B)^{n+2} \Big(\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} dx \Big)^{n+2}\\
&\quad + 2^{n+2} \cdot L \cdot c_1^{n+2} \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac{n+2}{2}}\\
&\le c_2(B) + c_3 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac{n+2}{2}}
\end{aligned}$$

with appropriate constants $c_2(B) > 0$ and $c_3 > 0$. From Lemmas 4.1 and 3.5, we thus obtain $c_4 > 0$, $c_5(B) > 0$, and $c_6 > 0$ such that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \quad &+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\
&+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&\le \quad c_5(B) + c_6 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2 \Big)^{\frac{n+2}{2}} dx \qquad \text{for all } t \in (0,T).
\end{aligned} \tag{4.37}$$

With the above constants being fixed, we consider the solution $y \equiv y_{A,B}$ of the initial-value problem

$$\begin{cases} y'(t) = c_5(B) + c_6 y^{\frac{n+2}{2}}(t), \qquad t > 0, \\ y(0) = A. \end{cases}$$

It is then clearly possible to fix some sufficiently small $T_0(A,B) \in (0,1)$ such that $y(t) \le A+1$ for all $t \in (0, T_0(A,B))$, and a comparison argument for ordinary differential equations, applied to (4.37), shows that

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t) dx \le A+1 \qquad \text{for all } t \in (0,T),\ T < T_0(A,B).$$

Inserting this into (4.37) and integrating, we readily arrive at (4.33).
From this, the estimate (4.35) easily follows upon recalling (2.12), (4.36), and Lemma 2.1 and applying Lemma 3.7 and Lemma 3.1. □

5 Local existence in the approximate problems

The a priori estimate of Lemma 4.2 allows us to prove the local existence of a classical solution to the approximate problem (2.10) for smooth initial data u_0 with compactly supported derivative u_{0x}.

Lemma 5.1 (Local existence for smooth data) *Let $\varepsilon_0 = \min\{1, \sqrt{\frac{L}{2}}\}$ and $\varepsilon \in (0, \varepsilon_0)$, and let $u_0 \in C^\infty(\bar\Omega)$ be positive and such that $u_{0x} \in C_0^\infty(\Omega)$. Then there exist $T_{max} \in (0, \infty]$ and a unique positive classical solution $u \in C^{4,1}(\bar\Omega \times [0, T_{max}))$ of (2.10) in $\Omega \times (0, T_{max})$ with $u(x,0) = u_0(x)$ for all $x \in \Omega$. Moreover, T_{max} has the property that*

$$\textit{if } T_{max} < \infty \textit{ then either } \liminf_{t \nearrow T_{max}} \Big(\inf_{x\in\Omega} u(x,t) \Big) = 0 \quad \textit{or } \limsup_{t \nearrow T_{max}} \Big(\sup_{x\in\Omega} u(x,t) \Big) = \infty. \tag{5.1}$$

Proof. For $k \in \mathbb{N}$, we let $f_k \in C^\infty(\mathbb{R})$ be a smooth nondecreasing truncation function on $\mathbb{R}$ such that $f_k(s) = s$ for all $s \in [\frac{1}{k}, k]$ and $\frac{1}{2k} \le f_k \le 2k$ on $\mathbb{R}$. Then each of the problems

$$\begin{cases} u_{kt} = \frac{1}{(x+\varepsilon)^\beta} \cdot \Big\{ - g_\varepsilon(x) f_k^n(u_k) u_{kxx} + 2 g_\varepsilon(x) f_k^{n-1}(u_k) u_{kx}^2 \Big\}_{xx}, \qquad x \in \Omega,\ t > 0, \\ u_{kx} = u_{kxxx} = 0, \qquad x \in \partial\Omega,\ t > 0, \\ u_k(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{5.2}$$

is non-degenerate, and since u_{0x} has compact support in Ω, standard parabolic theory [13] yields a uniquely determined global solution $u \in C^{4,1}(\bar{\Omega} \times [0,\infty))$.
Now for sufficiently large $k_0 \in \mathbb{N}$ and each $k \geq k_0$, it follows from the continuity of u_k and the positivity of u_0 in $\bar{\Omega}$ that

$$T_k := \sup \Big\{ T > 0 \ \Big| \ \frac{1}{k} \leq u_k \leq k \text{ in } \Omega \times (0,T) \Big\}$$

is a well-defined element of $(0,\infty]$, and by uniqueness in (5.2), it is clear that the sequence $(T_k)_{k \geq k_0}$ is nondecreasing, and that $u_{k_2} \equiv u_{k_1}$ in $\Omega \times (0, T_{k_1})$ whenever $k_2 \geq k_1 \geq k_0$. Consequently, the definition $T_{max} := \lim_{k\to\infty} T_k \in (0,\infty]$ is meaningful, and the trivially existing pointwise limit $u(x,t) := \lim_{k\to\infty} u_k(x,t)$, $(x,t) \in \bar{\Omega} \times [0, T_{max})$, satisfies $u \equiv u_k$ in $\Omega \times (0, T_k)$ for each $k \geq k_0$. It is therefore evident from (5.2) and the definition of f_k that u actually solves (2.10) in $\Omega \times (0, T_{max})$ with $u|_{t=0} = u_0$ in Ω.
It remains to verify (5.1). To this end, we assume on the contrary that $T_{max} < \infty$, but that both $\liminf_{t \nearrow T_{max}} (\inf_{x\in\Omega} u(x,t)) > 0$ and $\limsup_{t\nearrow T_{max}} (\sup_{x\in\Omega} u(x,t)) < \infty$. Then for some $k \geq k_0$, we would have $\frac{2}{k} \leq u \leq \frac{k}{2}$ in $\Omega \times (0, T_{max})$, implying that $u \equiv u_k$ in $\Omega \times (0, T_{max})$ by uniqueness. But since u_k is continuous at $t = T_{max}$, this would entail that $T_k > T_{max}$ and hence contradict the definition of T_{max}. $\square$

The following result rules out the occurrence of the first alternative in (5.1); that is, solutions to the approximate problem (2.10) cannot develop a dead core within finite time.

