GLOBAL RENORMALIZED SOLUTIONS TO REACTION-CROSS-DIFFUSION SYSTEMS

XIUQING CHEN AND ANSGAR JÜNGEL

Abstract. The global-in-time existence of renormalized solutions to reaction-cross-diffusion systems for an arbitrary number of variables in bounded domains with no-flux boundary conditions is proved. The cross-diffusion part describes the segregation of population species and is a generalization of the Shigesada-Kawasaki-Teramoto model. The diffusion matrix is not diagonal and generally neither symmetric nor positive semi-definite, but the system possesses a formal gradient-flow or entropy structure. The reaction part includes reversible reactions of mass-action kinetics and does not obey any growth condition. The existence result generalizes both the condition on the reaction part required in the boundedness-by-entropy method and the proof of J. Fischer for reaction-diffusion systems with diagonal diffusion matrices.

1. Introduction

Multi-species systems from thermodynamics, population dynamics, and cell biology, for instance, are often modeled by reaction-cross-diffusion equations. Cross diffusion occurs when the gradient of the density of one species induces a flux of another species. Therefore, cross-diffusion systems are strongly coupled, and only weak solutions can be expected. When the reaction terms grow too fast with the densities, there is no control of these terms and the definition of a weak or distributional solution is generally impossible. For this reason, growth restrictions have been imposed on the reactions in the literature [17, 20]. Roughly speaking, the reaction terms cannot grow faster than linear. The cross-diffusion systems from physics, biology, and chemistry often allow for entropy (or free energy) estimates that prevent the global blowup of solutions, but the bounds are not sufficient to define weak solutions. This suggests the concept of *renormalized* solutions. This approach was successfully realized by J. Fischer [14] for reaction-diffusion systems, i.e. weakly coupled equations. In this paper, we extend his approach to strongly coupled systems.

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More specifically, we investigate cross-diffusion systems from population dynamics, which extend the well-known model of Shigesada, Kawasaki, and Teramoto [26]. The evolution of the density $u_i = u_i(x,t)$ of the ith population species is governed by the equation

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\right) = f_i(u) \quad \text{in } \Omega,\ i=1,\ldots,n, \tag{1}$$

where $A_{ij}(u)$ are the density-dependent diffusion coefficients, $u=(u_1,\ldots,u_n)$ is the density vector, $b_i=(b_{i1},\ldots,b_{in})$ is a given vector which describes the environmental potential acting on the ith species, $f_i(u)$ is a reaction term describing the population growth dynamics, $\Omega\subset\mathbb{R}^n$ ($d\ge 1$) is a bounded domain, and $n\in\mathbb{N}$ is the number of species. We impose no-flux boundary and initial conditions,

$$\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\right)\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad u_i(\cdot,0)=u_i^0 \quad \text{in } \Omega,\ i=1,\ldots,n, \tag{2}$$

where ν is the exterior unit normal vector on $\partial\Omega$. The diffusion coefficients are given by

$$A_{ij}(u) = \delta_{ij}\left(a_{i0} + \sum_{k=1}^n a_{ik}u_k\right) + a_{ij}u_i, \quad i,j=1,\ldots,n, \tag{3}$$

where $a_{i0}\ge 0$, $a_{ij}\ge 0$ for $i,j=1,\ldots,n$, and δ_{ij} is the Kronecker delta. The reaction terms are often given by Lotka-Volterra-type expressions, but we allow for fast growing populations, and no growth condition on f_i will be imposed. Observe that (1) can be written in a more compact form as

$$\partial_t u - \operatorname{div}(A(u)\nabla u - ub) = f(u), \tag{4}$$

where the matrix ub is defined as $(ub)_{ij} := u_i b_{ij}$.

For $n=2$, we recover the population model of Shigesada, Kawasaki, and Teramoto [26], which descibes the segregation of two population species. Equations (1), (3) for an arbitrary number of species, $n\ge 2$, have been formally derived in [28] from a random-walk on a lattice in the diffusion limit.

For the analysis, we impose two key assumptions. First, we assume that the reaction terms f_i are continuous on $[0,\infty)^n$ and that there are numbers $\pi_1,\ldots,\pi_n>0$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$ such that for all $u\in(0,\infty)^n$,

$$\sum_{i=1}^n \pi_i f_i(u)(\log u_i + \lambda_i) \le 0. \tag{5}$$

These conditions imply the quasi-positivity property $f_i(u)\ge 0$ for all $u\in[0,\infty)^n$ with $u_i=0$, which is a necessary condition for having nonnegative solutions to (1). Moreover, it ensures that the so-called entropy density

$$h(u) = \sum_{i=1}^n \pi_i h_i(u_i), \quad h_i(s) = s(\log s - 1) + \lambda_i) + e^{-\lambda_i}, \tag{6}$$

is a Lyapunov functional for the reaction system $\partial_t u_i = f_i(u)$ if $\pi_i = 1$ for all i. Condition (5) (with $\pi_i = 1$) was also used in [14]. Compared to [14], we do not assume local Lipschitz continuity of f_i but only continuity.

Second, to ensure that the entropy (6) yields a Lyapunov functional also for the full system (1), we need to impose some conditions on the coefficients a_{ij}. It was shown in [3] that a sufficient requirement is either the weak cross-diffusion assumption

$$\alpha := \min_{i=1,\ldots,n} \left(a_{ii} - \frac{1}{4} \sum_{j=1}^{n} \left(\sqrt{a_{ij}} - \sqrt{a_{ji}}\right)^2 \right) > 0, \tag{7}$$

or the detailed-balance condition

$$\pi_i a_{ij} = \pi_j a_{ji} > 0 \quad \text{for all } i,j = 1,\ldots,n,\ i \neq j. \tag{8}$$

In the former case, we may choose $\pi_i = 1$ in (5) and (6). Condition (7) requires implicitly that $a_{ii} > 0$, while (8) requires that $a_{ij} > 0$ for $i \neq j$. A formal computation shows that

$$\frac{d}{dt} \int_\Omega h(u) dx + \int_\Omega \nabla u : \left(h''(u) A(u) \nabla u - h''(u) u b\right) dx \le 0,$$

where "':"' is the Frobenius matrix product and $h''(u)$ is the Hessian of the entropy density $h(u)$. The drift term $\nabla u : h''(u) u b = \nabla u : b$ can be estimated by using the Cauchy-Schwarz inequality, and the matrix $h''(u)A(u)$ is positive definite for $u_i > 0$. More precisely, assuming (5) and (7), it follows that

$$\frac{d}{dt} \int_\Omega h(u) dx + 4 \int_\Omega \sum_{i=1}^{n} a_{i0} |\nabla \sqrt{u_i}|^2 dx + \alpha \int_\Omega \sum_{i=1}^{n} |\nabla u_i|^2 dx \le 0, \tag{9}$$

while under the conditions (5) and (8), we have

$$\frac{d}{dt} \int_\Omega h(u) dx + 2 \int_\Omega \sum_{i=1}^{n} \pi_i \left(2a_{i0} |\nabla \sqrt{u_i}|^2 + a_{ii} |\nabla u_i|^2 + \sum_{j \neq i} a_{ij} |\nabla \sqrt{u_i u_j}|^2 \right) dx \le 0, \tag{10}$$

thus obtaining gradient estimates for $\sqrt{u_i}$ if $a_{i0} > 0$ and for u_i if $\alpha > 0$ or $a_{ii} > 0$.

Let us briefly comment on conditions (7) and (8); for details, we refer to [3]. If (a_{ij}) is symmetric and $a_{ii} > 0$, then (7) is fulfilled. Otherwise, the condition requires that the coefficient a_{ii} is larger than the "defect of symmetry" of the matrix (a_{ij}) or that the cross-diffusion coefficients a_{ij} are small compared to the self-diffusion coefficients a_{ii}. Assumption (8) is the detailed-balance condition for the Markov chain associated to (a_{ij}), and $(\pi_1, \ldots, \pi_n)$ is the reversible measure of the Markov chain. It turns out that this condition is equivalent to the symmetry of $h''(u)A(u)$ or, equivalently, of the so-called Onsager matrix $B = A(u)h''(u)^{-1}$. This indicates a close relationship between symmetry and reversibility, which is well known in nonequilibrim thermodynamics (also see [18, Section 4.3]).

Before stating our main result, we review the state of the art. The existence of global weak solutions to (1) was proved in [1, 2] for two species and in [3] for an arbitrary number of species. The global existence was also proved for diffusion matrices with nonlinear coefficients $A_{ij}(u)$. The case of sublinearly growing coefficients was treated in [8], whereas superlinear growth was analyzed in [9, 17] (for two species). The results were generalized to n species in [3, 20]. In [17, 20], the condition on the reaction terms is as follows (formulated here for the linear case): There exists $C > 0$ such that for all $u \in (0,\infty)^n$,

$$\sum_{i=1}^n f_i(u)\log u_i \le C(1+h(u)). \tag{11}$$

This inequality is satisfied for functions f_i which grow at most linearly. Comparing this condition with (5), written as $\sum_{i=1}^n f_i(u)\log u_i \le -\sum_{i=1}^n \lambda_i f_i(u)$, we see that for $\lambda_i < 0$, this inequality is usually weaker than (11).

When the diffusion matrix $A(u)$ is diagonal and constant, global existence results for (1) were shown in [7] and later extended to L^1 data in [22]. A more general result, assuming space-time dependent A_{ij} and mass action kinetics, was shown in [19]. In [6], strongly degenerate diffusion systems, still with diagonal diffusion matrices, were analyzed. When the reactions have quadratic growth and are dissipative in the sense $\sum_{i=1}^n f_i(u) \le 0$, classical solutions can be obtained [24]. If the diffusion coefficients are close to each other, even superquadratic growth in the reaction terms is possible [12]. On the other hand, it was shown in [23] that the L^∞ norm of the solutions to (1) with density-dependent diffusion coefficients may blow up in finite time, even if the total mass is controlled. As mentioned above, existence of renormalized solutions for general reaction terms, involving a single reversible reaction with mass-action kinetics, was proved in [14]. Furthermore, it was shown that the renormalized solutions satisfy a weak entropy-production inequality [15] and that they converge exponentially fast in the L^1 norm to the equilibrium [13].

In this paper, we combine the entropy method used in [3, 9, 20] and the concept of renormalized solutions of [14]. Our hypotheses are as follows.

(H1) Drift term: $b = (b_1,\ldots,b_n)$, $b_i \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^n))$, $i=1,\ldots,n$.

(H2) Reaction terms: $f = (f_1,\ldots,f_n) \in C^0([0,\infty)^n;\mathbb{R}^n)$.

(H3) Initial data: $u^0 = (u_1^0,\ldots,u_n^0)$ is measurable, $u_i^0 \ge 0$ in Ω, $i=1,\ldots,n$, and $\int_\Omega h(u^0)dx < \infty$, where h is defined in (6).

(H4) There exist numbers $\pi_i > 0$ and $\lambda_i \in \mathbb{R}$, $i=1,\ldots,n$, such that for all $u = (u_1,\ldots,u_n) \in (0,\infty)^n$, inequality (5) holds.

(H5') The weak cross-diffusion condition (7) holds and $\pi_i = 1$ for all $i=1,\ldots,n$.

(H5") The detailed-balance condition (8) and either $a_{i0} > 0$ for all $i=1,\ldots,n$ or $a_{ii} > 0$ for all $i=1,\ldots,n$ hold.

Definition 1 (Renormalized solution). *We call $u = (u_1,\ldots,u_n)$ a* renormalized solution *to (1)-(2) if for all $T > 0$, $u_i \in L^2(0,T;H^1(\Omega))$ or $\sqrt{u_i} \in L^2(0,T;H^1(\Omega))$, and for any $\xi \in C^\infty([0,\infty)^n)$ satisfying $\xi' \in C_0^\infty([0,\infty)^n;\mathbb{R}^n)$ and $\phi \in C_0^\infty(\overline{\Omega}\times[0,T))$, it holds that*

$$\begin{aligned} &-\int_0^T\int_\Omega \xi(u)\partial_t\phi dxdt - \int_\Omega \xi(u^0)\phi(\cdot,0)dx \\ &\quad = -\sum_{i,k=1}^n \int_0^T\int_\Omega \partial_i\partial_k\xi(u)\bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i\bigg)\cdot\nabla u_k\phi dxdt \end{aligned} \tag{12}$$

$$- \sum_{i=1}^n \int_0^T \int_\Omega \partial_i \xi(u) \bigg(\sum_{j=1}^n A_{ij}(u)\nabla u_j - u_i b_i \bigg) \cdot \nabla\phi dxdt$$
$$+ \sum_{i=1}^n \int_0^T \int_\Omega \partial_i \xi(u) f_i(u)\phi dxdt.$$

In the definition, ξ' is the gradient of ξ and $\partial_i\xi(u) = \partial\xi/\partial x_i$ the ith partial derivative. Note that ξ' is assumed to have compact support, so all integrals are well defined. If $\sqrt{u_i} \in L^2(0,T;H^1(\Omega))$, the expression $A_{ij}(u)\nabla u_j$ is interpreted as $2A_{ij}(u)\sqrt{u_j}\nabla\sqrt{u_j}$.

