A HIERARCHY OF HYDRODYNAMIC MODELS FOR PLASMAS ZERO-RELAXATION-TIME LIMITS

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Abstract – We present a model hierarchy of hydrodynamic and quasi-hydrodynamic equations for plasmas consisting of electrons and ions, and give a rigorous proof of the zero-relaxation-time limits in the hydrodynamic equations, described by the Euler equations coupled with a linear or nonlinear Poisson equation. The proof is based on the high energy estimates for the Euler equations together with compactness arguments.

1 Introduction

The mathematical study of Euler-Poisson systems for plasmas has attracted a lot of attention in the mathematical literature since several years (see, e.g.,[4, 6, 12, 19, 20, 22, 24, 25, 26]). In order to perform numerical simulations of the hyperbolic equations, a lot of computing power and special algorithms are needed [8, 10]. In some situations, however, the model equations can be approximated by simpler equations, like drift-diffusion models, in the sense that a small parameter appearing in the hyperbolic equations is set equal to zero. Considering a plasma composed of electrons and ions, the small parameters are, e.g., the electron mass ("zero-electron-mass limit") or the relaxation time ("zero-relaxation-time limit"). Therefore, letting the small parameters tend to zero we obtain a *hierarchy of hydrodynamic and quasi-hydrodynamic plasma models.*

We want to present this model hierarchy, make precise the connections between the corresponding systems, and prove rigorously the asymptotic limits as the small parameters tend to zero. In this paper we are concerned with the zero-relaxation-time limits in the hydrodynamic equations. The zero-electron-mass limits in the drift-diffusion equations and in the hydrodynamic equations are given in [15, 9].

We consider an unmagnetized plasma consisting of electrons with mass m_e and charge $q_e = -1$ and of a single species of ions with mass m_i and charge $q_i = 1$. Denote by $n_e = n_e(t,x)$, $u_e = u_e(t,x)$ (respectively, n_i, u_i) the scaled density and mean velocity of the electrons (respectively, ions) and by $\varphi = \varphi(t,x)$ the scaled electric potential at time $t > 0$ and position $x \in I\!R^d$. These variables satisfy the following scaled Euler-Poisson system **(HD-EI)**:

$$m_\alpha \partial_t n_\alpha + m_\alpha \mathrm{div}(n_\alpha u_\alpha) = 0, \tag{1.1}$$

$$m_\alpha \partial_t (n_\alpha u_\alpha) + m_\alpha \mathrm{div}(n_\alpha u_\alpha \otimes u_\alpha) + \nabla p_\alpha(n_\alpha) = -q_\alpha n_\alpha \nabla\varphi - m_\alpha \frac{n_\alpha u_\alpha}{\tau_\alpha}, \tag{1.2}$$

$$-\lambda^2 \Delta\varphi = n_i - n_e, \tag{1.3}$$

where $\alpha = e, i$ and $(t,x) \in I\!R^+ \times I\!R^d$. Here, $u_\alpha \otimes u_\alpha$ denotes the tensor product with components $u_{\alpha,j} u_{\alpha,k}$ for $j,k = 1,\ldots,d$, $\lambda > 0$ is the scaled Debye length, and $\tau_e > 0$ and $\tau_i > 0$ are the scaled relaxation time constants for electrons and ions, respectively.

The pressure functions are usually of the form

$$p_\alpha(n_\alpha) = a_\alpha^2 n_\alpha^{\gamma_\alpha}, \quad n_\alpha \geq 0,$$

where $\gamma_\alpha \geq 1$ and $a_\alpha > 0$ are constants. The fluid is called *isothermal* if $\gamma_\alpha = 1$ ($\alpha = e$ or $\alpha = i$) and *adiabatic* if $\gamma_\alpha > 1$.

The system (1.1)-(1.3) is complemented by initial conditions for n_α and u_α and by boundary conditions for φ :

$$t = 0: \; n_\alpha = n_{\alpha 0}(x), \; u_\alpha = u_{\alpha 0}(x), \quad x \in I\!R^d, \tag{1.4}$$

$$\lim_{|x|\to\infty} \varphi(t,x) = 0, \quad \text{a.e.} \quad t > 0. \tag{1.5}$$

The homogeneous boundary condition (1.5) means that the plasma is in equilibrium at infinity. In one-dimensional case, the problem (1.1)-(1.5) has been studied in [4, 19, 22, 25].

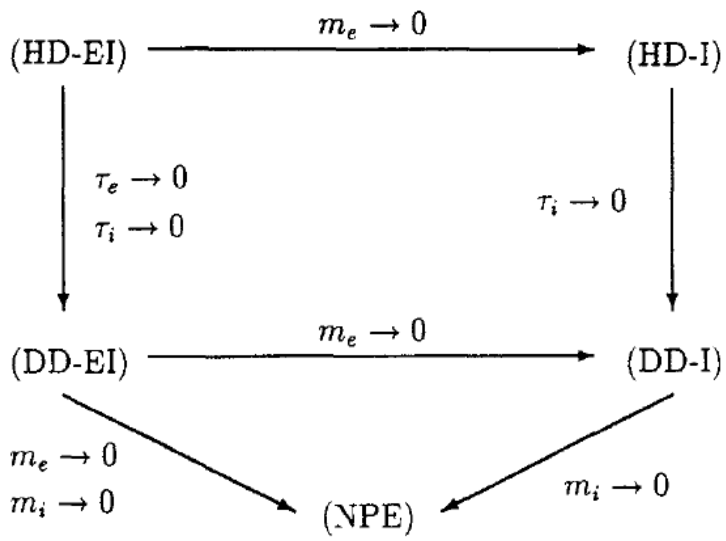

Figure 1: A hierarchy of plasma models.

Usually, the ions are heavy compared to the electrons, i.e. $m_i \gg m_e$. Therefore, letting $m_e \to 0$ in the equations (1.1)-(1.3), we obtain formally the model **(HD-I)** :

$$m_i \partial_t n + m_i \mathrm{div}(nu) = 0, \tag{1.6}$$

$$m_i \partial_t (nu) + m_i \mathrm{div}(nu \otimes u) + \nabla p(n) = -n\nabla\varphi - m_i \frac{nu}{\tau}, \tag{1.7}$$

$$-\lambda^2 \Delta\varphi = n - f(\varphi), \tag{1.8}$$

where we have used the notations $n = n_i$, $u = u_i$ and $\tau = \tau_i$ etc. See a physical explanation of this process given in [15]. The function $f = f_e$ is defined by

$$q_\alpha^{-1}(\varphi) := f_\alpha(\varphi) \geq 0, \;\; q'_\alpha(s) = p'_\alpha(s)/s, \;\; \alpha = e, i, \tag{1.9}$$

where q_α^{-1} is the inverse function of the enthalpy q_α. This zero-electron-mass limit will be studied in [9]. The existence of global weak entropic solutions to (1.6)-(1.8) is shown in [6, 24] when $d = 1$.

Another set of equations is obtained in the zero-relaxation-time limit of the model (HD-EI) and (HD-I). Indeed, introduce a scaling of time $s = \tau t$ and define

$$N_\alpha(x,s) = n_\alpha(x, \frac{s}{\tau}), \;\; U_\alpha(x,s) = \frac{1}{\tau} u_\alpha(x, \frac{s}{\tau}), \;\; \Phi(x,s) = \varphi(x, \frac{s}{\tau}), \qquad (1.10)$$

where, for simplicity, we take $\tau = \tau_e = \tau_i$. Setting again $t = s$, then the problem (1.1)-(1.3) become, for $\alpha = e, i$,

$$m_\alpha \partial_t N_\alpha + m_\alpha \text{div}(N_\alpha U_\alpha) = 0, \qquad (1.11)$$

$$\begin{aligned} &\tau^2 m_\alpha \partial_t (N_\alpha U_\alpha) + \tau^2 m_\alpha \text{div}(N_\alpha U_\alpha \otimes U_\alpha) + \nabla p_\alpha(N_\alpha) \\ &= -q_\alpha N_\alpha \nabla \Phi - m_\alpha N_\alpha U_\alpha, \end{aligned} \qquad (1.12)$$

$$-\lambda^2 \Delta \Phi = N_i - N_e. \qquad (1.13)$$

Letting formally $\tau \to 0$, we obtain the model **(DD-EI)** :

$$m_\alpha \partial_t N_\alpha - \text{div}(\nabla p_\alpha(N_\alpha) + q_\alpha N_\alpha \nabla \Phi) = 0, \;\; \alpha = e, i, \qquad (1.14)$$