Lemma 5.2 (Absence of dead core formation) *Let $n > 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then for all $\varepsilon \in (0, \varepsilon_0), \delta > 0, M > 0$, and $T > 0$ there exists $C(\varepsilon, \delta, M, T) > 0$ such that if $u \in C^{4,1}(\bar{\Omega} \times [0,T))$ is a positive classical solution of (2.10) in $\Omega \times (0,T)$ satisfying*

$$u(x,0) \geq \delta \qquad \textit{for all } x \in \Omega \tag{5.3}$$

and

$$u(x,t) \leq M \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T), \tag{5.4}$$

we have the inequality

$$\int_\Omega \frac{1}{u^2(x,t)} dx \leq C(\varepsilon, \delta, M, T) \qquad \textit{for all } t \in (0,T). \tag{5.5}$$

Proof. Our goal is to conclude (5.5) from a differential inequality for $\int_\Omega (x+\varepsilon)^\beta u^{-2}$ which we shall thus derive first. To this end, we twice integrate by parts over Ω to compute, using J as defined in (2.11),

$$\begin{aligned}
\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \frac{1}{u^2} dx &= -2 \int_\Omega (x+\varepsilon)^\beta \frac{u_t}{u^2} dx = -2 \int_\Omega \frac{1}{u^3} J_{xx} dx \\
&= -6 \int_\Omega \frac{u_x}{u^4} J_x dx = 6 \int_\Omega \Big(\frac{u_x}{u^4} \Big)_x J dx \\
&= 6 \int_\Omega \Big[\frac{u_{xx}}{u^4} - 4 \frac{u_x^2}{u^5} \Big] \Big[- g_\varepsilon(x) u^n u_{xx} + 2 g_\varepsilon(x) u^{n-1} u_x^2 \Big] dx \\
&= -6 \int_\Omega g_\varepsilon(x) u^{n-4} u_{xx}^2 dx + 36 \int_\Omega g_\varepsilon(x) u^{n-5} u_x^2 u_{xx} dx - 48 \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx
\end{aligned}$$

for all $t \in (0,T)$, because $u_x = J_x = 0$ on $\partial\Omega$ according to Lemma 2.2. Since one more integration by parts yields

$$36 \int_\Omega g_\varepsilon(x) u^{n-5} u_x^2 u_{xx} dx = 12(5-n) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx - 12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx,$$

this shows that

$$\begin{aligned} \frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx &= -6 \int_\Omega g_\varepsilon(x) u^{n-4} u_{xx}^2 dx - 12(n-1) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx \\ &\quad - 12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx \end{aligned} \tag{5.6}$$

for all $t \in (0,T)$. Here, since $n > 1$, the second term on the right-hand side is nonpositive, and by means of Young's inequality and Lemma 2.1, we can find $c_1 > 0$ and $c_2 > 0$ fulfilling

$$\begin{aligned} -12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx &\le 12(n-1) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx + c_1 \int_\Omega \frac{g_{\varepsilon x}^4(x)}{g_\varepsilon^3(x)} \cdot u^{n-2} dx \\ &\le c_2 \int_\Omega (x+\varepsilon)^{\alpha-4} u^{n-2} dx, \end{aligned}$$

whence (5.6) in particular entails that

$$\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \le c_2 \int_\Omega (x+\varepsilon)^{\alpha-4} u^{n-2} dx \qquad \text{for all } t \in (0,T). \tag{5.7}$$

Now if $n \ge 2$, writing $c_3(\varepsilon) := \int_\Omega (x+\varepsilon)^{\alpha-4} dx$ and using (5.4), from (5.7) we obtain

$$\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \le c_2 c_3(\varepsilon) M^{n-2} \qquad \text{for all } t \in (0,T),$$

which after integration implies that

$$\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \le \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,0)} dx + c_2 c_3(\varepsilon) M^{n-2} T \qquad \text{for all } t \in (0,T).$$

As a consequence of (5.3), we thereby find that

$$\begin{aligned} \int_\Omega \frac{1}{u^2(x,t)} dx &\le c_4(\varepsilon) \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \\ &\le c_4(\varepsilon) \Big\{ \frac{c_5(\varepsilon)}{\delta^2} + c_2 c_3(\varepsilon) M^{n-2} T \Big\} \qquad \text{for all } t \in (0,T) \end{aligned}$$

with $c_4(\varepsilon) := \max\{\varepsilon^{-\beta}, (L+\varepsilon)^{-\beta}\}$ and $c_5(\varepsilon) := \int_\Omega (x+\varepsilon)^\beta dx$.

In the remaining case $n < 2$, we first apply the Hölder inequality with $p = \frac{2}{n}$ and $p' = \frac{2}{2-n}$ to the right-hand side in (5.7) to obtain

$$\begin{aligned} \frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx &\le c_2 \int_\Omega (x+\varepsilon)^{\alpha-4-(2-n)\frac{\beta}{2}} \big((x+\varepsilon)^\beta u^{-2}\big)^{\frac{2-n}{2}} dx \\ &\le c_2 c_6(\varepsilon) \Big(\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \Big)^{\frac{2-n}{2}} \qquad \text{for all } t \in (0,T) \end{aligned}$$

with

$$c_6(\varepsilon) := \Big(\int_\Omega (x+\varepsilon)^{\frac{2(\alpha-4)-(2-n)\beta}{n}} dx \Big)^{\frac{n}{2}}.$$

Integrating this in time shows that in this case,

$$\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \le \Big\{ \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,0)} dx + \frac{n}{2} c_2 c_6(\varepsilon) T \Big\}^{\frac{2}{n}} \qquad \text{for all } t \in (0,T),$$

and hence,

$$\int_\Omega \frac{1}{u^2(x,t)} dx \le c_4(\varepsilon) \cdot \Big\{ \frac{c_5(\varepsilon)}{\delta^2} + \frac{n}{2} c_2 c_6(\varepsilon) T \Big\}^{\frac{2}{n}} \qquad \text{for all } t \in (0,T),$$

according to (5.4). $\square$

6 Hölder continuity

We next derive a spatio-temporal Hölder estimate for the above solutions to the approximate problems. This will allow us to construct a continuous weak solution of (1.3) along a uniformly convergent sequence of appropriate solutions of (2.10) as $\varepsilon \to 0$.

Lemma 6.1 (Hölder estimate) *With $n_\star$ as in Lemma 4.1, assume that $n \in (n_\star, 3)$ and that $\alpha > 0$, $\beta < \alpha - 4$, and $\gamma < 1$ are such that $\alpha - \beta + \gamma > 5$. Moreover, let $A > 0$ and $B > 0$, and let $\varepsilon_\star$ and $T_0(A,B) \in (0,1)$ be as given by Lemma 4.1 and Lemma 4.2, respectively. Then there exists $C(A,B) > 0$ such that, whenever $u \in C^{4,1}(\bar\Omega \times [0,T))$ is a positive classical solution of (2.10) in $\bar\Omega \times (0,T)$ for some $T \in (0, T_0(A,B))$ with*

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,0) dx \le A \qquad \text{and} \qquad \int_\Omega (x+\varepsilon)^\beta u(x,0) dx \le B,$$

the estimate

$$|u(x_2,t_2) - u(x_1,t_1)| \le C(A,B) \cdot \Big(|x_2 - x_1|^\theta + |t_2 - t_1|^{\frac{\theta}{2\theta+3}} \Big) \tag{6.1}$$

holds for all x_1, $x_2 \in \Omega$ and t_1, $t_2 \in (0,T)$ with $\theta := \min\{\frac{1}{2}, \frac{1-\gamma}{2}\}$.

