WEAK-STRONG UNIQUENESS FOR MAXWELL–STEFAN SYSTEMS

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Abstract. The weak-strong uniqueness for Maxwell–Stefan systems and some generalized systems is proved. The corresponding parabolic cross-diffusion equations are considered in a bounded domain with no-flux boundary conditions. The key points of the proofs are various inequalities for the relative entropy associated to the systems and the analysis of the spectrum of a quadratic form capturing the frictional dissipation. The latter task is complicated by the singular nature of the diffusion matrix. This difficulty is addressed by proving its positive definiteness on a subspace and using the Bott–Duffin matrix inverse. The generalized Maxwell–Stefan systems are shown to cover several known cross-diffusion systems for the description of tumor growth and physical vapor deposition processes.

1. Introduction

The Maxwell–Stefan equations describe the diffusive transport of the components of gaseous mixtures. Applications arise in, e.g., sedimentation, dialysis, electrolysis, and ion exchange [34]. They were suggested in 1866 by James Maxwell [31] for dilute gases and in 1871 by Josef Stefan [33] for fluids. While there are several works on the existence of local-in-time smooth solutions [4, 20, 21] and global-in-time weak solutions [27] in the case of vanishing barycentric velocity, the problem of the uniqueness of solutions is basically unsolved. The uniqueness of strong solutions has been shown in [21, 24], and uniqueness results for weak solutions in a very special case can be found in [10]. In this paper, we make a step forward in the uniqueness problem by showing that strong solutions are unique in the class of weak solutions to Maxwell–Stefan systems.

1.1. **Setting.** We consider an ideal gaseous mixture consisting of n components with volume fractions or concentrations $c_i(x,t)$, $i=1,\ldots,n$. The dynamics of the mixture is given by the mass balance equations and the relations between the driving forces and the fluxes,

$$\partial_t c_i + \operatorname{div}(c_i u_i) = 0, \quad \nabla c_i = -\sum_{j=1}^n \frac{c_i c_j}{D_{ij}}(u_i - u_j), \quad i=1,\ldots,n, \tag{1}$$

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where $u_i(x,t)$ are the partial velocities and $D_{ij} = D_{ji} > 0$ are diffusion coefficients. The equations are solved in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$), supplemented by the initial and no-flux boundary conditions

$$c_i(0) = c_i^0 \quad \text{in } \Omega, \quad \nabla c_i \cdot \nu = 0 \quad \text{on } \partial\Omega,\ t > 0,\ i = 1, \ldots, n, \tag{2}$$

where ν is the exterior unit normal vector to $\partial\Omega$.

We assume that the barycentric velocity vanishes, which implies that the sum of all fluxes vanishes, $\sum_{i=1}^n c_i u_i = 0$. Then, supposing that $c_i^0 \geq 0$ and $\sum_{i=1}^n c_i^0 = 1$ in Ω, we deduce from mass conservation that

$$\sum_{i=1}^n c_i = 1 \quad \text{in } \Omega \text{ for all } t > 0.$$

This constraint is necessary to invert the force-flux relations in (1), i.e. to express the flux $c_i u_i$ as a linear combination of the driving forces ∇c_j.

The global existence analysis for (1)–(2) is based on the property that the system is endowed with the entropy functional

$$H(\boldsymbol{c}) = \sum_{i=1}^n \int_\Omega c_i(\log c_i - 1)dx, \tag{3}$$

where $\boldsymbol{c} = (c_1, \ldots, c_n)$ solves (1)–(2) and satisfies the entropy dissipation inequality [27, (1.14)]

$$\frac{dH}{dt}(\boldsymbol{c}) + C\sum_{i=1}^n \int_\Omega |\nabla\sqrt{c_i}|^2 dx \leq 0, \tag{4}$$

with $C > 0$ depending only on (D_{ij}). The aim of this paper is to prove the weak-strong uniqueness for (1)–(2) and generalized systems. Weak-strong uniqueness means that any weak solution coincides with a strong solution emanating from the same initial data as long as the latter exists. In other words, the strong solutions must be unique within the class of weak solutions. To achieve this aim, we use ideas from our previous work [23] and establish a relative entropy inequality. This leads to a stability estimate for the difference of a weak and a strong solution and eventually to the weak-strong uniqueness property. Here, the relative entropy functional is given by

$$H(\boldsymbol{c}|\bar{\boldsymbol{c}}) = \sum_{i=1}^n \int_\Omega \left(c_i \log\frac{c_i}{\bar{c}_i} - (c_i - \bar{c}_i)\right)dx, \tag{5}$$

where $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ are suitable solutions to (1)–(2).

In the literature, relative entropies are known to be useful to prove the weak-strong uniqueness of solutions. First results were achieved for systems of hyperbolic conservation laws [14] and later for the compressible Navier–Stokes equations [16, 17] and general hyperbolic-parabolic systems endowed with an entropy [13]. The relative entropy technique was applied to, for instance, entropy-dissipating reaction-diffusion equations [18], reaction-cross-diffusion systems [11], energy-reaction-diffusion systems [22], nonlocal cross-diffusion systems [26], and quantum Euler systems [8, 19]. Compared to the results of, e.g. [11, 22], the diffusion matrix in these works is assumed to be positive definite if $c_i > 0$ for all $i = 1, \dots, n$, which is not satisfied for the Maxwell–Stefan system.

1.2. **Definitions and assumptions.** We impose the following assumptions:

(A1) Domain: $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ is a bounded domain.
(A2) Coefficients: $D_{ij} > 0$ and $D_{ij} = D_{ji}$ for all $i, j = 1, \dots, n$, $i \neq j$.
(A3) Initial data: $0 \leq c_i^0 \in L^1(\Omega)$ for $i = 1, \dots, n$, $H(\boldsymbol{c}^0) < \infty$, and $\sum_{i=1}^n c_i^0 = 1$ in Ω.

Next, we define the concept of weak and strong solutions employed in this paper.

We call $\boldsymbol{c} = (c_1, \dots, c_n)$ a *weak solution* to (1)–(2) if $\boldsymbol{c}$ satisfies the initial condition (2), $c_i \geq 0$, $\sum_{i=1}^n c_i = 1$ in $\Omega \times (0, \infty)$,

$$\sqrt{c_i} \in L^2_{\rm loc}(0, \infty; H^1(\Omega)), \quad c_i \in C^0_{\rm loc}([0, \infty); \mathcal{V}'), \quad i = 1, \dots, n,$$

where $\mathcal{V}'$ is the dual space of $\mathcal{V} = \{w \in H^2(\Omega) : \nabla w \cdot \nu = 0 \text{ on } \partial\Omega\}$, and $\boldsymbol{c}$ satisfies (1)–(2) in the weak sense, i.e., for any $\phi_i \in C^1_{\rm loc}([0, \infty); C^1(\overline{\Omega}))$ satisfying $\nabla \phi_i \cdot \nu = 0$ on $\partial\Omega$ and any $T > 0$, $i = 1, \dots, n$, we have

$$\int_\Omega c_i(T)\phi_i(T)dx - \int_\Omega c_i^0 \phi_i(0)dx - \int_0^T \int_\Omega c_i \partial_t \phi_i dxdt - \int_0^T \int_\Omega c_i u_i \cdot \nabla \phi_i dxdt = 0,$$

where u_i satisfies the force-flux relations in (1). The last integral is well defined, since the gradient bound for $\sqrt{c_i}$ implies that $\sqrt{c_i} u_i \in L^2_{\rm loc}(0, \infty; L^2(\Omega))$ (see Lemma 7 below) and thus, because of the property $0 \leq c_i \leq 1$, $c_i u_i \in L^2_{\rm loc}(0, \infty; L^2(\Omega))$. Finally, a weak solution is required to satisfy the entropy inequality

$$H(\boldsymbol{c}(t)) + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \int_\Omega \frac{c_i c_j}{D_{ij}} |u_i - u_j|^2 dxds \leq H(\boldsymbol{c}^0). \tag{6}$$

For the Maxwell-Stefan system this is not an additional requirement as it is guaranteed by the existence theory of [27]; see Section 3.1.

We will use the term *strong solution* to (1)–(2) to mean that $\bar{\boldsymbol{c}} = (\bar{c}_i, \dots, \bar{c}_n)$ with $0 < \bar{c}_i < 1$ is a weak solution satisfying additional regularity properties. The necessary regularity is stated precisely in context. In certain cases, $\bar{c}_i$ satisfies (1)–(2) pointwise, as is the traditional notion of strong solutions.

1.3. **Main results and key ideas of the proofs.** Our first main result is concerned with the Maxwell–Stefan system (1)–(2).

Theorem 1 (Weak-strong uniqueness). *Let Assumptions (A1)–(A2) hold. Let $\boldsymbol{c}$ be a weak solution to* (1)–(2) *and let $\bar{\boldsymbol{c}}$ be a strong solution to* (1)–(2) *satisfying $0 < \bar{c}_i < 1$ in Ω, $t > 0$, the regularity properties*

$$\log \bar{c}_i \in H^1_{\rm loc}(\Omega \times (0, \infty)), \quad \bar{u}_i \in L^\infty_{\rm loc}(\Omega \times (0, \infty)),$$

and $\bar c_i$ does not have anomalous dissipation, i.e., it satisfies the entropy identity

$$H(\bar{\boldsymbol{c}}(t)) + \frac12\int_0^t\int_\Omega \sum_{i,j=1}^n \frac{\bar c_i\bar c_j}{D_{ij}}|\bar u_i-\bar u_j|^2dxds = H(\bar{\boldsymbol{c}}^0) \quad \text{for } t>0.$$

The initial data for $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ satisfy Assumption (A3). Then for any $t>0$, there exists a constant $C(t)>0$, depending on t, Ω, n, and (D_{ij}), such that

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) + \sum_{i=1}^n\int_0^t\int_\Omega c_i|u_i-\bar u_i|^2dxds \le C(t)H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0). \tag{7}$$

If the initial data coincide, i.e. $\boldsymbol{c}^0=\bar{\boldsymbol{c}}^0$ in Ω, then $\boldsymbol{c}(t)=\bar{\boldsymbol{c}}(t)$ in Ω for $t>0$.

We verify in Section 3.1 that solutions with the stated regularity exist. To prove Theorem 1 we develop a relative entropy identity and use it as a yardstick to control the distance between two solutions. First, it is shown that the relative entropy (5) satisfies the inequality

$$\begin{aligned}\frac{dH}{dt}(\boldsymbol{c}|\bar{\boldsymbol{c}}) &+ \frac12\sum_{i,j=1,\,i\neq j}^n\int_\Omega\frac{c_ic_j}{D_{ij}}|(u_i-\bar u_i)-(u_j-\bar u_j)|^2dx \\ &\le -\sum_{i,j=1,\,i\neq j}^n\int_\Omega\frac{c_i}{D_{ij}}(c_j-\bar c_j)(u_i-\bar u_i)\cdot(\bar u_i-\bar u_j)dx.\end{aligned} \tag{8}$$

(see Section 3.2). Next, we study how the frictional dissipation (the second term in (8)) controls the L^2 norm of $u_i-\bar u_i$. The quadratic form in (8) captures the dissipative effect of friction in the following way:

$$\begin{aligned}\frac12\sum_{i,j=1,\,i\neq j}^n\frac{c_ic_j}{D_{ij}}|(u_i-\bar u_i)-(u_j-\bar u_j)|^2 &= \sum_{i=1}^n c_i(u_i-\bar u_i)\cdot\sum_{j=1}^n\frac{1}{D_{ij}}c_j\big((u_i-\bar u_i)-(u_j-\bar u_j)\big) \\ &= \sum_{i,j=1}^n A_{ij}(c)(\sqrt{c_i}(u_i-\bar u_i))\cdot(\sqrt{c_j}(u_j-\bar u_j)) = \boldsymbol{Y}^TA(\boldsymbol{c})\boldsymbol{Y},\end{aligned} \tag{9}$$

where the matrix $A(\boldsymbol{c})=(A_{ij}(\boldsymbol{c}))\in\mathbb{R}^{n\times n}$ is defined by

$$A_{ij}(\boldsymbol{c}) = \begin{cases}\sum_{k=1,\,k\neq i}^n c_k/D_{ik} & \text{if } i=j,\\ -\sqrt{c_ic_j}/D_{ij} & \text{if } i\neq j,\end{cases} \tag{10}$$

and $\boldsymbol{Y}=(Y_1,\ldots,Y_n)$ with $Y_i=\sqrt{c_i}(u_i-\bar u_i)$. The matrix $A(\boldsymbol{c})$ is singular and thus not positive definite. However, we can show that it is positive definite on the subspace $L:=\{\boldsymbol{z}\in\mathbb{R}^n:\sqrt{\boldsymbol{c}}\cdot\boldsymbol{z}=0\}$ (here, $\sqrt{\boldsymbol{c}}$ is the vector with components $\sqrt{c_i}$) and the quadratic form satisfies

$$\boldsymbol{Y}^TA(\boldsymbol{c})\boldsymbol{Y}\ge\mu|P_L\boldsymbol{Y}|^2,$$

where $\mu>0$ is a uniform lower bound for the positive eigenvalues of $A(\boldsymbol{c})$ and P_L is the projection on L. This inequality and a careful estimate of the right-hand side of (8) implies (7) and the weak-strong uniqueness property.

The L^∞ bound on the partial velocities $\bar u_i$ in Theorem 1 can be avoided at the expense of assuming $\nabla\sqrt{c_i}\in L^\infty$ and $\bar c_i$ is uniformly bounded from below by a positive constant. The uniform lower bound is not needed in Theorem 1, where only positivity is required.

Corollary 2. *Let the assumptions of Theorem 1 hold, replacing $\bar u_i\in L^\infty(\Omega\times(0,\infty))$ by $\nabla\sqrt{\bar c_i}\in L^\infty(\Omega\times(0,\infty))$, $i=1,\ldots,n$. Suppose additionally that there exists $m>0$ such that $\bar c_i(t)\ge m$ in Ω, $t>0$, $i=1,\ldots,n$. Then there exist constants $C_1>0$ and $C_2(t)>0$ (depending on t, Ω, n, and (D_{ij})) such that the following inequality holds for $t>0$:*

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t))+C_1\sum_{i=1}^n\int_0^t\int_\Omega|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2dxds\le C_2(t)H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0). \tag{11}$$

The relative entropy inequality (11) is the analogue of the entropy estimate (4). It can be achieved by working with the square roots $\sqrt{c_i}$ as the main variables. More precisely, we write the force-flux relations in (1) as

$$2\nabla\sqrt{c_i}=-\sum_{j=1}^n A_{ij}(\boldsymbol{c})\sqrt{c_j}u_j,\quad i=1,\ldots,n, \tag{12}$$

subject to $\sum_{i=1}^n c_iu_i=0$, where $A(\boldsymbol{c})$ is defined in (10). This system cannot be directly inverted, since $\ker A(\boldsymbol{c})=\operatorname{span}\{\sqrt{\boldsymbol{c}}\}$. However, introducing the Bott–Duffin inverse $A^{BD}(\boldsymbol{c})$ of $A(\boldsymbol{c})$ with respect to $L:=(\operatorname{span}\{\sqrt{\boldsymbol{c}}\})^\perp$ (see Section 2 and Appendix A), we can invert (12), leading to

$$\sqrt{c_i}u_i=-2\sum_{j=1}^n A_{ij}^{BD}(\boldsymbol{c})\nabla\sqrt{c_j},\quad i=1,\ldots,n, \tag{13}$$

and system (1) can be formulated in the concise form

$$\partial_t c_i=2\operatorname{div}\bigg(\sum_{j=1}^n\sqrt{c_i}A_{ij}^{BD}(\boldsymbol{c})\nabla\sqrt{c_j}\bigg),\quad i=1,\ldots,n. \tag{14}$$

The Bott–Duffin inverse $A^{BD}(\boldsymbol{c})$ equals the group inverse studied in [6], since $L=\operatorname{ran}A(\boldsymbol{c})$. Compared to [6], we work here with the square roots $\sqrt{c_i}$ instead of the chemical potentials $\log c_i$ (see [6, (4.25)]). The relative entropy inequality (8) is rewritten in the form (see Lemma 9)

$$\begin{aligned}\frac{dH}{dt}(\boldsymbol{c}|\bar{\boldsymbol{c}})&+4\sum_{i,j=1}^n\int_\Omega A_{ij}^{BD}(\boldsymbol{c})Z_i\cdot Z_jdx\\&\le 4\sum_{i,j=1}^n\int_\Omega Z_i\cdot\nabla\sqrt{\bar c_j}\bigg(\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol{c}})-A_{ij}^{BD}(\boldsymbol{c})\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg)dx,\end{aligned} \tag{15}$$

where $Z_i=\nabla\sqrt{c_i}-\sqrt{c_i/\bar c_i}\nabla\sqrt{\bar c_i}$, $i=1,\ldots,n$. We prove in Lemma 4 that the Bott–Duffin inverse is symmetric and positive definite on L,

$$\boldsymbol{Z}^TA^{BD}(\boldsymbol{c})\boldsymbol{Z}\ge\lambda|P_L\boldsymbol{Z}|^2,\quad \boldsymbol{Z}=(Z_1,\ldots,Z_n),$$

where $\lambda > 0$ is a uniform lower bound for the positive eigenvalues of $A^{BD}(\boldsymbol{c})$. Inequality (11) now follows from this property and suitable estimates for the right-hand side of (15).

The above-mentioned techniques can be extended to a class of generalized Maxwell–Stefan systems, which includes several examples of cross-diffusion systems occurring in applications (see Section 5):

$$\partial_t c_i + \operatorname{div}(c_i u_i) = 0, \quad \sum_{j=1}^n c_j u_j = 0, \tag{16}$$

$$-\sum_{j=1}^n K_{ij}(\boldsymbol{c}) c_j u_j = c_i \nabla \frac{\delta H}{\delta c_i}(\boldsymbol{c}) - c_i \sum_{j=1}^n c_j \nabla \frac{\delta H}{\delta c_j}(\boldsymbol{c}), \quad i = 1, \ldots, n, \tag{17}$$

together with the initial and boundary conditions (2), where $\delta H/\delta c_i$ denotes the variational derivative of H. Again $\sum_{i=1}^n c_i^0 = 1$ implies that $\sum_{i=1}^n c_i(t) = 1$ in Ω, $t > 0$. We assume that

$$H(\boldsymbol{c}) = \sum_{i=1}^n \int_\Omega h_i(c_i) dx,$$

which gives $\delta H/\delta c_i = h_i'$, and $(K_{ij}) \in \mathbb{R}^{n\times n}$ satisfies $\sum_{i=1}^n K_{ij}(\boldsymbol{c}) = 0$ for all $\boldsymbol{c} \in \mathbb{R}_+^n$. This model was proposed in [23] and can be obtained as the high-friction limit of multicomponent Euler systems. It can also be derived from elementary thermodynamic considerations; see Appendix C. If the entropy $H(\boldsymbol{c})$ equals (3) and $K_{ij}(\boldsymbol{c}) = \sqrt{c_i} A_{ij}(\boldsymbol{c})/\sqrt{c_j}$, where $A_{ij}(\boldsymbol{c})$ is defined in (10), then system (16)–(17) reduces to (1). We refer to [5, 12, 30] for multicomponent diffusion models that account for other factors, such as thermal conduction, viscous stresses, chemical reactions, etc.