The main result reads as follows.

Theorem 1 (Global existence)**.** *Let (H1)-(H4) and either (H5') or (H5") hold. Then there exists a renormalized solution $u = (u_1,\ldots,u_n)$ to* (1)*-*(3) *satisfying $u_i \geq 0$ in Ω and $\int_\Omega h(u(t))dx < \infty$ for all $t > 0$.*

The proof is based on the entropy method of [17] and the approximation scheme of [14]. In the following, we sketch the key ideas.

Step 1: Approximation scheme. Introducing the entropy variable $w = (w_1,\ldots,w_n)$ with $w_i = \partial h/\partial u_i = \log u_i + \lambda_i$, equations (1) can be equivalently written as

$$\partial_t u(w) - \operatorname{div}(B\nabla w - u(w)b) = f(u(w)), \tag{13}$$

where $B = A(u)h''(u)^{-1}$ is positive definite if condition (7) or (8) is assumed, and $u(w) = (h')^{-1}(w)$ with components $u_i(w) = \exp(w_i - \lambda_i)$ is interpreted as a function of the entropy variable w. System (13) is approximated by an implicit Euler scheme with time step $\tau > 0$, an elliptic regularization of the type $\varepsilon((-\Delta)^m w + w)$ for some $\varepsilon > 0$ and $m \in \mathbb{N}$, and a regularized reaction term

$$f_i^\delta(u) = \frac{f_i(u)}{1 + \delta|f(u)|}$$

with parameter $\delta > 0$. The Euler scheme avoids issues with the (low) time regularity; the elliptic regularization yields the regularity $w \in H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ if $m > d/2$; and the regularized reaction term is bounded, which allows us to apply the entropy method of [17].

Step 2: Limit $(\varepsilon,\tau) \to 0$. A discrete version of the entropy-production inequality (9) or (10) yields estimates uniform in τ and ε but not in δ. Using the Aubin-Lions lemma, we can pass to the limit $\varepsilon = \tau \to 0$, and we infer the strong convergence $u^{(\tau)} \to u$ in $L^1(\Omega \times (0,T))$. In order to pass to the limit in equations (1) with the right-hand side replaced by $f_i^\delta(u)$, we need to distinguish between the cases $a_{ii} > 0$, which yields uniform estimates for $\nabla u_i^{(\tau)}$, and $a_{i0} > 0$, which gives estimates only for $\nabla(u_i^{(\tau)})^{1/2}$. In the latter case, we need to exploit the uniform bound for $\nabla(u_i^{(\tau)}u_j^{(\tau)})^{1/2}$ to be able to pass to the limit in the expression $A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)}$.

Step 3: Limit $\delta \to 0$. For the limit $\delta \to 0$, we proceed as in [14]. The idea is to truncate u_i by a smooth function $\varphi_i^L(u)$ that equals u_i if $\sum_{j=1}^n u_j < L$ and which is constant if $\sum_{j=1}^n u_j > 2L$. A discrete version of the entropy-production inequality (9) or (10) (i.e.

using the test function of the type $\sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})\psi$, where $\partial_j = \partial/\partial u_j$ and $\psi \in C^\infty$) gives δ-uniform estimates. The integral involving the reaction term

$$\sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})\frac{f_i(u^{(\delta)})\psi}{1+\delta|f(u^{(\delta)})|}$$

can be bounded independently of δ since the support of $\partial_j\varphi_i^L$ is bounded. The Aubin-Lions lemma shows that a subsequence of $(\varphi_i^L(u^{(\delta)}))$ converges strongly in L^2 to some function v_i^L; see the proof of the key Lemma 11. By a diagonal argument, which will be made explicit in the proof of Lemma 11, the subsequence is independent of L. The properties of φ_i^L allow us to prove that $u_i^{(\delta)} \to u_i$ a.e. in $\Omega\times(0,T)$.

The limit $\delta\to 0$ in the equations satisfied by $u^{(\delta)}$ yields a defect measure due to the integral involving quadratic gradients of $u_i^{(\delta)}$ (or $(u_i^{(\delta)})^{1/2}$),

$$\int_0^T\int_\Omega \partial_j\partial_k\varphi_i^L(u^{(\delta)})A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}\cdot\nabla u_k^{(\delta)}\psi dxdt, \tag{14}$$

where ψ is some test function. As $\delta\to 0$, this expression converges (up to a subsequence) to $\int_0^T\int_\Omega \psi d\mu_i^L(x,t)$, where μ_i^L is a signed Radon measure.

Step 4: Limit $L\to\infty$. It turns out (as in [14]) that μ_i^L converges weak* to zero as $L\to\infty$ in the sense of measures. This follows since the squared gradient in (14) is uniformly bounded, which is a consequence of the entropy-production inequality (9) or (10), and $\|\partial_j\partial_k\varphi_i^L\|_{L^\infty}$ converges to zero as $L\to\infty$. Then we take $\xi(\varphi_i^L(u))$ as a test function in the equations satisfied by u_i, where the gradient of $\xi\in C^\infty$ has a compact support, and pass to the limit $L\to\infty$ in the equations. For this step, we use the chain-rule lemma of [14].

Compared to [14], we allow for strongly coupled reaction-diffusion equations with indefinite diffusion matrices and density-dependent coefficients $A_{ij}(u)$. Crucial is the linear dependence of $A_{ij}(u)$ on u_k. In fact, when the diffusion coefficients are nonlinear, say of type u_k^m with $m>0$, we need to define a different entropy density, $h_i(s)=s^m$. We believe that the existence of renormalized solutions can be proved also in this situation, using the ideas of [3, 20]. We expect that the proof can be also extended to cross-diffusion systems of volume-filling type [18]. Indeed, in this case, the solutions are bounded such that the expression $A_{ij}(u)\nabla u_j$ can be easily defined. Since our proof is already quite technical, we focus on the population model with linear diffusion coefficients and leave details for more general models to the reader. Finally, we mention that mixed Dirichlet-Neumann boundary conditions may be treated as well, as long as the entropy-production inequality can be shown; we refer to [14] for details in the diagonal case.

The paper is organized as follows. The existence of a weak solution to the approximate problem is shown in Section 2. Section 3 is concerned with the derivation of estimates uniform in (ε,τ). The limit $(\varepsilon,\tau)\to 0$ is proved in Section 4, while the limits $\delta\to 0$ and $L\to\infty$ as well as the proof of Theorem 1 are performed in Section 5.

2. Existence for an approximate problem

Let $T > 0$, $N \in \mathbb{N}$, set $\tau = T/N$, and let $\delta > 0$, and $m \in \mathbb{N}$ with $m > d/2$. Then the embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact. Let Hypothesis (H3) on the initial datum hold. To obtain strictly positive initial data, we need to truncate. For this, let $0 < \varepsilon < \min\{1, e^{-\lambda_1}, \ldots, e^{-\lambda_n}\}$ and introduce the cut-off function

$$Q_\varepsilon(y) = \begin{cases} \varepsilon & \text{if } 0 \le y < \varepsilon, \\ y & \text{if } \varepsilon \le y < \varepsilon^{-1}, \\ \varepsilon^{-1} & \text{if } y \ge \varepsilon^{-1}. \end{cases}$$

A computation shows that $h_i(Q_\varepsilon(y)) \le e^{-\lambda_i} + h_i(y)$ and $\lim_{\varepsilon\to 0} Q_\varepsilon(y) = y$ for all $y \ge 0$. We set $u_\varepsilon^0 := (Q_\varepsilon(u_1^0), \ldots, Q_\varepsilon(u_n^0))$. Then $u_\varepsilon^0(x) \in [\varepsilon, \varepsilon^{-1}]^n$ for $x \in \Omega$, and $w^0 = h'(u_\varepsilon^0) \in L^\infty(\Omega; \mathbb{R}^n)$ is well-defined.

Let $k \ge 1$ and let $w^{k-1} \in L^\infty(\Omega; \mathbb{R}^n)$ be given. We wish to find $w^k \in H^m(\Omega; \mathbb{R}^n)$ such that for all $\phi = (\phi_1, \ldots, \phi_n) \in H^m(\Omega; \mathbb{R}^n)$,

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega \big(u(w^k) - u(w^{k-1})\big)\cdot\phi dx + \int_\Omega \nabla\phi : B(w^k)\nabla w^k dx - \int_\Omega \sum_{i=1}^n u_i(w^k) b_i^k \cdot \nabla\phi_i dx \\ &\quad + \varepsilon\int_\Omega \bigg(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha\phi + w^k\cdot\phi\bigg) dx = \int_\Omega \frac{f(u(w^k))\cdot\phi}{1+\delta|f(u(w^k))|} dx. \end{aligned} \tag{15}$$

Here, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n = m$ is a multiindex, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n})$ is a partial derivative of order $|\alpha|$. Moreover, $u(w^k) = (h')^{-1}(w^k)$ (i.e. $u_i(w^k) = \exp(w_i^k - \lambda_i)$), $B(w^k) = A(u(w^k))h''(u(w^k))^{-1}$, and $b^k = \tau^{-1}\int_{(k-1)\tau}^{k\tau} b(\cdot, t)dt$. In particular, it follows from (H1) that

$$\|b^k\|_{L^\infty(\Omega)} \le \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} \|b(\cdot,t)\|_{L^\infty(\Omega)} dt \le \|b\|_{L^\infty(Q_T)},$$

where we have set $Q_T = \Omega \times (0, T)$. First, we show that there exists a solution to (15).

Lemma 2. *Let (H1), (H2), (H4), and either (H5') or (H5") hold. In case (H5") with $a_{i0} > 0$ for all i, we choose $\tau > 0$ sufficiently small. Then there exists a solution $w^k \in H^m(\Omega; \mathbb{R}^n)$ to* (15).

Proof. We follow the lines of [3] but some estimates simplify. For clarity, we present the full proof.

Step 1. Let (H5') hold and let $\bar w \in L^\infty(\Omega; \mathbb{R}^n)$. We claim that there exists a unique solution $w = (w_1, \ldots, w_n) \in H^m(\Omega; \mathbb{R}^n)$ to the linear problem

$$a(w, \phi) = F(\phi) \quad \text{for all } \phi \in H^m(\Omega; \mathbb{R}^n), \tag{16}$$

where

$$a(w,\phi) = \tau\int_\Omega \nabla\phi : B(\bar w)\nabla w dx + \varepsilon\tau\int_\Omega\bigg(\sum_{|\alpha|=m} D^\alpha w\cdot D^\alpha\phi + w\cdot\phi\bigg)dx,$$

$$F(\phi) = -\int_\Omega \big(u(\bar w) - u(w^{k-1})\big)\cdot \phi dx + \tau\int_\Omega \sum_{i=1}^n u_i(\bar w) b_i^k \cdot \nabla\phi_i dx \\ + \tau\int_\Omega \frac{f(u(\bar w))\cdot\phi}{1+\delta|f(u(\bar w))|}dx.$$

Since $\bar w$, $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^n)$, we have $u_i(\bar w) = \exp(\bar w_i - \lambda_i)$, $u_i(w^{k-1}) \in L^\infty(\Omega)$. Then each coefficient of $B(\bar w)$,

$$B_{ij}(\bar w) = \delta_{ij}\bigg(a_{i0} + \sum_{k=1}^n a_{ik}u_k(\bar w)\bigg)u_i(\bar w) + a_{ij}u_i(\bar w)u_j(\bar w),$$

is bounded. In view of $b_i^k \in L^\infty(\Omega;\mathbb{R}^n)$, it follows that the bilinear form a and the linear form F are bounded. The matrix $B(\bar w)$ is positive semidefinite since $h''(u)A(u)$ is positive semidefinite, by Lemma 6 in [3]. Consequently, for all $z \in \mathbb{R}^n$,

$$z^T B(\bar w) z = \big(h''(u(\bar w))^{-1}z\big)^T h''(u(\bar w))A(u(\bar w))\big(h''(u(\bar w))^{-1}z\big) \geq 0.$$

Hence, we infer from the generalized Poincaré inequality [27, Chapter 2, Section 1.4] that the bilinear form a is coercive,

$$a(w,w) \geq \tau\varepsilon \int_\Omega \bigg(\sum_{|\alpha|=m} |D^\alpha w|^2 + |w|^2\bigg)dx \geq \tau\varepsilon C\|w\|^2_{H^m(\Omega)}$$

for $w \in H^m(\Omega;\mathbb{R}^n)$. By the Lax-Milgram lemma, we conclude the existence of a unique solution $w \in H^m(\Omega;\mathbb{R}^n)$ to (16).