Proof. According to Lemma 4.2, we can pick c_1, as well as all constants $c_2, \dots$ below, possibly depending on n, α, β, γ, A, B, and L but independent from ε and u, such that

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2 \le c_1 \qquad \text{for all } t \in (0,T).$$

Hence, Lemma 3.6 provides $c_2 > 0$ such that we have the spatial Hölder estimate

$$|u(x_2,t_0) - u(x_1,t_0)| \le c_2 |x_2 - x_1|^\theta \qquad \text{for all } x_1, x_2 \in \Omega \text{ and } t_0 \in (0,T). \tag{6.2}$$

Using this, a corresponding Hölder estimate with respect to the time variable, that is, the inequality

$$|u(x_0,t_2) - u(x_0,t_1)| \le M |t_2 - t_1|^{\frac{\theta}{2\theta+3}} \qquad \text{for all } x_0 \in \Omega \text{ and } t_1, t_2 \in (0,T) \tag{6.3}$$

with suitably large $M > 1$, can be derived by adapting a standard technique due to Gilding and Kružkov ([15], cf. also [5] for a related procedure in a fourth-order setting). Indeed, following [5], let us assume that (6.3) be false, meaning that for some $x_0 \in \Omega$ and $t_1, t_2 \in (0,T)$ we have

$$u(x_0,t_2) - u(x_0,t_1) > M|t_2 - t_1|^{\frac{\theta}{2\theta+3}}, \tag{6.4}$$

where for definiteness we may suppose that $t_1 < t_2$. We then fix any $\zeta \in C_0^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $\mathbb{R}$, $\zeta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\zeta \equiv 0$ in $\mathbb{R} \setminus [-1,1]$, and let

$$\psi(x) := \zeta\Big(\frac{x - x_0}{\eta}\Big), \qquad x \in \bar{\Omega},$$

with

$$\eta := \Big(\frac{M}{16c_2}\Big)^{\frac{1}{\theta}} (t_2 - t_1)^{\frac{1}{2\theta+3}}. \tag{6.5}$$

Furthermore, we introduce the functions ξ_δ, $\delta > 0$, given by

$$\xi_\delta(t) := \frac{1}{\delta}\int_{-\infty}^t \Big\{\zeta\Big(\frac{s-t_2}{\delta}\Big) - \zeta\Big(\frac{s-t_1}{\delta}\Big)\Big\} ds, \qquad t \in (0,T), \tag{6.6}$$

which belong to $C_0^\infty((0,T))$ and satisfy $0 \ge \xi_\delta \ge -c_3$ with $c_3 := \int_{-1}^t \zeta(\sigma)d\sigma$, provided that $\delta < \delta_0 := \min\{t_1, T - t_2\}$ (this ensures that $\xi_\delta(0) = \xi_\delta(T) = 0$). Therefore, testing (2.10) against $\psi(x)\xi_\delta(t)$, $(x,t) \in \Omega \times (0,T)$, we obtain

$$\int_0^T \int_\Omega u(x,t)\psi(x)\xi_\delta'(t)dxdt = \int_0^T \int_\Omega \Big[(x+\varepsilon)^{-\beta}\psi(x)\xi_\delta(t)\Big]_x J_x(x,t)dxdt \qquad \text{for all } \delta \in (0,\delta_0) \tag{6.7}$$

with J as defined in (2.11), where we again have used that $J_x = 0$ on $\partial\Omega$ by (2.12).
We insert the definition of $\xi_\delta'(t)$, substitute $\sigma = \frac{t-t_i}{\delta}$, $i \in \{1,2\}$, and perform the limit $\delta \to 0$, to estimate the left-hand side in (6.7) from below according to

$$\begin{aligned}
\frac{1}{c_3}\lim_{\delta\searrow 0}\int_0^T\int_\Omega u(x,t)\psi(x)\xi_\delta'(t)dxdt &= \frac{1}{c_3}\lim_{\delta\searrow 0}\int_\Omega\int_{-1}^1 \Big[u(x,t_2+\delta\sigma) - u(x,t_1+\delta\sigma)\Big]\cdot\zeta(\sigma)d\sigma\cdot\psi(x)dx \\
&= \int_\Omega \Big[u(x,t_2) - u(x,t_1)\Big]\cdot\psi(x)dx \\
&\ge \int_\Omega \Big[u(x_0,t_2) - u(x_0,t_1)\Big]\cdot\psi(x)dx \\
&\quad - \int_\Omega \Big[|u(x,t_2) - u(x_0,t_2)| + |u(x_0,t_1) - u(x,t_1)|\Big]\cdot\psi(x)dx,
\end{aligned}$$

whence using (6.4) and (6.2) yields

$$\begin{aligned}
\frac{1}{c_3}\lim_{\delta\searrow 0}\int_0^T\int_\Omega u(x,t)\psi(x)\xi_\delta'(t)dxdt &\ge M(t_2-t_1)^{\frac{\theta}{2\theta+3}}\int_\Omega\psi(x)dx - 2c_2\int_\Omega|x-x_0|^\theta\cdot\psi(x)dx \\
&\ge M(t_2-t_1)^{\frac{\theta}{2\theta+3}}\cdot\frac{\eta}{2} - 2c_2\cdot\eta^\theta\cdot 2\eta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\eta}{2} \cdot \Big\{ M(t_2-t_1)^{\frac{\theta}{2\theta+3}} - 8c_2\eta^\theta \Big\} \\
&= \frac{\eta}{2} \cdot \frac{1}{2} M(t_2-t_1)^{\frac{\theta}{2\theta+3}} \\
&= c_4 M^{1+\frac{1}{\theta}} (t_2-t_1)^{\frac{\theta+1}{2\theta+3}} \qquad (6.8)
\end{aligned}
$$

with $c_4 := [4 \cdot (16c_2)^{\frac{1}{\theta}}]^{-1}$. On the right-hand side of (6.7), by the Cauchy-Schwarz inequality and Lemma 4.2, we find $c_5 > 0$ fulfilling