$$-\lambda^2 \Delta \Phi = N_i - N_e. \qquad (1.15)$$

Equations of this type are treated in [13, 14]. Furthermore, using the relaxation-time scaling (1.10) in equations (1.6)-(1.8) and letting $\tau \to 0$, we get the model **(DD-I)**:

$$m_i \partial_t N - \text{div}(\nabla p(N) + N \nabla \Phi) = 0, \qquad (1.16)$$

$$-\lambda^2 \Delta \Phi = N - f(\Phi). \qquad (1.17)$$

Similarly, after the same scaling $s = \tau t$ and setting again $t = s$, the model (HD-I) can be written as :

$$m_i \partial_t N + m_i \text{div}(NU) = 0, \qquad (1.18)$$

$$\tau^2 m_i \partial_t N + \tau^2 m_i \text{div}(NU \otimes U) + \nabla p(N) = -N \nabla \Phi - m_i NU, \qquad (1.19)$$

$$-\lambda^2 \Delta \Phi = N - f(\Phi), \qquad (1.20)$$

of which the formal limit as $\tau \to 0$ is still the model (DD-I). Finally, the formal limits as $m_e \to 0$ and $m_i \to 0$ in the system (DD-EI) lead to the following nonlinear Poisson equation **(NPE)**:

$$-\lambda^2\Delta\Phi = f_i(C-\Phi) - f(\Phi), \tag{1.21}$$

where $C \in I\!R$ is a constant. This equation can also be obtained formally from the system (DD-I) by letting $m_i \to 0$ in equation (1.16).

We observe that the one-dimensional equation (NPE) has been considered in [26]. The limit as $\lambda \to 0$ in the equations (1.18)-(1.20) has been partially performed in [5] for the smooth solutions. The rigorous proofs of the limits (DD-EI) $\longrightarrow$ (DD-I) $\longrightarrow$ (NPE) and (DD-EI) $\longrightarrow$ (NPE) have been given in [15]. In particular, these two limits are commutative. The zero-relaxation-time limit (HD-EI) $\longrightarrow$ (DD-EI) has been obtained in [20, 22] in the adiabatic case, under assumption of existence of a priori L^∞ estimate. The isothermal case has been proved in [12]. See Figure 1 for a summary of the above models and limits. For the theory of relaxation to the hyperbolic systems of conservation laws, we refer to the works [3, 11, 18, 23, 29].

In one space dimension, the existence of global weak solutions of these models has been investigated by many authors. Two frameworks may be applied. The isothermal case $\gamma_i = \gamma_e = 1$ is usually treated by the fractional step Glimm scheme, see for example [4, 6, 25]. The existence of weak solutions can be proved for arbitrarily large data in $BV(I\!R)$, because of the diminution in time of the total variation of the quantity $\log n(t,.)$ in the Glimm scheme. The adiabatic case $\gamma_i > 1$ and $\gamma_e > 1$ is treated by the fractional step Lax-Friedrichs scheme or Godunov scheme together with the compensated compactness argument, see [19, 20, 24].

In this paper, we are devoted to the zero-relaxation-time limits in the models (HD-EI) and (HD-I). In the next section, we recall the result of existence of weak solutions to the model (1.18)-(1.20) and state the main results of this paper. Section 3 is concerned with the entropy inequalities for the Euler equations. The main goal is to construct a family of positive and convex entropies to deduce high energy estimates of solutions. The results followed are based on the characterization of the convexity of the weak entropies proved in [17]. The uniform L^p estimates of solutions will be given in section 4 by using the high energy estimates of the system. Finally, in section 5, we prove the convergence of a subsequence of $(N_\tau, J_\tau, \Phi_\tau)_{\tau>0}$ to a solution of (DD-I). The proof is accomplished by applying the div-curl lemma, Aubin's compactness theorem and monotonicity argument.

2 Statement of the main results

We consider the zero-relaxation-time limits in the above hydrodynamic models in one-dimensional case $d = 1$. Our main goal is to give rigorous proofs of the limits (HD-EI) $\longrightarrow$ (DD-EI) and (HD-I) $\longrightarrow$ (DD-I) as the relaxation times $\tau_i \to 0$ and $\tau_e \to 0$. Since the proof of the first one is completely contained in that of the second, we will focus our study in the analysis of the problem (HD-I) $\longrightarrow$ (DD-I).

For simplicity, we suppose throughout this paper that $\lambda = m_i = 1$. Let $J_\tau = N_\tau U_\tau$. Then the equations (1.18)-(1.20) can be described by

$$\partial_t N_\tau + \partial_x J_\tau = 0, \quad t > 0, \quad x \in I\!R, \tag{2.1}$$

$$\tau^2 \partial_t N_\tau + \partial_x \Big(\frac{\tau^2 J_\tau^2}{N_\tau} + p(N_\tau) \Big) = -N_\tau \partial_x \Phi_\tau - J_\tau, \quad t > 0, \quad x \in I\!R, \tag{2.2}$$

$$\partial_x E_\tau = -\partial_{xx} \Phi_\tau = N_\tau - f(\Phi_\tau), \quad t > 0, \quad x \in I\!R, \tag{2.3}$$

with the initial and boundary conditions, followed from (1.4)-(1.5) :

$$N_\tau(0, x) = n_0(x), \quad J_\tau(0, x) = \frac{1}{\tau} j_0(x), \quad x \in I\!R, \tag{2.4}$$

$$\lim_{|x| \to \infty} \Phi_\tau(t, x) = 0, \quad \text{a.e.} \quad t > 0, \tag{2.5}$$

where $E_\tau = -\partial_x \Phi_\tau$ is the electric field and $j_0 = n_0 u_0$. The fluid of ions is assumed to be adiabatic, then the state equation is given by

$$p(n) = a^2 n^\gamma, \tag{2.6}$$

where $a > 0$ and $\gamma > 1$ are constants.

The existence of global weak solutions to (2.1)-(2.6) has been considered in [24] by using a fractional step Lax-Friedrichs scheme together with a compensated compactness argument. The main assumptions needed are :

(H1) $f \in C^1(I\!R^+)$, $f(0) = 0$, $f(+\infty) = +\infty$, $f'(s) > 0$, $\forall\, s > 0$,

(H2) $0 \le n_0$, $u_0 \in L^\infty(I\!R)$, $n_0 = 0$ for $|\, x\,| \ge L$,

where $L > 0$ is a given constant. We now summarize the result of existence of solutions as follow :

Theorem 1 *Let* $T > 0$ *and* $Q_T = [0, T[\times I\!R$. *Assume (H1)-(H2) hold and* $\gamma > 1$. *Then for any* $\tau > 0$, *the problem (2.1)-(2.6) has a global weak entropic solution* $(N_\tau, J_\tau, \Phi_\tau)$ *such that* $0 \le N_\tau$, $J_\tau/N_\tau \in L^\infty(Q_T)$, $0 \le \Phi_\tau \in L^\infty([0, T]; W^{2,\infty}(I\!R))$, *satisfying the weak formulation, i.e. for any test functions* ϕ_1 *and* ϕ_2 *of class* $C^1(Q_T)$ *with compact support in* Q_T *:*

$$\iint_{Q_T} (N_\tau \partial_t \phi_1 + J_\tau \partial_x \phi_1)(t, x)\, dx\, dt + \int_{I\!R} n_0(x)\phi_1(0, x)\, dx = 0, \tag{2.7}$$

$$\begin{aligned}&\iint_{Q_T} \left[\tau^2 J_\tau \partial_t \phi_2 + (\tau^2 J_\tau^2/N_\tau + p(N_\tau))\partial_x \phi_2 - (J_\tau + N_\tau \partial_x \Phi_\tau)\phi_2\right](t, x) dx\, dt \\ &+ \ \tau \int_{I\!R} j_0(x)\phi_2(0, x)\, dx = 0,\end{aligned} \tag{2.8}$$

$$-\partial_{xx}\Phi_\tau + f(\Phi_\tau) = N_\tau, \ \ in \ \ L^\infty(Q_T), \tag{2.9}$$

$$\lim_{|x|\to\infty} \Phi_\tau(t, x) = 0, \ \ a.e. \ \ t \in [0, T], \tag{2.10}$$

and the entropy condition for any pair of continuous entropy-flux (η_τ, q_τ) *of (2.1)-(2.3), with* η_τ *convex in* (N_τ, J_τ) *:*

$$\begin{aligned}&\partial_t \eta_\tau(N_\tau, \frac{J_\tau}{N_\tau}) + \partial_x q_\tau(N_\tau, \frac{J_\tau}{N_\tau}) + \frac{1}{\tau^2}(J_\tau + N_\tau \partial_x \Phi_\tau)\frac{\partial \eta_\tau(N_\tau, \frac{J_\tau}{N_\tau})}{\partial J_\tau} \\ &\le 0, \ \ in \ \ \mathcal{D}'(Q_T).\end{aligned} \tag{2.11}$$