Step 2. Define the mapping $\Phi : L^\infty(\Omega;\mathbb{R}^n) \to L^\infty(\Omega;\mathbb{R}^n)$ by $\Phi(\bar w) = w$, where $w \in H^m(\Omega;\mathbb{R}^n)$ is the unique solution to (16). Standard arguments (see, for instance, the proof of Lemma 5 in [17]), together with Hypothesis (H2), show that Φ is continuous. Then the compactness of the embedding $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$ implies the compactness of Φ. In order to apply the Leray-Schauder fixed-point theorem [16, Theorem 10.3], it remains to show that the set $\Lambda = \{w \in L^\infty(\Omega;\mathbb{R}^n) : w = \sigma\Phi(w) \text{ for some } \sigma \in (0,1]\}$ is bounded in $L^\infty(\Omega;\mathbb{R}^n)$.

Let $w \in \Lambda$. Then $a(w,\phi) = \sigma F(\phi)$ for all $\phi \in H^m(\Omega;\mathbb{R}^n)$, with $\bar w$ is replaced by w. Taking $\phi = w$ as a test function, we find that

$$\begin{aligned} &\tau\int_\Omega \nabla w : B(w)\nabla w dx + \varepsilon\tau\int_\Omega \bigg(\sum_{|\alpha|=m}|D^\alpha w|^2 + |w|^2\bigg) \\ &\quad = -\sigma\int_\Omega \big(u(w) - u(w^{k-1})\big)\cdot w dx + \sigma\tau\int_\Omega \frac{f(u(w))\cdot w}{1+\delta|f(u(w))|}dx \\ &\qquad + \sigma\tau\int_\Omega \sum_{i=1}^n u_i(w)b_i^k\cdot\nabla w_i dx. \end{aligned} \tag{17}$$

We estimate both sides term by term.

The identities $B(w) = A(u(w))h''(u(w))^{-1}$ and $\nabla w = h''(u(w))\nabla u(w)$ imply that

$$\tau \int_\Omega \nabla w : B(w)\nabla w dx = \tau \int_\Omega \nabla u(w) : h''(u(w))A(u(w))\nabla u(w) dx.$$

It is shown in [3, Lemma 6] that the matrix $h''(u(w))A(u(w))$ is positive definite:

$$J_1 := \tau \int_\Omega \nabla w : B(w)\nabla w dx \geq 2\eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)|^2 dx$$

for some constant $\eta > 0$. The convexity of h implies that $h(y) - h(z) \leq h'(y)\cdot(y-z)$ for all x, $y \in (0,\infty)^n$. Choosing $y = u(w)$, $z = u(w^{k-1})$ and using the property $h'(u(w)) = w$, we infer that

$$-\sigma \int_\Omega \big(u(w) - u(w^{k-1})\big)\cdot w dx \leq -\sigma \int_\Omega \big(h(u(w)) - h(u(w^{k-1}))\big) dx.$$

By Hypothesis (H4), we have

$$\sigma\tau \int_\Omega \frac{f(u(w))\cdot w}{1+\delta|f(u(w))|} dx = \sigma\tau \int_\Omega \sum_{i=1}^n \frac{f_i(u(w))(\log u_i(w) + \lambda_i)}{1+\delta|f(u(w))|} dx \leq 0.$$

It remains to estimate the last integral in (17). We use the Cauchy-Schwarz inequality to find that

$$\begin{aligned} J_2 &:= \sigma\tau \int_\Omega \sum_{i=1}^n u_i(w) b_i^k \cdot \nabla w_i dx = \sigma\tau \int_\Omega \sum_{i=1}^n b_i^k \cdot \nabla u_i(w) dx \\ &\leq \eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)|^2 dx + \frac{\tau}{4\eta}\|b\|^2_{L^\infty(Q_T)}. \end{aligned}$$

Hence, we conclude from (17) that

$$\begin{aligned} &\sigma \int_\Omega h(u(w)) dx + \eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)|^2 dx + \varepsilon\tau \int_\Omega \Big(\sum_{|\alpha|=m} |D^\alpha w|^2 + |w|^2\Big) \\ &\qquad \leq \sigma \int_\Omega h(u(w^{k-1})) dx + C(\eta, b)\tau. \end{aligned} \tag{18}$$

Recalling that $u(w^{k-1}) \in L^\infty(\Omega;\mathbb{R}^n)$, we infer that $\int_\Omega h(u(w^{k-1}))dx < \infty$ and hence $\|w\|_{L^\infty(\Omega)} \leq C\|w\|_{H^m(\Omega)} \leq C(\eta,\varepsilon,b,\tau)$, which is the desired uniform bound. By the Leray-Schauder theorem, there exists a solution $w^k := w \in H^m(\Omega;\mathbb{R}^n)$ to (15).

Step 3. Finally, let (H5") hold. The proof is almost identical to the previous steps except the estimation of J_1 and J_2. The matrix $h''(u)A(u)$ is positive definite and for all $z \in \mathbb{R}^n$ (see [3, Lemma 4]),

$$z : h''(u)A(u)z \geq \sum_{i=1}^n \pi_i a_{i0}\frac{z_i^2}{u_i} + 2\sum_{i=1}^n \pi_i a_{ii} z_i^2 + \frac{1}{2}\sum_{i,j=1,\ i\neq j}^n \pi_i a_{ij}\left(\sqrt{\frac{u_j}{u_i}} z_i + \sqrt{\frac{u_i}{u_j}} z_j\right)^2.$$

Therefore, in case $a_{ii} > 0$ for all $i = 1, \ldots, n$, we can proceed as in the previous step, obtaining inequality (18). In case $a_{i0} > 0$ for all $i = 1, \ldots, n$, there exists $\eta > 0$, only depending on the coefficients a_{ij}, such that

$$J_1 \geq 4\eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)^{1/2}|^2 dx + 2\eta\tau \int_\Omega \sum_{i,j=1,\ i\neq j}^n \left|\nabla (u_i(w)u_j(w))^{1/2}\right|^2 dx.$$

We use the Cauchy-Schwarz inequality and the elementary inequality $u_i \leq h_i(u_i) + C$ for some constant $C > 0$ to infer that

$$\begin{aligned} J_2 &= 2\sigma\tau \int_\Omega \sum_{i=1}^n u_i(w)^{1/2} b_i^k \cdot \nabla u_i(w)^{1/2} dx \\ &\leq \eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)^{1/2}|^2 dx + \frac{\sigma\tau}{\eta} \|b\|^2_{L^\infty(Q_T)} \int_\Omega \sum_{i=1}^n u_i(w) dx \\ &\leq \eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)^{1/2}|^2 dx + C(\eta, b)\sigma\tau \int_\Omega h(u(w)) dx + C(\eta, b)\tau. \end{aligned}$$

Then (17) can be written as

$$\begin{aligned} &\sigma(1 - C(\eta, b)\tau) \int_\Omega h(u(w)) dx + \eta\tau \int_\Omega \sum_{i=1}^n |\nabla u_i(w)^{1/2}|^2 dx \\ &\quad + 2\eta\tau \sum_{i,j=1\, i\neq j}^n \left\|\nabla (u_i(w)u_j(w))^{1/2}\right\|^2_{L^2(\Omega)} + \varepsilon\tau \bigg(\sum_{|\alpha|=m} \|D^\alpha w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)} \bigg) \\ &\leq \sigma \int_\Omega h(u(w^{k-1})) dx + C(\eta, b)\tau, \end{aligned} \tag{19}$$

and we obtain the desired L^∞ bound for w by choosing $\tau < 1/C(\eta, b)$. This ends the proof. $\square$

3. Uniform estimates

The next step is the derivation of estimates which are uniform in the approximation parameters. Let (H1)-(H4) and either (H5') or (H5") hold. Applying Lemma 2 iteratively, we obtain a sequence of solutions $w^k \in H^m(\Omega; \mathbb{R}^n)$ to (15) with $u^k := u(w^k) \in L^\infty(\Omega; (0, \infty)^n)$ for $k = 1, \ldots, N$. The first bounds are a consequence of the discrete entropy estimate (18) or (19), respectively.

Lemma 3. (i) *Let (H5') or (H5") with $a_{ii} > 0$ for all $i = 1, \ldots, n$ hold and let $k \in \{1, \ldots, N\}$. Then*

$$\int_\Omega h(u^k) dx + \eta\tau \sum_{j=1}^k \sum_{i=1}^n \|\nabla u_i^j\|^2_{L^2(\Omega)}$$

$$+ \varepsilon\tau \sum_{j=1}^{k} \bigg(\sum_{|\alpha|=m} \|D^\alpha w^j\|_{L^2(\Omega)}^2 + \|w^j\|_{L^2(\Omega)}^2 \bigg) \le C(u^0, b, T).$$

(ii) *Let $k \in \{1, \ldots, N\}$. If (H5") with $a_{i0} > 0$ for all $i = 1, \ldots, n$ holds, then for sufficiently small $\tau > 0$,*

$$\int_\Omega h(u^k)dx + \eta\tau \sum_{j=1}^{k} \sum_{i=1}^{n} \|\nabla (u_i^j)^{1/2}\|_{L^2(\Omega)}^2 + \eta\tau \sum_{j=1}^{k} \sum_{i,\ell=1,\, i\ne\ell}^{n} \|\nabla (u_i^j u_\ell^j)^{1/2}\|_{L^2(\Omega)}^2$$
$$+ \varepsilon\tau \sum_{j=1}^{k} \bigg(\sum_{|\alpha|=m} \|D^\alpha w^j\|_{L^2(\Omega)}^2 + \|w^j\|_{L^2(\Omega)}^2 \bigg) \le C(u^0, b, T).$$

In both cases, $\eta > 0$ and $C(u^0, b, T) > 0$ are constants which are independent of δ, ε, and τ.

Proof. (i) We have $a_{ii} > 0$ for all $i = 1, \ldots, n$. We take $\sigma = 1$, $w = w^k$ in (18) and sum the equations. This yields

$$\int_\Omega h(u^k)dx + \eta\tau \sum_{j=1}^{k} \sum_{i=1}^{n} \|\nabla u_i^j\|_{L^2(\Omega)}^2$$
$$+ \varepsilon\tau \sum_{j=1}^{k} \bigg(\sum_{|\alpha|=m} \|D^\alpha w^j\|_{L^2(\Omega)}^2 + \|w^j\|_{L^2(\Omega)}^2 \bigg) \le \int_\Omega h(u(w^0))dx + C(\eta, b)T.$$

Since $h_i(Q_\varepsilon(y)) \le e^{-\lambda_i} + h_i(y)$ on $[0, \infty)$, we have

$$h(u(w^0)) = h(u_\varepsilon^0) = \sum_{i=1}^{n} h_i(Q_\varepsilon(u_i^0)) \le h(u^0) + C,$$

concluding the proof.

(ii) Let $a_{i0} > 0$ for $i = 1, \ldots, n$. We choose $\sigma = 1$, $\tau < 1/(2C(\eta, b))$, and $w = w^k$ in (19) and sum the equations, yielding

$$(1 - C(\eta, b)\tau) \int_\Omega h(u^k)dx + \eta\tau \sum_{j=1}^{k} \sum_{i=1}^{n} \|\nabla (u_i^j)^{1/2}\|_{L^2(\Omega)}^2$$
$$+ 2\eta\tau \sum_{i,\ell=1,\, i\ne\ell}^{n} \big\|\nabla (u_i^j u_\ell^j)^{1/2}\big\|_{L^2(\Omega)}^2 + \varepsilon\tau \sum_{j=1}^{k} \bigg(\sum_{|\alpha|=m} \|D^\alpha w^j\|_{L^2(\Omega)}^2 + \|w^j\|_{L^2(\Omega)}^2 \bigg)$$
$$\le (1 - C(\eta, b)\tau) \int_\Omega h(u(w^0))dx + C(\eta, b)\tau \sum_{j=1}^{k} \int_\Omega h(u^{j-1})dx + C(\eta, b)T$$
$$\le \int_\Omega h(u^0) + C(\eta, b)\tau \sum_{j=1}^{k} \int_\Omega h(u^{j-1})dx + C(\eta, b)T.$$

Observing that $1 - C(\eta, b)\tau \geq 1/2$ and applying the discrete Gronwall inequality, this proves (ii). $\square$

We also need a uniform estimate for the discrete time derivative.

Lemma 4. *Let (H1)-(H3) and (H5') or (H5") hold. Then*

$$\tau \sum_{k=1}^{N} \big\|\tau^{-1}(u^k - u^{k-1})\big\|^r_{H^{m+1}(\Omega)'} \leq C(\delta, u^0, b, T), \tag{20}$$

where $r = (d+2)/(d+1)$ if (H5') or (H5") with $a_{ii} > 0$ (case (i)) and $r = (2d+2)/(2d+1)$ if (H5") with $a_{i0} > 0$ (case (ii)).