$$
\begin{aligned}
&\Big| \int_0^T \int_\Omega \Big[(x+\varepsilon)^{-\beta} \psi(x) \xi_\delta(t) \Big]_x \cdot J_x dx dt \Big| \\
&\le \Big(\int_0^T \int_\Omega (x+\varepsilon)^{-\alpha-\beta+\gamma} J_x^2 dx dt \Big)^{\frac{1}{2}} \Big(\int_0^T \xi_\delta^2(t) dt \Big)^{\frac{1}{2}} \Big(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma} \Big[(x+\varepsilon)^{-\beta}\psi(x) \Big]_x^2 dx \Big)^{\frac{1}{2}} \\
&\le c_5 c_3 (t_2 - t_1 + 2\delta)^{\frac{1}{2}} \cdot \Big(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma} \Big[(x+\varepsilon)^{-\beta}\psi(x) \Big]_x^2 dx \Big)^{\frac{1}{2}}, \qquad (6.9)
\end{aligned}
$$

since $\xi_\delta(t) = 0$ for $t \le t_1 - \delta$ or $t \ge t_2 + \delta$ and $\xi_\delta^2 \le c_3^2$. We use $\alpha - \beta - \gamma > 5 - 2\gamma > 3$ according to our assumptions and $\zeta' \equiv 0$ on $\mathbb{R}\backslash[-1,1]$, and we recall the definition (6.5) of η to find $c_6 > 0$ satisfying

$$
\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma} \cdot (x+\varepsilon)^{-2\beta} \psi_x^2(x) dx &= \frac{1}{\eta^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta-\gamma} \cdot \zeta\Big(\frac{x-x_0}{\eta}\Big) dx \\
&\le \frac{1}{\eta^2} \cdot (L+1)^{\alpha-\beta-\gamma} \cdot 2\eta \cdot \|\zeta'\|_{L^\infty(\mathbb{R})} \\
&= c_6 M^{-\frac{1}{\theta}} (t_2-t_1)^{-\frac{1}{2\theta+3}} \\
&\le c_6 (t_2-t_1)^{-\frac{1}{2\theta+3}},
\end{aligned}
$$

because of $M > 1$ and thus $M^{-\frac{1}{\theta}} < 1$. Similarly, with some $c_7 > 0$ we have

$$
\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma} \cdot (x+\varepsilon)^{-2\beta-2} \psi^2(x) dx &= \int_\Omega (x+\varepsilon)^{\alpha-\beta-\gamma-2} \psi^2(x) dx \\
&\le (L+1)^{\alpha-\beta-\gamma-2} \cdot 2\eta \\
&= c_7 \cdot M^{\frac{1}{\theta}} (t_2-t_1)^{\frac{1}{2\theta+3}},
\end{aligned}
$$

whence altogether

$$
\Big(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma} \Big[(x+\varepsilon)^{-\beta}\psi(x) \Big]_x^2 dx \Big)^{\frac{1}{2}} \le c_8 \Big\{ (t_2-t_1)^{-\frac{1}{2(2\theta+3)}} + M^{\frac{1}{2\theta}} (t_2-t_1)^{\frac{1}{2(2\theta+3)}} \Big\}
$$

holds with some $c_8 > 0$. Therefore, (6.7), (6.8), and (6.9) in the limit $\delta \searrow 0$ yield $c_9 > 0$ such that

$$
\begin{aligned}
\frac{c_3}{4} M^{1+\frac{1}{\theta}} (t_2-t_1)^{\frac{\theta+1}{2\theta+3}} &\le c_9 \Big\{ (t_2-t_1)^{\frac{1}{2}-\frac{1}{2(2\theta+3)}} + M^{\frac{1}{2\theta}} (t_2-t_1)^{\frac{1}{2}+\frac{1}{2(2\theta+3)}} \Big\} \\
&= c_9 \Big\{ (t_2-t_1)^{\frac{\theta+1}{2\theta+3}} + M^{\frac{1}{2\theta}} (t_2-t_1)^{\frac{\theta+2}{2\theta+3}} \Big\},
\end{aligned}
$$

which implies the inequality

$$
\begin{aligned}
c_4 M^{1+\frac{1}{\theta}} &\le c_9 \Big\{ 1 + M^{\frac{1}{2\theta}} (t_2 - t_1)^{\frac{1}{2\theta+3}} \Big\} \\
&\le c_9 \Big\{ 1 + M^{\frac{1}{2\theta}} (T_0(A,B))^{\frac{1}{2\theta+3}} \Big\}.
\end{aligned}
$$

Since $1+\frac{1}{\theta} > \frac{1}{2\theta}$, this gives an upper bound for M and thus yields the desired contradiction if M has been chosen suitably large initially. This proves the Hölder estimate (6.3) in time, and combining the latter with (6.2) completes the proof. $\square$

7 Proof of Theorem 1.1

Let $u_0 \in W^{1,2}_\gamma(\Omega)$, where $\gamma < 1$, and let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be a sequence satisfying $\varepsilon_j \to 0$ as $j \to \infty$. By a standard approximation argument, we may construct a sequence of functions $(u_{0\varepsilon_j})_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that

$$
u_{0\varepsilon} > 0 \quad \text{in } \bar\Omega \qquad \text{and} \qquad u_{0\varepsilon x} \in C_0^\infty(\Omega) \qquad\qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}, \tag{7.1}
$$

and such that

$$
u_{0\varepsilon} \to u_0 \quad \text{in } W^{1,2}_\gamma(\Omega) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{7.2}
$$

The following lemma asserts that under the assumptions on n, α, β, and γ required in Lemma 4.2, the corresponding solutions of (2.10) emanating from $u_{0\varepsilon_j}$ have their maximal existence time bounded from below for all sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, and moreover they accumulate at some continuous weak solution of (1.3).

Lemma 7.1 *Let n, α, β, and γ be as in Theorem 1.1. Then for all $A > 0$ and $B > 0$, there exists $T(A,B) \in (0,1)$ such that, whenever $u_0 \in W^{1,2}_{loc}(\Omega)$ is nonnegative and satisfies*

$$
\int_\Omega x^\gamma u_{0x}^2(x)dx \le A \qquad \text{and} \qquad \int_\Omega x^\beta u_0(x)dx \le B, \tag{7.3}
$$

the following holds: For any $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\varepsilon_0)$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and each $(u_{0\varepsilon_j})_{j\in\mathbb{N}} \subset C^\infty(\bar\Omega)$ fulfilling (7.1) and (7.2), the problem (2.10) possesses a unique positive classical solution $u_\varepsilon \in C^{4,1}(\bar\Omega\times [0,T(A,B)])$ for all sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, and moreover, there exists a subsequence $(\varepsilon_{j_l})_{l\in\mathbb{N}}$ such that

$$
u_\varepsilon \to u \quad \text{in } C^0(\bar\Omega \times [0,T(A,B)]) \qquad \text{as } \varepsilon = \varepsilon_{j_l} \to 0 \tag{7.4}
$$

with some continuous weak solution u of (1.3) in $\Omega \times (0,T(A,B))$.