We introduce the matrix $B(\boldsymbol{c}) = (B_{ij}(\boldsymbol{c})) \in \mathbb{R}^{n\times n}$ by

$$B_{ij}(\boldsymbol{c}) = \frac{1}{\sqrt{c_i}} K_{ij}(\boldsymbol{c}) \sqrt{c_j}, \quad i, j = 1, \ldots, n, \tag{18}$$

and we assume that $B(c)$ is symmetric and as before, we set $L := \{\boldsymbol{z} \in \mathbb{R}^n : \sqrt{\boldsymbol{c}} \cdot \boldsymbol{z} = 0\}$ and $L^\perp = \operatorname{span}\{\sqrt{\boldsymbol{c}}\}$. We write (17) as (see the beginning of Section 4)

$$-\sum_{j=1}^n B_{ij}(\boldsymbol{c}) \sqrt{c_j} u_j = \sum_{j=1}^n (P_L)_{ij} \sqrt{c_j} \nabla h_j'(c_j), \quad i = 1, \ldots, n.$$

We show in Lemma 11 that this system can be inverted, leading to

$$\sqrt{c_i} u_i = -\sum_{j=1}^n B_{ij}^{BD}(\boldsymbol{c}) \sqrt{c_j} \nabla h_j'(c_j),$$

where $B^{BD}(\boldsymbol{c})$ is the Bott–Duffin inverse of $B(\boldsymbol{c})$, and system (16)–(17) can be formulated as

$$\partial_t c_i = \operatorname{div}\left(\sum_{j=1}^n \sqrt{c_i} B_{ij}^{BD}(\boldsymbol{c}) \sqrt{c_j} \nabla \frac{\delta H}{\delta c_j}(\boldsymbol{c}) \right), \quad i = 1, \ldots, n,$$

which, by the way, equals (14) if $H(\boldsymbol{c})$ is given by (3) and $B(\boldsymbol{c}) = A(\boldsymbol{c})$.

We suppose for all $\boldsymbol{c} \in [0,1]^n$ the following conditions on the matrix $B(\boldsymbol{c})$:

(B1) $B(\boldsymbol{c})$ is symmetric and $L = \operatorname{ran} B(\boldsymbol{c})$, $L^\perp = \ker(B(\boldsymbol{c})P_L)$.

(B2) For all $i,j = 1,\ldots,n$ and $s > 0$, $B_{ij}(\boldsymbol{c})$ is bounded and Lipschitz continuous for all $\boldsymbol{c} \in [s,1]^n$.

(B3) There exists a function $\gamma : (0,\infty) \to (0,\infty)$ such that for all $m > 0$ and all $s \geq m$, it holds that $\gamma(s) \leq \gamma(m)$ and $\|B(\boldsymbol{c})\|_F \leq \gamma(\min_{i=1,\ldots,n} c_i)$.

(B4) All nonzero eigenvalues of $B(\boldsymbol{c})$ are not smaller than a positive constant $\mu > 0$.

The partial free energy functions $h_i(c_i)$ are associated to the pressures $p_i(c_i)$ via the thermodynamic relations

$$p_i'(c_i) = c_i h_i''(c_i), \quad p_i(c_i) = c_i h_i'(c_i) - h_i(c_i). \tag{19}$$

For $h_i(c_i)$ and $p_i(c_i)$, we assume that, for some constants K_1, $K_2 > 0$, it holds that

$$h_i \in C^3((0,1]), \quad 0 < c_i h_i''(c_i) \leq K_1, \ |p_i''(c_i)| \leq K_2 h_i''(c_i) \text{ for } c_i \in (0,1] \tag{H}$$

for $i = 1,\ldots,n$. This hypothesis implies that $h_i(c_i)$ is strictly convex, $p_i(c_i)$ is Lipschitz on $(0,1]$. The functions $h_i(c_i) = c_i \log c_i - c_i$ and $h_i(c_i) = c_i^\gamma$, $\gamma > 1$, satisfy (H).

Our final main result is the weak-strong uniqueness property for (16)–(17).

Theorem 3 (Weak-strong uniqueness for the generalized system)**.** *Let Assumptions (A1)–(A3) and (B1)–(B4) hold, and let h_i satisfy Hypothesis (H). Let $\boldsymbol{c}$ be a weak solution and $\bar{\boldsymbol{c}}$ be a strong solution to* (2), (16)–(17). *We suppose that $\bar{\boldsymbol{c}}$ satisfies $\bar{c}_i(t) \geq m$ in Ω, $t > 0$ for some constant $m > 0$,*

$$h_i'(\bar{c}_i) \in H^1_{\mathrm{loc}}(\Omega \times (0,\infty)) \cap L^\infty_{\mathrm{loc}}(0,\infty; W^{2,\infty}(\Omega)), \quad i = 1,\ldots,n,$$

and the entropy identity

$$H(\bar{\boldsymbol{c}}(t)) + \sum_{i,j=1}^n \int_0^t \int_\Omega \sqrt{\bar{c}_i \bar{c}_j} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \nabla h_i'(\bar{c}_i) \cdot \nabla h_j'(\bar{c}_j) dx ds = H(\bar{\boldsymbol{c}}^0) \tag{20}$$

for $t > 0$. Then there exists a constant $C(t) > 0$, depending on t, m, and (D_{ij}), such that

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) \leq C(t) H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) \quad \textit{for } t > 0.$$

If the initial data coincide, then $\boldsymbol{c}(t) = \bar{\boldsymbol{c}}(t)$ in Ω for $t > 0$.

We do not explore the existence of solutions with the stated regularity. The existence of weak solutions to (2), (16)–(17) can be shown by the techniques detailed in [28, 29] under suitable assumptions on K_{ij} and h_i' that guarantee nonlinear gradient estimates. The existence of (local-in-time) strong solutions can be shown by following the approach of [21] by formulating (16)–(17) as

$$\partial_t c_i = \mathcal{M}_i(\boldsymbol{c}) := \operatorname{div}\left(\sum_{j=1}^n M_{ij}(\boldsymbol{c}) \nabla c_j \right), \quad i = 1,\ldots,n,$$

where $M_{ij}(\boldsymbol{c})$ depends on $B^{BD}(\boldsymbol{c})$ and $h_i''(c_i)$, and verifying that the principal part of the operator $\mathcal{M}(\boldsymbol{c}) = (\mathcal{M}_1,\ldots,\mathcal{M}_n)(\boldsymbol{c})$, defined on suitable spaces, is normally elliptic and satisfies the Lopatinski–Shapiro condition. By [15, Theorem 8.2], the operator $\mathcal{M}(\boldsymbol{c})$ has maximal regularity of type L^p and the local existence result follows from [21, Theorem A1].

The strategy of the proof of Theorem 3 is similar to that one of Theorem 1, but it is more involved. First, we show a relative entropy inequality. The terms of this inequality are estimated by splitting the domain into two regions: $c_*(x,t) := \min_{i=1,\ldots,n} c_i(x,t) \le m/2$ and $c_*(x,t) > m/2$, where $m > 0$ is the uniform lower bound for $\bar{c}_i$. The final estimate reads

$$\begin{aligned} \frac{dH}{dt}(\boldsymbol{c}|\bar{\boldsymbol{c}}) &+ \frac{1}{2}\sum_{i,j=1}^n \int_\Omega (1-\chi(\boldsymbol{c}))\sqrt{c_ic_j}B_{ij}^{BD}(\boldsymbol{c})\nabla h_i'(c_i)\cdot\nabla h_j'(c_j)dx \\ &+ C(m)\sum_{i=1}^n \int_\Omega \chi(\boldsymbol{c})|\nabla(c_i-\bar{c}_i)|^2dx \le CH(\boldsymbol{c}|\bar{\boldsymbol{c}}), \end{aligned} \tag{21}$$

where $\chi(\boldsymbol{c})$ a cutoff function that vanishes if $c_i \le m/2$ for some i (see (65) for details). An application of Gronwall's lemma completes the proof. Notice, however, that we do not obtain a gradient estimate as in (11).

By specifying the coefficients $K_{ij}(\boldsymbol{c})$ and the entropy densities h_i, we prove the weak-strong property for cross-diffusion systems describing physical vapor deposition processes [1] and for the tumor-growth model suggested in [25] and analyzed in [27] and the Maxwell-Stefan system considering different molar masses that is derived in [3, 5]; see Section 5.

The main contributions of this work are, first, the derivation of the relative entropy inequality (8) for the Maxwell-Stefan system and (21) for generalized Maxwell-Stefan systems. Second, the introduction of the Bott–Duffin inverse provides an efficient way to reduce the Maxwell–Stefan system to a (degenerate) parabolic system formulated in the square roots $\sqrt{c_i}$. (Related formulations using the chemical potentials $\delta H/\delta c_i$ can be found in [6].) Third, we show that our technique can be extended to more general Maxwell–Stefan systems which may have degeneracy at zero.

The paper is organized as follows. We study the properties of the matrix $A(\boldsymbol{c})$, defined in (10), and its Bott–Duffin inverse $A^{BD}(\boldsymbol{c})$ in Section 2. In Section 3, we recall the existence results for global weak and local strong solutions to (1)–(2), prove the relative entropy inequalities (7) and (8) as well as Theorem 1 and Corollary 2. Section 4 is devoted to the existence of the Bott–Duffin inverse of $B(\boldsymbol{c})$, defined in (18), and the proof of the relative entropy inequality (21) eventually leading to the weak-strong uniqueness Theorem 3. In Section 5, we present some examples that fit into our framework. Finally, we recall the definition and some properties of the Bott–Duffin inverse in Appendix A, show two simple inequalities for the Boltzmann entropy density in Appendix B, and derive the generalized model (16)–(17) from thermodynamic principles in Appendix C.

Notation. We set $\mathbb{R}_+ = [0,\infty)$. Elements of the matrix $A \in \mathbb{R}^{n\times n}$ are denoted by A_{ij}, $i,j = 1,\ldots,n$, and the elements of a vector $\boldsymbol{c} \in \mathbb{R}^n$ are $c_1,\ldots,c_n$. If $f : \mathbb{R} \to \mathbb{R}$ is any function, we define $f(\boldsymbol{c}) = (f(c_1),\ldots,f(c_n))$ for $\boldsymbol{c} \in \mathbb{R}^n$. In the whole paper, $C > 0$, $C_i > 0$ denote generic constants whose values change from line to line.

2. Properties of the matrix $A(\boldsymbol{c})$

The properties of the matrix $A(\boldsymbol{c})$, defined in (10), have been studied in [4, 21, 27] under the assumption $c_i > 0$ for all $i = 1, \ldots, n$. Our results are valid for nonnegative concentrations $c_i \geq 0$, including vacuum.

Let $\boldsymbol{c} \in \mathbb{R}^n_+$. Since (D_{ij}) is symmetric, we have for all $\boldsymbol{z} \in \mathbb{R}^n$,

$$0 = \sum_{i,j=1}^n A_{ij}(\boldsymbol{c}) z_j = \sum_{i,j=1,\, j\neq i}^n \frac{c_j}{D_{ij}} z_i - \sum_{i,j=1,\, j\neq i}^n \frac{\sqrt{c_i c_j}}{D_{ij}} z_j = \sum_{i,j=1,\, j\neq i}^n \frac{\sqrt{c_j}}{D_{ij}} (\sqrt{c_j} z_i - \sqrt{c_i} z_j),$$

showing that $\operatorname{span}\{\sqrt{\boldsymbol{c}}\} = \ker A(\boldsymbol{c})$. We set

$$\operatorname{ran} A(\boldsymbol{c}) = L := \{\boldsymbol{x} \in \mathbb{R}^n : \sqrt{\boldsymbol{c}} \cdot \boldsymbol{x} = 0\},$$
$$\ker A(\boldsymbol{c}) = (\operatorname{ran} A(\boldsymbol{c}))^\perp = L^\perp = \operatorname{span}\{\sqrt{\boldsymbol{c}}\},$$

and note that $\sum_{i=1}^n c_i = 1$ implies that $|\sqrt{\boldsymbol{c}}|^2 = c_1 + \cdots + c_n = 1$. The projection matrices P_L on L and $P_{L^\perp}$ on $L^\perp$ are given by

$$(P_L)_{ij} = \delta_{ij} - \sqrt{c_i c_j}, \quad (P_{L^\perp})_{ij} = \delta_{ij} - (P_L)_{ij} = \sqrt{c_i c_j}, \quad i,j = 1, \ldots, n. \tag{22}$$

Lemma 4. *Let $\boldsymbol{c} \in \mathbb{R}^n_+$ be such that $\sum_{i=1}^n c_i = 1$. Then*

$$\boldsymbol{z}^T A(\boldsymbol{c}) \boldsymbol{z} \geq \mu |P_L \boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{23}$$

where $\mu = \min_{i\neq j}(1/D_{ij})$. Moreover, the Bott–Duffin inverse

$$A^{BD}(\boldsymbol{c}) = P_L (A(\boldsymbol{c}) P_L + P_{L^\perp})^{-1}$$

is well defined, symmetric, and satisfies

$$\boldsymbol{z}^T A^{BD}(\boldsymbol{c}) \boldsymbol{z} \geq \lambda |P_L \boldsymbol{z}|^2 \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^n, \tag{24}$$

where $\lambda = (2 \sum_{i\neq j}(1/D_{ij} + 1))^{-1}$.

Proof. We first prove (23). Let $0 < \alpha \leq \mu$ and suppose that $c_i \neq 0$ for $i = 1, \ldots, M$ and $c_i = 0$ for $i = M+1, \ldots, n$. If necessary, we may rearrange the indices to achieve this ordering. Since $\sum_{i=1}^n c_i = 1$, it holds that $M > 0$. Thus, we can write $-A(\boldsymbol{c}) - \alpha P_{L^\perp}$ in block diagonal form as

$$-A(\boldsymbol{c}) - \alpha P_{L^\perp} = \left(\begin{array}{c|ccc} \widetilde{A} & 0 & 0 & 0 \\ \hline 0 & a_{M+1} & & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & a_n \end{array}\right),$$

where $\widetilde{A} \in \mathbb{R}^{M\times M}$ has the coefficients $\widetilde{A}_{ii} = -A_{ii}(\boldsymbol{c}) - \alpha c_i$ and $\widetilde{A}_{ij} = (1/D_{ij} - \alpha)\sqrt{c_i c_j}$ for $i,j = 1, \ldots, M$, $i \neq j$, and $a_j = -\sum_{k=1,\, k\neq j}^n c_k / D_{kj}$ for $j = M+1, \ldots, n$. Because of $\alpha \leq \mu$, the matrix $\widetilde{A}$ is quasi-positive and irreducible. Hence, by the Perron–Frobenius theorem [32, Chapter 8], the spectral radius of $\widetilde{A}$ is less than or equal to the Perron–Frobenius eigenvalue that is a simple eigenvalue of $\widetilde{A}$ associated with a strictly positive eigenvector, and all other eigenvalues of $\widetilde{A}$ have no positive eigenvector. In the present case, the Perron–Frobenius eigenvalue is given by $\lambda_{\rm PF} = -\alpha$ and is associated with the eigenvector $(\sqrt{c_1},\ldots,\sqrt{c_M})$, recalling that $c_i > 0$ for all $i = 1,\ldots,M$ (also see the proof of Lemma 2.1 in [27]). Since all eigenvalues of $\widetilde{A}$ are not larger than $\lambda_{\rm PF}$, we have

$$\widetilde{\boldsymbol{z}}^T(-\widetilde{A})\widetilde{\boldsymbol{z}} \ge \alpha|\widetilde{\boldsymbol{z}}|^2 \quad \text{for } \widetilde{\boldsymbol{z}} = (z_1,\ldots,z_M) \in \mathbb{R}^M.$$

This leads, for any $\alpha \le \mu$ and $\boldsymbol{z} \in \mathbb{R}^n$, to the inequality

$$\begin{aligned}\boldsymbol{z}^T(A(\boldsymbol{c}) + \alpha P_{L^\perp})\boldsymbol{z} &= \widetilde{z}^T(-\widetilde{A})\widetilde{\boldsymbol{z}} + \sum_{i=M+1}^n \sum_{j=1,\,j\ne i}^n \frac{c_j}{D_{ij}} z_i^2 \\ &\ge \alpha|\widetilde{\boldsymbol{z}}|^2 + \min_{\substack{k,\ell=1,\ldots,M\\ k\ne\ell}} \frac{1}{D_{k\ell}} \sum_{j=1,\,j\ne i}^n c_j \sum_{i=M+1}^n z_i^2 \ge \alpha|\widetilde{\boldsymbol{z}}|^2 + \alpha \sum_{i=M+1}^n z_i^2 = \alpha|\boldsymbol{z}|^2,\end{aligned}$$

where we have used the fact that $\sum_{j=1,\,j\ne i}^n c_j = 1$ for $i = M+1,\ldots,n$, since $c_i = 0$ for exactly these indices. This inequality implies that for all $\boldsymbol{z} = P_L\boldsymbol{z} + P_{L^\perp}\boldsymbol{z} \in \mathbb{R}^n$,

$$\boldsymbol{z}^T A(\boldsymbol{c})\boldsymbol{z} + \alpha|P_{L^\perp}\boldsymbol{z}|^2 = \boldsymbol{z}^T(A(\boldsymbol{c}) + \alpha P_{L^\perp})\boldsymbol{z} \ge \alpha|P_L\boldsymbol{z}|^2 + \alpha|P_{L^\perp}\boldsymbol{z}|^2,$$

which shows (23).

The invertibility of $A(\boldsymbol{c})P_L + P_{L^\perp}$ is a consequence of Lemma 17 in the appendix. Consequently, the Bott–Duffin inverse $A^{BD}(\boldsymbol{c}) = P_L(A(\boldsymbol{c})P_L + P_{L^\perp})^{-1}$ exists.

It remains to show (24). The spectral radius $r(A(\boldsymbol{c})P_L + P_{L^\perp})$ is bounded by the Frobenius norm. Thus, because of $A(\boldsymbol{c})P_L = A(\boldsymbol{c})$ (see Lemma 17 in Appendix A) and $0 \le c_i \le 1$,

$$\begin{aligned} r(A(\boldsymbol{c})P_L + P_{L^\perp}) &\le \|A(\boldsymbol{c}) + P_{L^\perp}\|_F = \bigg(\sum_{i,j=1}^n (A_{ij}(\boldsymbol{c}) + \sqrt{c_ic_j})^2\bigg)^{1/2} \\ &= \bigg\{\sum_{i=1}^n \bigg(\sum_{j=1,\,j\ne i}^n \frac{c_j}{D_{ij}} + c_i\bigg)^2 + \sum_{i,j=1,\,i\ne j}^n \bigg(1 - \frac{1}{D_{ij}}\bigg)^2 c_ic_j\bigg\}^{1/2} \\ &\le 2\sum_{i,j=1,\,i\ne j} \bigg(\frac{1}{D_{ij}} + 1\bigg) = \frac{1}{\lambda}.\end{aligned}$$

We infer that the eigenvalues of $(A(\boldsymbol{c})P_L + P_{L^\perp})^{-1}$ are larger than or equal to λ. Thus, in view of (81), we find that for all $\boldsymbol{z} \in \mathbb{R}^n$,

$$\boldsymbol{z}^T A^{BD}(\boldsymbol{c})\boldsymbol{z} = (P_L\boldsymbol{z})^T(A(\boldsymbol{c})P_L + P_{L^\perp})^{-1}P_L\boldsymbol{z} \ge \lambda|P_L\boldsymbol{z}|^2,$$

finishing the proof. □

Since $(\nabla\sqrt{c_1},\ldots,\nabla\sqrt{c_n}) \in L$, the existence of the Bott–Duffin inverse guarantees that the solution of (12) can be expressed via the formula (13); see Appendix A.