Proof. We reformulate (15) as

$$\begin{aligned} &\int_\Omega \tau^{-1}(u^k - u^{k-1}) \cdot \phi dx + \int_\Omega \nabla\phi : A(u^k)\nabla u^k dx - \int_\Omega \sum_{i=1}^n u_i^k b_i^k \cdot \nabla\phi_i dx \\ &\quad + \varepsilon \int_\Omega \bigg(\sum_{|\alpha|=m} D^\alpha w^k \cdot D^\alpha \phi + w^k \cdot \phi \bigg) dx = \int_\Omega \frac{f(u^k)\cdot\phi}{1 + \delta|f(u^k)|} dx, \end{aligned} \tag{21}$$

where $\phi \in H^{m+1}(\Omega; \mathbb{R}^n) \hookrightarrow W^{1,\infty}(\Omega; \mathbb{R}^n)$. Therefore,

$$\begin{aligned} \bigg| \int_\Omega \tau^{-1}(u^k - u^{k-1}) \cdot \phi dx \bigg| &\leq \sum_{i,j=1}^n \|A_{ij}(u^k)\nabla u_j^k\|_{L^1(\Omega)} \|\nabla\phi_i\|_{L^\infty(\Omega)} \\ &\quad + \|b\|_{L^\infty(Q_T)} \|u^k\|_{L^1(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} + \varepsilon\|w^k\|_{H^m(\Omega)} \|\phi\|_{H^m(\Omega)} + \delta^{-1}\|\phi\|_{L^1(\Omega)} \\ &\leq \bigg(\sum_{i,j=1}^n \|A_{ij}(u^k)\nabla u_j^k\|_{L^1(\Omega)} + C(b)\|u^k\|_{L^1(\Omega)} + \varepsilon\|w^k\|_{H^m(\Omega)} + C(\delta) \bigg) \|\phi\|_{H^{m+1}(\Omega)}. \end{aligned} \tag{22}$$

Observe that the entropy controls the L^1 norm such that, by Lemma 3,

$$\|u^k\|_{L^1(\Omega)} \leq C \quad \text{for all } k = 1, \ldots, N \tag{23}$$

and some $C > 0$ which is independent of k, δ, ε, and τ.

We estimate now the L^1 norm of $A_{ij}(u^k)\nabla u_j^k$. For this, we need to distinguish the cases (i) and (ii). In case (i), we take $\theta = d(d+2) \in (0,1)$. Then the Gagliardo-Nirenberg inequality and estimate (23) give

$$\|u^k\|_{L^2(\Omega)} \leq C\|\nabla u^k\|^\theta_{L^2(\Omega)} \|u^k\|^{1-\theta}_{L^1(\Omega)} + \|u^k\|_{L^1(\Omega)} \leq C\big(1 + \|\nabla u^k\|^\theta_{L^2(\Omega)}\big).$$

Since $A_{ij}(u^k)$ depends linearly on u_i^k, this shows that

$$\sum_{i,j=1}^n \|A_{ij}(u^k)\nabla u_j^k\|_{L^1(\Omega)} \leq \sum_{i,j=1}^n \|A_{ij}(u^k)\|_{L^2(\Omega)} \|\nabla u_j^k\|_{L^2(\Omega)} \leq C\big(1 + \|\nabla u^k\|^{1+\theta}_{L^2(\Omega)}\big).$$

We deduce from (22) that

$$\big\|\tau^{-1}(u^k - u^{k-1})\|_{H^{m+1}(\Omega)'} \leq C\|\nabla u^k\|^{1+\theta}_{L^2(\Omega)} + \varepsilon\|w^k\|_{H^m(\Omega)} + C(\delta, b).$$

For $r=(d+2)/(d+1)$, we have $(1+\theta)r=2$ and $r<2$ and consequently, after summing (22) from $k=1,\ldots,N$,

$$\begin{aligned}
\bigg(\tau\sum_{k=1}^N\big\|\tau^{-1}(u^k-u^{k-1})\big\|_{H^{m+1}(\Omega)'}^r\bigg)^{1/r}&\\
\le C\bigg(\tau\sum_{k=1}^N\|\nabla u^k\|_{L^2(\Omega)}^{(1+\theta)r}\bigg)^{1/r}&+\varepsilon\bigg(\tau\sum_{k=1}^N\|w^k\|_{H^m(\Omega)}^r\bigg)^{1/r}+C(\delta,b,T)\\
\le C\bigg(\tau\sum_{k=1}^N\|\nabla u^k\|_{L^2(\Omega)}^2\bigg)^{1/r}&+C(T)\varepsilon\bigg(\tau\sum_{k=1}^N\|w^k\|_{H^m(\Omega)}^2\bigg)^{1/2}+C(\delta,b,T).
\end{aligned}$$

Lemma 3 (i) implies that the right-hand side is uniformly bounded, which shows (20).

In case (ii), we need the L^2 bound for $\nabla(u_i^ku_j^k)^{1/2}$ and the special structure of $A_{ij}(u^k)$. Since a similar argument was presented in [3, Remark 12], we give only a sketch of the proof. First, we observe that

$$\sum_{j=1}^n A_{ij}(u^k)\nabla u_j^k=\nabla\bigg(a_{i0}u_i^k+\sum_{j=1}^n a_{ij}u_i^ku_j^k\bigg).$$

Lemma 3 (ii) shows that $\tau\sum_{k=1}^N\|\nabla(u_i^ku_j^k)^{1/2}\|_{L^2(\Omega)}^2$ and $\sup_{k=1,\ldots,N}\|(u_i^ku_j^k)^{1/2}\|_{L^1(\Omega)}$ are uniformly bounded. Hence, by the Gagliardo-Nirenberg inequality, $\tau\sum_{k=1}^N\|(u_i^ku_j^k)^{1/2}\|_{L^p(\Omega)}^p$ is uniformly bounded, where $p=2+2/d$. Then the Hölder inequality with $r=(2d+2)/(2d+1)$ and $r'=2d+2$ gives the bound

$$\begin{aligned}
\bigg(\tau\sum_{k=1}^N\big\|\nabla(u_i^ku_j^k)\big\|_{L^r(\Omega)}^r\bigg)^{1/r}&=2\bigg(\tau\sum_{k=1}^n\big\|(u_i^ku_j^k)^{1/2}\nabla(u_i^ku_j^k)^{1/2}\big\|_{L^r(\Omega)}^r\bigg)^{1/r}\\
\le 2\bigg(\tau\sum_{k=1}^N\big\|(u_i^ku_j^k)^{1/2}\big\|_{L^p(\Omega)}^p\bigg)^{1/p}&\bigg(\tau\sum_{k=1}^N\big\|\nabla(u_i^ku_j^k)^{1/2}\big\|_{L^2(\Omega)}^2\bigg)^{1/2}\le C.
\end{aligned}$$

Consequently,

$$\bigg(\tau\sum_{k=1}^N\sum_{i,j=1}^n\|A_{ij}(u^k)\nabla u_j^k\|_{L^1(\Omega)}^r\bigg)^{1/r}\le C(u^0,b,T).$$

This proves (20) with $r=(2d+2)/(2d+1)$. $\square$

We define the piecewise constant functions in time $u^{(\tau)}(x,t)=u(w^k(x))$, $w^{(\tau)}(x,t)=w^k(x)$, and $b^{(\tau)}(x,t)=b^k(x)$ for $x\in\Omega$ and $t\in((k-1)\tau,\tau k]$, $k=1,\ldots,N$. Furthermore, we need the discrete time derivative $\partial_t^{(\tau)}u^{(\tau)}(x,t)=\tau^{-1}(u(w^k(x))-u(w^{k-1}(x)))$ for $x\in\Omega$, $t\in((k-1)\tau,\tau k]$. Recall that for $k=1$, we have $u(w^0)=u_\varepsilon^0$. We conclude from Lemmas 3 and 4 the following bounds.

Corollary 5. (i) *Let (H5') or (H5") with $a_{ii} > 0$ for $i = 1, \ldots, n$ holds. Then for all $i = 1, \ldots, n$:*

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))} + \varepsilon^{1/2}\|w_i^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \le C(u^0, b, T), \tag{24}$$

$$\|\partial_t^{(\tau)} u_i^{(\tau)}\|_{L^r(0,T;H^{m+1}(\Omega)')} \le C(\delta, u^0, b, T), \quad r = \frac{d+2}{d+1}. \tag{25}$$

(ii) *Let (H5") with $a_{i0} > 0$ for $i = 1, \ldots, n$ holds. Then for all $i \neq j$,*

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))} + \|(u_i^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\Omega))} \tag{26}$$

$$+ \|\nabla(u_i^{(\tau)} u_j^{(\tau)})^{1/2}\|_{L^2(Q_T)}^2 + \varepsilon^{1/2}\|w_i^{(\tau)}\|_{L^2(0,T;H^m(\Omega))} \le C(u^0, b, T), \tag{27}$$

$$\|\partial_t^{(\tau)} u_i^{(\tau)}\|_{L^r(0,T;H^{m+1}(\Omega)')} \le C(\delta, u^0, b, T), \quad r = \frac{2d+2}{2d+1}. \tag{28}$$

4. The limit $(\varepsilon, \tau) \to 0$

The uniform bounds of Corollary 5 are sufficient to pass to the simultaneous limit $(\varepsilon, \tau) \to 0$. First, consider case (i), i.e. (H5') or (H5") with $a_{ii} > 0$. The gradient bound in (24) and estimate (25) of the discrete time derivative allow us to apply the Aubin-Lions lemma in the version of [10], yielding the existence of a subsequence, which is not relabeled, such that, as $(\varepsilon, \tau) \to 0$,

$$\begin{aligned} u^{(\tau)} &\to u \quad \text{strongly in } L^2(Q_T), \\ u^{(\tau)} &\rightharpoonup u \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \end{aligned}$$

where $u = (u_1, \ldots, u_n)$ and $u_i \ge 0$. In Case (ii), i.e. (H5") with $a_{i0} > 0$, we have only a gradient estimate for the square root of $u_i^{(\tau)}$, by (27). Therefore, we need to apply the nonlinear Aubin-Lions lemma of [4, Theorem 3] to find that, again for a subsequence,

$$\begin{aligned} u^{(\tau)} &\to u \quad \text{strongly in } L^1(Q_T), \\ (u^{(\tau)})^{1/2} &\rightharpoonup u^{1/2} \quad \text{weakly in } L^2(0, T; H^1(\Omega)). \end{aligned}$$

In both cases, possibly for a subsequence,

$$\varepsilon w^{(\tau)} \to 0 \quad \text{strongly in } L^2(0, T; H^m(\Omega)).$$

In the following, we focus on the case (H5') or (H5") with $a_{ii} > 0$ since the other case, (H5") with $a_{i0} > 0$, can be presented in a similar way. In fact, the existence proof works as long as $A_{ij}(u)\nabla u_j$ is bounded in $L^s(Q_T)$ for some $s > 1$, and this holds in both cases, as shown in the proof of Lemma 4.

Lemma 6. *Let (H5') or (H5") with $a_{ii} > 0$ for $i = 1, \ldots, n$ hold. Then*

$$\int_\Omega h(u)dx + \|u\|_{L^2(0,T;H^1(\Omega))} + \|u\|_{L^{2+2/d}(Q_T)} + \|A(u)\nabla u\|_{L^s(Q_T)} \le C(u^0, b, T),$$

where $s=(2d+2)/(2d+1)$, $\partial_t u_i \in L^s(0,T;W^{1,2d+2}(\Omega)')$, *for all* $\phi\in C_0^\infty(\overline{\Omega}\times[0,T);\mathbb{R}^n)$,

$$\begin{aligned}-\int_0^T\int_\Omega u\cdot\partial_t\phi dxdt-\int_\Omega u^0\cdot\phi(\cdot,0)dx+\int_0^T\int_\Omega\nabla\phi:(A(u)\nabla u-ub)dxdt\\ =\int_0^T\int_\Omega\frac{f(u)\cdot\phi}{1+\delta|f(u)|}dxdt,\end{aligned}\tag{29}$$

and $u_i(\cdot,0)=u_i^0$ *is satisfied in the sense of* $W^{1,2d+2}(\Omega)'$.

The notation $\psi\in C_0^\infty(\overline{\Omega}\times[0,T))$ means that $\psi\in C^\infty(\overline{\Omega}\times[0,T])$ and $\psi(\cdot,T)=0$. Furthermore, recall that $\nabla\phi:(ub)=\sum_{i=1}^n u_ib_i\cdot\nabla\phi_i$.