Proof. We claim that the statement is valid if we let $T \equiv T(A,B) := T_0(A+1,B+1)$ with T_0 as provided by Lemma 4.2. To verify this, we first note that, according to (7.3) and upon passing to subsequences, we may assume that

$$
\int_\Omega (x+\varepsilon_j)^\gamma u^2_{0\varepsilon_j x}(x) \le A+1 \qquad \text{for all } j \in \mathbb{N} \tag{7.5}
$$

and

$$
\int_\Omega (x+\varepsilon_j)^\beta u_{0\varepsilon_j}(x)dx \le B+1 \qquad \text{for all } j \in \mathbb{N}. \tag{7.6}
$$

For $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, we then let u_ε denote the corresponding positive classical solution of (2.10), that is, of the initial-boundary value problem

$$\begin{cases} u_{\varepsilon t} = \frac{1}{(x+\varepsilon)^\beta} \cdot \Big\{ - g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} + 2 g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big\}_{xx}, \qquad x \in \Omega,\ t>0, \\ u_{\varepsilon x} = u_{\varepsilon xxx} = 0, \qquad x \in \partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), \qquad x \in \Omega, \end{cases}$$

which according to Lemma 5.1 exists up to a maximal time $T_\varepsilon \in (0,\infty]$ having the property stated in (5.1). We divide the proof into four steps.

Step 1. We first show that actually $T_\varepsilon \ge T$.
To see this, we apply Lemma 4.2 to find, upon passing to a subsequence if necessary, that for some $c_1>0$ and all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, we have

$$\begin{aligned} &\sup_{t\in(0,\hat{T}_\varepsilon)} \int_\Omega (x+\varepsilon)^\gamma u_{\varepsilon x}^2(x,t)dx \\ &+ \int_0^{\hat{T}_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u_\varepsilon^{n-2} u_{\varepsilon x}^2 u_{\varepsilon xx}^2 dxdt + \int_0^{\hat{T}_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u_\varepsilon^{n-4} u_{\varepsilon x}^6 dxdt \\ &+ \int_0^{\hat{T}_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^{n} u_{\varepsilon xx}^2 dxdt + \int_0^{\hat{T}_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^{n-2} u_{\varepsilon x}^4 dxdt \\ &\le c_1, \end{aligned} \tag{7.7}$$

where $\hat{T}_\varepsilon := \min\{T_\varepsilon, T\}$. Since

$$\sup_{t\in(0,\hat{T}_\varepsilon)} \int_\Omega (x+\varepsilon)^\beta u_\varepsilon(x,t)dx = \int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x)dx \le B+1$$

by Lemma 2.3, Lemma 3.7 yields $c_2>0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\sup_{t\in(0,\hat{T}_\varepsilon)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2. \tag{7.8}$$

Moreover, for fixed $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we may apply Lemma 3.6 with γ replaced by 0 to see that (7.7) entails that with some $c_3(\varepsilon)>0$, the spatial Hölder estimate

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \le c_3(\varepsilon)|x-y|^{\frac{1}{2}} \tag{7.9}$$

is valid for all $x,y \in \bar\Omega$ and any $t \in (0,\hat{T}_\varepsilon)$. Now assuming that $T_\varepsilon < T$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, in view of the extensibility criterion in Lemma 5.1 and the inequality (7.8), we would have

$$u_\varepsilon(x_k,t_k) \to 0 \qquad \text{as } k\to\infty$$

with some $(x_k)_{k\in\mathbb{N}} \subset \Omega$ and $(t_k)_{k\in\mathbb{N}} \subset (0,T_\varepsilon)$, where we may assume that $x_k \to x_0$ and $t_k \nearrow T_\varepsilon$ as $k\to\infty$ with some $x_0 \in \bar\Omega$. According to (7.8), (7.9), and the Arzelà-Ascoli theorem, we may pass to subsequences to achieve that with some $v \in C^0(\bar\Omega)$ we have

$$u_\varepsilon(\cdot,t_k) \to v \quad \text{in } C^0(\bar\Omega) \qquad \text{as } k\to\infty, \tag{7.10}$$

and conclude that $v(x_0) = 0$ and hence, again by (7.9), that

$$0 \le v(x) \le c_3(\varepsilon)|x - x_0|^{\frac{1}{2}} \qquad \text{for all } x \in \Omega. \tag{7.11}$$

This, however, contradicts the outcome of Lemma 5.2: The latter, namely, along with (7.8) implies that with some $c_4(\varepsilon) > 0$ we have

$$\int_\Omega \frac{1}{u_\varepsilon^2(x, t_k)} dx \le c_4(\varepsilon) \qquad \text{for all } k \in \mathbb{N},$$

so that Fatou's lemma and (7.11) give

$$\frac{1}{c_3^2(\varepsilon)} \int_\Omega \frac{1}{|x - x_0|} dx \le \int_\Omega \frac{1}{v^2(x)} dx \le c_4(\varepsilon),$$

which is impossible.

Step 2. We next construct the limit function u.
To achieve this, we observe that since $T_\varepsilon \ge T$ according to the above arguments, we may replace $\hat{T}_\varepsilon$ by T in (7.7) and (7.8) and apply Lemma 6.1 to derive the ε-independent estimate

$$\|u_\varepsilon\|_{C^{\theta, \frac{\theta}{2\theta+3}}(\bar{\Omega} \times [0,T])} \le c_5 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$$

with a certain $c_5 > 0$. Therefore, the Arzelà-Ascoli theorem yields a subsequence, again denoted by $(\varepsilon_j)_{j \in \mathbb{N}}$, and a nonnegative function $u \in C^0(\bar{\Omega} \times [0, T])$ such that

$$u_\varepsilon \to u \qquad \text{in } C^0(\bar{\Omega} \times [0, T]) \tag{7.12}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, interior parabolic regularity theory [13] shows that $(u_{\varepsilon_j})_{j \in \mathbb{N}}$ is relatively compact in $C^{4,1}_{loc}(((0, L] \times (0, T]) \cap \{u > 0\})$, and hence we may assume that as $\varepsilon = \varepsilon_j \searrow 0$, we also have

$$u_\varepsilon \to u \qquad \text{in } C^{4,1}_{loc}(\mathcal{P}), \qquad \text{where } \mathcal{P} := \Big((0, L] \times (0, T]\Big) \cap \{u > 0\}. \tag{7.13}$$