3. Weak-strong uniqueness for Maxwell–Stefan systems

3.1. **Existence theory.** We discuss the existence of weak and strong solutions to the Maxwell–Stefan system (1)–(2). First, we recall the existence theorem for weak solutions, which was proved in [27].

Theorem 5 (Global existence for Maxwell–Stefan systems)**.** *Let Assumptions (A1)–(A3) hold. Then there exists a weak solution to* (1)–(2) *satisfying the entropy inequality* (6) *for* $t > 0$*, or equivalently,*

$$H(\boldsymbol{c}(t)) + 4\sum_{i,j=1}^{n}\int_0^t\int_\Omega A_{ij}^{BD}(\boldsymbol{c})\nabla\sqrt{c_i}\cdot\nabla\sqrt{c_j}dxds \le H(\boldsymbol{c}_0).$$

The existence of strong solutions was proved in [4, Theorem 1] and [21, Theorem 3.2].

Theorem 6 (Strong solutions for Maxwell–Stefan systems)**.** *Let* $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ *be a bounded domain with* $\partial\Omega \in C^2$ *and let* $\boldsymbol{c}^0 \in W^{2-2/p,p}(\Omega;\mathbb{R}^n)$ *with* $c_i^0 \ge 0$, $\sum_{i=1}^n c_i^0 = 1$ *in* Ω*, where* $p > d+2$*. Then there exists* $T^* > 0$ *and a unique solution* $\boldsymbol{c}$ *to* (1)–(2) *satisfying*

$$c_i \in C^1((0,T^*);W^{2-2/p,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$$

for $i = 1,\ldots,n$*.*

The strong solution of Theorem 6 has the property of immediate positivity: If $c_i^0 \ge 0$ in Ω then $c_i(t) > 0$ in Ω for $0 < t < T'$, where $T' \le T$ depends on $\boldsymbol{c}^0$. Moreover, if the initial data is close to a constant vector, the strong solution can be extended globally: Let $\boldsymbol{c}^* \in \mathbb{R}_+^n$. Then there exists $\varepsilon > 0$ such that if the initial data satisfies $\|\boldsymbol{c}^0 - \boldsymbol{c}^*\|_{W^{2-2/p,p}(\Omega)} \le \varepsilon$, then the strong solution exists globally in time.

If $c_i^0 > 0$ in Ω for $i = 1,\ldots,n$, the continuity of the strong solution implies that there exists $0 < T'' \le T^*$ and $m > 0$ such that $c_i(t) \ge m > 0$ in Ω for $i = 1,\ldots,n$. Therefore, because of the embedding $W^{2-2/p,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, we have $\sqrt{c_i} \in L^\infty(0,T'';W^{1,\infty}(\Omega))$, and $T'' = \infty$ if $\|\boldsymbol{c}^0 - \boldsymbol{c}^*\|_{W^{2-2/p,p}(\Omega)}$ is sufficiently small. This shows that the strong solution satisfies the regularity assumptions of Corollary 2.

The assumption $\sqrt{c_i} \in L^\infty_{\rm loc}(0,\infty;W^{1,\infty}(\Omega))$ and the property $c_i(t) \ge m$ in Ω imply that the regularity condition $u_i \in L^\infty_{\rm loc}(\Omega\times(0,\infty))$ of Theorem 1 is satisfied. This is a consequence of the following lemma and $m\sum_{i=1}^n |u_i|^2 \le \sum_{i=1}^n c_i|u_i|^2 \le C\sum_{i=1}^n |\nabla\sqrt{c_i}|^2$. Moreover, the assumption $\log c_i \in H^1_{\rm loc}(\Omega\times(0,\infty))$ follows from $|\nabla\log c_i| \le |\nabla c_i|/m \in C^0((0,T'');C^0(\overline{\Omega}))$.

Lemma 7. *Let* $0 \le c_i \le 1$ *and let* u_i *be given by the force-flux relations in* (1) *satisfying* $\sum_{i=1}^n c_iu_i = 0$*. Then there exists a constant* $C > 0$*, only depending on* (D_{ij}) *such that*

$$\sum_{i=1}^n c_i|u_i|^2 \le C\sum_{i=1}^n |\nabla\sqrt{c_i}|^2.$$

Proof. It follows from (12) and the symmetry of $A(\boldsymbol{c})$, defined in (10), that

$$4\sum_{i=1}^n |\nabla\sqrt{c_i}|^2 = \sum_{i=1}^n\bigg|\sum_{j=1}^n A_{ij}(\boldsymbol{c})\sqrt{c_j}u_j\bigg|^2 = \sum_{i,j=1}^n \sqrt{c_i}u_i(A(\boldsymbol{c})^2)_{ij}\sqrt{c_j}u_j.$$

Since the eigenvalues of $A(\boldsymbol{c})^2$ are the square of the eigenvalues of $A(\boldsymbol{c})$, we deduce from (23) that $\boldsymbol{z}^T A(\boldsymbol{c})^2\boldsymbol{z} \geq \mu^2|P_L\boldsymbol{z}|^2$ for all $\boldsymbol{z}\in\mathbb{R}^n$. This yields

$$4\sum_{i=1}^n|\nabla\sqrt{c_i}|^2 \geq \mu^2|P_L(\sqrt{c_i}u_i)_{i=1,\ldots,n}|^2.$$

Because of $\sum_{i=1}^n\sqrt{c_i}(\sqrt{c_i}u_i)=0$, we have $(\sqrt{c_i}u_i)_i\in L$ and hence, $P_L(\sqrt{c_i}u_i)_i=(\sqrt{c_i}u_i)_i$. The statement of the lemma follows after setting $C=4/\mu^2$. □

3.2. **Relative entropy inequality.** We first derive a relative entropy inequality via a formal computation. Using (5) and (1), we obtain

$$\begin{aligned}\frac{d}{dt}H(\boldsymbol{c}|\bar{\boldsymbol{c}}) &= \sum_{i=1}^n\int_\Omega\bigg(\log\frac{c_i}{\bar{c}_i}\partial_t c_i + \bigg(1-\frac{c_i}{\bar{c}_i}\bigg)\partial_t\bar{c}_i\bigg)dx \\ &= \sum_{i=1}^n\int_\Omega\bigg(\nabla\log\frac{c_i}{\bar{c}_i}\cdot(c_iu_i) - \nabla\log\frac{c_i}{\bar{c}_i}\cdot(c_i\bar{u}_i)\bigg)dx \\ &= \sum_{i=1}^n\int_\Omega c_i\nabla(\log c_i - \log\bar{c}_i)\cdot(u_i-\bar{u}_i)dx. \end{aligned} \tag{25}$$

To reformulate the integrand of the right-hand side, we insert the second equation of (1), and use the symmetry of (D_{ij}):

$$\begin{aligned}\sum_{i=1}^n c_i\nabla(\log c_i - \log\bar{c}_i)\cdot(u_i-\bar{u}_i) &= -\sum_{i=1}^n c_i(u_i-\bar{u}_i)\cdot\sum_{j\neq i}\frac{1}{D_{ij}}\big(c_j(u_i-u_j)-\bar{c}_j(\bar{u}_i-\bar{u}_j)\big) \\ &= -\sum_{i=1}^n c_i(u_i-\bar{u}_i)\cdot\sum_{j\neq i}\frac{c_j}{D_{ij}}\big((u_i-\bar{u}_i)-(u_j-\bar{u}_j)\big) \\ &\quad - \sum_{i,j=1,\,i\neq j}^n\frac{1}{D_{ij}}c_i(u_i-\bar{u}_i)\cdot\big((c_j-\bar{c}_j)(\bar{u}_i-\bar{u}_j)\big) \\ &= -\sum_{i,j=1,\,i\neq j}^n\frac{c_ic_j}{2D_{ij}}\big|(u_i-\bar{u}_i)-(u_j-\bar{u}_j)\big|^2 - \sum_{i,j=1,\,i\neq j}^n\frac{c_i}{D_{ij}}(c_j-\bar{c}_j)(u_i-\bar{u}_i)\cdot(\bar{u}_i-\bar{u}_j). \end{aligned} \tag{26}$$

This shows that

$$\begin{aligned}\frac{d}{dt}H(\boldsymbol{c}|\bar{\boldsymbol{c}}) &+ \frac{1}{2}\sum_{i,j=1,\,i\neq j}^n\int_\Omega\frac{c_ic_j}{D_{ij}}\big|(u_i-\bar{u}_i)-(u_j-\bar{u}_j)\big|^2dx \\ &= -\sum_{i,j=1,\,i\neq j}^n\int_\Omega\frac{c_i}{D_{ij}}(c_j-\bar{c}_j)(u_i-\bar{u}_i)\cdot(\bar{u}_i-\bar{u}_j)dx. \end{aligned} \tag{27}$$

Our aim is to make this computation rigorous. Since the computation in (26) is purely algebraic, it holds without any regularity restrictions. In principle, one would expect that (27) holds under the condition that all the terms are well defined, which would cover the class of weak solutions subject to the condition $u_i \in L^\infty$ (to ensure integrability of the right-hand side). However, we have not been able to establish (25) for such a class of solutions, and stricter conditions on one of the solutions are required.

Lemma 8. *Let* $\boldsymbol{c}$ *be a weak solution to* (1)–(2) *and let* $\bar{\boldsymbol{c}}$ *be a strong solution to* (1)–(2) *satisfying* $0 < \bar{c}_i(t) < 1$ *in* Ω, *the regularity*

$$\log \bar{c}_i \in L^2_{\mathrm{loc}}(0,\infty; H^1(\Omega)), \ \partial_t \log \bar{c}_i \in L^2_{\mathrm{loc}}(\Omega \times (0,\infty)), \ \bar{u}_i \in L^\infty_{\mathrm{loc}}(0,\infty; L^\infty(\Omega)),$$

and the entropy identity

$$H(\bar{\boldsymbol{c}}(t)) + \sum_{i,j=1}^n \frac{1}{2}\int_0^t \int_\Omega \frac{\bar{c}_i\bar{c}_j}{D_{ij}}|u_i - \bar{u}_i|^2 dxds = H(\bar{\boldsymbol{c}}^0) \quad \textit{for } t > 0. \tag{28}$$

Then

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) + \sum_{i,j=1}^n \int_0^t \int_\Omega A_{ij}(\boldsymbol{c})(\sqrt{c_i}(u_i - \bar{u}_i)) \cdot (\sqrt{c_j}(u_j - \bar{u}_j))dxds$$
$$\le H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) - \sum_{i,j=1,\, i\neq j}^n \int_0^t \int_\Omega \frac{c_i}{D_{ij}}(c_j - \bar{c}_j)(u_i - \bar{u}_i)\cdot(\bar{u}_i - \bar{u}_j)dxds. \tag{29}$$

Proof. Since

$$H(\boldsymbol{c}|\bar{\boldsymbol{c}}) = H(\boldsymbol{c}) - H(\bar{\boldsymbol{c}}) - \int_\Omega \sum_{i=1}^n (c_i - \bar{c}_i)\log \bar{c}_i dx,$$

we need to formulate the time evolution of each of these terms. According to Theorem 5, the weak solution $\boldsymbol{c}$ satisfies $\nabla\sqrt{c_i}$, $\sqrt{c_i}u_i \in L^2_{\mathrm{loc}}(0,\infty; L^2(\Omega))$ and

$$H(\boldsymbol{c}(t)) + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \int_\Omega \frac{c_ic_j}{D_{ij}}|u_i - u_j|^2 dxds \le H(\boldsymbol{c}^0) \quad \text{for } t > 0.$$

The symmetry of (D_{ij}) and the force-flux relations in (1) give

$$\sum_{i=1}^n c_iu_i \cdot \nabla \log c_i = -\sum_{i,j=1}^n c_iu_i \cdot \frac{c_j}{D_{ij}}(u_i - u_j) = -\frac{1}{2}\sum_{i,j=1}^n \frac{c_ic_j}{D_{ij}}|u_i - u_j|^2,$$

and we formulate the entropy inequality as

$$H(\boldsymbol{c}(t)) - H(\boldsymbol{c}^0) \le \sum_{i=1}^n \int_0^t \int_\Omega c_iu_i \cdot \nabla \log c_i dxds \quad \text{for } t > 0. \tag{30}$$

The expression $c_iu_i \cdot \nabla \log c_i$ has to be understood as $2\nabla\sqrt{c_i}\cdot(\sqrt{c_i}u_i)$, which is well defined since $\nabla\sqrt{c_i}$, $\sqrt{c_i}u_i \in L^2(\Omega \times (0,T))$ (see Lemma 7). In a similar way, we express the entropy identity (28) as

$$H(\bar{\boldsymbol{c}}(t)) - H(\bar{\boldsymbol{c}}^0) = \sum_{i=1}^n \int_0^t \int_\Omega \bar{c}_i\bar{u}_i \cdot \nabla \log \bar{c}_i dxds \quad \text{for } t > 0. \tag{31}$$

Next, the difference of the weak formulations for $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ gives

$$\int_\Omega (c_i - \bar{c}_i)(t)\phi_i(t)dx - \int_\Omega (c_i^0 - \bar{c}_i^0)\phi_i(0)dx = \int_0^t \int_\Omega (c_i - \bar{c}_i)\partial_t\phi_i dxds + \int_0^t \int_\Omega (c_iu_i - \bar{c}_i\bar{u}_i)\cdot\nabla\phi_i dxds$$

for test functions $\phi_i \in C^1_{\rm loc}([0,\infty); C^1(\overline{\Omega}))$. Using a density argument we see that the test function ϕ_i can be taken in the class $H^1(\Omega\times(0,T))$ for $T>0$, in which case $\phi_i(t)$, $\phi_i(0)$ are well defined by the trace theorem. Selecting $\phi_i = \log\bar{c}_i$, we obtain

$$\int_\Omega (c_i - \bar{c}_i)(t)\log\bar{c}_i(t)dx - \int_\Omega (c_i^0 - \bar{c}_i^0)\log\bar{c}_i^0 dx = \int_0^t \int_\Omega (c_i - \bar{c}_i)\frac{\partial_t\bar{c}_i}{\bar{c}_i}dxds + \int_0^t \int_\Omega (c_iu_i - \bar{c}_i\bar{u}_i)\cdot\nabla\log\bar{c}_i dxds. \tag{32}$$

Taking into account the regularity properties of $\bar{c}_i$, we insert $\partial_t\bar{c}_i = -\operatorname{div}(\bar{c}_i\bar{u}_i)$ in the third term and integrate by parts:

$$\int_0^t \int_\Omega (c_i - \bar{c}_i)\frac{\partial_t\bar{c}_i}{\bar{c}_i}dxds = \int_0^t \int_\Omega \nabla\left(\frac{c_i}{\bar{c}_i}\right)\cdot(\bar{c}_i\bar{u}_i)dxds.$$

We wish to write the integrand on the right-hand side as

$$\nabla\left(\frac{c_i}{\bar{c}_i}\right)\cdot(\bar{c}_i\bar{u}_i) = \left(\nabla c_i - \frac{c_i}{\bar{c}_i}\nabla\bar{c}_i\right)\cdot\bar{u}_i = c_i\nabla\log\left(\frac{c_i}{\bar{c}_i}\right)\cdot\bar{u}_i.$$

Since $c_i \ge 0$ only, the expression $\log c_i$ may be not integrable. Therefore, we define

$$\nabla\log\left(\frac{c_i}{\bar{c}_i}\right) := \frac{1}{\sqrt{c_i}}(2\nabla\sqrt{c_i} - \sqrt{c_i}\nabla\log\bar{c}_i) \quad \text{if } c_i > 0$$

as the product of two functions and $\nabla\log(c_i/\bar{c}_i)$ arbitrary if $c_i = 0$. Although this product may be not integrable, the expression $c_i\nabla\log(c_i/\bar{c}_i)$ lies in $L^2(\Omega\times(0,T))$ and consequently, $c_i\nabla\log(c_i/\bar{c}_i)\cdot\bar{u}_i$ lies in the same space. Therefore, we can formulate (32) as

$$\int_\Omega (c_i - \bar{c}_i)(t)\log\bar{c}_i(t)dx - \int_\Omega (c_i^0 - \bar{c}_i^0)\log\bar{c}_i^0 dx = \int_0^t \int_\Omega c_i\nabla\log\left(\frac{c_i}{\bar{c}_i}\right)\cdot\bar{u}_i dxds + \int_0^t \int_\Omega (c_iu_i - \bar{c}_i\bar{u}_i)\cdot\nabla\log\bar{c}_i dxds. \tag{33}$$

Subtracting (31) and (33) from (30) leads to

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) - H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) \le \sum_{i=1}^n \int_0^t \int_\Omega c_i\nabla\log\left(\frac{c_i}{\bar{c}_i}\right)\cdot(u_i - \bar{u}_i)dxds. \tag{34}$$

Finally, using (26) and the form (9) of the friction, we obtain (29). □

3.3. **Proof of Theorem 1.** We proceed to estimate (29). We set $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ with $Y_i = \sqrt{c_i}(u_i - \bar{u}_i)$, $i = 1, \ldots, n$. Then, using (23), we have

$$\sum_{i,j=1}^{n} A_{ij}(\boldsymbol{c})(\sqrt{c_i}(u_i - \bar{u}_i)) \cdot (\sqrt{c_j}(u_j - \bar{u}_j)) = \boldsymbol{Y}^T A(\boldsymbol{c})\boldsymbol{Y} \geq \mu |P_L \boldsymbol{Y}|^2. \tag{35}$$

It follows from the constraints $\sum_{i=1}^n c_i u_i = \sum_{i=1}^n \bar{c}_i \bar{u}_i = 0$ that

$$\begin{aligned}
(P_{L^\perp}\boldsymbol{Y})_i &= \sum_{j=1}^n \sqrt{c_i} c_j (u_j - \bar{u}_j) = \sqrt{c_i} \sum_{j=1}^n (\bar{c}_j - c_j)\bar{u}_j, \\
|P_L \boldsymbol{Y}|^2 = |\boldsymbol{Y}|^2 - |P_{L^\perp}\boldsymbol{Y}|^2 &= \sum_{i=1}^n c_i |u_i - \bar{u}_i|^2 - \sum_{i=1}^n c_i \bigg| \sum_{j=1}^n (c_j - \bar{c}_j)\bar{u}_j \bigg|^2 \\
&\geq \sum_{i=1}^n c_i |u_i - \bar{u}_i|^2 - n\|\bar{u}\|_{L^\infty} \sum_{j=1}^n (c_j - \bar{c}_j)^2,
\end{aligned}$$

where $\|\bar{u}\|_{L^\infty} := \max_{j=1,\ldots,n} \|u_j\|_{L^\infty(\Omega\times(0,T))}$, and we used $\sum_{i=1}^n c_i = 1$.