Proof. The strong convergence of $(u^{(\tau)})$ in $L^2(Q_T)$ implies that, for a subsequence, $u^{(\tau)}\to u$ a.e. in Q_T and, because of the continuity of h, $h(u^{(\tau)})\to h(u)$ a.e. in Q_T. Then Lemma 3 and Fatou's lemma show that $\int_\Omega h(u)dx\le C(u^0,b,T)$. The uniform estimate for $(u^{(\tau)})$ in $L^2(0,T;H^1(\Omega))$ and the weakly lower semi-continuity of the norm imply that $\|u\|_{L^2(0,T;H^1(\Omega))}\le C(u^0,b,T)$. Next, we apply the Gagliardo-Nirenberg inequality with $p=2+2/d$ and $\theta=d/(d+1)$ (such that $\theta p=2$):

$$\begin{aligned}\|u_i^{(\tau)}\|_{L^p(Q_T)}^p&=\int_0^T\|u_i^{(\tau)}\|_{L^p(\Omega)}^p dx\le C\int_0^T\|u_i^{(\tau)}\|_{H^1(\Omega)}^{\theta p}\|u_i^{(\tau)}\|_{L^1(\Omega)}^{(1-\theta)p}dt\\ &\le C\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega)}^{(1-\theta)p}\int_0^T\|u_i^{(\tau)}\|_{H^1(\Omega)}^2dt\le C(u^0,b,T).\end{aligned}$$

Then, by Hölder's inequality, since $1/s=1/p+1/2$,

$$\|u_i^{(\tau)}\nabla u_j^{(\tau)}\|_{L^s(Q_T)}\le C\|u_i^{(\tau)}\|_{L^p(Q_T)}\|\nabla u_i^{(\tau)}\|_{L^2(Q_T)}\le C(u^0,b,T).$$

Consequently, since $A_{ij}(u^{(\tau)})$ depends on $u_i^{(\tau)}$ linearly, $\|A_{ij}(u^{(\tau)})\nabla u_j^{(\tau)}\|_{L^s(Q_T)}\le C(u^0,b,T)$. It holds that $A_{ij}(u^{(\tau)})\to A_{ij}(u)$ strongly in $L^2(Q_T)$ and $\nabla u_j^{(\tau)}\rightharpoonup\nabla u_j$ weakly in $L^2(Q_T)$. Then the product converges weakly, $A_{ij}(u^{(\tau)})\nabla u^{(\tau)}\rightharpoonup A_{ij}(u)\nabla u$ weakly in $L^1(Q_T)$, and this convergence holds even in $L^s(Q_T)$. Hence, the limit $(\varepsilon,\tau)\to 0$ leads to the estimate $\|A_{ij}(u)\nabla u_j\|_{L^s(Q_T)}\le C(u^0,b,T)$.

Summing the weak formulation (21) from $k=1,\ldots,N$, we may reformulate it as

$$\begin{aligned}&\int_0^T\int_\Omega\partial_t^{(\tau)}u^{(\tau)}\cdot\phi dxdt+\int_0^T\int_\Omega\nabla\phi:\big(A(u^{(\tau)})\nabla u^{(\tau)}-u^{(\tau)}b^{(\tau)}\big)dxdt\\ &\quad+\varepsilon\int_0^T\int_\Omega\Bigg(\sum_{|\alpha|=m}D^\alpha w^{(\tau)}\cdot D^\alpha\phi+w^{(\tau)}\cdot\phi\Bigg)dxdt=\int_0^T\int_\Omega\frac{f(u^{(\tau)})\cdot\phi}{1+\delta|f(u^{(\tau)})|}dxdt.\end{aligned}\tag{30}$$

Using the arguments of [5, pp. 2792-2793], we can show that

$$\begin{aligned}\int_0^T\int_\Omega\partial_t^{(\tau)}u^{(\tau)}\cdot\phi dxdt\to-\int_0^T\int_\Omega u\cdot\partial_t\phi dxdt-\int_\Omega u^0\cdot\phi(\cdot,0)dx,\\ \partial_t^{(\tau)}u^{(\tau)}\rightharpoonup\partial_t u\quad\text{weakly in } L^r(0,T;H^{m+1}(\Omega)').\end{aligned}$$

Since $A(u^{(\tau)})\nabla u^{(\tau)} \rightharpoonup A(u)\nabla u$ weakly in $L^s(Q_T)$ (see the above argumentation) and $b^{(\tau)} \to b$ in $L^2(Q_T)$, we can pass to the limit $(\varepsilon,\tau)\to 0$ in the second integral of (30). The third integral vanishes in the limit, and the integral on the right-hand side of (30) converges to

$$\int_0^T \int_\Omega \frac{f(u)\cdot\phi}{1+\delta|f(u)|}dxdt,$$

since $f(u^{(\tau)})/(1+\delta|f(u^{(\tau)})|)$ is bounded independently of (ε,τ) (but depending on δ). Thus, in the limit $(\varepsilon,\tau)\to 0$, we infer formulation (29). Since $A_{ij}(u)\nabla u_j \in L^s(Q_T)$ and $f_i(u)/(1+\delta|f(u)|) \in L^\infty(Q_T)$ for any fixed $\delta > 0$, a density argument shows that the weak formulation (29) holds for all $\phi \in L^{2d+2}(0,T;W^{1,2d+2}(\Omega;\mathbb{R}^n))$. This implies that $\partial_t u_i \in L^s(0,T;W^{1,2d+2}(\Omega)')$. $\square$

5. The limit $\delta \to 0$, $L \to \infty$

We use two results from [14], a truncation and an approximate chain rule. For the truncation, let $\varphi \in C^\infty(\mathbb{R})$ be a nonincreasing function satisfying $\varphi(x) = 1$ for $x < 0$ and $\varphi(x) = 0$ for $x \geq 1$. We define for $i = 1,\ldots,n$ and $L \in \mathbb{N}$ the truncation function

$$\varphi_i^L(v) = v_i\varphi\bigg(\frac{1}{L}\sum_{k=1}^n v_k - 1\bigg) + 2L\bigg(1-\varphi\bigg(\frac{1}{L}\sum_{k=1}^n v_k - 1\bigg)\bigg) \quad \text{for } v \in [0,\infty)^n. \tag{31}$$

Clearly, $\varphi_i^L \in C^\infty([0,\infty)^n)$. Moreover, the following properties hold.

Lemma 7. *It holds that*

(L1) *For all $L \in \mathbb{N}$, $v \in [0,\infty)^n$, $i = 1,\ldots,n$,*

$$0 \leq \varphi_i^L(v) \leq v_i + 2\sum_{k=1}^n v_k.$$

(L2) *For all $v \in [0,\infty)^n$ with $\sum_{k=1}^n v_k < L$, we have $\varphi_i^L(v) = v_i$.*

(L3) *For any fixed $L \in \mathbb{N}$, $\operatorname{supp}(\varphi_i^L)'$ is a compact subset of $[0,\infty)^n$.*

(L4) *For all $v \in [0,\infty)^n$, $j = 1,\ldots,n$, we have $\lim_{L\to\infty}\partial_j\varphi_i^L(v) = \delta_{ij}$.*

(L5) *There exists $K_1 > 0$ such that for all $L \in \mathbb{N}$, $v \in [0,\infty)^n$, $j = 1,\ldots,n$,*

$$|\partial_j\varphi_i^L(v)| \leq K_1.$$

(L6) *For all $j,k = 1\ldots,n$,*

$$\lim_{L\to\infty}\sup_{v\in[0,\infty)^n}|\partial_j\partial_k\varphi_i^L(v)| = 0.$$

(L7) *There exists $K_2 > 0$ such that for all $L \in \mathbb{N}$, $v \in [0,\infty)^n$, $j,k,\ell = 1,\ldots,n$,*

$$(1+v_\ell)|\partial_j\partial_k\varphi_i^L(v)| + v_j^{1/2}v_k^{1/2}|\partial_j\partial_k\varphi_i^L(v)| \leq K_2.$$

(L8) *Let $L > L_0 > 0$ and $v \in [0,\infty)^n$. If $\sum_{i=1}^n v_i \geq L_0$ then $\sum_{i=1}^n \varphi_i^L(v) \geq L_0$.*

Property (L4) is not used in the proof but it clarifies the role of φ_i^L, when we compute the limit $L\to\infty$ in $\sum_{j=1}^n \partial_j\varphi_i^L(u)$, which gives u_i.

Proof. Ad (L1): Let $v_s := \frac{1}{L}\sum_{k=1}^n v_k - 1$. If $v_s < 0$ then $\varphi(v_s) = 1$ and $\varphi_i^L(v) = v_i$. If $v_s > 1$ (which is equivalent to $2L < \sum_{k=1}^n v_k$) then $\varphi(v_s) = 0$ and $\varphi_i^L(v) = 2L < \sum_{k=1}^n v_k$. Finally, if $0 \le v_s \le 1$ (or $L \le \sum_{k=1}^n v_k \le 2L$), we have $\varphi_i^L(v) \le v_i + 2L \le v_i + 2\sum_{k=1}^n v_k$.

Ad (L2): $\sum_{k=1}^n v_k < L$ implies that $v_s < 0$ and $\varphi_i^L(v) = v_i$.

Ad (L3): This is clear since φ is constant on $(-\infty, 0)$ and $(1,\infty)$.

Ad (L4) and (L5): See (E4) and (E5), respectively, in [14].

Ad (L6): This is (E7) in [14] except that the supremum is computed on $(0,\infty)^n$ (which is possible since $\partial_j\partial_k\varphi_i^L$ has compact support).

Ad (L7): The second inequality can be found in (E2) of [14]. The first one is new and can be readily verified.

Ad (L8): By definition of φ_i^L, we have

$$\sum_{i=1}^n \varphi_i^L(v) - \sum_{i=1}^n v_i = \bigg(2nL - \sum_{i=1}^n v_i\bigg)\bigg(1 - \varphi\bigg(\frac{1}{L}\sum_{k=1}^n v_k - 1\bigg)\bigg).$$

If $\sum_{i=1}^n v_i \le 2nL$ then $\sum_{i=1}^n \varphi_i^L(v) \ge \sum_{i=1}^n v_i \ge L_0$; otherwise $\frac{1}{L}\sum_{k=1}^n v_k - 1 > 2n - 1 \ge 1$. Hence, $\varphi_i^L(v) = 2L > L_0$. □

The second result concerns an approximate chain rule. Let $\mathcal{M}(\overline{\Omega}\times[0,T))$ denote the set of Radon measures on the Borel sets of $\overline{\Omega}\times[0,T)$ and let S denote the $(d-1)$-dimensional Hausdorff measure.

Lemma 8 (Lemma 4 of [14])**.** *Let $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ be a bounded domain with Lipschitz boundary, $T > 0$, $v_{0,i} \in L^1(\Omega)$, $w_i \in L^1(0,T;L^1(\Omega))$, $z_i \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, $q_i \in L^1(0,T;L^1(\partial\Omega))$, and $\mu_i \in \mathcal{M}(\overline{\Omega}\times[0,T))$, $i = 1,\ldots,n$. Assume that $v \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ solves for all $\psi \in C_0^\infty(\overline{\Omega}\times[0,T))$,*

$$\begin{aligned} -\int_0^T\int_\Omega v_i\partial_t\psi dxdt - \int_\Omega v_{0,i}\psi(\cdot,0)dx &= \int_{\overline{\Omega}\times[0,T)}\psi d\mu_i + \int_0^T\int_\Omega w_i\psi dxdt \\ &\quad + \int_0^T\int_{\partial\Omega} q_i\psi dSdt + \int_0^T\int_\Omega z_i\cdot\nabla\psi dxdt. \end{aligned}$$

Finally, let $\xi \in C^\infty(\mathbb{R}^n)$ be given with compactly supported first derivatives. Then there exists $C(\Omega) > 0$ such that for all $\psi \in C_0^\infty(\overline{\Omega}\times[0,T))$,

$$\begin{aligned} &\bigg| -\int_0^T\int_\Omega \xi(v)\partial_t\psi dxdt - \int_\Omega \xi(v_0)\psi(\cdot,0)dx \\ &\quad - \int_0^T\int_\Omega\sum_{i=1}^n \partial_i\xi(v)w_i\psi dxdt - \int_0^T\int_{\partial\Omega}\sum_{i=1}^n\partial_i\xi(v)q_i\psi dSdt \\ &\quad - \int_0^T\int_\Omega\sum_{i=1}^n\partial_i\xi(v)z_i\cdot\nabla\psi dxdt - \int_0^T\int_\Omega\sum_{j,k=1}^n\partial_j\partial_k\xi(v)z_j\cdot\nabla v_k\psi dxdt\bigg| \\ &\le C(\Omega)\sup_u|\xi'(u)|\|\psi\|_{L^\infty(Q_T)}\sum_{i=1}^n\|\mu_i\|_{\mathcal{M}(\overline{\Omega}\times[0,T))}. \end{aligned}$$

We apply this lemma to (29).