Furthermore, N_τ *and* J_τ/N_τ *have compact support in* $[0, T] \times [-L_\tau(T), L_\tau(T)]$, *where* $L_\tau(T) > 0$ *is a constant depending on the given data and* L. □

The proof of Theorem 1 is given in [24] for the value $\gamma \in]1, 5/3]$. The case $\gamma > 1$ can be treated in the same way. Indeed, it is shown in [24] that for any pair of weak entropy-flux (η, q), the sequence $(\partial_t \eta(N^h, U^h) + \partial_x q(N^h, U^h))_{h>0}$ lies in a compact set of $H^{-1}_{loc}(Q_T)$, valid for any $\gamma > 1$, where $(N^h, U^h = J^h/N^h, \Phi^h)_{h>0}$ is the approximate solution constructed by the fractional step Lax-Friedrichs scheme or Godunov scheme with the space mesh size $h > 0$. Therefore, applying the result in [16], we have the strong convergence of a subsequence of $(N^h, J^h)_{h>0}$, which implies the strong convergence of the sequence $(\Phi^h)_{h>0}$. The consistency of the schemes [24] shows that the limit of (N^h, J^h, Φ^h) is a weak solution of the problem (2.1)-(2.6).

If a priori $L^\infty(Q_T)$ estimate is available for the sequence $(N_\tau)_{\tau>0}$, it is easier (than the argument below) to pass to the limit in (2.1)-(2.6) in the weak sense.

The weak limit (N, J, Φ) of $(N_\tau, J_\tau, \Phi_\tau)$, with $N \geq 0$ and $\Phi \geq 0$, satisfies the following problem in the sense of distributions :

$$\partial_t N - \partial_x(\partial_x p(N) + N\partial_x \Phi) = 0, \;\; t > 0, \; x \in I\!R, \tag{2.12}$$

$$-\partial_{xx}\Phi = N - f(\Phi), \;\; t > 0, \; x \in I\!R, \tag{2.13}$$

$$J = -\partial_x p(N) - N\partial_x \Phi, \;\; t > 0, \; x \in I\!R, \tag{2.14}$$

with initial and boundary conditions

$$N(0, x) = n_0(x), \;\; x \in I\!R, \;\; \lim_{|x|\to\infty} \Phi(t, x) = 0, \;\; \text{a.e. } t > 0. \tag{2.15}$$

In particular, the convergence of a subsequence of $(N_\tau)_{\tau>0}$ is strong in $L^p_{loc}(Q_T)$ for any $p \in [1, \infty[$. This is due to the pointwise convergence deduced from the Young measures and the $L^\infty(Q_T)$ bound of $(N_\tau)_{\tau>0}$, see [20, 22].

Unfortunately, this $L^\infty(Q_T)$ bound of $(N_\tau)_{\tau>0}$ has not been justified and we really don't know if it exists. To remedy this, we try to establish L^p estimates for $(N_\tau, J_\tau)_{\tau>0}$ and show the strong convergence in $L^p_{loc}(Q_T)$ of a subsequence of $(N_\tau)_{\tau>0}$ for any $p \in [1, \gamma + 1]$. This is achieved by using the well-known div-curl lemma and the monotonicity method. The strong convergence of (N_τ) is sufficient to pass to the limit in (2.7)-(2.8).

The L^p estimates of $(N_\tau, J_\tau)_{\tau>0}$ are then the main task of the proof. They will be obtained by using the entropy conditions (2.11). The key point is to choose a family of positive and convex entropies. Here we will construct a sequence of positive and convex entropies $(\eta_\tau^{(k)})_{k\in I\!N}$, belonging to the weak entropy family of the system (2.1)-(2.3). The construction is based on the characterization of the convexity of the weak entropies proved in [17]. As we will see, this sequence of entropies represents the high energy of the system. We show then that this sequence of entropies gives the desired estimates.

We observe that the Poisson equation (2.3) is nonlinear. Therefore, we have to prove the strong compactness of the sequence $(\Phi_\tau)_{\tau>0}$, which follows from Aubin's theorem [1]. Indeed, by the hypothesis (H1') below, we are able to show that the sequences $(\Phi_\tau)_{\tau>0}$ and $(\partial_t \Phi_\tau)_{\tau>0}$ are bounded respectively in $L^\infty(0, T; H^1(I\!R))$ and $L^2(0, T; H^1(I\!R))$.

This study needs two further hypotheses :

(H1') $f \in C^1(I\!R^+)$, $f(0) = 0$, $f(s) \geq a_0^2 s^{\gamma_0}$, $f'(s) > f_0 > 0$, $\forall\, s \geq 0$, (H3) $\gamma = 1 + 2/m$, with $1 \leq m \in I\!N$,

where $f_0, a_0 > 0$ and $\gamma_0 \geq 1$ are constants. Condition (H1') is stronger than (H1) and fulfilled, for example, by functions $f(s) = a_0^2 s$ and $f(s) = e^s - 1$, which correspond respectively to the state equations for electrons $p_e(n_e) = a_e^2 n_e^2$ and $p_e(n_e) = n_e - \log(n_e + 1)$. It is sufficient to get the compactness of $(\Phi_\tau)_{\tau>0}$. Condition (H3) has been first imposed by DiPerna [7] for $m \geq 5$ to prove the global existence of solutions for the isentropic gas dynamics equations. DiPerna's result has been generalized by Chen [2] for $\gamma \in]1, 5/3]$, and then by Lions-Perthame-Tadmor [17], Lions-Perthame-Souganidis [16] for any value $\gamma > 1$. Condition (H3) is only needed to obtain the $L^2(Q_T)$ estimate for $(J_\tau)_{\tau>0}$. Unfortunately, we do not know how to obtain it for any value $\gamma > 1$ in our problem. Therefore, our results of zero-relaxation-time limits are essentially valid only for $\gamma \in]1, 3]$.

Now we can state the main result of this paper.

Theorem 2 *((HD-I) $\longrightarrow$ (DD-I)) Let $(N_\tau, J_\tau, \Phi_\tau)$ be a weak entropic solution of (2.1)-(2.6). Suppose the conditions (H1'), (H2)-(H3) hold. Then, as $\tau \to 0$, passing if necessary to subsequences, $(N_\tau, J_\tau, \Phi_\tau)$ converges to (N, J, Φ) in the following sense :*

$$N_\tau \longrightarrow N, \quad \text{in } L^p_{loc}(Q_T) \text{ strongly, for any } p \in [1, \gamma + 1],$$

$$J_\tau \rightharpoonup J, \quad \text{in } L^2(Q_T) \text{ weakly,}$$

$$\frac{\tau^2 J_\tau^2}{N_\tau} \longrightarrow 0, \quad \text{in } L^2(Q_T) \text{ strongly,}$$

$$\Phi_\tau \longrightarrow \Phi, \quad \text{in } L^2_{loc}(Q_T) \text{ strongly,}$$

where (N, J, Φ), satisfying $(N, J) \in L^\infty(0, T; L^p(I\!R)) \times L^2(Q_T)$ for any $p \in]1, +\infty[$, and $\Phi \in L^\infty(0, T; W^{2,p}(I\!R))$ for any $p \in [2, +\infty[$, is a solution of (2.12)-(2.15). □

Remark 1 *The similar result holds for the zero-relaxation-time limit (HD-EI) $\longrightarrow$ (DD-EI) in one-dimensional case. Indeed, the Poisson equation* (1.13) *is linear. Then the $L^\infty(Q_T)$ estimate for the electric field E_τ of the model (HD-EI) can be easily obtained by the formulas*

$$E_\tau(t,x) = E^-(t) + \int_{-\infty}^{x} \left(N_i^\tau(t,y) - N_e^\tau(t,y)\right) dy,$$

and

$$\int_{I\!R} N_\alpha^\tau(t,x)dx = \int_{I\!R} n_{\alpha 0}(x)dx, \quad \alpha = e, i,$$

where E^- is a given function. The remainder of the analysis is completely contained in the proof of Theorem 2.