Step 3. We proceed to verify that there exists a null set $N \subset (0, T)$ such that for all $t \in (0, T) \setminus N$, $u(\cdot, t)$ is differentiable at $x = L$ with $u_x(L, t) = 0$ for all $t \in (0, T) \setminus N$.
To this end, we note that (7.7) in particular implies that for some $c_6 > 0$, we have

$$\int_0^T \int_{\frac{L}{2}}^L u_\varepsilon^{n-2} u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 dx dt + \int_0^T \int_{\frac{L}{2}}^L u_\varepsilon^{n-4} u_{\varepsilon x}^6 dx dt \le c_6 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$$

and since

$$\begin{aligned} \left\{ \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x^2 \right\}_x^2 &= \Big(\frac{n+2}{4}\Big)^4 \cdot \Big\{ 2 u_\varepsilon^{\frac{n-2}{2}} u_{\varepsilon x} u_{\varepsilon xx} + \frac{n-2}{2} u_\varepsilon^{\frac{n-4}{2}} u_{\varepsilon x}^3 \Big\}^2 \\ &\le 2 \Big(\frac{n+2}{4}\Big)^4 \cdot \Big\{ 4 u_\varepsilon^{n-2} u_{\varepsilon x}^2 u_{\varepsilon xx}^2 + \Big(\frac{n-2}{2}\Big)^2 u_\varepsilon^{n-4} u_{\varepsilon x}^6 \Big\} \qquad \text{in } \Omega \times (0, T), \end{aligned}$$

we thus find $c_7 > 0$ fulfilling

$$\int_0^T \int_{\frac{L}{2}}^L \left\{ \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x^2 \right\}_x^2 dxdt \le c_7 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{7.14}$$

Since $u_{\varepsilon x}(L,t) = 0$ and hence $(u_\varepsilon^{\frac{n+2}{4}})_x(L,t) = 0$ for all $t \in (0,T)$ by (2.10), using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x^2 (x,t) &= -\int_x^L \left\{ \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x^2 \right\}_x (y,t)dy \\ &\le (L-x)^{\frac{1}{2}} \cdot a_\varepsilon(t) \qquad \text{for all } x \in \left(\frac{L}{2}, L\right) \text{ and } t \in (0,T), \end{aligned} \tag{7.15}$$

where

$$a_\varepsilon(t) := \int_{\frac{L}{2}}^L \left\{ \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x^2 \right\}_x^2 (y,t)dy, \qquad t \in (0,T).$$

Again by the Cauchy-Schwarz inequality, (7.15) in turn implies that

$$\begin{aligned} \left| u_\varepsilon^{\frac{n+2}{4}}(L-x,-t) - u_\varepsilon^{\frac{n+2}{4}}(x,t) \right| &= \left| \int_x^L \left(u_\varepsilon^{\frac{n+2}{4}} \right)_x (y,t)dy \right| \\ &\le (L-x)^{\frac{1}{2}} \cdot \left\{ \int_x^L (L-y)^{\frac{1}{2}} \cdot a_\varepsilon^{\frac{1}{2}}(t)dy \right\}^{\frac{1}{2}} \\ &= \sqrt{\frac{2}{3}} (L-x)^{\frac{5}{4}} \cdot a_\varepsilon^{\frac{1}{4}}(t) \qquad \text{for all } x \in \left(\frac{L}{2}, L\right) \text{ and } t \in (0,T), \end{aligned}$$

by (7.14) and the definition of a_ε meaning that

$$\int_0^T \sup_{x\in(\frac{L}{2},L)} \frac{\left| u_\varepsilon^{\frac{n+2}{4}}(L,t) - u_\varepsilon^{\frac{n+2}{4}}(x,t) \right|^4}{(L-x)^5} dt \le \frac{4}{9} c_7.$$

Using (7.12) and Fatou's lemma, from this we conclude that

$$\int_0^T \sup_{x\in(\frac{L}{2},L)} \frac{\left| u^{\frac{n+2}{4}}(L,t) - u^{\frac{n+2}{4}}(x,t) \right|^4}{(L-x)^5} dt \le \frac{4}{9} c_7,$$

so that in particular we can find a null set $N \subset (0,T)$ such that for all $t \in (0,T) \setminus N$,

$$b(t) := \sup_{x\in(\frac{L}{2},L)} \frac{\left| u^{\frac{n+2}{4}}(L,t) - u^{\frac{n+2}{4}}(x,t) \right|^4}{(L-x)^5}$$

is finite.
Now if $t \in (0,T) \setminus N$ is such that $u(L,t) > 0$, then from (7.13) we clearly infer the existence of

$u_x(L,t) = \lim_{\varepsilon=\varepsilon_j \searrow 0} u_{\varepsilon x}(L,t) = 0$. On the other hand, if $t \in (0,T) \setminus N$ is such that $u(L,t) = 0$, then according to the definition of $b(t)$, we obtain

$$u^{n+2}(x,t) \le b(t) \cdot (L-x)^5 \qquad \text{for all } x \in \Big(\frac{L}{2}, L\Big),$$

that is,

$$\begin{aligned}
\left|\frac{u(L,t)-u(x,t)}{L-x}\right| &= \frac{u(x,t)}{L-x} \\
&\le \frac{\Big\{b(t)\cdot(L-x)^5\Big\}^{\frac{1}{n+2}}}{L-x} \\
&= b^{\frac{1}{n+2}}(t)\cdot(L-x)^{\frac{3-n}{n+2}} \qquad \text{for all } x \in \Big(\frac{L}{2}, L\Big),
\end{aligned}$$

so that, since $n<3$, we infer that also in this case $u_x(L,t)$ exists and vanishes.

Step 4. We finally show that u furthermore satisfies the integral identity (1.6).
To prepare this, let us first derive two further estimates from (7.7): Namely, for any $q \in [1,2)$, (7.7) along with Lemma 2.1 and the Hölder inequality implies that for all measurable $\Omega_0 \subset \Omega$ and any measurable $Q \subset \Omega_0 \times (0,T)$, we have

$$\begin{aligned}
\iint_Q \Big|g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx}\Big|^q dxdt &\le \iint_Q (x+\varepsilon)^{q\alpha} u_\varepsilon^{nq} |u_{\varepsilon xxx}|^q dxdt \\
&\le \Big(\iint_Q (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^n u_{\varepsilon xxx}^2 dxdt\Big)^{\frac{q}{2}} \Big(\iint_Q (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} u_\varepsilon^{\frac{nq}{2-q}} dxdt\Big)^{\frac{2-q}{2}} \\
&\le c_1^{\frac{q}{2}} T^{\frac{2-q}{2}} \cdot \Big(\int_{\Omega_0} (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx\Big)^{\frac{2-q}{2}} \cdot \|u_\varepsilon\|_{L^\infty(Q)}^{\frac{nq}{2}},
\end{aligned} \tag{7.16}$$