Lemma 9. *Let $u^{(\delta)}$ be a weak solution to* (29). *Then, for all $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$, $L \in \mathbb{N}$, and $i = 1, \ldots, n$,*

$$
\begin{aligned}
&- \int_0^T \int_\Omega \varphi_i^L(u^{(\delta)}) \partial_t \psi dx dt - \int_\Omega \varphi_i^L(u^0)\psi(\cdot,0)dx \\
&\quad = - \int_0^T \int_\Omega \sum_{j,k=1}^n \partial_j \partial_k \varphi_i^L(u^{(\delta)}) \bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)}) \nabla u_\ell^{(\delta)} - u_j^{(\delta)} b_j \bigg) \cdot \nabla u_k^{(\delta)} \psi dx dt \\
&\qquad - \int_0^T \int_\Omega \sum_{j=1}^n \partial_j \varphi_i^L(u^{(\delta)}) \bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)}) \nabla u_\ell^{(\delta)} - u_j^{(\delta)} b_j \bigg) \cdot \nabla \psi dx dt \\
&\qquad + \int_0^T \int_\Omega \sum_{j=1}^n \partial_j \varphi_i^L(u^{(\delta)}) \frac{f_j(u^{(\delta)})\psi}{1+\delta|f(u^{(\delta)})|} dx dt,
\end{aligned} \tag{32}
$$

where we recall definition (31) *of φ_i^L.*

Proof. Taking the test function $\phi = (\phi_1, \ldots, \phi_n)$ with $\phi_j = \delta_{ij}\psi$ and $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$, we see that $u^{(\delta)}$ solves

$$
\begin{aligned}
&- \int_0^T \int_\Omega u_i^{(\delta)} \partial_t \psi dx dt - \int_\Omega u_i^0 \psi(\cdot,0) dx \\
&\quad + \int_0^T \int_\Omega \nabla\psi \cdot \bigg(\sum_{j=1}^n A_{ij}(u^{(\delta)}) \nabla u_j^{(\delta)} - u_i^{(\delta)} b_i \bigg) dx dt = \int_0^T \int_\Omega \frac{f_i(u^{(\delta)})\psi}{1+\delta|f(u^{(\delta)})|} dx dt.
\end{aligned} \tag{33}
$$

In Lemma 8, we choose $\xi(v) = \varphi_i^L(v)$, $v_i = u_i^{(\delta)} \in L^2(0,T;H^1(\Omega))$, $v_{0,i} = u_i^0 \in L^1(\Omega)$, $w_i = f_i(u^{(\delta)})/(1+\delta|f(u^{(\delta)})|) \in L^1(Q_T)$, $\mu_i = 0$, $q_i = 0$, and $z_i = -(\sum_{j=1}^n A_{ij}(u^{(\delta)})\nabla u_j^{(\delta)} - u_i^{(\delta)} b_i) \in L^s(0,T;L^s(\Omega;\mathbb{R}^n))$ with $s = (2d+2)/(2d+1)$, where the regularity statements for $u_i^{(\delta)}$ and z_i are obtained from Lemma 6. If $z_i \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, the weak formulation (32) is a direct result of Lemma 8. We claim that the lemma can be applied also in the present situation. Indeed, as the support of $(\varphi_i^L)'$ is bounded and $\nabla u_\ell^{(\delta)} \in L^2(Q_T)$, we have

$$
\partial_j \partial_k \varphi_i^L(u^{(\delta)}) z_j = \partial_j \partial_k \varphi_i^L(u^{(\delta)}) \bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)}) \nabla u_\ell^{(\delta)} - u_j^{(\delta)} b_j \bigg) \in L^2(Q_T;\mathbb{R}^n),
$$

and this regularity is sufficient for the proof of Lemma 8. More precisely, let [14, page 579]

$$
\widehat{v}_i = \rho_\varepsilon * u_i^{(\delta)}, \quad \widehat{w}_i = \rho_\varepsilon * w_i + \operatorname{div} \bigg(\sum_{\ell=1}^n A_{i\ell}(\rho_\varepsilon * u^{(\delta)}) \nabla(\rho_\varepsilon * u_\ell^{(\delta)}) - (\rho_\varepsilon * u_i^{(\delta)}) b_i \bigg),
$$

where ρ_ε is the standard mollifier on Ω. Then the second line on page 580 in [14] can be replaced by

$$\sum_{j,k=1}^n \int_0^T \int_\Omega \partial_j\partial_k\varphi_i^L(\rho_\varepsilon * u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(\rho_\varepsilon * u^{(\delta)})\nabla(\rho_\varepsilon * u_\ell^{(\delta)}) - (\rho_\varepsilon * u_j^{(\delta)})b_j\bigg) \times \nabla(\rho_\varepsilon * u_k^{(\delta)})\psi dxdt.$$

The regularity $\nabla u_\ell^{(\delta)} \in L^2(Q_T)$ and condition (L3) are sufficient to pass to the limit $\varepsilon \to 0$. The corresponding term on page 582, line 5 in [14] can be treated in a similar way. Consequently, Lemma 8 implies (32). □

Remark 10. Note that if (H5") with $a_{i0} > 0$ holds, the argumentation of the previous proof is slightly different. Because of the L^2 bound of $\nabla(u_i^{(\delta)})^{1/2}$, we write

$$\partial_j\partial_k\varphi_i^L(u^{(\delta)})z_j = \partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(2\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})(u_\ell^{(\delta)})^{1/2}\nabla(u_\ell^{(\delta)})^{1/2} - u_j^{(\delta)}b_j\bigg),$$

and this expression is still in $L^2(Q_T;\mathbb{R}^n)$ taking into account the compact support of $\partial_k\varphi_i^L$. This argument can be also used in the following proofs. □

Now, we can perform the limit $\delta \to 0$. The following result is the key lemma.

Lemma 11. *There exists a subsequence of $(u^{(\delta)})$ (not relabeled) and a nonnegative function $u \in L^2(0,T;\ H^1(\Omega;\mathbb{R}^n))$ satisfying $\int_\Omega h(u)dx < \infty$ such that, as $\delta \to 0$,*

$$u^{(\delta)} \to u \quad \text{strongly in } L^2(Q_T), \tag{34}$$

$$u^{(\delta)} \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^1(\Omega)). \tag{35}$$

Moreover, u solves for all $L \in \mathbb{N}$, $i = 1,\dots,n$, and $\psi \in C_0^\infty(\overline{\Omega}\times[0,T))$,

$$\begin{aligned} -\int_0^T\int_\Omega \varphi_i^L(u)\partial_t\psi dxdt - \int_\Omega \varphi_i^L(u^0)\psi(\cdot,0)dx &= -\int_0^T\int_\Omega \psi d\mu_i^L(x,t) \\ &\quad - \int_0^T\int_\Omega \sum_{j=1}^n \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla\psi dxdt \\ &\quad + \int_0^T\int_\Omega \sum_{j=1}^n \partial_j\varphi_i^L(u)f_j(u)\psi dxdt, \end{aligned} \tag{36}$$

where $(\mu_i^L)_{L\in\mathbb{N}}$ is a sequence of signed Radon measures satisfying

$$\lim_{L\to\infty} |\mu_i^L|\big(\overline{\Omega}\times[0,T)\big) = 0. \tag{37}$$

Proof. Step 1. The weak formulation (36) holds for all test functions which vanish at $t = T$. We wish to derive a weak formulation valid for all test functions $\psi \in C^\infty(\overline{\Omega}\times[0,T])$. To this end, we introduce for $m\in\mathbb{N}$ the functions

$$g_m(t)=\begin{cases} 1 & \text{if } t\in(-1,T-\frac{1}{m}],\\ 2m(T-\frac{1}{m}-t)+1 & \text{if } t\in(T-\frac{1}{m},T-\frac{1}{2m}),\\ 0 & \text{if } t\in[T-\frac{1}{2m},T+1). \end{cases}$$

Then the function g_m is continuous on $(-1,T+1)$, its weak derivative equals

$$g_m'(t)=\begin{cases} -2m & \text{if } t\in(T-\frac{1}{m},T-\frac{1}{2m}),\\ 0 & \text{if } t\in(-1,T-\frac{1}{m}]\cup[T-\frac{1}{2m},T+1), \end{cases}$$

$\lim_{m\to\infty}g_m(t)=1$ for $t\in[0,T)$, and $\lim_{m\to\infty}g_m(t)=0$ for $t\in[T,T+1)$. Set $g_m^\varepsilon:=\eta_\varepsilon*g_m$, where η_ε is the standard mollifier on $\mathbb{R}$; see the definition in [11, Section C.4]. In particular (see [11, Section 5.3.1, Theorem 1]), $g_m^\varepsilon\in C_0^\infty([0,T))\cap C^\infty([0,T])$, $(g_m^\varepsilon)'=\eta_\varepsilon*g_m'$ on $(-1+\varepsilon,T+1-\varepsilon)$ for sufficiently small $\varepsilon>0$, and

$$g_m^\varepsilon\to g_m \quad\text{in } C^0([0,T]),\qquad (g_m^\varepsilon)'\to g_m' \quad\text{in } L^2(0,T) \text{ as } \varepsilon\to 0. \tag{38}$$

Let $\psi\in C^\infty(\overline{\Omega}\times[0,T])$. Then $\psi g_m^\varepsilon\in C_0^\infty(\overline{\Omega}\times[0,T))$, and we can use this function as a test function in (32):

$$\begin{aligned} -\int_0^T\int_\Omega \varphi_i^L(u^{(\delta)})\partial_t(\psi g_m^\varepsilon)dxdt-\int_\Omega \varphi_i^L(u^0)(\psi g_m^\varepsilon)(\cdot,0)dx \\ =-\int_0^T\int_\Omega\sum_{j,k=1}^n\partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}-u_j^{(\delta)}b_j\bigg)\cdot\nabla u_k^{(\delta)}(\psi g_m^\varepsilon)dxdt \\ -\int_0^T\int_\Omega\sum_{j=1}^n\partial_j\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}-u_j^{(\delta)}b_j\bigg)\cdot\nabla\psi g_m^\varepsilon dxdt \\ +\int_0^T\int_\Omega\sum_{j=1}^n\partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})\psi g_m^\varepsilon}{1+\delta|f(u^{(\delta)})|}dxdt. \end{aligned}$$

Taking into account the compact support of $(\varphi_i^L)'$, by (L3), the uniform bounds from Lemma 6, and the convergence properties (38), we can pass to the limit $\varepsilon\to 0$ in the previous equation, leading to

$$\begin{aligned} -\int_0^T\int_\Omega \varphi_i^L(u^{(\delta)})\partial_t\psi g_m dxdt-\int_0^T\int_\Omega\varphi_i^L(u^{(\delta)})\psi g_m'dx-\int_\Omega \varphi_i^L(u^0)(\psi g_m)(\cdot,0)dx \\ =-\int_0^T\int_\Omega\sum_{j,k=1}^n\partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}-u_j^{(\delta)}b_j\bigg)\cdot\nabla u_k^{(\delta)}(\psi g_m)dxdt \\ -\int_0^T\int_\Omega\sum_{j=1}^n\partial_j\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}-u_j^{(\delta)}b_j\bigg)\cdot\nabla(\psi g_m)dxdt \\ +\int_0^T\int_\Omega\sum_{j=1}^n\partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})\psi g_m}{1+\delta|f(u^{(\delta)})|}dxdt. \end{aligned}$$

Next, we perform the limit $m \to \infty$. The only delicate term is the integral involving g_m':

$$-\int_0^T \int_\Omega \varphi_i^L(u^{(\delta)})\psi g_m' dxdt = 2m \int_{T-1/m}^{T-1/(2m)} \int_\Omega \varphi_i^L(u^{(\delta)})\psi dxdt$$
$$\to \int_\Omega \varphi_i^L(u^{(\delta)}(\cdot,T))\psi(\cdot,T)dx \quad \text{as } m \to \infty.$$

For the other terms, we employ the uniform bounds in Lemma 6, the pointwise convergence $g_m(t) \to 1$ for $t \in [0,T)$, and Lebesgue's dominated convergence theorem. Then, in the limit $m \to \infty$,

$$\begin{aligned} -\int_0^T \int_\Omega \varphi_i^L(u^{(\delta)})\partial_t\psi dxdt &+ \int_\Omega \varphi_i^L(u^{(\delta)}(\cdot,T))\psi(\cdot,T)dx - \int_\Omega \varphi_i^L(u^0)\psi(,\cdot 0)dx \\ &= -\int_0^T \int_\Omega \sum_{j,k=1}^n \partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)} - u_j^{(\delta)}b_j\bigg)\cdot\nabla u_k^{(\delta)}\psi dxdt \\ &\quad - \int_0^T \int_\Omega \sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)} - u_j^{(\delta)}b_j\bigg)\cdot\nabla\psi dxdt \\ &\quad + \int_0^T \int_\Omega \sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})\psi}{1+\delta|f(u^{(\delta)})|}dxdt. \end{aligned} \tag{39}$$

This holds for all $\psi \in C^\infty(\overline{\Omega}\times[0,T])$. In fact, by a density argument, the weak formulation also holds for all $\psi \in C^0([0,T];H^p(\Omega))$, where $p > d/2+1$ (such that the embedding $H^p(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ is continuous).