3 Construction of positive and convex entropies

In this section, we investigate the entropy inequalities of the system (2.1)-(2.3). For simplicity, we drop the subscript τ when there is no confusion. Then the homogeneous hyperbolic system which appears in (2.1)-(2.3) is :

$$\partial_t N + \partial_x J = 0, \quad t > 0, \quad x \in I\!R, \tag{3.1}$$

$$\partial_t J + \partial_x (J^2/N + p_\tau(N)) = 0, \quad t > 0, \quad x \in I\!R, \tag{3.2}$$

where

$$p_\tau(N) = a_\tau^2 N^\gamma, \quad a_\tau = a/\tau. \tag{3.3}$$

A pair of functions (η, q) is called entropy-flux of the system (3.1)-(3.2) if for any smooth solution $(N, U = J/N)$, we have

$$\partial_t \eta(N,U) + \partial_x q(N,U) = 0. \tag{3.4}$$

This condition is equivalent to

$$\frac{\partial q}{\partial N} = U\frac{\partial \eta}{\partial N} + \frac{p_\tau'(N)}{N}\frac{\partial \eta}{\partial U}, \quad \frac{\partial q}{\partial U} = N\frac{\partial \eta}{\partial N} + U\frac{\partial \eta}{\partial U} \tag{3.5}$$

which yields

$$\frac{\partial^2 \eta}{\partial N^2} = \frac{p_\tau'(N)}{N}\frac{\partial^2 \eta}{\partial U^2}. \tag{3.6}$$

We say that η is a weak entropy if $\eta(0,U) = 0$ for any $U \in I\!R$. In particular, the physical entropy (energy) and its flux are given by

$$
\begin{cases} \eta^*(N,U) = \dfrac{1}{2}NU^2 + \dfrac{a_\tau^2 N^\gamma}{\gamma - 1} \ , \\ q^*(N,U) = \dfrac{1}{2}NU^3 + \dfrac{\gamma a_\tau^2 N^\gamma U}{\gamma - 1} \ . \end{cases} \tag{3.7}
$$

It is known that η^* is a weak entropy, positive and convex in conservative variables (N, J) for any value $\gamma > 1$.

Let w_1 and w_2 be two classical Riemann invariants of the system, defined by

$$
w_1 = \frac{J}{N} - A_\tau N^{\frac{\gamma-1}{2}}, \ \ w_2 = \frac{J}{N} + A_\tau N^{\frac{\gamma-1}{2}}, \tag{3.8}
$$

with

$$
A_\tau = \frac{A}{\tau}, \ \ A = \frac{2a\sqrt{\gamma}}{\gamma - 1}. \tag{3.9}
$$

According to Lions-Perthame-Tadmor [17], the weak entropy-flux and the convexity of the weak entropies can be characterized as (see also [2] for the expressions (3.10) and (3.11) below) :

Lemma 1 *Let $\gamma > 1$. Then any pair of weak entropy-flux of (3.1)-(3.2) can be written as*

$$
\eta = \int_{w_1}^{w_2} g(\xi)[(w_2 - \xi)(\xi - w_1)]^\lambda \, d\xi, \tag{3.10}
$$

$$
q = \int_{w_1}^{w_2} g(\xi)(\theta\xi + (1-\theta)U)[(w_2 - \xi)(\xi - w_1)]^\lambda \, d\xi, \tag{3.11}
$$

where $g \in L^1(I\!R)$ is arbitrary, and

$$
\theta = \frac{\gamma - 1}{2} \ , \ \ \lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \ \ \gamma > 1 \ . \tag{3.12}
$$

Moreover, η is a convex function in (N, J) if and only if g is convex. □

By the change of variables

$$
\xi = \frac{J}{N} + A_\tau y N^{\frac{\gamma-1}{2}}, \tag{3.13}
$$

(3.10) can be expressed as (up to a constant) :

$$
\eta(N, J) = N \int_{-1}^{1} g\left(\frac{J}{N} + A_\tau y N^{\frac{\gamma-1}{2}}\right)(1 - y^2)^\lambda dy. \tag{3.14}
$$

Lemma 1 and (3.14) allow us to construct a sequence of entropies by choosing positive and convex functions $g(\xi) = \xi^{2k}$ for $k \in I\!N$. We denote by $(\eta_\tau^{(k)}, q_\tau^{(k)})$ the corresponding pair of entropy-flux. From (3.14), we have

$$\eta_\tau^{(k)}(N,J) = N\int_{-1}^{1}\left(\frac{J}{N} + A_\tau y N^{\frac{\gamma-1}{2}}\right)^{2k}(1-y^2)^\lambda dy, \quad k \in I\!N. \tag{3.15}$$

From Lemma 1, it is clear that $\eta_\tau^{(k)}$ is positive and convex for any $k \in I\!N$. Moreover, it is easy to check that $\eta_\tau^{(0)}$ and $\eta_\tau^{(1)}$ give respectively the density N and the usual energy η^*. Therefore, we call $\eta_\tau^{(k)}$ high energy of order k of the system.

We now seek an explicit expression of $\eta_\tau^{(k)}$. Since

$$\begin{aligned}\left(\frac{J}{N} + A_\tau y N^{\frac{\gamma-1}{2}}\right)^{2k} &= \sum_{i=0}^{2k} C_{2k}^i \left(\frac{J}{N}\right)^i A_\tau^{2k-i} y^{2k-i} N^{(2k-i)(\gamma-1)/2} \\ &= \sum_{i=0}^{2k} C_{2k}^i A_\tau^{2k-i} y^{2k-i} N^{(2k-i)(\gamma-1)/2-i} J^i,\end{aligned}$$

with $C_k^i = \frac{k!}{(k-i)!\,i!}$, and

$$\int_{-1}^{1} y^i (1-y^2)^\lambda d\,y = 0, \quad \text{for } i \text{ odd},$$

we deduce that

$$\begin{aligned}\eta_\tau^{(k)}(N,J) &= N\int_{-1}^{1} \sum_{i\le 2k,\ i\ even} C_{2k}^i A_\tau^{2k-i} y^{2k-i}(1-y^2)^\lambda N^{(2k-i)(\gamma-1)/2-i} J^i d\,y \\ &= N\int_{-1}^{1}\sum_{i=0}^{k} C_{2k}^{2i} A_\tau^{2(k-i)} y^{2(k-i)} (1-y^2)^\lambda N^{(k-i)(\gamma-1)-2i} J^{2i} d\,y.\end{aligned}$$

Hence

$$\eta_\tau^{(k)}(N,J) = \sum_{i=0}^{k} \beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i}, \tag{3.16}$$

where

$$\beta_i^{(k)} = C_{2k}^{2i}\int_{-1}^{1} y^{2(k-i)}(1-y^2)^\lambda d\,y, \quad 0 \le i \le k, \tag{3.17}$$

$$\alpha_i^{(k)} = (k-i)(\gamma-1) - 2i + 1, \quad 0 \le i \le k, \tag{3.18}$$

which are constants independent of τ. Obviously, we have $0 < \beta_i^{(k)} \le 2C_{2k}^{2i}$ and

$$\frac{\partial \eta_\tau^{(k)}}{\partial J} = \sum_{i=1}^{k} 2i\beta_i^{(k)} A_\tau^{2(k-i)} N^{\alpha_i^{(k)}} J^{2i-1},$$

or equivalently

$$\frac{\partial \eta_\tau^{(k)}}{\partial J} = 2\sum_{i=0}^{k-1}(i+1)\beta_{i+1}^{(k)} A_\tau^{2(k-i-1)} N^{\alpha_{i+1}^{(k)}} J^{2i+1}. \tag{3.19}$$

Finally, we remark that the pair of entropy-flux $(\eta_\tau^{(k)}, q_\tau^{(k)})$ as well as $\frac{\partial \eta_\tau^{(k)}}{\partial J}$ are well defined for $(N,J) \in I\!R^+ \times I\!R$ and $k \in I\!N$, since $N_\tau, J_\tau/N_\tau \in L^\infty(Q_T)$.

We conclude this section by the following entropy inequalities of the problem (2.1)-(2.3).