We turn to the last term in (29), which is estimated

$$\begin{aligned}
&\bigg| \int_0^t \int_\Omega \sum_{i,j=1,\, i\neq j}^n \frac{c_i}{D_{ij}} (c_j - \bar{c}_j)(u_i - \bar{u}_i) \cdot (\bar{u}_i - \bar{u}_j) dxds \bigg| \\
&\quad \leq \int_0^t \int_\Omega \sum_{i=1}^n \big(\sqrt{c_i}(u_i - \bar{u}_i)\big) \bigg(\sqrt{c_i} \sum_{j=1}^n \frac{|c_j - \bar{c}_j|}{D_{ij}} |\bar{u}_i - \bar{u}_j| \bigg) dxds \\
&\quad \leq \frac{2\|\bar{u}\|_{L^\infty}}{\min_{i\neq j} D_{ij}} \int_0^t \int_\Omega \bigg(\sum_{i=1}^n c_i |u_i - \bar{u}_i|^2 \bigg)^{1/2} \bigg(n \sum_{j=1}^n |c_j - \bar{c}_j|^2 \bigg)^{1/2} dxds \\
&\quad \leq \frac{\mu}{2} \int_0^t \int_\Omega \sum_{i=1}^n c_i |u_i - \bar{u}_i|^2 dxds + C(\mu) \int_0^t \int_\Omega \sum_{i=1}^n (c_i - \bar{c}_i)^2 dxds,
\end{aligned} \tag{36}$$

where the constant $C(\mu) > 0$ also depends on $\min_{i\neq j} D_{ij}$ and $\|\bar{u}\|_{L^\infty}$. Inserting (35)–(36) into (29) and taking into account Lemma 18 in Appendix B, we find that

$$\begin{aligned}
&H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) + \frac{\mu}{2} \sum_{i=1}^n \int_0^t \int_\Omega c_i |u_i - \bar{u}_i|^2 dxds \\
&\quad \leq H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + C(\mu) \int_0^t \int_\Omega \sum_{i=1}^n (c_i - \bar{c}_i)^2 dxds \leq H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + 2C(\mu) \int_0^t H(\boldsymbol{c}|\bar{\boldsymbol{c}}) ds,
\end{aligned}$$

and an application of Gronwall's lemma finishes the proof.

3.4. **Proof of Corollary 2.** In this section, we express the relative entropy via the Bott–Duffin inverse $A^{BD}(\boldsymbol{c})$.

Lemma 9. *Let the assumptions of Lemma 8 hold with $\bar u_i \in L^\infty_{\rm loc}(0,\infty;L^\infty(\Omega))$ replaced by $\sqrt{\bar c_i} \in L^\infty_{\rm loc}(0,\infty;W^{1,\infty}(\Omega))$. Then, setting $\boldsymbol{Z} = (Z_1,\ldots,Z_n)$ with $Z_i = \nabla\sqrt{c_i} - (\sqrt{c_i}/\sqrt{\bar c_i})\nabla\sqrt{\bar c_i}$ for $i=1,\ldots,n$,*

$$H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) + 4\sum_{i,j=1}^n\int_0^t\int_\Omega A_{ij}^{BD}(\boldsymbol{c})Z_i\cdot Z_j dxds \le H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + 4\sum_{i,j=1}^n\int_0^t\int_\Omega Z_i\cdot\nabla\sqrt{\bar c_j}\bigg(\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol{c}}) - A_{ij}^{BD}(\boldsymbol{c})\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg)dxds. \tag{37}$$

Proof. Starting with the relative entropy inequality in the form (34), we express its right-hand side by using (13):

$$\begin{aligned}
&\sum_{i=1}^n c_i\nabla(\log c_i - \log\bar c_i)\cdot(u_i-\bar u_i) \\
&\quad = -\sum_{i=1}^n\bigg(\nabla c_i - \frac{c_i}{\bar c_i}\nabla\bar c_i\bigg)\cdot\sum_{j=1}^n\bigg(\frac{1}{\sqrt{c_i}}A_{ij}^{BD}(\boldsymbol{c})\frac{1}{\sqrt{c_j}}\nabla c_j - \frac{1}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol{c}})\frac{1}{\sqrt{\bar c_j}}\nabla\bar c_j\bigg) \\
&\quad = -4\sum_{i,j=1}^n Z_i\cdot A_{ij}^{BD}(\boldsymbol{c})Z_j - 4\sum_{i,j=1}^n Z_i\cdot\bigg(A_{ij}^{BD}(\boldsymbol{c})\sqrt{\frac{c_j}{\bar c_j}} - A_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\frac{c_i}{\bar c_i}}\bigg)\nabla\sqrt{\bar c_j},
\end{aligned}$$

which gives (37). □

We continue with the proof of Corollary 2. We estimate the two integrals of the relative entropy inequality (37). The integrand of the second term is estimated, because of (24), as

$$\sum_{i,j=1}^n A_{ij}^{BD}(\bar c)Z_i\cdot Z_j \ge \lambda|P_L\boldsymbol{Z}|^2. \tag{38}$$

The definitions of P_L and $\boldsymbol{Z}$ yield

$$\begin{aligned}
(P_L\boldsymbol{Z})_i &= \bigg(\nabla\sqrt{c_i} - \frac{\sqrt{c_i}}{\sqrt{\bar c_i}}\nabla\sqrt{\bar c_i}\bigg) - \sum_{j=1}^n\sqrt{c_ic_j}\bigg(\nabla\sqrt{c_j} - \frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\nabla\sqrt{\bar c_j}\bigg) \\
&= \nabla(\sqrt{c_i}-\sqrt{\bar c_i}) + \frac{\sqrt{\bar c_i}-\sqrt{c_i}}{\sqrt{\bar c_i}}\nabla\sqrt{\bar c_i} - \sqrt{c_i}\sum_{j=1}^n\frac{(\sqrt{\bar c_j})^2-(\sqrt{c_j})^2}{\sqrt{\bar c_j}}\nabla\sqrt{\bar c_j}.
\end{aligned}$$

Using Young's inequality $(A+B+C)^2 \ge A^2/2 - 4B^2 - 4C^2$ and the bounds $\bar c_i \ge m$ and $\sqrt{c_i}+\sqrt{\bar c_i}\le 2$, we infer that

$$\begin{aligned}
|P_L\boldsymbol{Z}|^2 \ge \sum_{i=1}^n\Bigg(&\frac12|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2 - 4\bigg|\frac{\sqrt{\bar c_i}-\sqrt{c_i}}{\sqrt{\bar c_i}}\nabla\sqrt{\bar c_i}\bigg|^2 \\
&- 4\bigg|\sqrt{c_i}\sum_{j=1}^n\frac{(\sqrt{\bar c_j})^2-(\sqrt{c_j})^2}{\sqrt{\bar c_j}}\nabla\sqrt{\bar c_j}\bigg|^2\Bigg)
\end{aligned}$$

$$\geq \frac{1}{2}\sum_{i=1}^n |\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2 - \frac{4(n+1)}{m}\sum_{i=1}^n(\sqrt{c_i}-\sqrt{\bar c_i})^2|\nabla\sqrt{\bar c_i}|^2.$$

With this estimate, (38) becomes, after integration over $\Omega\times(0,t)$,

$$\begin{aligned}(39)\qquad 4\sum_{i,j=1}^n\int_0^t\int_\Omega A_{ij}^{BD}(\bar c)Z_i\cdot Z_j dxds &\geq 2\lambda\sum_{i=1}^n\int_0^t\int_\Omega|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2dxds \\ &- \frac{16(n+1)}{m}\max_{j=1,\ldots,n}\|\nabla\sqrt{\bar c_j}\|^2_{L^\infty(\Omega\times(0,T))}\sum_{i=1}^n\int_0^t\int_\Omega(\sqrt{c_i}-\sqrt{\bar c_i})^2dxds.\end{aligned}$$

Next, we consider the last integral in (37). By Young's inequality,

$$\begin{aligned}(40)\quad 4\sum_{i,j=1}^n\int_0^t\int_\Omega Z_i\cdot\nabla\sqrt{\bar c_j}\bigg(\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol c})-A_{ij}^{BD}(\boldsymbol c)\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg)dxds &\leq \frac{\lambda}{2}\sum_{i=1}^n\int_0^t\int_\Omega|Z_i|^2dxds \\ + C\max_{j=1,\ldots,n}\|\nabla\sqrt{\bar c_j}\|^2_{L^\infty(\Omega\times(0,T))}\sum_{i,j=1}^n\int_0^t\int_\Omega\bigg(\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol c})&-A_{ij}^{BD}(\boldsymbol c)\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg)^2dxds,\end{aligned}$$

and the constant $C>0$ depends on λ and n. The first term on the right-hand side is estimated according to

$$\begin{aligned}(41)\qquad |Z_i|^2 &= \bigg|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})-\frac{\sqrt{c_i}-\sqrt{\bar c_i}}{\sqrt{\bar c_i}}\nabla\sqrt{\bar c_i}\bigg|^2 \\ &\leq 2|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2+\frac{2}{m}|\sqrt{c_i}-\sqrt{\bar c_i}|^2|\nabla\sqrt{\bar c_i}|^2.\end{aligned}$$

To estimate the second term on the right-hand side of (40), we need some preparations. We write

$$A^{BD}(\boldsymbol c) = P_L(A(\boldsymbol c)+P_{L^\perp})^{-1} = \frac{P_L\operatorname{adj}(A(\boldsymbol c)+P_{L^\perp})}{\det(A(\boldsymbol c)+P_{L^\perp})} =: \frac{R(\sqrt{\boldsymbol c})}{S(\sqrt{\boldsymbol c})},$$

where "adj" denotes the adjugate matrix. We know that the elements of $A(\boldsymbol c)$, P_L, and $P_{L^\perp}$ are polynomials of $\sqrt{\boldsymbol c}$. Therefore, $R(\sqrt{\boldsymbol c})$ and $S(\sqrt{\boldsymbol c})$ are also polynomials of $\sqrt{\boldsymbol c}$. Any eigenvalue of $A(\boldsymbol c)$ is also an eigenvalue of $A(\boldsymbol c)+P_{L^\perp}$ (since $L^\perp=\ker A(\boldsymbol c)$). As $A(\boldsymbol c)$ has the eigenvalue 0 with eigenvector $\sqrt{\boldsymbol c}$, $A(\boldsymbol c)+P_{L^\perp}$ has the eigenvalue 1 with the same eigenvector. Moreover, all other eigenvalues of $A(\boldsymbol c)+P_{L^\perp}$ are larger than or equal to μ. Since the determinant of a matrix is the product of its eigenvalues, we conclude that $S(\sqrt{\boldsymbol c})\geq\mu^{n-1}>0$. This shows that $S(\sqrt{\boldsymbol c})$ is uniformly bounded from below. Thus, we can estimate as follows, denoting the elements of the matrix $R(\sqrt{\boldsymbol c})$ by $R_{ij}(\sqrt{\boldsymbol c})$:

$$\begin{aligned}\bigg|\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol c})-A_{ij}^{BD}(\boldsymbol c)\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg| &= \bigg|\frac{\sqrt{c_i}R_{ij}(\sqrt{\bar{\boldsymbol c}})}{\sqrt{\bar c_i}S(\sqrt{\bar{\boldsymbol c}})}-\frac{R_{ij}(\sqrt{\boldsymbol c})\sqrt{c_j}}{S(\sqrt{\boldsymbol c})\sqrt{\bar c_j}}\bigg| \\ &= \frac{1}{S(\sqrt{\boldsymbol c})S(\sqrt{\bar{\boldsymbol c}})\sqrt{\bar c_i\bar c_j}}\big|\big(\sqrt{c_i}S(\sqrt{\boldsymbol c})-\sqrt{\bar c_i}S(\sqrt{\bar{\boldsymbol c}})\big)R_{ij}(\sqrt{\bar{\boldsymbol c}})\sqrt{\bar c_j} \\ &\quad -\big(R_{ij}(\sqrt{\boldsymbol c})\sqrt{c_j}-R_{ij}(\sqrt{\bar{\boldsymbol c}})\sqrt{\bar c_j}\big)\sqrt{\bar c_i}S(\sqrt{\bar{\boldsymbol c}})\big|\end{aligned}$$

$$\leq C(m)\sum_{i=1}^n |\sqrt{c_i}-\sqrt{\bar c_i}|,$$

where $C(m)>0$ depends on the Lipschitz constants of the polynomials $\sqrt{c_i}R_{ij}(\sqrt{\boldsymbol{c}})$ and $\sqrt{c_i}S(\sqrt{\boldsymbol{c}})$. Inserting this estimate into (40), we obtain

$$\begin{aligned}(42)\qquad 4\sum_{i,j=1}^n\int_0^t\int_\Omega Z_i\cdot\nabla\sqrt{\bar c_j}\bigg(\frac{\sqrt{c_i}}{\sqrt{\bar c_i}}A_{ij}^{BD}(\bar{\boldsymbol{c}})-A_{ij}^{BD}(\boldsymbol{c})\frac{\sqrt{c_j}}{\sqrt{\bar c_j}}\bigg)dxds \\ \leq \lambda\sum_{i=1}^n\int_0^t\int_\Omega|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2dxds + C\sum_{i=1}^n\int_0^t\int_\Omega(\sqrt{c_i}-\sqrt{\bar c_i})^2dxds,\end{aligned}$$

where $C>0$ also depends on the L^∞ norm of $\nabla\sqrt{\bar c_i}$ through (41).

Finally, we use estimates (39) and (42) in the relative entropy inequality (37), together with Lemma 18 in Appendix B, to find that

$$\begin{aligned}H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) + \lambda\sum_{i=1}^n\int_0^t\int_\Omega|\nabla(\sqrt{c_i}-\sqrt{\bar c_i})|^2dxds \\ \leq C\sum_{i=1}^n\int_0^t\int_\Omega(\sqrt{c_i}-\sqrt{\bar c_i})^2dxds \leq C\sum_{i=1}^n\int_0^t H(\boldsymbol{c}|\bar{\boldsymbol{c}})ds,\end{aligned}$$

and an application of Gronwall's lemma finishes the proof.

Remark 10 (Nonhomogeneous total mass)**.** The condition $\sum_{i=1}^n c_i^0(x)=1$ for $x\in\Omega$ on the initial total mass can be relaxed to $\sum_{i=1}^n c_i^0(x)=M(x)$ for $x\in\Omega$ and some strictly positive function $M\in L^\infty(\Omega)$. In this situation, the force-flux relations in (1) change to

$$\nabla c_i - \frac{c_i}{\sum_{j=1}^n c_j}\sum_{j=1}^n\nabla c_j = -\sum_{j=1,\,j\neq i}^n\frac{c_ic_j}{D_{ij}}(u_i-u_j),\quad i=1,\ldots,n.$$

Notice that the total mass $\sum_{j=1}^n c_j = M$ is preserved in time. The previous equation can be expressed in terms of the matrix $A(\boldsymbol{c})$, defined in (10), by

$$\sum_{j=1}^n(P_L)_{ij}\nabla\sqrt{c_j} = -\sum_{j=1}^n A_{ij}(\boldsymbol{c})\sqrt{c_i}u_j,$$

where the projection matrix P_L is now given by $(P_L)_{ij}=\delta_{ij}-\sqrt{c_ic_j}/M(x)$, $i,j=1,\ldots,n$. Lemma 4 still holds in this situation with $\alpha\leq\inf_{x\in\Omega}M(x)\min_{i\neq j}(1/D_{ij})$. The relative entropy inequalities (29) and (37) do not depend on the assumption $\sum_{i=1}^n c_i=1$ such that the relative entropy inequalities in Theorem 1 and Corollary 2 still hold but with constants depending on M. □

4. Weak-strong uniqueness for generalized Maxwell–Stefan systems

We consider the generalized Maxwell–Stefan system (16)–(17). First, we rewrite (17) in terms of the Bott–Duffin inverse of $B(\boldsymbol{c})$. To this end, we recall the definition of $(P_L)_{ij} = \delta_{ij} - \sqrt{c_ic_j}$ and rewrite the right-hand side of (17),

$$\begin{aligned} c_i\nabla h_i'(c_i) - c_i\sum_{j=1}^n c_j\nabla h_j'(c_j) &= \sqrt{c_i}\bigg(\sqrt{c_i}\nabla h_i'(c_i) - \sum_{j=1}^n \sqrt{c_ic_j}\sqrt{c_j}\nabla h_j'(c_j)\bigg) \\ &= \sqrt{c_i}\sum_{j=1}^n (P_L)_{ij}\sqrt{c_j}\nabla h_j'(c_j), \end{aligned}$$

as well as the left-hand side of (17), using definition (18) of $B(\boldsymbol{c})$,

$$-\sum_{j=1}^n K_{ij}(\boldsymbol{c})c_ju_j = -\sqrt{c_i}\sum_{j=1}^n B_{ij}(\boldsymbol{c})\sqrt{c_j}u_j, \quad i=1,\ldots,n,$$

showing that (17) is equivalent to

$$-\sum_{j=1}^n B_{ij}(\boldsymbol{c})\sqrt{c_j}u_j = \sum_{j=1}^n (P_L)_{ij}\sqrt{c_j}\nabla h_j'(c_j), \quad i=1,\ldots,n.$$

We prove in Lemma 11 below that the Bott–Duffin inverse $B^{BD}(\boldsymbol{c})$ of $B(\boldsymbol{c})$ exists. Thus, we can invert the previous system:

$$\sqrt{c_i}u_i = -\sum_{j=1}^n (B^{BD}(\boldsymbol{c})P_L)_{ij}\sqrt{c_j}\nabla h_j'(c_j) = -\sum_{j=1}^n B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j}\nabla h_j'(c_j),$$

where we have used the relation $B^{BD}(\boldsymbol{c})P_L = B^{BD}(\boldsymbol{c})$ (see (81) in Appendix A). This equation generalizes (13). We conclude that system (16)–(17) can be written as

$$\partial_t c_i = \operatorname{div}\bigg(\sum_{j=1}^n \sqrt{c_i}B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j}\nabla h_j'(c_j)\bigg), \quad i=1,\ldots,n. \tag{43}$$

4.1. Properties of the matrix $B(\boldsymbol{c})$.

We prove the following lemma.

Lemma 11. *Let Assumptions (B1)–(B4) hold for $B(\boldsymbol{c})$, defined in* (18)*. Then the Bott–Duffin inverse $B^{BD}(\boldsymbol{c}) = P_L(B(\boldsymbol{c})P_L + P_{L^\perp})^{-1}$ of $B(\boldsymbol{c})$ exists, is symmetric and satisfies the following properties:*

- *Let $s>0$. Then the elements $B_{ij}^{BD}(\boldsymbol{c})$ are bounded and Lipschitz continuous for all $\boldsymbol{c}\in[s,1]^n$.*
- *Let $m>0$. Then there exists $\lambda(m)>0$ such that for all $\boldsymbol{z}\in\mathbb{R}^n$ and $\boldsymbol{c}\in[m,1]^n$,*

$$\boldsymbol{z}^TB^{BD}(\boldsymbol{c})\boldsymbol{z} \ge \lambda(m)|P_L\boldsymbol{z}|^2. \tag{44}$$

- *The matrix $B^{BD}(\mathbf{c})$ satisfies for all $\boldsymbol{z}\in\mathbb{R}^n$ and $\boldsymbol{c}\in[0,1]^n$,*

$$\boldsymbol{z}^TB^{BD}(\boldsymbol{c})\boldsymbol{z} \le \frac{1}{\mu}|\boldsymbol{z}|^2, \tag{45}$$

recalling that $\mu > 0$ is a lower bound for the nonzero eigenvalues of $B(\boldsymbol{c})$; see Assumption (B4).

Proof. Assumption (B1) and Lemma 16 (ii) in Appendix A imply that

$$\ker(B(\boldsymbol{c})P_L + P_{L^\perp}) = \ker(B(\boldsymbol{c})P_L) \cap L = L^\perp \cap L = \{0\}.$$

Hence, $B(\boldsymbol{c})P_L + P_{L^\perp}$ is invertible and the Bott–Duffin inverse is well defined and symmetric.