Step 2. We claim that a subsequence of $(\varphi_i^L(u^{(\delta)}))$ is convergent in the limit $\delta \to 0$. Observing that the dual space of $C^0([0,T];H^p(\Omega))$ is $\mathcal{M}([0,T];H^p(\Omega)')$, where $\mathcal{M}$ denotes the space of Radon measures, we find that

$$\|\partial_t\varphi_i^L(u^{(\delta)})\|_{\mathcal{M}([0,T];H^p(\Omega)')} = \sup_{\|\psi\|_{C^0([0,T];H^p(\Omega))}\le 1}\bigg| -\int_0^T \int_\Omega \varphi_i^L(u^{(\delta)})\partial_t\psi dxdt$$
$$+ \int_\Omega \varphi_i^L(u^{(\delta)}(\cdot,T))\psi(\cdot,T)dx - \int_\Omega \varphi_i^L(u^0)\psi(\cdot,0)dx\bigg|.$$

We insert (39) and take into account the uniform bounds in Lemma 6 and the compact support of $(\varphi_i^L)'$. Then the right-hand side of (39) can be bounded uniformly in δ:

$$\|\partial_t\varphi_i^L(u^{(\delta)})\|_{\mathcal{M}([0,T];H^p(\Omega)')} \le C(L,u^0,b,T). \tag{40}$$

By (L1) and (L5), the function φ_i^L is growing at most linearly and its gradient is bounded. Therefore, the gradient bound for $u^{(\delta)}$ shows that

$$\|\varphi_i^L(u^{(\delta)})\|_{L^2(0,T;H^1(\Omega))} \le C(u^0,b,T). \tag{41}$$

Estimates (40) and (41) allow us to apply the Aubin-Lions lemma in the version of [25, Section 7.3, Corollary 7.9] to obtain the existence of a subsequence of $(\varphi_i^L(u^{(\delta)}))$, which is not relabeled, such that, as $\delta \to 0$,

$$\varphi_i^L(u^{(\delta)}) \to v_i^L \quad \text{strongly in } L^2(Q_T) \tag{42}$$

for some nonnegative function v_i^L.

We claim that the subsequence can be chosen in such a way that it is independent of $i \in \{1,\ldots,n\}$ and $L \in \mathbb{N}$. Since the set $\{1,\ldots,n\}$ is finite, we have to prove this statement only for $L \in \mathbb{N}$. The idea is to apply a diagonal argument. Let $i \in \{1,\ldots,n\}$ be fixed. By the Aubin-Lions lemma, there exists a subsequence (δ_k^1) such that $\varphi_i^1(u^{(\delta_k^1)}) \to v_i^1$ as $k \to \infty$, and there exists a subsequence (δ_k^2) of (δ_k^1) such that $\varphi_i^2(u^{(\delta_k^2)}) \to v_i^2$. Continuing this argument, we find a subsequence (δ_k^L) of (δ_k^{L-1}) such that $\varphi_i^L(u^{(\delta_k^L)}) \to v_i^L$. In other words,

$$\begin{aligned}
&\varphi_i^1(u^{(\delta_1^1)}),\ \varphi_i^1(u^{(\delta_2^1)}),\ \varphi_i^1(u^{(\delta_3^1)}),\ldots \to v_i^1, \\
&\varphi_i^1(u^{(\delta_1^2)}),\ \varphi_i^1(u^{(\delta_2^2)}),\ \varphi_i^1(u^{(\delta_3^2)}),\ldots \to v_i^1, \\
&\varphi_i^1(u^{(\delta_1^3)}),\ \varphi_i^1(u^{(\delta_2^3)}),\ \varphi_i^1(u^{(\delta_3^3)}),\ldots \to v_i^1, \text{ etc.}
\end{aligned}$$

Thus, taking the diagonal terms as a new subsequence $(\varphi_i^1(u^{(\delta_k^k)}))$, we have the convergence $\varphi_i^1(u^{(\delta_k^k)}) \to v_i^1$ as $k \to \infty$. For $L = 2$, we argue in a similar way,

$$\begin{aligned}
&\varphi_i^2(u^{(\delta_2^2)}),\ \varphi_i^2(u^{(\delta_3^2)}),\ \varphi_i^2(u^{(\delta_4^2)}),\ldots \to v_i^2, \\
&\varphi_i^2(u^{(\delta_2^3)}),\ \varphi_i^2(u^{(\delta_3^3)}),\ \varphi_i^2(u^{(\delta_4^3)}),\ldots \to v_i^2, \\
&\varphi_i^2(u^{(\delta_2^4)}),\ \varphi_i^2(u^{(\delta_3^4)}),\ \varphi_i^2(u^{(\delta_4^4)}),\ldots \to v_i^2, \text{ etc.}
\end{aligned}$$

Then the diagonal sequence converges, $\varphi_i^2(u^{(\delta_k^k)}) \to v_i^2$ as $k \to \infty$. This argument can be continued, and we obtain a universal subsequence $(u^{(\delta_k^k)})$, which is independent of L, such that (42) holds, and we call this subsequence simply $u^{(\delta)}$.

Step 3. We prove that, up to a subsequence, $u_i^{(\delta)} \to u_i$ a.e. First, we claim that $(v_i^L)_{L\in\mathbb{N}}$ is a Cauchy sequence. Let K, $L \in \mathbb{N}$ with $K > L$ be given. Then, by definition of φ_i^L and setting $v_\delta^L := \frac{1}{L}\sum_{k=1}^n u_k^{(\delta)} - 1$,

$$\begin{aligned}
\big\|\varphi_i^L(u^{(\delta)}) - \varphi_i^K(u^{(\delta)})\big\|_{L^1(Q_T)} &\le \big\|u_i^{(\delta)}\big(\varphi(v_\delta^L) - \varphi(v_\delta^K)\big)\big\|_{L^1(Q_T)} \\
&\quad + 2\big\|L\big(1-\varphi(v_\delta^L)\big) - K\big(1-\varphi(v_\delta^K)\big)\|_{L^1(Q_T)} \\
&=: I_1 + I_2.
\end{aligned} \tag{43}$$

By the mean value theorem, Hölder's inequality, and the uniform bounds in Lemma 6, we find that

$$\begin{aligned}
I_1 &\le \max_{s\in\mathbb{R}} |\varphi'(s)| \bigg\| u_i^{(\delta)}\bigg(\frac{1}{L}\sum_{k=1}^n u_k^{(\delta)} - \frac{1}{K}\sum_{k=1}^n u_k^{(\delta)}\bigg)\bigg\|_{L^1(Q_T)} \\
&\le \max_{s\in\mathbb{R}} |\varphi'(s)| \bigg(\frac{1}{L} - \frac{1}{K}\bigg)\sum_{k=1}^n \|u_i^{(\delta)}u_k^{(\delta)}\|_{L^1(Q_T)}
\end{aligned}$$

$$\le \max_{s\in\mathbb{R}}|\varphi'(s)|\frac{1}{L}\|u_i^{(\delta)}\|_{L^2(Q_T)}\sum_{k=1}^n\|u_k^{(\delta)}\|_{L^2(Q_T)} \le \frac{C(u^0,b,T)}{L}.$$

For the expression I_2, we use the property $\varphi(-1)=1$ and the mean value theorem again:

$$\begin{aligned}
I_2 &= 2\big\|L\big(\varphi(-1)-\varphi(v_\delta^L)\big) - K\big(\varphi(-1)-\varphi(v_\delta^K)\big)\big\|_{L^1(Q_T)} \\
&= 2\bigg\|\sum_{k=1}^n u_k^{(\delta)}\bigg\{\varphi'\bigg(-1+\frac{\theta_1}{L}\sum_{k=1}^n u_k^{(\delta)}\bigg) - \varphi'\bigg(-1+\frac{\theta_2}{K}\sum_{k=1}^n u_k^{(\delta)}\bigg)\bigg\}\bigg\|_{L^1(Q_T)} \\
&= 2\bigg\|\bigg(\sum_{k=1}^n u_k^{(\delta)}\bigg)^2\varphi''(\xi)\bigg(\frac{\theta_1}{L}-\frac{\theta_2}{K}\bigg)\bigg\|_{L^1(Q_T)},
\end{aligned}$$

where $\theta_1=\theta_1(x,t)$, $\theta_2=\theta_2(x,t)\in(0,1)$ and $\xi\in\mathbb{R}$. Therefore,

$$I_2 \le 2\max_{s\in\mathbb{R}}|\varphi''(s)|\frac{1}{L}\bigg\|\sum_{k=1}^n u_k^{(\delta)}\bigg\|_{L^1(Q_T)}^2 \le \frac{C(u^0,b,T)}{L}.$$

We infer from (43) that

$$\big\|\varphi_i^L(u^{(\delta)}) - \varphi_i^K(u^{(\delta)})\big\|_{L^1(Q_T)} \le \frac{C(u^0,b,T)}{L}.$$

This estimate and the convergence (42) allow us to conclude that (v_i^L) is a Cauchy sequence. Indeed, we find that

$$\begin{aligned}
\|v_i^L - v_i^K\|_{L^1(Q_T)} &\le \big\|v_i^L - \varphi_i^L(u^{(\delta)})\big\|_{L^1(Q_T)} + \big\|\varphi_i^L(u^{(\delta)}) - \varphi_i^K(u^{(\delta)})\big\|_{L^1(Q_T)} \\
&\quad + \big\|\varphi_i^K(u^{(\delta)}) - v_i^K\big\|_{L^1(Q_T)} \\
&\le \big\|v_i^L - \varphi_i^L(u^{(\delta)})\big\|_{L^1(Q_T)} + \frac{C(u^0,b,T)}{L} + \big\|\varphi_i^K(u^{(\delta)}) - v_i^K\big\|_{L^1(Q_T)}.
\end{aligned}$$

In the limit $\delta\to 0$, we infer from (42) that for all K, $L\in\mathbb{N}$ with $K>L$,

$$\|v_i^L - v_i^K\|_{L^1(Q_T)} \le \frac{C(u^0,b,T)}{L},$$

which proves the claim. Consequently, there exist functions $u_i\in L^1(Q_T)$ with $u_i\ge 0$ in Q_T such that $v_i^L\to u_i$ strongly in $L^1(Q_T)$ as $L\to\infty$.

Next, we prove that $u_i^{(\delta)}\to u_i$ a.e. For this, we proceed similarly as in [14, p. 572] and show that $(u_i^{(\delta)})$ converges in measure. Since

$$|u_i^{(\delta)} - u_i| \le \big|u_i^{(\delta)} - \varphi_i^L(u^{(\delta)})\big| + \big|\varphi_i^L(u^{(\delta)}) - v_i^L\big| + |v_i^L - u_i|,$$

we have for any $\varepsilon>0$,

$$\begin{aligned}
&\operatorname{meas}\big(\big\{(x,t)\in Q_T : |u_i^{(\delta)}(x,t) - u_i(x,t)| > \varepsilon\big\}\big) \le \operatorname{meas}(\{u_i^{(\delta)} \ne \varphi_i^L(u^{(\delta)})\}) \\
&\qquad + \operatorname{meas}\bigg(\bigg\{|\varphi_i^L(u^{(\delta)}) - v_i^L| > \frac{\varepsilon}{2}\bigg\}\bigg) + \operatorname{meas}\bigg(\bigg\{|v_i^L - u_i| > \frac{\varepsilon}{2}\bigg\}\bigg).
\end{aligned} \tag{44}$$