Lemma 2 *For any $k \in I\!N^*$, the weak solutions given by Theorem 1 satisfy the entropy inequalities :*

$$\begin{aligned} &\frac{d}{dt}\int_{I\!R}\sum_{i=0}^{k}\beta_i^{(k)} A^{2(k-i)}\tau^{2i}\left(N_\tau^{\alpha_i^{(k)}} J_\tau^{2i}\right)(t,x)dx \\ &\le 2\sum_{i=0}^{k-1}(i+1)\beta_{i+1}^{(k)} A^{2(k-i-1)}\tau^{2i}\int_{I\!R}\left(E_\tau N_\tau^{\alpha_{i+1}^{(k)}+1} J_\tau^{2i+1} - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)}\right)(t,x)dx. \ \Box \end{aligned} \tag{3.20}$$

Proof. Applying the entropy inequalities (2.11) to $(\eta_\tau^{(k)}, q_\tau^{(k)})$, we obtain

$$\partial_t \eta_\tau^{(k)}(N_\tau, J_\tau) + \partial_x q_\tau^{(k)}(N_\tau, J_\tau) \le \frac{1}{\tau^2}(N_\tau E_\tau - J_\tau)\frac{\partial \eta_\tau^{(k)}(N_\tau, J_\tau)}{\partial J_\tau}, \tag{3.21}$$

which implies using (3.16) and (3.19) :

$$\begin{aligned} &\partial_t\left(\sum_{i=0}^{k}\beta_i^{(k)} A_\tau^{2(k-i)} N_\tau^{\alpha_i^{(k)}} J_\tau^{2i}\right) + \partial_x q_\tau^{(k)} \\ \le\ & \frac{2}{\tau^2}(E_\tau N_\tau - J_\tau)\sum_{i=0}^{k-1}(i+1)\beta_{i+1}^{(k)} A_\tau^{2(k-i-1)} N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2i+1} \\ \le\ & \frac{2}{\tau^2}\sum_{i=0}^{k-1}(i+1)\beta_{i+1}^{(k)} A_\tau^{2(k-i-1)}\left(E_\tau N_\tau^{\alpha_{i+1}^{(k)}+1} J_\tau^{2i+1} - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)}\right). \end{aligned}$$

Since N_τ and J_τ/N_τ have compact support in $[0,T]\times[-L_\tau(T), L_\tau(T)]$, so has $q_\tau^{(k)}$. Multiplying the last relation by τ^{2k} and integrating it over $I\!R$, we get (3.20). $\Box$

4 Uniform estimates of solutions

This section is devoted to the uniform estimates for the sequence of solutions $(N_\tau, J_\tau, \Phi_\tau)_{\tau>0}$. We prove that $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^p(I\!R))$ for any $p \in [1,+\infty[$ and $(J_\tau)_{\tau>0}$ is bounded in $L^2(Q_T)$. However, the bounds of the first estimate is not uniform in p. Therefore, the uniform $L^\infty(Q_T)$ estimate for $(N_\tau)_{\tau>0}$ is not a direct consequence. To obtain the $L^2(Q_T)$ estimate for $(J_\tau)_{\tau>0}$, we have to take the discrete value for γ, i.e. (H3).

Let us first prove

Lemma 3 *The sequences $(N_\tau)_{\tau>0}$ and $(E_\tau)_{\tau>0} = (\partial_x \Phi_\tau)_{\tau>0}$, which are solutions of the problem (2.1)-(2.6), are respectively bounded in $L^\infty(0,T;L^1(I\!R))$ and $L^\infty(Q_T)$.* □

Proof. Integrating equation (2.1) over $I\!R$, we have

$$\partial_t \int_{I\!R} N_\tau(t,x)dx = -\partial_x \int_{I\!R} J_\tau(t,x)dx.$$

Since the support of J_τ is included in $[0,T] \times [-L_\tau(T), L_\tau(T)]$, by $(H2)$, we deduce that

$$\| N_\tau(t,.) \|_{L^1(I\!R)} = \| n_0 \|_{L^1(I\!R)} \le 2L \, \| n_0 \|_{L^\infty(I\!R)} . \tag{4.1}$$

Thus, the sequence $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^1(I\!R))$.

Since $\Phi_\tau \in L^\infty(0,T;W^{2,\infty}(I\!R)) \subset L^\infty(0,T;C^1(I\!R))$, $\Phi_\tau(t,\pm\infty) = 0$ and $\Phi_\tau \ge 0$, there exists a point $x_\tau(t) \in I\!R$ such that

$$\partial_x \Phi_\tau(t, x_\tau(t)) = 0,$$

and

$$\partial_x \Phi_\tau(t,+\infty) \le 0, \;\; \partial_x \Phi_\tau(t,-\infty) \ge 0, \;\; \text{a.e. } t \in [0,T].$$

Integrating the Poisson equation (2.3), we have

$$E_\tau(t,x) = \partial_x \Phi_\tau(t,x) = \int_{x_\tau(t)}^x \left(f(\Phi_\tau(t,y)) - N_\tau(t,y)\right) dy, \;\; \forall \, (t,x) \, \in Q_T.$$

Therefore

$$| \, E_\tau(t,x) \, | = | \, \partial_x \Phi_\tau(t,x) \, | \le \int_{I\!R} (f(\Phi_\tau(t,x)) + N_\tau(t,x)) \, dx, \;\; \forall \, (t,x) \in Q_T. \tag{4.2}$$

On the other hand, by $(H1')$

$$0 \leq \int_{I\!R} f(\Phi_\tau(t,x))\,dx = \int_{I\!R} N_\tau(t,x)\,dx + \partial_x \Phi_\tau(t,+\infty) - \partial_x \Phi_\tau(t,-\infty) \qquad (4.3)$$
$$\leq \| N_\tau(t,.) \|_{L^1(I\!R)} .$$

Thus, the results follow from (4.1) and (4.2). □

We define now, for any $k \in I\!N^*$, the following functions of time $t \in [0,T]$:

$$F_\tau^{(k)}(t) = \sum_{i=0}^{k} \beta_i^{(k)} A^{2(k-i)} \tau^{2i} \int_{I\!R} \left(N_\tau^{\alpha_i^{(k)}} J_\tau^{2i} \right)(t,x)dx \qquad (4.4)$$

$$G_\tau^{(k)}(t) = 2\sum_{i=0}^{k-1} (i+1)\beta_{i+1}^{(k)} A^{2(k-i-1)} \tau^{2i} \int_{I\!R} \left(N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)} \right)(t,x)dx. \qquad (4.5)$$

Clearly, we have $F_\tau^{(k)}(t) \geq 0$ and $G_\tau^{(k)}(t) \geq 0$ for all $t \in [0,T]$ and all $k \in I\!N^*$.

The key estimate of proving the zero-relaxation-time limit is contained in the following lemma :

Theorem 3 *For any $k \in I\!N^*$, there is a constant $D^{(k)}(T) > 0$, independent of $\tau \in]0,1]$, such that*

$$F_\tau^{(k)}(t) \leq D^{(k)}(T), \quad \forall\, t \in [0,T], \qquad (4.6)$$

$$\int_0^t G_\tau^{(k)}(s)ds \leq D^{(k)}(T), \quad \forall\, t \in [0,T]. \ \square \qquad (4.7)$$

Proof. The proof of (4.6) and (4.7) is carried out by induction in k. For $k = 1$, Lemma 2 gives the usual energy estimate corresponding to the high energy of order 1 :

$$\frac{d}{dt} \int_{I\!R} \left(\beta_0^{(1)} A^2 N_\tau^{\alpha_0^{(1)}} + \beta_1^{(1)} \tau^2 N_\tau^{\alpha_1^{(1)}} J_\tau^2 \right)(t,x)dx$$
$$\leq 2\beta_1^{(1)} \int_{I\!R} \left(E_\tau N_\tau^{\alpha_1^{(1)}+1} J_\tau - N_\tau^{\alpha_1^{(1)}} J_\tau^2 \right)(t,x)dx. \qquad (4.8)$$

By (3.18),

$$\alpha_0^{(1)} = \gamma, \quad \alpha_1^{(1)} = -1.$$

Therefore, (4.8) can be rewritten under form

$$\frac{d}{dt}\int_{\mathbb{R}}\left(\beta_0^{(1)}A^2N_\tau^\gamma+\beta_1^{(1)}\tau^2N_\tau^{-1}J_\tau^2\right)(t,x)dx\le 2\beta_1^{(1)}\int_{\mathbb{R}}\left(E_\tau J_\tau-N_\tau^{-1}J_\tau^2\right)(t,x)dx.$$

Since E_τ is bounded in $L^\infty(Q_T)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\left(E_\tau J_\tau-N_\tau^{-1}J_\tau^2\right)(t,x)dx &\le \int_{\mathbb{R}}\left[\frac{1}{2}\left(\overline{E}^2N_\tau+N_\tau^{-1}J_\tau^2\right)-N_\tau^{-1}J_\tau^2\right](t,x)dx\\ &\le \frac{1}{2}\int_{\mathbb{R}}\left(\overline{E}^2N_\tau-N_\tau^{-1}J_\tau^2\right)(t,x)dx,\end{aligned}$$

where $\overline{E}=\| E_\tau \|_{L^\infty(Q_T)}$ is a constant independent of τ. From Lemma 3, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\beta_0^{(1)}A^2N_\tau^\gamma+\beta_1^{(1)}\tau^2N_\tau^{-1}J_\tau^2\right)(t,x)dx\\ &\le \beta_1^{(1)}\,\overline{E}^2\,\| n_0 \|_{L^1(\mathbb{R})}-\beta_1^{(1)}\int_{\mathbb{R}}\left(N_\tau^{-1}J_\tau^2\right)(t,x)dx.\end{aligned}$$