We continue by studying the eigenvalues of $B(\boldsymbol{c})P_L + P_{L^\perp}$. A computation shows that for $\sqrt{\boldsymbol{c}} \in \ker(B(\boldsymbol{c})P_L) = L^\perp$ we have $(B(\boldsymbol{c})P_L + P_{L^\perp})\sqrt{\boldsymbol{c}} = P_{L^\perp}\sqrt{\boldsymbol{c}} = \sqrt{\boldsymbol{c}}$, i.e., $\sqrt{\boldsymbol{c}}$ is an eigenvector of $B(\boldsymbol{c})P_L + P_{L^\perp}$ with eigenvalue 1. Let $\xi \notin L^\perp$, $\xi \neq 0$, be another eigenvector,

$$(B(\boldsymbol{c})P_L + P_{L^\perp})\xi = \rho\xi.$$

Then $\rho \neq 0$. Applying P_L on both sides, we obtain $P_L B(\boldsymbol{c})P_L\xi = B(\boldsymbol{c})(P_L\xi) = \rho P_L\xi$, i.e., $P_L\xi \neq 0$ is an eigenvector of $B(\boldsymbol{c})$ with eigenvalue ρ. Due to Assumption (B4), we conclude that $\rho \geq \mu > 0$.

We claim that the elements $B^{BD}_{ij}(\boldsymbol{c})$ are bounded and Lipschitz continuous for all $\boldsymbol{c} \in [s,1]^n$. Indeed, observe that

$$B^{BD}(\boldsymbol{c}) = P_L(B(\boldsymbol{c})P_L + P_{L^\perp})^{-1} = \frac{P_L \operatorname{adj}(B(\boldsymbol{c})P_L + P_{L^\perp})}{\det(B(\boldsymbol{c})P_L + P_{L^\perp})}, \tag{46}$$

where "adj" denotes the adjugate. Since the determinant of a matrix is the product of its eigenvalues, $\det(B(\boldsymbol{c})P_L + P_{L^\perp}) \geq \mu^{n-1} > 0$ and the denominator in (46) is bounded from below. Assumption (B2) implies that the elements of $B(\boldsymbol{c})$ are bounded. Hence, all elements of $\operatorname{adj}(B(\boldsymbol{c})P_L + P_{L^\perp})$ are bounded too. We conclude from (46) that the elements of $B^{BD}(\boldsymbol{c})$ are bounded. Since the product of Lipschitz continuous functions is Lipschitz continuous, Assumption (B2) further implies that the elements of $B^{BD}(\boldsymbol{c})$ are Lipschitz continuous for all $\boldsymbol{c} \in [s,1]^n$ for any $s > 0$.

We wish to verify (44). Since the spectral radius r of a matrix is bounded by its Frobenius norm $\|\cdot\|_F$ and the Frobenius norm is submultiplicative, we have

$$\begin{aligned} r(B(\boldsymbol{c})P_L + P_{L^\perp}) &\leq \|B(\boldsymbol{c})P_L + P_{L^\perp}\|_F \leq \|B(\boldsymbol{c})P_L\|_F + \|P_{L^\perp}\|_F \\ &\leq \|B(\boldsymbol{c})\|_F\|P_L\|_F + \|P_{L^\perp}\|_F. \end{aligned}$$

The Frobenius norms of P_L and $P_{L^\perp}$ are estimated according to

$$\begin{aligned} \|P_L\|_F^2 &= \sum_{i,j=1}^n (\delta_{ij} - \sqrt{c_ic_j})^2 \leq \sum_{i,j=1}^n 1 = n^2, \\ \|P_{L^\perp}\|_F^2 &= \sum_{i,j=1}^n (\sqrt{c_ic_j})^2 = \bigg(\sum_{i=1}^n c_i\bigg)^2 = 1. \end{aligned}$$

Assumption (B3) guarantees that $\|B(\boldsymbol{c})\|_F \leq \gamma(m)$. This shows that $r(B(\boldsymbol{c})P_L + P_{L^\perp}) \leq \gamma(m)n + 1$. We infer that the smallest eigenvalue of $(B(\boldsymbol{c})P_L + P_{L^\perp})^{-1}$ is larger than $\lambda(m) := 1/(\gamma(m)n + 1)$ proving (44).

It remains to prove (45). First, we show that the nonzero eigenvalues of $B(\boldsymbol{c})$ and $B^{BD}(\boldsymbol{c})$ are reciprocal to each other. Let $\ell \in \mathbb{R}$ be a nonzero eigenvalue of $B^{BD}(\boldsymbol{c})$. Then the corresponding eigenvector $\boldsymbol{y} \in L$ satisfies $B^{BD}(\boldsymbol{c})\boldsymbol{y} = \ell\boldsymbol{y}$, which is $P_L(B(\boldsymbol{c})P_L + P_{L^\perp})^{-1}\boldsymbol{y} = \ell\boldsymbol{y}$. Hence, $\boldsymbol{z} := (B(\boldsymbol{c})P_L + P_{L^\perp})^{-1}\boldsymbol{y}$ satisfies $P_L\boldsymbol{z} = \ell(B(\boldsymbol{c})P_L + P_{L^\perp})\boldsymbol{z}$. Applying P_L on both sides yields $P_L\boldsymbol{z} = \ell P_L B(\boldsymbol{c})P_L\boldsymbol{z} = \ell B(\boldsymbol{c})P_L\boldsymbol{z}$. Thus, $P_L\boldsymbol{z}$ is an eigenvector of $B(\mathbf{c})$ with eigenvalue $1/\ell$. Similarly, we can reverse the above argument and verify that if $\boldsymbol{z}$ is a nonzero eigenvector of $B(\boldsymbol{c})$ with eigenvalue ℓ, then $(B(\boldsymbol{c})P_L + P_{L^\perp})\boldsymbol{z}$ is an eigenvector of $B^{BD}(\boldsymbol{c})$ with eigenvalue $1/\ell$. We conclude that the largest eigenvalue of $B^{BD}(\boldsymbol{c})$ is the reciprocal of the smallest eigenvalue of $B(\boldsymbol{c})$, and Assumption (B4) implies that $\boldsymbol{z}^T B^{BD}(\boldsymbol{c})\boldsymbol{z} \le |\boldsymbol{z}|^2/\mu$ for all $\boldsymbol{z} \in \mathbb{R}^n$. □

4.2. **Weak and strong solutions.** We call $\boldsymbol{c}$ a *weak solution* to (2), (16)–(17) if

$$c_i \in C^0_{\rm loc}([0,\infty);\mathcal{V}') \cap L^2_{\rm loc}(0,\infty;H^1(\Omega)), \quad i = 1,\ldots,n,$$

where $\mathcal{V}'$ is the dual space of $\mathcal{V} = \{w \in H^2(\Omega) : \nabla w \cdot \nu = 0 \text{ on } \partial\Omega\}$, it holds for any test function $\phi_i \in C^1_{\rm loc}([0,\infty);C^1(\overline{\Omega}))$ with $\nabla\phi_i \cdot \nu = 0$ on $\partial\Omega$ and all $t > 0$ that

$$\begin{aligned}\int_\Omega c_i(t)\phi_i(t)dx - \int_\Omega c_i^0\phi_i(0)dx - \int_0^t\int_\Omega c_i\partial_t\phi_i dxds \\ + \sum_{j=1}^n\int_0^t\int_\Omega \sqrt{c_i}B^{BD}_{ij}(\boldsymbol{c})\sqrt{c_j}\nabla h_j'(c_j)\cdot\nabla\phi_i dxds = 0, \quad i = 1,\ldots,n,\end{aligned}$$

and the entropy dissipation inequality

$$H(\boldsymbol{c}(t)) + \sum_{i,j=1}^n\int_0^t\int_\Omega \sqrt{c_ic_j}B^{BD}_{ij}(\boldsymbol{c})\nabla h_i'(c_i)\cdot\nabla h_j'(c_j)dxds \le H(\boldsymbol{c}^0) \tag{47}$$

is satisfied for $t > 0$.

Furthermore, we call $\bar{\boldsymbol{c}}$ a strong solution to (2), (16)–(17) if $c_i \in C^1_{\rm loc}([0,\infty);C^1(\overline{\Omega}))$ for $i = 1,\ldots,n$, if (2), (16)–(17) are satisfied pointwise, and the entropy identity (20) is fulfilled.

4.3. **Relative entropy inequality.** The partial free energies $h_i(c_i)$ and pressures $p_i(c_i)$ are associated through (19). Define the associated relative free energy density and relative pressure via

$$\begin{aligned}h_i(c_i|\bar{c}_i) &:= h_i(c_i) - h_i(\bar{c}_i) - h_i'(\bar{c}_i)(c_i - \bar{c}_i), \\ p_i(c_i|\bar{c}_i) &:= p_i(c_i) - p_i(\bar{c}_i) - p_i'(\bar{c}_i)(c_i - \bar{c}_i).\end{aligned} \tag{48}$$

We prove a relative entropy inequality associated to the generalized Maxwell–Stefan system.

Lemma 12. *Let* $\boldsymbol{c}$ *be a weak solution to* (2), (16)–(17) *and let* $\bar{\boldsymbol{c}}$ *be a strong solution to* (2), (16)–(17) *satisfying*

$$h_i'(\bar{c}_i) \in L^2_{\rm loc}(0,\infty;H^2(\Omega)),\ h_i''(\bar{c}_i) \in L^\infty_{\rm loc}(0,\infty;L^\infty(\Omega)),\ \partial_t\bar{c}_i \in L^2_{\rm loc}(0,\infty;L^2(\Omega)).$$

Then the following relative entropy inequality holds:

$$\begin{aligned} H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) - H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + \sum_{i,j=1}^n \int_0^t \int_\Omega B_{ij}^{BD}(\boldsymbol{c}) Y_i \cdot Y_j dxds \\ \le - \sum_{i,j=1}^n \int_0^t \int_\Omega \left(B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} - \frac{\sqrt{c_i} B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \right) Y_j \cdot \nabla h_j'(\bar{c}_j) dxds \\ + \sum_{i=1}^n \int_0^t \int_\Omega p_i(c_i|\bar{c}_i) \operatorname{div} \left(\sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) \right) dxds, \end{aligned} \tag{49}$$

where

$$Y_i = \sqrt{c_i} \nabla (h_i'(c_i) - h_i'(\bar{c}_i)), \quad i = 1, \ldots, n. \tag{50}$$

Note that the definition for Y_i differs from that one used in Section 3.

Proof. We proceed as in the proof of Lemma 9, but re-arrange the terms in a different fashion. The difference $c_i - \bar{c}_i$ satisfies the weak formulation

$$\begin{aligned} 0 = \int_\Omega (c_i - \bar{c}_i)(t)\phi_i(t)dx - \int_\Omega (c_i^0 - \bar{c}_i^0)\phi_i(0)dx - \int_0^t \int_\Omega (c_i - \bar{c}_i)\partial_t \phi_i dxds \\ + \sum_{j=1}^n \int_0^t \int_\Omega \left(\sqrt{c_i} B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} \nabla h_j'(c_j) - \sqrt{\bar{c}_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \sqrt{\bar{c}_j} \nabla h_j'(\bar{c}_j) \right) \cdot \nabla \phi_i dxds \end{aligned}$$

for $i = 1, \ldots, n$. We wish to use $\phi_i = h_i'(\bar{c}_i)$ as a test function. Strictly speaking, this is not possible, but, as in the proof of Lemma 8, we can use a density argument. Then, using (43) for the third term and adding over $i = 1, \ldots, n$, we obtain

$$\begin{aligned} 0 = \sum_{i=1}^n \int_\Omega (c_i - \bar{c}_i)(t) h_i'(\bar{c}_i(t)) dx - \sum_{i=1}^n \int_\Omega (c_i^0 - \bar{c}_i^0) h_i'(\bar{c}_{i0}) dx \\ - \sum_{i=1}^n \int_0^t \int_\Omega (c_i - \bar{c}_i) h_i''(c_i) \operatorname{div} \left(\sum_{j=1}^n \sqrt{\bar{c}_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \sqrt{\bar{c}_j} \nabla h_j'(\bar{c}_j) \right) dxds \\ + \sum_{i,j=1}^n \int_0^t \int_\Omega \left(\sqrt{c_i} B_{ij}^{BD}(\boldsymbol{c}) \sqrt{c_j} \nabla h_j'(c_j) - \sqrt{\bar{c}_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \sqrt{\bar{c}_j} \nabla h_j'(\bar{c}_j) \right) \cdot \nabla h_i'(\bar{c}_i) dxds. \end{aligned} \tag{51}$$

Subtracting (51) and the entropy identity (20) for $\bar{\boldsymbol{c}}$ from the entropy inequality (47) for $\boldsymbol{c}$, we find that

$$\begin{aligned} H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) \le H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) - \sum_{i,j=1}^n \int_0^t \int_\Omega B_{ij}^{BD}(\boldsymbol{c}) \sqrt{c_i c_j} \nabla (h_i'(c_i) - h_i'(\bar{c}_i)) \cdot \nabla h_j'(c_j) dxds \\ - \sum_{i=1}^n \int_0^t \int_\Omega (c_i - \bar{c}_i) h_i''(\bar{c}_i) \operatorname{div} \left(\sum_{j=1}^n \sqrt{\bar{c}_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \sqrt{\bar{c}_j} \nabla h_j'(\bar{c}_j) \right) dxds. \end{aligned} \tag{52}$$

In turn, using (50), this is rewritten as

$$
\begin{aligned}
&H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) - H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + \sum_{i,j=1}^n \int_0^t \int_\Omega B_{ij}^{BD}(\boldsymbol{c}) Y_i \cdot Y_j dxds \\
&\quad \le - \sum_{i,j=1}^n \int_0^t \int_\Omega \left(B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} - \sqrt{c_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \right) Y_i \cdot \nabla h_j'(\bar{c}_j) dxds \\
&\qquad - \sum_{i,j=1}^n \int_0^t \int_\Omega \sqrt{c_i} B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} Y_i \cdot \nabla h_j'(\bar{c}_j) dxds \\
&\qquad - \int_0^t \int_\Omega \sum_{i=1}^n (c_i - \bar{c}_i) \bar{c}_i h_i''(\bar{c}_i) \operatorname{div} \left(\sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) \right) dxds \\
&\qquad - \int_0^t \int_\Omega \sum_{i=1}^n (c_i - \bar{c}_i) h_i''(\bar{c}_i) \nabla \bar{c}_i \cdot \sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) dxds \\
&\quad =: J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

The sum $J_2 + J_4$ becomes

$$
\begin{aligned}
J_2 + J_4 &= - \int_0^t \int_\Omega \sum_{i=1}^n \big(c_i \nabla h_i'(c_i) - \bar{c}_i \nabla h_i'(\bar{c}_i) \big) \cdot \sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) dxds \\
&= \int_0^t \int_\Omega \sum_{i=1}^n (p_i(c_i) - p_i(\bar{c}_i)) \operatorname{div} \left(\sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) \right) dxds.
\end{aligned}
$$

Combining this expression with J_3 and using definition (48) finally leads to (49). □

4.4. **The entropy dissipation structure.** We state an auxiliary lemma that provides some control of the entropy inequality (47) and the relative entropy inequality (49).

Lemma 13. *Let $\boldsymbol{c}$ be a weak solution and $\bar{\boldsymbol{c}}$ be a strong solution to* (2), (16)–(17), *satisfying the hypotheses of Lemma 12 and $\bar{c}_i(t) \ge m$ in Ω, $t > 0$ for some constant $m > 0$.*

(i) *Assume that $c_i \ge m/2$ for all $i = 1 \ldots, n$ and let $Z_i = \sqrt{c_i}\nabla h_i'(c_i)$. Then, for some constant $\beta(m) > 0$, we have*

$$
\sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c}) Z_i \cdot Z_j \ge 2\beta(m) \sum_{i=1}^n |\nabla c_i|^2. \tag{53}
$$

(ii) *Assume that $c_i \ge m/2$ for all $i = 1, \ldots, n$ and let $Y_i = \sqrt{c_i}\nabla(h_i'(c_i) - h_i'(\bar{c}_i))$. Then, for some $\beta(m) > 0$ and $C > 0$, we have*

$$
\sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c}) Y_i \cdot Y_j \ge \beta(m) \sum_{i=1}^n |\nabla c_i - \nabla \bar{c}_i|^2 - C \sum_{i=1}^n |c_i - \bar{c}_i|^2. \tag{54}
$$

(iii) *Let the weak solution* $\mathbf{c}$ *satisfy* (47) *and set*

$$c_*(x,t) := \min_{i=1,\ldots,n} c_i(x,t). \tag{55}$$

Then

$$H(\boldsymbol{c}(t)) + 2\beta(m)\int_0^t\int_\Omega 1_{\{c_*>m/2\}}\sum_{i=1}^n|\nabla c_i|^2dxds \le H(\boldsymbol{c}^0). \tag{56}$$

Note that (56) provides a partial control of the gradients, which however might degenerate as m tends to zero.

Proof. Proof of (i). Inequality (44) in Lemma 11 implies that

$$\begin{aligned}\sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c})Z_i\cdot Z_j &\ge \lambda(m/2)|P_L\boldsymbol{Z}|^2 = \lambda(m/2)\boldsymbol{Z}^TP_L^TP_L\boldsymbol{Z} \\ &= \lambda(m/2)\sum_{i,j=1}^n(P_L)_{ij}Z_i\cdot Z_j.\end{aligned} \tag{57}$$

Before we can estimate the right-hand side, we need some preparations.