whereas similarly,

$$\begin{aligned}
\iint_Q \Big|g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2\Big|^q dxdt &\le \iint_Q (x+\varepsilon)^{q\alpha} u_\varepsilon^{(n-1)q} |u_{\varepsilon x}|^{2q} dxdt \\
&\le \Big(\iint_Q (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^{n-2} u_{\varepsilon x}^4 dxdt\Big)^{\frac{q}{2}} \\
&\quad \times \Big(\iint_Q (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} u_\varepsilon^{\frac{nq}{2-q}} dxdt\Big)^{\frac{2-q}{2}} \\
&\le c_1^{\frac{q}{2}} T^{\frac{2-q}{2}} \cdot \Big(\int_{\Omega_0} (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx\Big)^{\frac{2-q}{2}} \cdot \|u_\varepsilon\|_{L^\infty(Q)}^{\frac{nq}{2}}.
\end{aligned} \tag{7.17}$$

In view of our assumptions $\alpha>3$, $\beta>-1$, and $\gamma<1$, we infer that $\alpha+\beta-\gamma+2>3$. Hence, picking any $q\in(1,2)$ we know that

$$c_8 := \sup_{\varepsilon\in(0,1)} \Big(\int_\Omega (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx\Big)^{\frac{2-q}{2}}$$

is finite. Then (7.16) and (7.17), applied to $Q := Q_T := \Omega \times (0,T)$, show that because of $q > 1$, we may pass to a further subsequence to achieve that

$$g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} \rightharpoonup w \qquad \text{in } L^q(\Omega \times (0,T))$$

and

$$g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \rightharpoonup z \qquad \text{in } L^q(\Omega \times (0,T))$$

as $\varepsilon = \varepsilon_j \to 0$ with some w and z belonging to $L^q(\Omega \times (0,T))$. In view of the pointwise convergence properties $u_{\varepsilon x} \to u_x$ and $u_{\varepsilon xxx} \to u_{xxx}$ inside $\mathcal{P}$, as guaranteed by (7.13), we may identify these limits to obtain that actually

$$g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} \rightharpoonup x^\alpha u^n u_{xxx} \qquad \text{in } L^q(\mathcal{P}) \tag{7.18}$$

and

$$g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \rightharpoonup x^\alpha u^{n-1} u_x^2 \qquad \text{in } L^q(\mathcal{P}), \tag{7.19}$$

because $n > 1$.
Next, outside the set $\mathcal{P}$, we may use that $u_\varepsilon \to 0$ uniformly in $Q_T \setminus \mathcal{P}$ to infer upon another application of (7.13) and (7.16) to $q := 1$ that

$$\iint_{Q_T \setminus \mathcal{P}} g_\varepsilon(x) u_\varepsilon^n |u_{\varepsilon xxx}| dx dt \to 0 \tag{7.20}$$

and

$$\iint_{Q_T \setminus \mathcal{P}} g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 dx dt \to 0 \tag{7.21}$$

as $\varepsilon = \varepsilon_j \to 0$, noting here again that $\alpha + \beta - \gamma + 2 > 0$ by assumption.

Now for the verification of (1.6), we fix any $\phi \in C_0^\infty(\bar{\Omega} \times [0,T))$ such that $\phi_x = 0$ at $x = L$. We then approximate ϕ by letting

$$\phi_\delta(x,t) := \phi(0,t) + \int_0^x \zeta_\delta(y) \phi_x(y,t) dy, \qquad (x,t) \in \bar{\Omega} \times [0,T), \tag{7.22}$$

for $\delta \in (0, \frac{L}{2})$, where

$$\zeta_\delta(x) := \zeta\Big(\frac{x}{\delta}\Big), \qquad x \in \bar{\Omega},$$

with a fixed cut-off function $\zeta \in C^\infty\mathbb{R})$ such that $\zeta \equiv 0$ in $(-\infty, 1]$, $\zeta \equiv 1$ in $[2, \infty)$ and $0 \le \zeta' \le 2$ on $\mathbb{R}$. This construction ensures that $\phi_{\delta x}$ vanishes at both $x = L$ and $x = 0$, so that upon multiplying (2.10) by $(x+\varepsilon)^\beta \phi_\delta$, we may integrate by parts, again using Lemma 2.2, to obtain

$$\begin{aligned} -\int_0^T \int_\Omega (x+\varepsilon)^\beta u_\varepsilon \phi_{\delta t} dx dt \quad &- \quad \int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x) \phi_\delta(x,0) dx \\ &= \int_0^T \int_\Omega \Big[- g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} + 2 g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big] \cdot \phi_{\delta xx} dx dt \end{aligned} \tag{7.23}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and all $\delta \in (0, \frac{L}{2})$. Here, from (7.12) and the fact that $u_{0\varepsilon} \to u_0$ in $C^0(\bar{\Omega})$ by (7.2) and the restriction $\gamma < 1$, it is clear that

$$-\int_0^T \int_\Omega (x+\varepsilon)^\beta u_\varepsilon \phi_{\delta t} dxdt \to -\int_0^T \int_\Omega x^\beta u \phi_t dxdt$$

and

$$-\int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x)\phi_\delta(x,0)dx \to -\int_\Omega x^\beta u_0(x)\phi_\delta(x,0)dx$$

as $\varepsilon = \varepsilon_j \to 0$, whereas (7.18)-(7.21) warrant that

$$\int_0^T \int_\Omega \Big[- g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xx} + 2g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big] \cdot \phi_{\delta xx} dxdt \to \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt$$

as $\varepsilon = \varepsilon_j \to 0$; hence, (7.23) yields

$$-\int_0^T \int_\Omega x^\beta u \phi_{\delta t} dxdt - \int_\Omega x^\beta u_0(x)\phi_\delta(x,0)dx = \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt \qquad (7.24)$$

for all $\delta \in (0, \frac{L}{2})$. Now, taking $\delta \searrow 0$, we observe that by (7.22),

$$\phi_{\delta xx}(x,t) = \zeta_\delta(x) \cdot \phi_{xx}(x,t) + \frac{1}{\delta}\zeta'\Big(\frac{x}{\delta}\Big) \cdot \phi_x(x,t) \qquad \text{for all } (x,t) \in \Omega \times (0,T),$$