Integrating this relation over $[0,t]$ and using the initial condition (2.4), we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\left(\beta_0^{(1)}A^2N_\tau^\gamma+\beta_1^{(1)}\tau^2N_\tau^{-1}J_\tau^2\right)(t,x)dx\\ &\le \int_{\mathbb{R}}\left[\beta_0^{(1)}A^2n_0^\gamma(x)+\beta_1^{(1)}n_0(x)u_0^2(x)\right]dx\\ &+\beta_1^{(1)}T\,\overline{E}^2\,\| n_0 \|_{L^1(\mathbb{R})}-\beta_1^{(1)}\int_0^t\int_{\mathbb{R}}\left(N_\tau^{-1}J_\tau^2\right)(s,x)dxds.\end{aligned}$$

There exists then a constant $D(T)>0$ such that

$$F_\tau^{(1)}(t)=\int_{\mathbb{R}}\left(\beta_0^{(1)}A^2N_\tau^\gamma+\beta_1^{(1)}\tau^2N_\tau^{-1}J_\tau^2\right)(t,x)dx\le D(T),\quad \forall\, t\in[0,T],$$

and

$$\int_0^t G_\tau^{(1)}(s,x)ds=2\beta_1^{(1)}\int_0^t\int_{\mathbb{R}}\left(N_\tau^{-1}J_\tau^2\right)(s,x)dxds\le D(T),\quad \forall\, t\in[0,T].$$

This proves (4.6) and (4.7) for $k=1$.

Suppose now (4.6) and (4.7) hold for $k-1\ge 1$, namely, there is a constant $D^{(k-1)}(T)>0$ such that

$$F_\tau^{(k)}(t)\le D^{(k-1)}(T),\quad \int_0^t G_\tau^{(k-1)}(s)ds\le D^{(k-1)}(T),\quad \forall\, t\in[0,T]. \tag{4.9}$$

Observing that

$$\alpha_{i+1}^{(k)} + 2 = \alpha_i^{(k-1)}, \quad 0 \le i \le k-1. \tag{4.10}$$

Indeed, from (3.18), we have

$$\begin{aligned} \alpha_{i+1}^{(k)} + 2 &= (k-i-1)(\gamma-1) - 2(i+1) + 1 + 2 \\ &= ((k-1)-i)(\gamma-1) - 2i + 1 = \alpha_i^{(k-1)}. \end{aligned}$$

Therefore, the term in the right hand side of the integral (3.20) can be estimated as

$$\begin{aligned} & E_\tau N_\tau^{\alpha_{i+1}^{(k)}+1} J_\tau^{2i+1} - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)} \\ &= \left(N_\tau^{\frac{\alpha_{i+1}^{(k)}}{2}} J_\tau^{i+1} \right) \left(E_\tau N_\tau^{\frac{\alpha_i^{(k-1)}}{2}} J_\tau^i \right) - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)} \\ &\le \frac{1}{2} \left(\overline{E}^2 N_\tau^{\alpha_i^{(k-1)}} J_\tau^{2i} - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)} \right). \end{aligned}$$

It follows from (3.20) that

$$\begin{aligned} & \frac{d}{dt} \int_{I\!R} \sum_{i=0}^{k} \beta_i^{(k)} A^{2(k-i)} \tau^{2i} \left(N_\tau^{\alpha_i^{(k)}} J_\tau^{2i} \right)(t,x) dx \\ \le & \sum_{i=0}^{k-1} (i+1) \beta_{i+1}^{(k)} A^{2(k-i-1)} \tau^{2i} \int_{I\!R} \left(\overline{E}^2 N_\tau^{\alpha_i^{(k-1)}} J_\tau^{2i} - N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)} \right)(t,x) dx, \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt} F_\tau^{(k)}(t) \le\ & \overline{E}^2 \sum_{i=0}^{k-1} (i+1) \beta_{i+1}^{(k)} A^{2(k-i-1)} \tau^{2i} \int_{I\!R} \left(N_\tau^{\alpha_i^{(k-1)}} J_\tau^{2i} \right)(t,x) dx \\ & - \frac{1}{2} G_\tau^{(k)}(t). \end{aligned} \tag{4.11}$$

On the other hand,

$$\begin{aligned} & \sum_{i=0}^{k-1} (i+1) \beta_{i+1}^{(k)} A^{2(k-i-1)} \tau^{2i} \int_{I\!R} \left(N_\tau^{\alpha_i^{(k-1)}} J_\tau^{2i} \right)(t,x) dx \\ = \ & \beta_1^{(k)} A^{2(k-1)} \int_{I\!R} N_\tau^{\alpha_0^{(k-1)}}(t,x) dx \\ & + \tau^2 \sum_{i=0}^{k-2} (i+2) \beta_{i+2}^{(k)} A^{2(k-i)} \tau^{2i} \int_{I\!R} \left(N_\tau^{\alpha_{i+1}^{(k-1)}} J_\tau^{2(i+1)} \right)(t,x) dx \\ \le \ & C_0^{(k)} \left(F_\tau^{(k-1)}(t) + \tau^2 G_\tau^{(k-1)}(t) \right), \end{aligned}$$

where $C_0^{(k)}$ is a constant depending only on k. Hence, (4.11) implies

$$\frac{d}{dt}F_\tau^{(k)}(t) + \frac{1}{2}G_\tau^{(k)}(t) \le C_0^{(k)}\ \overline{E}^2 \left(F_\tau^{(k-1)}(t) + \tau^2 G_\tau^{(k-1)}(t) \right).$$

By the hypothesis of induction (4.9), we deduce that

$$F_\tau^{(k)}(t) + \int_0^t G_\tau^{(k)}(s)ds \le F_0^{(k)}(0) + C^{(k)}(T),$$

where $C^{(k)}(T) > 0$ is a constant independent of τ. Observing that

$$\begin{aligned} F_0^{(k)}(0) &= \sum_{i=0}^{k} \beta_i^{(k)} A^{2(k-i)} \int_{I\!R} \left(N_\tau^{\alpha_i^{(k)}} (\tau J_\tau)^{2i} \right)(0,x)dx \\ &= \sum_{i=0}^{k} \beta_i^{(k)} A^{2(k-i)} \int_{I\!R} n_0^{\alpha_i^{(k)}}(x) j_0^{2i}(x)dx. \end{aligned}$$

By $(H2)$, we have

$$F_0^{(k)}(0) \le C^{(k)}(T).$$

The proof of Theorem 3 is finished. □

Lemma 4 *For any $k \in I\!N^*$,*

(i) the sequence $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T; L^{\alpha_0^{(k)}}(I\!R))$;

(ii) the sequence $(N_\tau^{\alpha_1^{(k)}} J_\tau^2)_{\tau>0}$ is bounded in $L^1(Q_T)$;

(iii) the sequence $(\tau^2 N_\tau^{\alpha_2^{(k)}} J_\tau^4)_{\tau>0}$ is bounded in $L^1(Q_T)$ for $k \ge 2$. □

Proof. Combining (4.4) and (4.6), the sequence $(\tau^{2i} N_\tau^{\alpha_i^{(k)}} J_\tau^{2i})_{\tau>0}$ is bounded in $L^\infty(0,T; L^1(I\!R))$ for any $k \in I\!N^*$ and any $0 \le i \le k$. In particular, we obtain (i) by taking $i = 0$. Similarly, the sequence $(\tau^{2i} N_\tau^{\alpha_{i+1}^{(k)}} J_\tau^{2(i+1)})_{\tau>0}$ is bounded in $L^1(Q_T)$ for any $k \in I\!N^*$ and any $0 \le i \le k-1$. Therefore (ii) and (iii) follow from the respective choices $i = 0$ and $i = 1$. □

Lemma 5 *For any $p \in [1, +\infty[$, the sequence of solutions $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T; L^p(I\!R))$.* □

Proof. By the definition (3.18) of $\alpha_i^{(k)}$, for any $p \ge 1$, there is a $k \in I\!N^*$ such that $\alpha_0^{(k)} \ge p$. Since $(N_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T; L^1(I\!R)) \cap L^\infty(0,T; L^{\alpha_0^{(k)}}(I\!R))$, we conclude the result by the interpolation inequality. □

Lemma 6 *For any* $\gamma = 1 + 2/m$ *with* $1 \le m \in I\!N$,

(i) the sequence $(J_\tau)_{\tau>0}$ *is bounded in* $L^2(Q_T)$*;*

(ii) the sequence $(\tau J_\tau^2/N_\tau)_{\tau>0}$ *is bounded in* $L^2(Q_T)$. $\square$