We define the vector $\widetilde{\boldsymbol{c}} := (c_1,\ldots,c_{n-1})$ without the last component and define the entropy density in $n-1$ variables according to

$$\widetilde{h}(\widetilde{\boldsymbol{c}}) = \sum_{i=1}^{n-1}h_i(c_i) + h_n\bigg(1-\sum_{j=1}^{n-1}c_j\bigg).$$

Its partial derivative is given by

$$\widetilde{h}_i'(\widetilde{\boldsymbol{c}}) := \frac{\partial\widetilde{h}(\widetilde{\boldsymbol{c}})}{\partial c_i} = h_i'(c_i) - h_n'\bigg(1-\sum_{j=1}^{n-1}c_j\bigg), \quad i=1,\ldots,n-1.$$

Next, introduce the matrix $E(\boldsymbol{c})$ with elements

$$E_{ij}(\boldsymbol{c}) = (P_L)_{ij}\sqrt{c_ic_j} = c_i\delta_{ij} - c_ic_j = \begin{cases} c_i - c_i^2 & \text{if } i=j. \\ -c_ic_j & \text{if } i\neq j.\end{cases}$$

The sum of its rows and columns vanishes, $\sum_{j=1}^n E_{ij}(\boldsymbol{c}) = \sum_{i=1}^n E_{ij}(\boldsymbol{c}) = 0$. We deduce from the symmetry of $E(\boldsymbol{c})$ that for all $z_i\in\mathbb{R}^d$,

$$\begin{aligned}\sum_{i,j=1}^n E_{ij}(\boldsymbol{c})z_i\cdot z_j &= \sum_{i=1}^n z_i\cdot\bigg(\sum_{j=1}^{n-1}E_{ij}(\boldsymbol{c})z_j + E_{in}(\boldsymbol{c})z_n\bigg) = \sum_{i=1}^n z_i\cdot\sum_{j=1}^{n-1}E_{ij}(\boldsymbol{c})(z_j-z_n) \\ &= \sum_{i=1}^{n-1}z_i\cdot\sum_{j=1}^{n-1}E_{ij}(\boldsymbol{c})(z_j-z_n) + z_n\cdot\sum_{j=1}^{n-1}E_{nj}(\boldsymbol{c})(z_j-z_n) \\ &= \sum_{i,j=1}^{n-1}E_{ij}(\boldsymbol{c})(z_i-z_n)\cdot(z_j-z_n).\end{aligned}$$

Choosing $z_i = \nabla h_i'(c_i)$ and observing that $Z_i = \sqrt{c_i} z_i$, we rewrite the right-hand side of (57):

$$\sum_{i,j=1}^{n} (P_L)_{ij} Z_i \cdot Z_j = \sum_{i,j=1}^{n-1} E_{ij}(\boldsymbol{c})(z_i - z_n) \cdot (z_j - z_n)$$
$$= \sum_{i,j=1}^{n-1} E_{ij}(\boldsymbol{c}) \nabla \widetilde{h}_i'(\widetilde{\boldsymbol{c}}) \cdot \nabla \widetilde{h}_j'(\widetilde{\boldsymbol{c}}).$$

Introducing the matrix $Q(\widetilde{\boldsymbol{c}})$ with elements $Q_{ij}(\widetilde{\boldsymbol{c}}) = \partial^2 \widetilde{h}(\widetilde{\boldsymbol{c}})/\partial c_i \partial c_j$ for $i, j = 1, \ldots, n-1$, this expression becomes

$$\sum_{i,j=1}^{n} (P_L)_{ij} Z_i \cdot Z_j = \sum_{i,j,k,\ell=1}^{n-1} E_{ij}(\boldsymbol{c}) Q_{ik}(\widetilde{\boldsymbol{c}}) \nabla c_k \cdot Q_{j\ell}(\widetilde{\boldsymbol{c}}) \nabla c_\ell. \tag{58}$$

We claim that there exists $\zeta(m) > 0$ such that for all $\boldsymbol{y} \in \mathbb{R}^{n-1}$,

$$\boldsymbol{y}^T \big(Q(\widetilde{\boldsymbol{c}})^T E(\boldsymbol{c}) Q(\widetilde{\boldsymbol{c}})\big) \boldsymbol{y} \geq \zeta(m) |\boldsymbol{y}|^2. \tag{59}$$

Then, letting $\boldsymbol{y} = \nabla \boldsymbol{c}$ in (59) and using (57) and (58) leads to (53) with $\beta(m) = \zeta(m)\lambda(m/2)/2$. The proof of (59) proceeds in several steps.

Consider first the matrix $Q(\widetilde{\boldsymbol{c}})$. Let $\eta := \min_{i=1,\ldots,n} \min_{m/2 \leq c_i \leq 1} h_i''(c_i) > 0$ and $\xi \in \mathbb{R}^{n-1}$ and compute

$$Q_{ij}(\widetilde{\boldsymbol{c}}) = h_i''(c_i)\delta_{ij} + h_n''\bigg(1 - \sum_{k=1}^{n-1} c_k\bigg),$$
$$\xi^T Q \xi = \sum_{j=1}^{n-1} h_j''(c_j)\xi_j^2 + h_n''\bigg(1 - \sum_{k=1}^{n-1} c_k\bigg)(\xi_1 + \cdots + \xi_{n-1})^2 \geq \frac{\eta}{2}|\xi|^2.$$

This implies that $Q(\widetilde{\boldsymbol{c}})$ is positive definite with eigenvalues larger than or equal to $\eta/2$. Consider next the $(n-1) \times (n-1)$ submatrix $\widetilde{P}_L = ((P_L)_{ij})_{i,j=1,\ldots,n-1}$ of P_L and note that for $\xi \in \mathbb{R}^{n-1}$, we have

$$\xi^T \widetilde{P}_L \xi = |\xi|^2 - (\xi \cdot \sqrt{\widetilde{c}})^2 \geq |\xi|^2 - |\xi|^2(c_1 + \cdots + c_{n-1}) = c_n |\xi|^2 \geq \frac{m}{2}|\xi|^2.$$

Finally, let $\widetilde{E}(\boldsymbol{c})$ be the first $(n-1) \times (n-1)$ submatrix of $E(\boldsymbol{c})$. Then $\widetilde{E}(\boldsymbol{c}) = S^T \widetilde{P}_L S$ with $S = \operatorname{diag}(\sqrt{c_1}, \ldots, \sqrt{c_{n-1}})$ and, for all $\boldsymbol{y} \in \mathbb{R}^{n-1}$,

$$\boldsymbol{y}^T \widetilde{E}(\boldsymbol{c}) \boldsymbol{y} = (S\boldsymbol{y})^T \widetilde{P}_L (S\boldsymbol{y}) \geq \frac{m}{2}|S\boldsymbol{y}|^2 = \frac{m}{2}\sum_{i=1}^{n-1} c_i y_i^2 \geq \frac{m^2}{4}\sum_{i=1}^{n} y_i^2,$$

i.e., the eigenvalues of $\widetilde{E}(\boldsymbol{c})$ are larger than or equal to $m^2/4$. Since $\widetilde{E}(\boldsymbol{c}) - (m^2/8)I_{n-1}$, with I_{n-1} is the unit matrix on $\mathbb{R}^{(n-1)\times(n-1)}$, and $Q(\widetilde{\boldsymbol{c}})$ are symmetric and positive definite, we deduce that

$$Q(\widetilde{\boldsymbol{c}})^T\widetilde{E}(\boldsymbol{c})Q(\widetilde{\boldsymbol{c}}) \geq Q(\widetilde{\boldsymbol{c}})^T\frac{m^2}{8}I_{n-1}Q(\widetilde{\boldsymbol{c}}) \geq \frac{m^2\eta^2}{32}I_{n-1}\,.$$

This proves (59) with $\zeta(m) = m^2\eta^2/32$.

Proof of (ii). Inequality (44) in Lemma 11 implies that

$$\sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c})Y_i\cdot Y_j \geq \lambda(m/2)|P_L\boldsymbol{Y}|^2 = \lambda(m/2)\sum_{i,j=1}^n (P_L)_{ij}Y_i\cdot Y_j. \tag{60}$$

Similarly as the derivation of (58), we compute

$$\begin{aligned}\sum_{i,j=1}^n (P_L)_{ij}Y_i\cdot Y_j &= \sum_{i,j,k,\ell=1}^{n-1} E_{ij}(\boldsymbol{c})\big(Q_{ik}(\widetilde{\boldsymbol{c}})\nabla c_k - Q_{ik}(\widetilde{\bar{\boldsymbol{c}}})\nabla \bar{c}_k\big)\cdot\big(Q_{j\ell}(\widetilde{\boldsymbol{c}})\nabla c_\ell - Q_{j\ell}(\widetilde{\bar{\boldsymbol{c}}})\nabla\bar{c}_\ell\big) \\ &= \sum_{i,j,k,\ell=1}^{n-1} E_{ij}(\boldsymbol{c})\big(Q_{ik}(\widetilde{\boldsymbol{c}})\nabla(c_k-\bar{c}_k) + (Q_{ik}(\widetilde{\boldsymbol{c}}) - Q_{ik}(\widetilde{\bar{\boldsymbol{c}}}))\nabla\bar{c}_k\big) \\ &\quad\times\big(Q_{j\ell}(\widetilde{\boldsymbol{c}})\nabla(c_\ell-\bar{c}_\ell) + (Q_{j\ell}(\widetilde{\boldsymbol{c}}) - Q_{j\ell}(\widetilde{\bar{\boldsymbol{c}}}))\nabla\bar{c}_\ell\big).\end{aligned}$$

We remark that if E is any symmetric positive definite matrix, then for any $\boldsymbol{z}_1$, $\boldsymbol{z}_2\in\mathbb{R}^n$, the Cauchy–Schwarz and Young's inequalities show that

$$\begin{aligned}(\boldsymbol{z}_1+\boldsymbol{z}_2)^TE(\boldsymbol{z}_1+\boldsymbol{z}_2) &= \boldsymbol{z}_1^TE\boldsymbol{z}_1 + \boldsymbol{z}_1^TE\boldsymbol{z}_2 + \boldsymbol{z}_2^TE\boldsymbol{z}_1 + \boldsymbol{z}_2^TE\boldsymbol{z}_2 \\ &\geq \boldsymbol{z}_1^TE\boldsymbol{z}_1 - \sqrt{\boldsymbol{z}_1^TE\boldsymbol{z}_1}\cdot\sqrt{\boldsymbol{z}_2^TE\boldsymbol{z}_2} - \sqrt{\boldsymbol{z}_1^TE\boldsymbol{z}_1}\cdot\sqrt{\boldsymbol{z}_2^TE\boldsymbol{z}_2} + \boldsymbol{z}_2^TE\boldsymbol{z}_2 \\ &\geq \boldsymbol{z}_1^TE\boldsymbol{z}_1 - \frac{1}{2}\boldsymbol{z}_1^TE\boldsymbol{z}_1 - 2\boldsymbol{z}_2^TE\boldsymbol{z}_2 + \boldsymbol{z}_2^TE\boldsymbol{z}_2 = \frac{1}{2}\boldsymbol{z}_1^TE\boldsymbol{z}_1 - \boldsymbol{z}_2^TE\boldsymbol{z}_2.\end{aligned}$$

Using this inequality, (58) is estimated as

$$\begin{aligned}\sum_{i,j=1}^n (P_L)_{ij}Y_i\cdot Y_j &\geq \frac{1}{2}\sum_{i,j,k,\ell=1}^{n-1} E_{ij}(\boldsymbol{c})Q_{ik}(\widetilde{\boldsymbol{c}})Q_{j\ell}(\widetilde{\boldsymbol{c}})\nabla(c_k-\bar{c}_k)\cdot\nabla(c_\ell-\bar{c}_\ell) \\ &\quad - \sum_{i,j,k,\ell=1}^{n-1} E_{ij}(\boldsymbol{c})\big((Q_{ik}(\widetilde{\boldsymbol{c}}) - Q_{ik}(\widetilde{\bar{\boldsymbol{c}}}))\nabla\bar{c}_k\big)\cdot\big((Q_{j\ell}(\widetilde{\boldsymbol{c}}) - Q_{j\ell}(\widetilde{\bar{\boldsymbol{c}}}))\nabla\bar{c}_\ell\big) =: J_5 + J_6.\end{aligned} \tag{61}$$

We infer from (59) that

$$J_5 \geq \frac{1}{2}\zeta(m)\sum_{i=1}^{n-1}|\nabla(c_i-\bar{c}_i)|^2. \tag{62}$$

It follows from $\nabla c_n = -\sum_{k=1}^{n-1}\nabla c_k$ that

$$|\nabla(c_n-\bar{c}_n)|^2 = \bigg|\sum_{k=1}^{n-1}\nabla(c_k-\bar{c}_k)\bigg|^2 \leq (n-1)\sum_{k=1}^{n-1}|\nabla(c_k-\bar{c}_k)|^2,$$

$$\sum_{k=1}^{n}|\nabla(c_k-\bar{c}_k)|^2 \le n\sum_{k=1}^{n-1}|\nabla(c_k-\bar{c}_k)|^2.$$

Inserting these estimates into (62) yields finally

$$J_5 \ge \frac{\zeta(m)}{2n}\sum_{i=1}^{n}|\nabla(c_i-\bar{c}_i)|^2. \tag{63}$$

The estimate of the term J_2 is easier. Since $E_{ij}(\boldsymbol{c})$ is bounded and the Hessian $Q(\widetilde{\boldsymbol{c}}) = D^2\widetilde{h}$ is Lipschitz continuous,

$$J_6 \le C\sum_{i,k=1}^{n-1}(Q_{ik}(\widetilde{\boldsymbol{c}})-Q_{ik}(\widetilde{\bar{\boldsymbol{c}}}))^2|\nabla\bar{c}_k|^2 \le C\sum_{i=1}^{n-1}(c_i-\bar{c}_i)^2|\nabla\bar{c}_i|^2 \le C\sum_{i=1}^{n-1}(c_i-\bar{c}_i)^2.$$

Combining the above inequality with (63) and (61) gives

$$\sum_{i,j=1}^{n}(P_L)_{ij}Y_i\cdot Y_j \ge \frac{\zeta(m)}{2n}\sum_{i=1}^{n}|\nabla(c_i-\bar{c}_i)|^2 - C\sum_{i=1}^{n-1}(c_i-\bar{c}_i)^2.$$

We conclude (54) after inserting the previous estimate into (60).

Proof of (iii). Let $c_*(x,t)$ be defined by (55) and split the domain of integration into the two subdomains

$$\Omega\times(0,t) = \Big\{c_* > \frac{m}{2}\Big\}\cup\Big\{c_*\le\frac{m}{2}\Big\}.$$

By Lemma 11, the matrix $B^{BD}(\boldsymbol{c})$ is symmetric and positive semi-definite. Using (53), the entropy inequality (47) yields (56). □

4.5. **Proof of Theorem 3.** Lemma 12 suggests that the relative entropy inequality can be expressed in two ways, using either (52) or (49):

$$\begin{aligned} H(\boldsymbol{c}|\bar{\boldsymbol{c}})(t) &\le H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + \int_0^t\int_\Omega (I_1+I_2)dxds \\ &= H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + \int_0^t\int_\Omega (I_3+I_4+I_5)dxds, \end{aligned} \tag{64}$$

where

$$I_1 = -\sum_{i,j=1}^{n} B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_ic_j}\nabla(h_i'(c_i)-h_i'(\bar{c}_i))\cdot\nabla h_j'(c_j),$$

$$I_2 = -\sum_{i=1}^{n}(c_i-\bar{c}_i)h_i''(\bar{c}_i)\,\mathrm{div}\left(\sum_{j=1}^{n}\sqrt{\bar{c}_i}B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar{c}_j}\nabla h_j'(\bar{c}_j)\right),$$

$$I_3 = -\sum_{i,j=1}^{n} Y_i\cdot B_{ij}^{BD}(\boldsymbol{c})Y_j,$$

$$I_4 = -\sum_{i,j=1}^n Y_i \cdot \left(B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} - \frac{\sqrt{c_i}B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \right) \nabla h_j'(\bar{c}_j),$$
$$I_5 = \sum_{i,j=1}^n p_i(c_i|\bar{c}_i) \operatorname{div} \left(\frac{B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j) \right),$$

and $Y_i = \sqrt{c_i}\nabla(h_i'(c_i) - h_i'(\bar{c}_i))$, $i = 1, \ldots, n$.

Step 1: Preparations. Recall that we have assumed that $\bar{c}_i(x,t) \geq m$ for $x \in \Omega$, $t > 0$ for some $m > 0$. Let $c_*(x,t) := \min_{i=1,\ldots,n} c_i(x,t)$. We split the estimations of the above integrals into two subdomains: one where $c_*(x,t) \leq m/2$ and another one where $c_*(x,t) > m/2$. To this end, we use a cutoff function. Let $\varepsilon > 0$ be sufficiently small and $\psi : [0,1] \to [0,1]$ be a C^2-function, which takes the values $\psi = 0$ on $[0, m/2]$, $\psi = 1$ on $[m/2 + \varepsilon, 1]$, and $\psi \in (0,1)$ on the complementary interval $(m/2, m/2 + \varepsilon)$. Define $\chi(\boldsymbol{c}) : \mathbb{R}^n \to [0,1]$ by

$$\chi(\boldsymbol{c}) := \prod_{i=1}^n \psi(c_i) = \begin{cases} 1 & \text{if } m/2 + \varepsilon \leq c_i \leq 1 \text{ for all } i = 1, \ldots, n, \\ 0 & \text{if } 0 < c_i \leq m/2 \text{ for some } i, \\ \alpha(\boldsymbol{c}) \in (0,1) & \text{else.} \end{cases} \tag{65}$$

We employ $\chi(\boldsymbol{c})$ to split the integral (64) into two parts:

$$\begin{aligned} H(\boldsymbol{c}|\bar{\boldsymbol{c}})(t) - H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) &\leq \int_0^t \int_\Omega (1 - \chi(\boldsymbol{c}))(I_3 + I_4 + I_5) dx ds \\ &\quad + \int_0^t \int_\Omega \chi(\boldsymbol{c})(I_3 + I_4 + I_5) dx ds =: J_L + J_H. \end{aligned} \tag{66}$$

In the sequel, we estimate J_L and J_H separately.

Step 2: Case $c_*(x,t) \leq m/2 + \varepsilon$. We estimate the term J_L in (66). By replacing $p(c_i|\bar{c}_i)$ in I_5 by definitions (19) and (48) and tracing backwards the derivation from (52) to (49), we can express the integral over $(1 - \chi(\boldsymbol{c})(I_3 + I_4 + I_5)$ by $(1 - \chi(\boldsymbol{c})(I_1 + I_2)$ except for a term accounting for the cutoff function:

$$\begin{aligned} J_L &= \int_0^t \int_\Omega (1 - \chi(\boldsymbol{c}))(I_1 + I_2) dx ds \\ &\quad + \int_0^t \int_\Omega \nabla\chi(\boldsymbol{c}) \cdot \sum_{i=1}^n (p_i(c_i) - p_i(\bar{c}_i)) \sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}}) \frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}} \nabla h_j'(\bar{c}_j). \end{aligned} \tag{67}$$

In the sequel, we estimate the right-hand side of (67) term-by-term. To estimate I_1, we set $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ with $Z_i := \sqrt{c}_i \nabla h_i'(c_i)$, $i = 1, \ldots, n$. By Lemma 11, the matrix $B^{BD}(\boldsymbol{c})$ is symmetric and positive semi-definite. Therefore, using Young's inequality and the boundedness of $B^{BD}(\boldsymbol{c})$ (see Assumption (B2)),

$$I_1 = -\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z} + \sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c}) Z_i \sqrt{c_j} \nabla h_j'(\bar{c}_j)$$

$$\leq -\frac{1}{2}\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z} + \frac{1}{2}\sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c})(\sqrt{c_i}\nabla h_i'(\bar{c}_i))\cdot(\sqrt{c_j}\nabla h_j'(\bar{c}_j))$$
$$\leq -\frac{1}{2}\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z} + C\sum_{i=1}^n |\nabla h_i'(\bar{c}_i)|^2,$$

where $C > 0$ depends on m and μ (defined in Assumption (B4)). For the term I_2, we use the regularity for $\bar{c}_i$ to conclude that

$$I_2 \leq \sum_{i=1}^n (c_i - \bar{c}_i)^2 + C\sum_{i,j=1}^n \left|h_i''(\bar{c}_i)\operatorname{div}\left(\sqrt{\bar{c}_i}B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar{c}_j}\nabla h_j'(\bar{c}_j)\right)\right|^2 \leq \sum_{i=1}^n (c_i - \bar{c}_i)^2 + C.$$

On the set $\chi(\boldsymbol{c}) < 1$, we have $c_*(x,t) < m/2 + \varepsilon$, and there exists $\ell \in \{1,\ldots,n\}$ such that $c_\ell(x,t) < m/2 + \varepsilon$. Thanks to Assumption (H) on page 7, we can apply Lemma 19 to find that

$$h_{i_0}(c_{i_0}|\bar{c}_{i_0}) \geq \kappa_m(c_{i_0} - \bar{c}_{i_0})^2 \geq \kappa_m\left(\frac{m}{2} - \varepsilon\right)^2 \quad \text{when } \chi(\boldsymbol{c}) < 1.$$