so that since $0 \le \zeta' \le 2$ we find that

$$\begin{aligned}
&\Big| \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt - \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{xx} dxdt \Big| \\
&\le \quad \|\phi_{xx}\|_{L^\infty(\Omega\times(0,T))} \cdot \Big| \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot (1-\zeta_\delta(x)) dxdt \Big| \\
&\quad + \frac{2}{\delta} \|\phi_x\|_{L^\infty(\Omega\times(0,T))} \cdot \iint_{S_\delta} x^\alpha u^n |u_{xx}| dxdt + \frac{4}{\delta} \|\phi_x\|_{L^\infty(\Omega\times(0,T))} \cdot \iint_{S_\delta} x^\alpha u^{n-1} u_x^2 dxdt \\
&=: \quad I_1(\delta) + I_2(\delta) + I_3(\delta),
\end{aligned}$$

where $S_\delta := ((0,2\delta) \times (0,T)) \cap \mathcal{P}$. Clearly,

$$I_1(\delta) \to 0 \qquad \text{as } \delta \searrow 0$$

by the dominated convergence theorem in conjunction with the integrability property $-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2 \in L^1(\mathcal{P})$ asserted by (7.18) and (7.19). Moreover, applying (7.13) and (7.16) to $Q := ((0,2\delta) \times (0,T)) \cap \mathcal{P}$ and $q := 1$ and once more recalling (7.18) and (7.19), we see that

$$\iint_{S_\delta} x^\alpha u^n |u_{xx}| dxdt \le c_9 \Big(\int_0^{2\delta} x^{\alpha+\beta-\gamma+2} dx \Big)^{\frac{1}{2}} \le c_{10} \delta^{\frac{\alpha+\beta-\gamma+3}{2}}$$

and similarly

$$\iint_{S_\delta} x^\alpha u^{n-1} u_x^2 dxdt \le c_{11} \delta^{\frac{\alpha+\beta-\gamma+3}{2}}$$

for all $\delta \in (0, \frac{L}{2})$ with positive constants c_9, c_{10}, and c_{11}. As our hypotheses $\alpha > 3$, $\beta > -1$, and $\gamma < 1$ guarantee that $\frac{\alpha+\beta-\gamma+3}{2} > 1$, we thus obtain that also

$$I_2(\delta) + I_3(\delta) \to 0 \qquad \text{as } \delta \searrow 0,$$

so that, since clearly $\phi_\delta \to \phi$ and $\phi_{\delta t} \to \phi_t$ uniformly in $\Omega \times (0, t)$, we conclude from (7.24) that indeed (1.6) is valid. $\Box$

We can now prove our main result.

PROOF of Theorem 1.1. According to Lemma 7.1 with $K > 0$ as in (4.1), we know that there exists $T > 0$ and a continuous weak solution u of (1.3) in $\Omega \times (0, T)$, which due to Lemma 4.1 and the approximation statement in Lemma 7.1 has the additional regularity property

$$u \in L^\infty((0, T); W_\gamma^{1,2}(\Omega)) \tag{7.25}$$

and satisfies

$$\int_\Omega x^\gamma u_x^2(x, t) dx \le \int_\Omega x^\gamma u_{0x}^2(x) dx + K \int_0^t \int_\Omega x^{\alpha-\beta+\gamma-6} u^{n+2} dxds \qquad \text{for a.e. } t \in (0, T). \tag{7.26}$$

From Lemma 2.3 combined with Lemma 7.1, we infer that moreover

$$\int_\Omega x^\beta u(x, t) dx = B_0 := \int_\Omega x^\beta u_0(x) dx \qquad \text{for all } t \in (0, T). \tag{7.27}$$

Therefore,

$$T_{max} := \sup \Big\{ T > 0 \ \Big| \ \begin{array}{l} \text{There exists a continuous weak solution } u \text{ of (1.3) in } \Omega \times (0, T) \\ \text{which satisfies (7.25), (7.26) and (7.27)} \end{array} \Big\} \le \infty$$

is well-defined, and it remains to show that (1.7) holds.
Indeed, let us assume on the contrary that $T_{max} < \infty$ but $u \le M$ in $\Omega \times (0, T_{max})$ for some $M > 0$. Then (7.26) would imply that

$$\int_\Omega x^\gamma u_x^2(x, t) dx \le A_0 := \int_\Omega x^\gamma u_{0x}^2(x) dx + K M^{n+2} T_{max} \int_\Omega x^{\alpha-\beta+\gamma-6} dx \quad \text{for a.e. } t \in (0, T_{max}),$$

where our assumption $\gamma > 5 - \alpha + \beta$ ensures that $\alpha - \beta + \gamma - 6 > -1$ and hence $A_0 < \infty$. We could thus pick some $t_0 \in (0, T_{max})$ such that

$$t_0 > T_{max} - \frac{1}{2} T(A_0, B_0) \qquad \text{and} \qquad \int_\Omega x^\gamma u_x^2(x, t_0) dx \le A_0,$$

to see upon another application of Lemma 7.1 to $A := A_0$, $B := B_0$ and

$$v_0(x) := u(x,t_0), \qquad x \in \Omega,$$

that the problem

$$\begin{cases} v_t = \frac{1}{x^\beta} \cdot \Big\{ x^\alpha[-v^n v_{xx} + 2v^{n-1}v_x^2] \Big\}_{xx}, & x \in \Omega,\ t > 0, \\ x^\alpha[-v^n u_{xx} + 2v^{n-1}u_x^2] = x^\alpha[-v^n v_{xx} + 2v^{n-1}v_x^2]_x = 0, & x = 0,\ t > 0, \\ v_x = v_{xxx} = 0, & x = L,\ t > 0, \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

would possess a continuous weak solution v in $\Omega \times (0, T(A_0, B_0))$ which, again by Lemma 4.1, Lemma 2.3, and Lemma 7.1, would satisfy $v \in L^\infty((0,T(A_0,B_0)); W^{1,2}_\gamma(\Omega))$ and

$$\int_\Omega x^\gamma v_x^2(x,t)dx \le \int_\Omega x^\gamma v_{0x}^2(x)dx + K \int_0^t \int_\Omega x^{\alpha-\beta+\gamma-6} v^{n+2} dxds \quad \text{for a.e. } t \in (0, T(A_0,B_0))$$

as well as

$$\int_\Omega x^\beta v(x,t)dx = B_0 \qquad \text{for all } t \in (0, T(A_0,B_0)).$$

It can therefore easily be checked that

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{if } x \in \Omega \text{ and } t \in (0,t_0), \\ v(x,t-t_0) & \text{if } x \in \Omega \text{ and } t \in [t_0, t_0 + T(A_0,B_0)), \end{cases}$$

would define a continuous weak solution $\tilde{u}$ of (1.3) in $\Omega \times (0, t_0 + T(A_0,B_0))$, yet fulfilling (7.25), (7.26), and (7.27). As $t_0 + T(A_0,B_0) > T_{max}$, this contradicts the definition of T_{max}. □

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