Proof. For any $k \in I\!N^*$, we have

$$2J_\tau^2 = 2\left(N_\tau^{\frac{\alpha_1^{(k)}}{2}} J_\tau\right)\left(N_\tau^{-\frac{\alpha_1^{(k)}}{2}} J_\tau\right) \le N_\tau^{\alpha_1^{(k)}} J_\tau^2 + N_\tau^{-\alpha_1^{(k)}} J_\tau^2.$$

By (ii) of Lemma 4, the sequence $(N_\tau^{\alpha_1^{(k)}} J_\tau^2)_{\tau>0}$ is bounded in $L^1(Q_T)$ for all $k \in I\!N^*$. Then to prove (i), it suffices to show the existence of $k_1, k_2 \in I\!N^*$ such that

$$-\alpha_1^{(k_2)} = \alpha_1^{(k_1)}.$$

By (3.18), this is equivalent to

$$(k_1 + k_2 - 2)(\gamma - 1) = 2.$$

This last equation implies that γ has necessarily the form $\gamma = 1 + 2/m$ with $m \ge 1$. This proves the existence of $k_1, k_2 \in I\!N^*$, since

$$k_1 + k_2 = m + 2 \ge 3.$$

The proof of (ii) is similar to that of (i). For any $2 \le k \in I\!N^*$, we have

$$2\tau^2 N_\tau^{-2} J_\tau^4 = 2\tau^2 \left(N_\tau^{\frac{\alpha_2^{(k)}}{2}} J_\tau^2\right)\left(N_\tau^{-(2+\frac{\alpha_2^{(k)}}{2})} J_\tau^2\right) \le \tau^2 N_\tau^{\alpha_2^{(k)}} J_\tau^4 + \tau^2 N_\tau^{-(4+\alpha_2^{(k)})} J_\tau^4.$$

By (iii) of Lemma 4, the sequence $(\tau^2 N_\tau^{\alpha_2^{(k)}} J_\tau^4)_{\tau>0}$ is bounded in $L^1(Q_T)$ for all $k \ge 2$. It suffices then to show the existence of $k_1' \ge 2$ and $k_2' \ge 2$ such that

$$-\left(4 + \alpha_2^{(k_1')}\right) = \alpha_2^{(k_2')}. \tag{4.12}$$

But

$$\alpha_2^{(k)} = (k-2)(\gamma - 1) - 4 + 1 = (k-2)(\gamma-1) - 3.$$

Therefore, (4.12) implies

$$k'_1 + k'_2 = \frac{2}{\gamma - 1} + 4 = m + 4 \geq 5.$$

This ends the proof. □

Now we establish uniform estimates for the sequence of the electric potential $(\Phi_\tau)_{\tau>0}$.

Lemma 7 *Under the assumptions (H1'), (H2)-(H3),*

(i) the sequence $(\Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(I\!R))$ *for any* $p \in [\gamma_0, +\infty]$*;*

(ii) the sequence $(\partial_x \Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(I\!R))$ *for any* $p \in [2, +\infty]$*;*

(iii) the sequence $(\partial_{xx} \Phi_\tau)_{\tau>0}$ *is bounded in* $L^\infty(0,T;L^p(I\!R))$ *for any* $p \in [1, +\infty[$*;*

(iv) the sequence $(\partial_t \Phi_\tau)_{\tau>0}$ *is bounded in* $L^2(0,T;H^1(I\!R))$*.* □

Proof. Multiplying (2.3) by Φ_τ and integrating over $I\!R$, we obtain

$$\int_{I\!R} |\,\partial_x \Phi_\tau(t,x)|^2 \, dx + \int_{I\!R} (f(\Phi_\tau)\Phi_\tau)(t,x)dx = \int_{I\!R} (N_\tau \Phi_\tau)(t,x)dx.$$

By (H1'), we have

$$\|\,\partial_x \Phi_\tau(t,.)\,\|^2_{L^2(I\!R)} + a_0^2 \,\|\,\Phi_\tau(t,.)\,\|^{\gamma_0+1}_{L^{\gamma_0+1}(I\!R)} \leq \|\,N_\tau(t,.)\,\|_{L^{\gamma_1}(I\!R)} \,.\, \|\,\Phi_\tau(t,.)\,\|_{L^{\gamma_0+1}(I\!R)},$$

where $\frac{1}{\gamma_1} + \frac{1}{\gamma_0+1} = 1$. It follows from Lemma 5 that $(\Phi_\tau)_{\tau>0}$ and $(\partial_x \Phi_\tau)_{\tau>0}$ are respectively bounded in $L^\infty(0,T;L^{\gamma_0+1}(I\!R))$ and in $L^\infty(0,T;L^2(I\!R))$. Therefore, (ii) is a consequence of Lemma 3 and an interpolation argument.

We deduce furthermore that $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;W^{1,\gamma_0+1}(I\!R))$ since $\gamma_0 + 1 \geq 2$. By the Sobolev embedding theorem, $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(Q_T)$. On the other hand, (4.3) and (H1') imply that $(\Phi_\tau)_{\tau>0}$ is bounded in $L^\infty(0,T;L^{\gamma_0}(I\!R))$. Thus, (i) follows from the interpolation inequality. Finally, (i) and (4.3) imply that $(f(\Phi_\tau))_{\tau>0}$ is bounded in $L^\infty(Q_T) \cap L^\infty(0,T;L^1(I\!R))$. Hence, $(f(\Phi_\tau))_{\tau>0}$ is bounded in $L^\infty(0,T;L^p(I\!R))$ for any $p \in [1,+\infty]$. Thus, (iii) follows from Lemma 5 and the Poisson equation (2.3).

To show (iv), let us define the quantity

$$z^h_\tau(t,x) = \frac{z_\tau(t+h,x) - z_\tau(t,x)}{h}, \quad \forall\, h > 0.$$

By (2.3)-(2.5), the following equations hold

$$-\partial_{xx}\Phi_\tau^h + \overline{f}_\tau^h \Phi_\tau^h = N_\tau^h, \quad t>0, \quad x \in I\!R, \tag{4.13}$$

$$\lim_{|x|\to\infty} \Phi_\tau^h(t,x) = 0, \quad \text{a.e. } t \in [0,T], \tag{4.14}$$

where

$$\overline{f}_\tau^h(t,x) := \frac{f(\Phi_\tau(t+h,x)) - f(\Phi_\tau(t,x))}{\Phi_\tau(t+h,x) - \Phi_\tau(t,x)},$$

which satisfies, by (H1'), $\overline{f}_\tau^h(t,x) \geq f_0 > 0$. We note that since $(J_\tau)_{\tau>0}$ is bounded in $L^2(Q_T)$, $(\partial_x J_\tau)_{\tau>0}$ is bounded in $L^2(0,T;H^{-1}(I\!R))$. Therefore, using (2.1), $(\partial_t N_\tau)_{\tau>0}$ is bounded in $L^2(0,T;H^{-1}(I\!R))$. This implies that $(N_\tau^h)_{\tau,h>0}$ is bounded in $L^2(0,T;H^{-1}(I\!R))$, and as $h \to 0$

$$N_\tau^h \longrightarrow \partial_t N_\tau \text{ in } L^2(0,T;H^{-1}(I\!R)).$$

Multiplying (4.13) by Φ_τ^h and integrating over $I\!R$, we have

$$\int_{I\!R} |\,\partial_x \Phi_\tau^h(t,x)\,|^2\, dx + \int_{I\!R} \overline{f}_\tau^h (\Phi_\tau^h)^2(t,x) dx \leq \| N_\tau^h(t,.)\|_{H^{-1}(I\!R)} \, . \, \| \Phi_\tau^h(t,.)\|_{H^1(I\!R)} \, .$$

Since $\overline{f}_\tau^h(t,x) \geq f_0 > 0$, we deduce

$$\| \Phi_\tau^h(t,.)\|_{L^2(0,T;H^1(I\!R))} \leq \max(1, 1/f_0) \, \| N_\tau^h(t,.)\|_{L^2(0,T;H^{-1}(I\!R))} \, .$$

Hence $(\Phi_\tau^h)_{\tau,h>0}$ is bounded in $L^2(0,T;H^1(I\!R))$, which yields (iv). □