We infer that

$$I_2 \leq \sum_{i=1}^n (c_i - \bar{c}_i)^2 + C\sum_{i=1}^n h_i(c_i|\bar{c}_i) \leq C\sum_{i=1}^n h_i(c_i|\bar{c}_i).$$

It remains to estimate the last term in (67), using the fact that $\nabla\chi(\boldsymbol{c})$ vanishes outside the set $c_* \in [m/2, m/2+\varepsilon]$, the Lipschitz continuity of $p_i(c_i)$, entropy inequality (56) and Lemma 19:

$$\left|\int_0^t\int_\Omega \nabla\chi(\boldsymbol{c})\cdot\sum_{i=1}^n (p_i(c_i) - p_i(\bar{c}_i))\sum_{j=1}^n B_{ij}^{BD}(\bar{\boldsymbol{c}})\frac{\sqrt{\bar{c}_j}}{\sqrt{\bar{c}_i}}\nabla h_j'(\bar{c}_j)dxds\right|$$
$$\leq C\left(\max_{j=1,\ldots,n}\sup_{m/2\leq c_j\leq m/2+\varepsilon}\left|\frac{\partial\chi}{\partial c_j}(\boldsymbol{c})\right|\right)\int_0^t\int_\Omega 1_{\{m/2<c_*<m/2+\varepsilon\}}\sum_{i=1}^n|\nabla c_i|\sum_{j=1}^n|c_j - \bar{c}_j|dxds$$
$$\leq C\int_0^t\int_\Omega 1_{\{c_*>m/2\}}\sum_{i=1}^n|\nabla c_i|^2dxds + C\int_0^t\int_\Omega 1_{\{c_*<m/2+\varepsilon\}}\sum_{j=1}^n|c_j - \bar{c}_j|^2dxds$$
$$\leq C + \int_0^t\int_\Omega 1_{\{c_*<m/2+\varepsilon\}}\sum_{j=1}^n|c_j - \bar{c}_j|^2dxds$$
$$\leq C\int_0^t\int_\Omega 1_{\{c_*<m/2+\varepsilon\}}\sum_{i=1}^n h_i(c_i|\bar{c}_i)dxds.$$

Note that the final constant C, depending on $\max_{j=1,\ldots,n}\sup_{m/2\leq c_j\leq m/2+\varepsilon}|(\partial\chi/\partial c_j)(\boldsymbol{c})|$, will blow up if we let $\varepsilon \to 0$. Therefore, we fix $\varepsilon > 0$. Combining the previous estimate, we end up with

$$J_L \leq -\frac{1}{2}\int_0^t\int_\Omega (1-\chi(\boldsymbol{c}))\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z}dxds + C\int_0^t\int_\Omega 1_{\{c_*<m/2+\varepsilon\}}\sum_{i=1}^n h_i(c_i|\bar{c}_i)dxds. \tag{68}$$

Step 3: Case $c_*(x,t) > m/2$. We proceed to estimate the term J_H in (66). The range of integration now consists of the sets $\{m/2 < c_* < m/2+\varepsilon\}$, where $0 < \chi(\boldsymbol{c}) < 1$, and the set $\{m/2+\varepsilon < c_* \le 1\}$, where $\chi(\boldsymbol{c}) = 1$.

For the term I_3, we use (54) in Lemma 13:

$$
\begin{aligned}
-\int_0^T\int_\Omega \chi(\boldsymbol{c}) I_3 dxds &= \int_0^t\int_\Omega \chi(\boldsymbol{c}) \sum_{i,j=1}^n B_{ij}^{BD}(\boldsymbol{c}) Y_i\cdot Y_j dxds \\
&\ge \beta(m)\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n |\nabla(c_i-\bar c_i)|^2 dxds - C\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n |c_i-\bar c_i|^2 dxds.
\end{aligned}
$$

By Young's inequality with $\delta > 0$, the term I_4 can be estimated as

$$
\begin{aligned}
\int_0^t\int_\Omega \chi(\boldsymbol{c}) I_4 dxds &\le \delta\sum_{i=1}^n \int_0^t\int_\Omega \chi(\boldsymbol{c})|Y_i|^2 dxds \\
&\quad + \frac{1}{4\delta}\sum_{i,j=1}^n \int_0^t\int_\Omega \chi(\boldsymbol{c})\bigg(B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} - \frac{\sqrt{c_i}B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar c_j}}{\sqrt{\bar c_i}}\bigg)^2 |\nabla h_j'(\bar c_j)|^2 dxds.
\end{aligned}
$$

Recall that we work in the range $c_i > m/2$ and $\bar c_i \ge m$. The boundedness and Lipschitz continuity of h_i'' imply that

$$
\begin{aligned}
\sum_{i=1}^n |Y_i|^2 &= \sum_{i=1}^n c_i|\nabla(h_i'(c_i)-h_i'(\bar c_i))|^2 \le \sum_{i=1}^n c_i|h_i''(c_i)\nabla(c_i-\bar c_i) + (h_i''(c_i)-h_i''(\bar c_i))\nabla\bar c_i|^2 \\
&\le C\sum_{i=1}^n \big(|\nabla(c_i-\bar c_i)|^2 + (c_i-\bar c_i)^2|\nabla\bar c_i|^2\big).
\end{aligned}
$$

Furthermore, the boundedness and Lipschitz continuity of B_{ij}^{BD} (see Lemma 11) yield

$$
\begin{aligned}
&\bigg|B_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j} - \frac{\sqrt{c_i}B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar c_j}}{\sqrt{\bar c_i}}\bigg| \\
&\qquad = \bigg|B_{ij}(\boldsymbol{c})\sqrt{c_j} - B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar c_j} + \frac{(\sqrt{\bar c_i}-\sqrt{c_i})B_{ij}^{BD}(\bar{\boldsymbol{c}})\sqrt{\bar c_j}}{\sqrt{\bar c_i}}\bigg| \\
&\qquad \le C\sum_{i=1}^n |c_i-\bar c_i| + |\sqrt{c_i}-\sqrt{\bar c_i}| \le C\sum_{i=1}^n |c_i-\bar c_i|.
\end{aligned}
$$

Thus, the choice $\delta = \beta(m)/(2C)$ gives

$$
\begin{aligned}
\int_0^t\int_\Omega \chi(\boldsymbol{c}) I_4 dxds &\le \frac{\beta(m)}{2}\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n |\nabla(c_i-\bar c_i)|^2 dxds \\
&\quad + C\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n |c_i-\bar c_i|^2 dxds.
\end{aligned}
$$

Finally, we use definition (48) of $p_i(c_i|\bar{c}_i)$ and Hypothesis (H) to estimate

$$|p_i(c_i|\bar{c}_i)| = \left|(c_i-\bar{c}_i)^2\int_0^1\int_0^s p_i''(\tau c_i+(1-\tau)\bar{c}_i)d\tau ds\right|$$
$$\le K_2(c_i-\bar{c}_i)^2\int_0^1\int_0^s h_i''(\tau c_i+(1-\tau)\bar{c}_i)d\tau ds = K_2 h_i(c_i|\bar{c}_i).$$

In turn, this implies that

$$\int_0^t\int_\Omega \chi(\boldsymbol{c})I_5 dxds \le C\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n h_i(c_i|\bar{c}_i)dxds.$$

Summarizing the previous computations and using Lemma 19, we conclude that

$$J_H \le -\frac{\beta(m)}{2}\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n|\nabla(c_i-\bar{c}_i)|^2 dxds + C\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n h_i(c_i|\bar{c}_i)dxds. \tag{69}$$

Step 4: End of the proof. We combine the differential inequality (66) with the estimations (68) and (69) to obtain

$$\begin{aligned} H(\boldsymbol{c}(t)|\bar{\boldsymbol{c}}(t)) &+ \frac{1}{2}\int_0^t\int_\Omega (1-\chi(\boldsymbol{c}))\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z} dxds \\ &+ \frac{\beta(m)}{2}\int_0^t\int_\Omega \chi(\boldsymbol{c})\sum_{i=1}^n |\nabla(c_i-\bar{c}_i)|^2 dxds \\ &\le H(\boldsymbol{c}^0|\bar{\boldsymbol{c}}^0) + C\int_0^t H(\boldsymbol{c}|\bar{\boldsymbol{c}})ds, \end{aligned} \tag{70}$$

The constant $C>0$ depends in particular on m and the $L^\infty(0,T;W^{2,\infty}(\Omega))$ norm of $\bar{c}_j$, $j=1,\ldots,n$. The proof of Theorem 3 finishes after applying Gronwall's inequality.

Remark 14. Inequality (70) leads to a slightly stronger version of the relative entropy inequality than stated in Theorem 3. However, we obtain gradient estimates only on the set $\{c_*>m/2\}$, while on $\{c_*\le m/2\}$, the quadratic form $\boldsymbol{Z}^T B^{BD}(\boldsymbol{c})\boldsymbol{Z}$ generally does not lead to a control of the L^2-norm of ∇c_i. ☐

5. Examples

We present some examples for the generalized Maxwell–Stefan system (16)–(17) satisfying Assumptions (B1)–(B4).

5.1. **A cross-diffusion system for thin-film solar cells.** Thin-film crystalline solar cells can be fabricated by the so-called physical vapor deposition process. This process produces a metal vapor that can be deposited on electrically conductive materials as a thin coating. It is shown in [1] that the evolution of the volume fractions of the thin-film components can be described by the cross-diffusion system

$$\partial_t c_i = \operatorname{div}\bigg(\sum_{j=1}^n a_{ij}(u_j\nabla u_i - u_i\nabla u_j)\bigg), \quad i=1,\ldots,n, \tag{71}$$

where $a_{ij}=a_{ji}>0$ for $i,j=1,\ldots,n$, and $\sum_{i=1}^n c_i = 1$. This model can be formulated as a generalized Maxwell–Stefan system. Indeed, let $h_i(c_i)=c_i(\log c_i - 1)$ and $K_{ij}(\boldsymbol{c}) = \sum_{j=1}^n \sqrt{c_i}A_{ij}^{BD}(\boldsymbol{c})\sqrt{c_j}$ for $i,j=1,\ldots,n$, where $A(\boldsymbol{c})$ is given by (10) with $D_{ij}=1/a_{ij}$. Then $B(\boldsymbol{c}) = A^{BD}(\boldsymbol{c})$ (see (18)) and hence $B^{BD}(\boldsymbol{c}) = A(\boldsymbol{c})$. Equation (43) becomes

$$\partial_t c_i = \operatorname{div}\bigg(\sum_{j=1}^n \sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j}\nabla\log c_j\bigg), \quad i=1,\ldots,n. \tag{72}$$

Because of (10), the mobility matrix $(\sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j})$ reads as

$$\sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j} = \begin{cases} \sum_{k\neq i} a_{ik}c_ic_k & \text{if } i=j, \\ -a_{ij}c_ic_j & \text{if } i\neq j, \end{cases}$$

and an elementary computation shows that (72) can actually be written as (71).

Although it can be checked that the matrix $B(\boldsymbol{c}) = A^{BD}(\boldsymbol{c})$ satisfies Assumptions (B1)–(B4), we can here directly verify the statements of Lemma 11. Definition (10) of $A(\boldsymbol{c})$ immediately implies that $B_{ij}^{BD}(\boldsymbol{c})$ is bounded and Lipschitz continuous on $[0,1]^n$. Property (44) follows from (23) in Lemma 4 with $\lambda(m)=\mu$. Hence, the weak-strong uniqueness property holds for this model.

5.2. **A tumor-growth model.** The growth of a symmetric avascular tumor can be modeled on the mechanical level by diffusion fluxes of the tumor cells, the extracellular matrix (ECM), and the interstitial fluid (water, nutrients, etc.). The model was suggested in [25] and analyzed in [27]. The evolution of the volume fractions c_i of the tumor cells, ECM, and interstitial fluid is given by (see, e.g., [29, Section 4.2])

$$\partial_t c_i + \operatorname{div}(c_iu_i) = 0, \quad i=1,\ldots,3, \tag{73}$$

$$\nabla(c_iP_i) + c_i\nabla p = -\sum_{j=1}^3 k_{ij}c_ic_j(u_i-u_j), \quad i=1,2, \tag{74}$$

$$c_3\nabla p = -\sum_{j=1}^3 k_{ij}c_ic_j(u_i-u_j), \tag{75}$$

where $k_{ij}=k_{ji}>0$ for $i,j=1,2,3$, the partial pressures P_1, P_2 and the phase pressure p are given by

$$P_1 = c_1, \quad P_2 = \beta c_2(1+\theta c_1), \quad p = -c_1P_1 - c_2P_2,$$

$\beta>0$, $\theta>0$ are suitable parameters, and it holds that $\sum_{i=1}^3 c_i = 1$.

We claim that (73)–(75) can be formulated as a generalized Maxwell–Stefan system. We define the entropy densities as in the previous example, $h_i(c_i) = c_i(\log c_i - 1)$, $i = 1, 2, 3$. With the matrix

$$W(\boldsymbol{c}) = \begin{pmatrix} 2c_1(1-c_1) - \beta\theta c_1 c_2^2 & -2\beta c_1 c_2(1+\theta c_1) & 0 \\ -2c_1c_2 + \beta\theta(1-c_2)c_2^2 & 2\beta c_2(1-c_2)(1+\theta c_1) & 0 \\ -2c_1c_3 - \beta\theta c_3 c_2^2 & -2\beta c_3 c_2(1+\theta c_1) & 0 \end{pmatrix},$$

the left-hand side of (74)–(75) can be written in a more concise form:

$$\begin{pmatrix} \nabla(c_1P_1) + c_1\nabla p \\ \nabla(c_2P_2) + c_2\nabla p \\ c_3\nabla p \end{pmatrix} = W(\boldsymbol{c}) \begin{pmatrix} \nabla c_1 \\ \nabla c_2 \\ \nabla c_3 \end{pmatrix}.$$

Let the matrix $A(\boldsymbol{c})$ be given by (10) with $D_{ij} = 1/k_{ij}$. Then the right-hand side of (74)–(75) equals (also see (12))

$$-\sum_{j=1,\, j\neq i}^{3} \frac{c_ic_j}{D_{ij}}(u_i - u_j) = -\sum_{j=1}^{3} \sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j}u_j.$$

Thus, inverting

$$\sum_{j=1}^{n} W_{ij}(\boldsymbol{c})\nabla c_j = -\sum_{j=1}^{3}\sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j}u_j, \quad i = 1,2,3,$$

(which is the same as (74)–(75)) yields

$$\sqrt{c_i}u_i = -\sum_{j,k=1}^{3} A_{ij}^{BD}(\boldsymbol{c})\frac{1}{\sqrt{c_j}}W_{jk}(\boldsymbol{c})\nabla c_k, \quad i = 1,2,3,$$

and system (73)–(75) can be written for $i = 1, 2, 3$ as

$$\begin{aligned} \partial_t c_i &= \operatorname{div}\left(\sum_{j,k=1}^{3}\sqrt{c_i}A_{ij}^{BD}(\boldsymbol{c})\frac{1}{\sqrt{c_j}}W_{jk}(\boldsymbol{c})\nabla c_k\right) \\ &= \operatorname{div}\left(\sum_{j,k,\ell=1}^{3}\sqrt{c_i}A_{ij}^{BD}(\boldsymbol{c})\frac{1}{\sqrt{c_j}}W_{jk}(\boldsymbol{c})\sqrt{c_k}(P_L)_{k\ell}\sqrt{c_\ell}\nabla\log c_\ell\right), \end{aligned}$$

recalling definition (22) of P_L. Thus,

$$\partial_t c_i = \operatorname{div}\left(\sum_{\ell=1}^{n}\sqrt{c_i}R_{i\ell}(\boldsymbol{c})\sqrt{c_\ell}\nabla\log c_\ell\right), \quad \text{where}$$

$$R_{i\ell}(\boldsymbol{c}) := \sum_{j,k=1}^{3} A_{ij}^{BD}(\boldsymbol{c})\frac{1}{\sqrt{c_j}}W_{jk}(\boldsymbol{c})\sqrt{c_k}(P_L)_{k\ell}, \quad i,\ell = 1,2,3.$$

Also in this case, it is more convenient to check the statements of Lemma 11 instead of Assumptions (B1)–(B4). Notice that $W(\boldsymbol{c})$ is not symmetric, so $R(\boldsymbol{c})$ is not symmetric either. The elements $R_{ij}(\boldsymbol{c})$ are bounded and Lipschitz continuous, since $A^{BD}_{ij}(\boldsymbol{c})$ and $W_{ij}(\boldsymbol{c})$ have these properties. The second statement of Lemma 11, namely property (44), is verified only for a special example that was considered in [27].

Lemma 15. *Let $k_{ij} = 1$ for $i,j = 1,2,3$ and $0 \le \theta < 4/\sqrt{\beta}$. Then, with $m = \min_{i=1,2,3} c_i > 0$, there exists $\lambda(m) > 0$ such that*

$$\boldsymbol{z}^T R(\boldsymbol{c})\boldsymbol{z} \ge \lambda(m)|P_L\boldsymbol{z}|^2 \quad \textit{for all } \boldsymbol{z} \in \mathbb{R}^3.$$

Proof. The assumption $k_{ij} = 1$ implies that $A(\boldsymbol{c}) = P_L$ and hence $A^{BD}(\boldsymbol{c}) = P_L(A(\boldsymbol{c})P_L + P_{L^\perp})^{-1} = P_L$. Suppose that for any $\boldsymbol{y} \in \mathbb{R}^3$ satisfying $\boldsymbol{y} \in L$ (i.e. $\sqrt{\boldsymbol{c}} \cdot \boldsymbol{y} = 0$), we have

$$\sum_{i,j=1}^3 \frac{1}{\sqrt{c_i}} W_{ij}(\boldsymbol{c})\sqrt{c_j}y_iy_j \ge \lambda(m)|\boldsymbol{y}|^2. \tag{76}$$

Then, for $\boldsymbol{z} \in \mathbb{R}^3$ and $\boldsymbol{y} = P_L\boldsymbol{z} \in L$,

$$\boldsymbol{z}^T R(\boldsymbol{c})\boldsymbol{z} = \sum_{i,j=1}^3 (P_L\boldsymbol{z})_i \frac{1}{\sqrt{c_i}} W_{ij}(\boldsymbol{c})\sqrt{c_j}(P_L\boldsymbol{z})_j \ge \lambda(m)|P_L\boldsymbol{z}|^2,$$

which proves the lemma.

It remains to verify (76). Using $y_3 = -(\sqrt{c_1}y_1 + \sqrt{c_2}y_2)/\sqrt{c_3}$, we calculate

$$\sum_{i,j=1}^3 \frac{1}{\sqrt{c_i}} W_{ij}(\boldsymbol{c})\sqrt{c_j}y_iy_j = 2\beta(1+\theta c_1)(\sqrt{c_2}y_2)^2 + \beta\theta c_2(\sqrt{c_1}y_1)(\sqrt{c_2}y_2) + 2(\sqrt{c_1}y_1)^2.$$

Since $0 \le \theta < 4/\sqrt{\beta}$, the discriminant of the quadratic form is negative and there exists $\kappa > 0$ such that

$$\begin{aligned}\sum_{i,j=1}^3 \frac{1}{\sqrt{c_i}} W_{ij}(\boldsymbol{c})\sqrt{c_j}y_iy_j &\ge \kappa(c_1y_1^2 + c_2y_2^2) \ge \frac{\kappa}{3}\big(c_1y_1^2 + c_2y_2^2 + (\sqrt{c_1}y_1 + \sqrt{c_2}y_2)^2\big) \\ &= \frac{\kappa}{3}(c_1y_1^2 + c_2y_2^2 + c_3y_3^2) \ge \frac{\kappa}{3}\Big(\min_{i=1,2,3} c_i\Big)(y_1^2 + y_2^2 + y_3^2),\end{aligned}$$

proving the claim with $\lambda(m) = \kappa m/3$. □

We deduce from the previous lemma that the weak-strong uniqueness property holds for the tumor-growth model if $k_{ij} = 1$ for $i,j = 1,2,3$ and $0 \le \theta < 4/\sqrt{\beta}$. The latter condition is necessary to achieve the global existence of weak solutions, since it guarantees the positive semidefiniteness of the mobility matrix; see [27] for details.

5.3. **A multi-species porous-medium-type model.** Another model is a generalization of the first example to illustrate that also non-logarithmic entropies may be considered. We choose $h_i(c_i) = c_i^\gamma/(\gamma-1)$ with $\gamma > 1$ and $A(\boldsymbol{c})$ as in (10). The partial pressure becomes $p_i = c_ih_i'(c_i) - h_i(c_i) = c_i^\gamma$, and equation (72) reads here as

$$\partial_t c_i = \operatorname{div}\bigg(\sum_{j=1}^n \sqrt{c_i}A_{ij}(\boldsymbol{c})\sqrt{c_j}\nabla h_j'(c_j)\bigg) = \frac{\gamma}{\gamma-1}\operatorname{div}\bigg(\sum_{j=1}^n \frac{1}{D_{ij}}(c_j\nabla c_i^{\gamma-1} - c_i\nabla c_j^{\gamma-1})\bigg).$$

Hence, the weak-strong property holds for this model.

5.4. **Maxwell-Stefan system with different molar masses.** In equations (1), we have implicitly assumed that all molar masses of the species are the same. We show that the weak-strong uniqueness property also holds for the model proposed in [3, 5] without this assumption. In the case of different molar masses M_i, we need to distinguish between the mass densities ρ_i and the molar concentrations $c_i = \rho_i/M_i$. The Maxwell–Stefan equations read for $i = 1, \ldots, n$ as

$$\partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0, \quad -\sum_{j=1}^n \frac{c_i c_j}{c^2 D_{ij}}(u_i - u_j) = \rho_i \nabla \frac{\delta H}{\delta \rho_i}(\boldsymbol{\rho}) - \rho_i \sum_{j=1}^n \rho_j \nabla \frac{\delta H}{\delta \rho_j}(\boldsymbol{\rho}), \tag{77}$$

where $c = \sum_{i=1}^n c_i$. As before, the restriction $\sum_{j=1}^n \rho_j = 1$ inherited from the initial data is imposed. The second equation can be rewritten as

$$-\sum_{j=1}^n \frac{\rho_i \rho_j}{\widetilde{D}_{ij}(\boldsymbol{\rho})}(u_i - u_j) = \rho_i \nabla \frac{\delta H}{\delta \rho_i}(\boldsymbol{\rho}) - \rho_i \sum_{j=1}^n \rho_j \nabla \frac{\delta H}{\delta \rho_j}(\boldsymbol{\rho}),$$

where $\widetilde{D}_{ij}(\boldsymbol{\rho}) = c^2 M_i M_j = (\sum_{k=1}^n \rho_k/M_k)^2 M_i M_j$. Due to

$$\sum_{k=1}^n \frac{\rho_k}{M_k} \geq \sum_{k=1}^n \frac{\rho_k}{\max_{\ell=1,\ldots,n} M_\ell} = \frac{1}{\max_{\ell=1,\ldots,n} M_\ell},$$

the coefficients $\widetilde{D}_{ij}(\boldsymbol{\rho})$ are uniformly bounded from below. Since the proof of Lemma 4 only relies on the uniform boundness of D_{ij}, Lemma 4 also holds for the following matrix $\widetilde{A}(\boldsymbol{c})$, defined similarly as in (10):

$$\widetilde{A}_{ij}(\boldsymbol{\rho}) = \begin{cases} \sum_{k=1,\, k\neq i}^n \rho_k/\widetilde{D}_{ik}(\boldsymbol{\rho}) & \text{if } i = j, \\ -\sqrt{\rho_i \rho_j}/\widetilde{D}_{ij}(\boldsymbol{\rho}) & \text{if } i \neq j. \end{cases}$$

Therefore, the weak-strong uniqueness holds if $H(\boldsymbol{c})$ satisfies the assumptions of Theorem 3.

Recalling that $H(\boldsymbol{\rho}) = \sum_{i=1}^n \int_\Omega h_i(\rho_i)dx$, we may formulate the entropy in terms of the concentrations $\boldsymbol{c}$ as $\eta(\boldsymbol{c}) = \sum_{i=1}^n \int_\Omega \eta_i(c_i)dx$, where $\eta_i(c_i) = h_i(\rho_i/M_i)$. Then we can rewrite the second equation in (77) as

$$-\sum_{j=1}^n \frac{c_i c_j}{c^2 D_{ij}}(u_i - u_j) = c_i \nabla \mu_i - c_i M_i \sum_{j=1}^n c_j \nabla \mu_j,$$

where $\mu_i = \eta_i'(c_i)$ is the molar-based chemical potential. Using the Gibbs–Duhem equation $\nabla p = \sum_{i=1}^n c_j \nabla \mu_j$, where p is the pressure, the above equation can be put into the form

$$-\sum_{j=1}^n \frac{c_i c_j}{c^2 D_{ij}}(u_i - u_j) = c_i \nabla \mu_i - \rho_i \nabla p,$$

which is [5, Formula (203)]. Yet another formulation in terms of the molar fractions $X_i = c_i/c$ is

$$-\sum_{j=1}^n \frac{c_ic_j}{c^2D_{ij}}(u_i - u_j) = c_i\nabla_p\widetilde{\mu}_i + (\phi_i - \rho_i)\nabla p,$$

where $\widetilde{\mu}_i$ is given by $\widetilde{\mu}_i(p, X_1, \ldots, X_n) = \mu_i(c_i)$, $\nabla_p\widetilde{\mu}_i := \sum_{j=1}^n(\partial\widetilde{\mu}_i/\partial X_j)\nabla X_j$, and $\phi_i := \partial\widetilde{\mu}_i/\partial p$ is the volume fraction.

A simple choice is the entropy

$$\eta_i(c_i) = c_i \log c_i - c_i, \quad i = 1, \ldots, n,$$

corresponding to $h_i(\rho_i) = (\rho_i/M_i)(\log(\rho_i/M_i) - 1)$. It leads to $\mu_i = \log c_i$, $p = c$, and the model

$$-\sum_{j=1}^n \frac{c_ic_j}{c^2D_{ij}}(u_i - u_j) = \nabla c_i - \rho_i\nabla c.$$

The existence of local strong solutions to this model can be proved as in [4], while the existence of global weak solutions was shown in [9].

Appendix A. The Bott–Duffin inverse

For the convenience of the reader, we recall the definition and some properties of the Bott–Duffin inverse. Let $A \in \mathbb{R}^{n\times n}$ be an arbitrary matrix and $L \subset \mathbb{R}^n$ be a subspace. The Bott–Duffin inverse is introduced in connection to the solution of the constrained inversion problem (see [7], [2, Ch 2.10])

$$Ax + y = b, \quad x \in L, \quad y \in L^\perp. \tag{78}$$

Let P_L and $P_{L^\perp}$ be the projection operators onto L and $L^\perp$, respectively. The set of solutions of (78) is the same as the set of solutions to $(AP_L + P_{L^\perp})z = b$, and (x, y) solves (78) if and only if $x = P_Lz$ and $y = P_{L^\perp}z = b - AP_Lz$. Then, if the matrix $AP_L + P_{L^\perp}$ is invertible, we define the *Bott–Duffin inverse* of A with respect to L by

$$A^{BD} := P_L(AP_L + P_{L^\perp})^{-1}. \tag{79}$$

and the solution to (78) is expressed in the form

$$x = A^{BD}b\,, \quad y = b - Ax. \tag{80}$$

If $L = \operatorname{ran}(A)$ and A is symmetric, the Bott–Duffin inverse is the same as the group inverse, which was investigated in the context of Maxwell–Stefan systems in [6].

Let A be symmetric. We call A *L-positive definite* if $\boldsymbol{z}^TA\boldsymbol{z} > 0$ for all $\boldsymbol{z} \in L \setminus \{\boldsymbol{0}\}$. For this class of matrices, a generalized Bott–Duffin inverse is defined in [35], which coincides with the classical Bott–Duffin inverse when $AP_L + P_{L^\perp}$ is invertible. The following result is proved in [35, Lemma 2c and 1b].

Lemma 16. *Let A be symmetric and L-positive definite. Then*

(i) $A^{BD}P_{L^\perp} = 0$,

(ii) $\operatorname{ran}(AP_L + P_{L^\perp}) = P_L\operatorname{ran}(A) \otimes L^\perp, \quad \ker(AP_L + P_{L^\perp}) = \ker(AP_L) \cap L.$

It follows from property (i) that A^{BD} can be formulated as

$$A^{BD} = P_L(AP_L + P_{L^\perp})^{-1}(P_L + P_{L^\perp}) = P_L(AP_L + P_{L^\perp})^{-1}P_L = A^{BD}P_L. \tag{81}$$

Lemma 17. *Let A be symmetric and $L = \operatorname{ran} A$, $L^\perp = \ker A$. Then $AP_L = A$, $P_LA = A$, A^{BD} is well defined and symmetric.*

Proof. The identities $AP_L = A$ and $P_LA = A$ follow immediately from $L = \operatorname{ran} A$, We infer from property (ii) that

$$\ker(AP_L + P_{L^\perp}) = \ker(AP_L) \cap L = \ker(A) \cap L = L^\perp \cap L = \{0\},$$

showing that $AP_L + P_{L^\perp}$ is invertible. The matrix $AP_L = P_LAP_L$ is symmetric, since P_L and A are symmetric. Also $P_{L^\perp}$ is symmetric, so $AP_L + P_{L^\perp}$ and its inverse are symmetric too. Taking into account (81), this implies that $A^{BD} = P_L(AP_L + P_{L^\perp})^{-1}P_L$ is also symmetric. □

In our context, we are interested in the constrained inversion

$$Ax = b, \quad x \in L,$$

where A is a symmetric positive semidefinite matrix, with $L = \operatorname{ran}(A)$ and thus $L^\perp = \ker(A)$, and $b \in L$. Lemma 17 implies that $AP_L + P_{L^\perp}$ is invertible and A^{BD} is well defined by (79). Because of (80), we can express the inverse as $x = A^{BD}b$ if $b \in L$.

Appendix B. Pointwise estimates for entropy functions

For the convenience of the reader, we recall the following lower bounds.

Lemma 18. *The following estimates hold for any c, $\bar c \in [0,1]$:*

$$c\log\frac{c}{\bar c} - (c-\bar c) \ge \frac12(c-\bar c)^2, \quad c\log\frac{c}{\bar c} - (c-\bar c) \ge (\sqrt{c}-\sqrt{\bar c})^2.$$

Proof. Let $f(c) = c\log c$. Then

$$f(c) - f(\bar c) = f(\theta(c-\bar c)+\bar c)\big|_{\theta=0}^1 = (c-\bar c)\int_0^1 f'(\theta(c-\bar c)+\bar c)d\theta$$

and

$$\begin{aligned} c\log\frac{c}{\bar c} - (c-\bar c) &= f(c) - f(\bar c) - f'(\bar c)(c-\bar c) \\ &= (c-\bar c)\int_0^1 \big(f'(\theta(c-\bar c)+\bar c) - f'(\bar c)\big)d\theta \\ &= (c-\bar c)\int_0^1 f'(s(c-\bar c)+\bar c)\big|_{s=0}^\theta d\theta = (c-\bar c)^2\int_0^1\int_0^\theta f''(s(c-\bar c)+\bar c)dsd\theta. \end{aligned}$$

The first inequality follows after observing that $f''(s(c-\bar c)+\bar c) = 1/(s(c-\bar c)+\bar c) \ge 1$.

For the second inequality, we define $g(c) = (c\log c - c + 1)/(\sqrt{c}-1)^2$ for $c \ne 1$ and $g(1) = 2$. Then g is continuous and increasing, which implies that $g(c) \ge g(0) = 1$ and proves the statement. □

Lemma 19. *Let $\boldsymbol{c}, \bar{\boldsymbol{c}} \in \mathbb{R}^n_+$ satisfy $0 \le c_i \le 1$, $m \le \bar{c}_i \le 1$, for $i = 1, \ldots, n$, and suppose that $h_i \in C([0,1]) \cap C^2((0,1])$ satisfies*

$$h_i''(c_i) > 0 \quad \text{for } 0 < c_i \le 1.$$

Then, for some $\kappa_m > 0$,

$$h_i(c_i|\bar{c}_i) = h_i(c_i) - h_i(\bar{c}_i) - h_i'(\bar{c}_i)(c_i - \bar{c}_i) \ge \kappa_m (c_i - \bar{c}_i)^2. \tag{82}$$

Proof. By Taylor expansion, the relative entropy density satisfies

$$\lim_{c_i \to \bar{c}_i} \frac{h_i(c_i|\bar{c}_i)}{(c_i - \bar{c}_i)^2} = \lim_{c_i \to \bar{c}_i} \int_0^1 \int_0^\theta h_i''(s(c_i - \bar{c}_i) + \bar{c}_i) ds d\theta = \frac{1}{2} h_i''(\bar{c}_i) > 0.$$

Therefore, $h_i(c_i|\bar{c}_i)/(c_i - \bar{c}_i)^2$ is a continuous function with a positive minimum:

$$\kappa_m := \min_{i=1,\ldots,n} \min_{c_i \in [0,1],\, \bar{c}_i \in [m,1]} \frac{h_i(c_i|\bar{c}_i)}{(c_i - \bar{c}_i)^2} > 0.$$

This shows that $h_i(c_i|\bar{c}_i) \ge \kappa_m (c_i - \bar{c}_i)^2$ for $c_i \in [0,1]$, $\bar{c}_i \in [m,1]$ and proves (82). □

Appendix C. Thermodynamic derivation of the generalized Maxwell–Stefan system

The aim of this section is to derive (16)–(17) from elementary thermodynamic principles. We assume that the evolution of the gaseous mixture is given by the conservation of mass and energy (without chemical reactions),

$$\begin{aligned} \partial_t(\rho c_i) + \operatorname{div}(\rho c_i v + J_i) &= 0, \\ \partial_t(\rho U) + \operatorname{div}(\rho U v + q) &= 0, \\ \partial_t \rho + \operatorname{div}(\rho v) &= 0, \quad i = 1, \ldots, n, \end{aligned} \tag{83}$$

where ρ_i is the partial density of the ith species, $\rho = \sum_{i=1}^n \rho_i$ the total density, $c_i = \rho_i/\rho$ the concentration of the ith species, v the barycentric velocity, J_i the ith flux, q the heat flux, and the internal energy U is given by the first law of thermodynamics in differential form by

$$dU = TdS - pdV + \sum_{i=1}^n \mu_i dc_i, \tag{84}$$

where S is the entropy, $V = 1/\rho$ the volume, and $\mu_i = \partial U/\partial c_i$ the ith chemical potential. By definition, it holds that $\sum_{i=1}^n c_i = 1$. Adding the first and last equation in (83), we see that $\operatorname{div} \sum_{i=1}^n J_i = 0$, which motivates us to assume that $\sum_{i=1}^n J_i = 0$.

The sum of the fluxes should vanish, $\sum_{i=1}^n J_i = 0$, to be consistent with the conservation laws.

With the material derivative $D_t f = \partial_t f + v \cdot \nabla f$, the conservation laws can be simplified to

$$\rho D_t c_i + \operatorname{div} J_i = 0, \quad \rho D_t U + \operatorname{div} q = 0, \quad D_t \rho + \rho \operatorname{div} v = 0.$$

Inserting these equations into equation (84), formulated as $D_tU = TD_tS - pD_tV + \sum_{i=1}^n \mu_i D_t c_i$, yields the entropy balance

$$\begin{aligned} \rho D_t S &= \frac{\rho}{T} D_t U + \frac{\rho}{T} p D_t \left(\frac{1}{\rho}\right) - \sum_{i=1}^n \frac{\mu_i}{T} D_t c_i \\ &= -\frac{1}{T} \operatorname{div} q + \frac{p}{T} \operatorname{div} v + \sum_{i=1}^n \frac{\mu_i}{T} \operatorname{div} J_i = -\operatorname{div} J_S + r_S, \end{aligned}$$

where

$$J_S = \frac{q}{T} - \sum_{i=1}^n \frac{\mu_i}{T} J_i, \quad r_S = q \cdot \nabla \frac{1}{T} + p \operatorname{div} v + \sum_{i=1}^n J_i \cdot \nabla \frac{\mu_i}{T}$$

are the entropy flux and entropy production, respectively.

In our Maxwell–Stefan model, we assume that $v = 0$ and $T = 1$. Then the entropy production simplifies to $r_S = \sum_{i=1}^n J_i \cdot \nabla \mu_i$. It can be reformulated by taking into account that $\sum_{i=1}^n J_i = 0$ and hence $J_i/\sqrt{c_i} \in L = \{\boldsymbol{x} \in \mathbb{R}^n : \sqrt{\boldsymbol{c}} \cdot \boldsymbol{x} = 0\}$:

$$r_S = -\sum_{i=1}^n \frac{J_i}{\sqrt{c_i}} \cdot \sqrt{c_i} \nabla \mu_i = -\sum_{i,j=1}^n \frac{J_i}{\sqrt{c_i}} \cdot (P_L)_{ij} \sqrt{c_j} \nabla \mu_j = -\sum_{i=1}^n \frac{J_i}{\sqrt{c_i}} \cdot \sum_{j=1}^n (P_L)_{ij} \sqrt{c_j} \nabla \mu_j,$$

where the projection P_L on L is defined in (22). By the second law of thermodynamics, it should hold that $r_S \geq 0$. To guarantee this property, we introduce a positive semidefinite matrix $B(\boldsymbol{c})$ such that

$$\sum_{j=1}^n (P_L)_{ij} \sqrt{c_j} \nabla \mu_j = -\sum_{j=1}^n B_{ij}(\boldsymbol{c}) \frac{J_j}{\sqrt{c_j}}, \quad i = 1, \ldots, n. \tag{85}$$

We claim that these equations correspond to the generalized Maxwell–Stefan equations (17) after setting $J_i = c_i u_i$ and $K_{ij}(\boldsymbol{c}) = \sqrt{c_i} B_{ij}(\boldsymbol{c})/\sqrt{c_j}$ (see (18)). Indeed, the left-hand side of (85), multiplied by $\sqrt{c_i}$, becomes

$$\sqrt{c_i} \sum_{j=1}^n (P_L)_{ij} \sqrt{c_j} \nabla \mu_j = c_i \nabla \mu_i - c_i \sum_{j=1}^n c_j \nabla \mu_j,$$

and the right-hand side of (85), multiplied by $\sqrt{c_i}$, equals

$$-\sqrt{c_i} \sum_{j=1}^n B_{ij}(\boldsymbol{c}) \frac{J_j}{\sqrt{c_j}} = -\sum_{j=1}^n K_{ij}(\boldsymbol{c}) J_j = -\sum_{j=1}^n K_{ij}(\boldsymbol{c}) c_j u_j.$$

Hence, observing that $\mu_j = \partial U/\partial c_j$ corresponds to $\delta H/\delta c_j$, (85) equals (17).

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