5 Proofs of the main results

We first pass to the limit in the Euler equation (2.1)-(2.2). From Lemmas 5-6, for any $p \in]1,+\infty[$, there exist functions $N \in L^\infty(0,T;L^p(I\!R))$, $J \in L^2(Q_T)$, $\overline{p} \in L^\infty(0,T;L^p(I\!R))$ and a subsequence, still denoted by $(N_\tau, J_\tau)_{\tau>0}$, such that as $\tau \to 0$,

$$N_\tau \rightharpoonup N \quad \text{in } L^p(Q_T) \text{ weakly }, \tag{5.1}$$

$$J_\tau \rightharpoonup J \quad \text{in } L^2(Q_T) \text{ weakly }, \tag{5.2}$$

$$p(N_\tau) = a^2 N_\tau^\gamma \rightharpoonup \overline{p} \quad \text{in } L^p(Q_T) \text{ weakly }, \tag{5.3}$$

and

$$\tau^2 J_\tau^2/N_\tau \longrightarrow 0 \quad \text{in} \quad L^2(Q_T) \ \text{ strongly} . \tag{5.4}$$

It follows that

$$\tau^2 J_\tau \longrightarrow 0 \quad \text{in} \quad L^2(Q_T) \ \text{ strongly} . \tag{5.5}$$

Moreover, by Lemma 7, for any $p \in [\gamma_0, +\infty]$, there exists $\Phi \in L^\infty(0,T;L^p(I\!R))$, such that as $\tau \to 0$,

$$\Phi_\tau \rightharpoonup \Phi \quad \text{in} \quad L^p(Q_T) \ \text{ weakly (or weakly} - * \text{ for } p = +\infty). \tag{5.6}$$

In order to pass to the limits in the terms $N_\tau \partial_x \Phi_\tau$ and $p(N_\tau)$, we use the well-known div-curl lemma. We have

Lemma 8 *There is a subsequence, still denoted by $(N_\tau, \Phi_\tau)_{\tau>0}$, such that as $\tau \to 0$,*

$$N_\tau \partial_x \Phi_\tau \rightharpoonup N \partial_x \Phi \quad in \quad \mathcal{D}'(Q_T). \ \Box \tag{5.7}$$

Proof. We define two sequences $U_\tau = (N_\tau, J_\tau)^t$ and $V_\tau = (\partial_x \Phi_\tau, 0)^t$. It is clear that $(U_\tau)_{\tau>0}$ and $(V_\tau)_{\tau>0}$ are bounded in $(L^2(Q_T))^2$. On the other hand,

$$div_{t,x} U_\tau = \partial_t N_\tau + \partial_x J_\tau = 0,$$

$$rot_{t,x} V_\tau = \begin{pmatrix} 0 & \partial_{xx}\Phi_\tau \\ -\partial_{xx}\Phi_\tau & 0 \end{pmatrix} .$$

From (iii) of Lemma 7, $(rot_{t,x} V_\tau)_{\tau>0}$ is bounded in $(L^2(Q_T))^4$. Since $div_{t,x} U_\tau = 0$, the div-curl lemma can be applied to the sequences $(U_\tau)_{\tau>0}$ and $(V_\tau)_{\tau>0}$ to obtain (5.7). $\Box$

It remains to pass to the limit in the nonlinear term $p(N_\tau)$. To this end, we have to show the strong convergence of a subsequence of $(N_\tau)_{\tau>0}$. This can be achieved by the monotonicity argument and the div-curl lemma.

Lemma 9 *There is a subsequence of $(N_\tau)_{\tau>0}$ (not relabeled), such that for any $p \in [1, \gamma+1]$, as $\tau \to 0$,*

$$N_\tau \longrightarrow N \quad in \quad L^p_{loc}(Q_T) \ \ strongly. \ \Box \tag{5.8}$$

Proof. We first show for any $p \in [1, \infty[$, as $\tau \to 0$,

$$N_\tau^{\gamma+1} \rightharpoonup N^{\gamma+1} \text{ in } L^p(Q_T) \text{ weakly.} \tag{5.9}$$

Indeed, let $W_\tau = (\tau^2 J_\tau^2/N_\tau + p(N_\tau), -\tau^2 J_\tau)^t$. Then from Lemmas 5-6, $(W_\tau)_{\tau>0}$ is bounded in $(L^2(Q_T))^2$. Moreover,

$$rot_{t,x} W_\tau = \begin{pmatrix} 0 & N_\tau \partial_x \Phi_\tau - J_\tau \\ -N_\tau \partial_x \Phi_\tau + J_\tau & 0 \end{pmatrix},$$

which is bounded in $(L^2(Q_T))^4$. By the div-curl lemma applied to $(U_\tau)_{\tau>0}$ and $(W_\tau)_{\tau>0}$, we have

$$\lim_{\tau\to 0} N_\tau p(N_\tau) = a^2 \lim_{\tau \to 0} N_\tau^{\gamma+1} = \lim_{\tau\to 0} U_\tau . W_\tau = N\overline{p}, \text{ in } \mathcal{D}'(Q_T), \tag{5.10}$$

or equivalently

$$\iint_{Q_T} N_\tau p(N_\tau) \phi dx dt \longrightarrow \iint_{Q_T} N \overline{p} \phi dx dt, \ \forall \phi \in \mathcal{D}(Q_T).$$

It follows that for any $v \in L^p(Q_T)$ with $p \in]1, \infty[$ and all $\phi \in \mathcal{D}(Q_T)$

$$\begin{aligned} A_\tau(\phi) &:= \iint_{Q_T} (N_\tau - v)(p(N_\tau) - p(v)) \phi dx dt \\ &\longrightarrow \iint_{Q_T} (N - v)(\overline{p} - p(v)) \phi dx dt := A(\phi). \end{aligned}$$

Since $s \to p(s)$ is monotone increasing on $I\!R^+$, we have $A_\tau(\phi) \ge 0$ for all $\phi \ge 0$. Hence, $A(\phi) \ge 0$ for all $\phi \in \mathcal{D}(Q_T)$ with $\phi \ge 0$. Let $z \in L^p(Q_T)$ with $p \in]1, \infty[$ and let $\lambda \in I\!R^*$. By taking $v = N - \lambda z \in L^p(Q_T)$, we obtain

$$\iint_{Q_T} (\overline{p} - p(N - \lambda z)) z \phi dx dt = A/\lambda \ge 0, \ \forall \lambda > 0 \text{ and } \forall \phi \ge 0.$$

Similarly

$$\iint_{Q_T} (\overline{p} - p(N - \lambda z)) z \phi dx dt = A/\lambda \le 0, \ \forall \lambda < 0 \text{ and } \forall \phi \ge 0.$$

Hence, for any $z \in L^p(Q_T)$ and any $\phi \ge 0$,

$$\iint_{Q_T} (\overline{p} - p(N)) z \phi dx dt = \lim_{\lambda \to 0} \iint_{Q_T} (\overline{p} - p(N - \lambda z)) z \phi dx dt = 0,$$

or equivalently, for any $z \in L^p(Q_T)$ and any $\phi \in \mathcal{D}(Q_T)$,

$$\iint_{Q_T} (\overline{p} - p(N))z\phi dxdt = \lim_{\lambda \to 0} \iint_{Q_T} (\overline{p} - p(N - \lambda z))z\phi dxdt = 0.$$

By (5.3), we have shown

$$\lim_{\tau \to 0} p(N_\tau) = \overline{p} = p(N), \quad \text{in} \quad L^p(Q_T) \text{ weakly.}$$

Thus, (5.9) follows from (2.6) and (5.10).

To prove (5.8), let $\Omega \in Q_T$ be an arbitrarily bounded set. Taking $\phi = 1_\Omega$ as test function in (5.9). we have

$$\lim_{\tau \to 0} \| N_\tau \|_{L^{\gamma+1}(\Omega)} = \| N \|_{L^{\gamma+1}(\Omega)} .$$

Therefore

$$N_\tau \longrightarrow N \quad \text{in} \quad L^{\gamma+1}_{loc}(Q_T) \quad \text{strongly.}$$

Thus we have proved (5.8). □

Now we pass to the limit in the nonlinear Poisson equation (2.3). From Lemma 7, the sequences $(\Phi_\tau)_{\tau>0}$ and $(\partial_t \Phi_\tau)_{\tau>0}$ are bounded in $L^2(0, T; H^1(I\!\!R))$. Hence, by Aubin's theorem, $(\Phi_\tau)_{\tau>0}$ is relatively compact in $L^2_{loc}(Q_T)$. This shows the strong convergence of a subsequence of $(\Phi_\tau)_{\tau>0}$. Thus, (2.13) holds. Finally, $\Phi \in L^\infty(0, T; H^1(I\!\!R))$ implies that $\lim_{|x| \to \infty} \Phi(t, x) = 0$, a.e. $t > 0$. The proof of Theorem 2 is completed.

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