By (L2), $u_i^{(\delta)} \neq \varphi_i^L(u^{(\delta)})$ implies that $\sum_{k=1}^n u_i^{(\delta)} \geq L$. Therefore, by the uniform L^1 bound for $u^{(\delta)}$, the first term on the right-hand side can be estimated as follows:

$$\begin{aligned}\mathrm{meas}(\{u_i^{(\delta)} \neq \varphi_i^L(u^{(\delta)})\}) &\leq \mathrm{meas}\bigg(\bigg\{\sum_{k=1}^n u_k^{(\delta)} \geq L\bigg\}\bigg) \\ &= \frac{1}{L}\int_0^T\int_\Omega \chi_{\{\sum_{k=1}^n u_k^{(\delta)} \geq L\}} L dxdt \leq \frac{1}{L}\int_0^T\int_\Omega \sum_{k=1}^n u_k^{(\delta)} dxdt \leq \frac{C(u^0,b,T)}{L},\end{aligned}$$

where χ_A is the characteristic function on the set A. Similarly,

$$\begin{aligned}\mathrm{meas}\bigg(\bigg\{|\varphi_i^L(u^{(\delta)}) - v_i^L| > \frac{\varepsilon}{2}\bigg\}\bigg) &\leq \frac{2}{\varepsilon}\big\|\varphi_i^L(u^{(\delta)}) - v_i^L\big\|_{L^1(Q_T)} \\ \mathrm{meas}\bigg(\bigg\{|v_i^L - u_i| > \frac{\varepsilon}{2}\bigg\}\bigg) &\leq \frac{2}{\varepsilon}\|v_i^L - u_i\|_{L^1(Q_T)}.\end{aligned}$$

Therefore, (44) gives for any $\varepsilon > 0$,

$$\mathrm{meas}(\{|u_i^{(\delta)} - u_i| > \varepsilon\}) \leq \frac{C(u^0,b,T)}{L} + \frac{2}{\varepsilon}\big\|\varphi_i^L(u^{(\delta)}) - v_i^L\big\|_{L^1(Q_T)} + \frac{2}{\varepsilon}\|v_i^L - u_i\|_{L^1(Q_T)}.$$

We infer from (42) that in the limit $\delta \to 0$,

$$\limsup_{\delta\to 0}\mathrm{meas}(\{u_i^{(\delta)} - u_i| > \varepsilon\}) \leq \frac{C(u^0,b,T)}{L} + \frac{2}{\varepsilon}\|v_i^L - u_i\|_{L^1(Q_T)}.$$

As $v_i^L \to u_i$ a.e., the limit $L \to \infty$ then gives

$$\lim_{\delta\to 0}\mathrm{meas}(\{|u_i^{(\delta)} - u_i| > \varepsilon\}) = 0.$$

This shows that $(u_i^{(\delta)})$ converges in measure. Hence, there exists a subsequence, which is not relabeled, such that $u_i^{(\delta)} \to u_i$ a.e. in Q_T. The uniform bound for $(u_i^{(\delta)})$ in $L^{2+2/d}(Q_T)$ (see Lemma 6) implies that $u_i^{(\delta)} \to u_i$ strongly in $L^2(Q_T)$, which proves (34). By the same lemma, also the weak convergence (35) follows (again up to a subsequence). Moreover, by Fatou's lemma, $\int_\Omega h(u)dx < \infty$.

Step 4. Next, we verify identity (36) by passing to the limit $\delta \to 0$ in the weak formulation (32). We observe that, using the mean value theorem and (L5),

$$\|\varphi_i^L(u^{(\delta)}) - \varphi_i^L(u)\|_{L^2(Q_T)} \leq \sup_{v\in(0,\infty)^n}|(\varphi_i^L)'(v)|\|u^{(\delta)} - u\|_{L^2(Q_T)} \leq C\|u^\delta - u\|_{L^2(Q_T)},$$

where $C > 0$ is here and in the following a constant which is independent of δ (and L). Consequently, the L^2 convergence of $(u^{(\delta)})$ shows that $\varphi_i^L(u^{(\delta)}) \to \varphi_i^L(u)$ strongly in $L^2(Q_T)$ as $\delta \to 0$, and the first integral in (32) converges:

$$\int_0^T\int_\Omega \varphi_i^L(u^{(\delta)})\partial_t\psi dxdt \to \int_0^T\int_\Omega \varphi_i^L(u)\partial_t\psi dxdt.$$

By (L3) and (L5), the sequence $(\partial_j\varphi_i^L(u^{(\delta)})A_{j\ell}(u^{(\delta)}))$ is bounded in $L^\infty(Q_T)$ with respect to δ. We conclude from the convergence (34) that $\partial_j\varphi_i^L(u^{(\delta)})A_{j\ell}(u^{(\delta)}) \to \partial_j\varphi_i^L(u)A_{j\ell}(u)$

strongly in $L^2(Q_T)$. Together with the weak convergence (35) of the gradients, we infer that

$$\int_0^T\int_\Omega \partial_j\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)}\bigg)\cdot\nabla\psi dxdt \\ \to \int_0^T\int_\Omega \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell\bigg)\cdot\nabla\psi dxdt.$$

Again using (L3), we have

$$\big\|\partial_j\varphi_i^L(u^{(\delta)})u_j^{(\delta)}\big\|_{L^\infty(Q_T)} + \bigg\|\partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})}{1+\delta|f(u^{(\delta)})|}\bigg\|_{L^\infty(Q_T)} \le C(L).$$

Consequently, by (34), $\partial_j\varphi_i^L(u^{(\delta)})u_j^{(\delta)} \to \partial_j\varphi_i^L(u)u_j$ and

$$\partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})}{1+\delta|f(u^{(\delta)})|} \to \partial_j\varphi_i^L(u)f_j(u) \quad \text{strongly in } L^2(Q_T).$$

This allows us to perform the limit $\delta\to 0$ in the drift and reaction terms:

$$\int_0^T\int_\Omega\sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})u_j^{(\delta)}b_j\cdot\nabla\psi dxdt \to \int_0^T\int_\Omega\sum_{j=1}^n \partial_j\varphi_i^L(u)u_jb_j\cdot\nabla\psi dxdt$$

$$\int_0^T\int_\Omega\sum_{j=1}^n \partial_j\varphi_i^L(u^{(\delta)})\frac{f_j(u^{(\delta)})\psi}{1+\delta|f(u^{(\delta)})|}dxdt \to \int_0^T\int_\Omega\sum_{j=1}^n \partial_j\varphi_i^L(u)f_j(u)\psi dxdt.$$

It remains to perform the limit $\delta\to 0$ in the integral involving the second derivatives of $\varphi_i^L(u^{(\delta)})$ in (32). Define the signed Radon measures

$$\mu_i^{(\delta,L)} := \sum_{j,k=1}^n \partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)} - u_j^{(\delta)}b_j\bigg)\cdot\nabla u_k^{(\delta)}dxdt.$$

It follows from (L7), the Cauchy-Schwarz inequality, and the uniform bounds in Lemma 6 that

$$|\mu_i^{(\delta,L)}|(Q_T) \le C\int_0^T\int_\Omega\sum_{k=1}^n\big(|\nabla u^{(\delta)}| + \|b\|_{L^\infty(Q_T)}\big)|\nabla u_k^{(\delta)}|dxdt \\ \le C\big(\|b\|_{L^\infty(Q_T)}^2 + \|\nabla u^{(\delta)}\|_{L^2(Q_T)}^2\big) \le C(u^0,b,T).$$

The weak-star compactness criterium for Radon measures [21, Corollary 4.34] then implies the existence of a subsequence (not relabeled) such that $\mu_i^{(\delta,L)} \rightharpoonup^* \mu_i^L$ as $\delta\to 0$ in the sense of measures, where μ_i^L is a signed Radon measure. Hence,

$$\int_0^T\int_\Omega\sum_{j,k=1}^n \partial_j\partial_k\varphi_i^L(u^{(\delta)})\bigg(\sum_{\ell=1}^n A_{j\ell}(u^{(\delta)})\nabla u_\ell^{(\delta)} - u_j^{(\delta)}b_j\bigg)\cdot\nabla u_k^{(\delta)}\psi dxdt$$

$$\to \int_0^T \int_\Omega \psi d\mu_i^L(x,t).$$

This shows (36).

Step 5. The final step is the proof of the convergence (37). For this, we write

$$\begin{aligned}|\mu_i^{(\delta,L)}|(Q_T) &\le \sum_{j,k,\ell=1}^n \int_0^T \int_\Omega |\partial_j\partial_k\varphi_i^L(u^{(\delta)})| \\ &\qquad \times \big(|A_{j\ell}(u^{(\delta)})|\,|\nabla u_\ell^{(\delta)}| + |u_j^{(\delta)}|\|b\|_{L^\infty(Q_T)}\big)|\nabla u_k^{(\delta)}|dxdt \\ &= \sum_{j,k,\ell=1}^n \sum_{K=1}^\infty \int_0^T \int_\Omega \chi_{\{K-1\le|u^{(\delta)}|<K\}}|\partial_j\partial_k\varphi_i^L(u^{(\delta)})| \\ &\qquad \times \big(|A_{j\ell}(u^{(\delta)})|\,|\nabla u_\ell^{(\delta)}| + |u_j^{(\delta)}|\|b\|_{L^\infty(Q_T)}\big)|\nabla u_k^{(\delta)}|dxdt \\ &\le C\sum_{K=1}^\infty F_K^{(\delta)}G_K^L,\end{aligned}$$

where

$$\begin{aligned}F_K^{(\delta)} &= \int_0^T \int_\Omega \chi_{\{K-1\le|u^{(\delta)}|<K\}}\big(|\nabla u^{(\delta)}| + \|b\|_{L^\infty(Q_T)}\big)|\nabla u^{(\delta)}|dxdt, \\ G_K^L &= \sum_{j,k=1}^n \sup_{K-1\le|v|<K}(1+|v|)|\partial_j\partial_k\varphi_i^L(v)|.\end{aligned}$$

For the last inequality, we have used the fact that $A_{j\ell}(u^{(\delta)})$ depends linearly on $u_i^{(\delta)}$, i.e. $A_{j\ell}(u^{(\delta)}) \le C(1+|u^{(\delta)}|)$. By (L3), for any fixed $L\in\mathbb{N}$ and sufficiently large $K\in\mathbb{N}$, it holds that $\partial_j\partial_k\varphi_i^L(v) = 0$ for all $K-1\le|v|<K$ and consequently $G_K^L = 0$. This means that for any fixed $L\in\mathbb{N}$, the sum $\sum_{K=1}^\infty F_K^{(\delta)}G_L^K$ contains only a finite number of nonvanishing terms. By the weak-star lower semicontinuity of the total variation of signed Radon measures on open sets [21, Prop. 4.29], we deduce from $\mu_i^{(\delta,L)} \rightharpoonup^* \mu_i^L$ as $\delta\to0$ that for some subsequence,

$$\begin{aligned}|\mu_i^L|(\overline{\Omega}\times[0,T)) &= |\mu_i^L|(Q_T) \le \liminf_{\delta\to0}|\mu_i^{(\delta,L)}|(Q_T) \\ &\le C\liminf_{\delta\to0}\sum_{K=1}^\infty F_K^{(\delta)}G_K^L = C\sum_{K=1}^\infty G_K^L\lim_{\delta\to0}F_K^{(\delta)}\end{aligned} \tag{45}$$

In the last equality, we have selected a common subsequence such that $(F_K^{(\delta)})$ converges. It follows from the Cauch-Schwarz inequality and the uniform bounds in Lemma 6 that

$$\begin{aligned}\sum_{K=1}^\infty F_K^{(\delta)} &= \int_0^T \int_\Omega \big(|\nabla u^{(\delta)}| + \|b\|_{L^\infty(Q_T)}\big)|\nabla u^{(\delta)}|dxdt \\ &\le C\big(\|\nabla u^{(\delta)}\|^2_{L^2(Q_T)} + \|b\|^2_{L^\infty(Q_T)}\big) \le C(u^0,b,T).\end{aligned}$$

By Fatou's lemma, we have for the same subsequence as in (45),

$$\sum_{K=1}^{\infty} \lim_{\delta\to 0} F_K^{(\delta)} \le \liminf_{\delta\to 0} \sum_{K=1}^{\infty} F_K^{(\delta)} \le C(u^0, b, T).$$

It follows from (L7) that $0 \le G_K^L \le C$ for some constant $C > 0$, and hence

$$0 \le G_K^L \lim_{\delta\to 0} F_K^{(\delta)} \le C \lim_{\delta\to 0} F_K^{(\delta)}.$$

Moreover, we infer from (L6) that

$$0 \le \lim_{L\to\infty} G_K^L \le (1+K) \sum_{j,k=1}^{n} \lim_{L\to\infty} \sup_{v\in[0,\infty)^n} |\partial_j\partial_k \varphi_i^L(v)| = 0,$$

i.e. $\lim_{L\to\infty} G_K^L = 0$. We deduce from Lebesgue's dominated convergence theorem that

$$0 \le \lim_{L\to\infty} |\mu_i^L|(\overline{\Omega}\times[0,T)) \le C \lim_{L\to\infty} \sum_{K=1}^{\infty} G_K^L \lim_{\delta\to 0} F_K^{(\delta)} = C \sum_{K=1}^{\infty} \lim_{L\to\infty} G_K^L \lim_{\delta\to 0} F_K^{(\delta)} = 0.$$

This ends the proof of (37). □

Proof of Theorem 1. We apply Lemma 8 with $v_i = \varphi_i^L(u) \in L^2(0,T;H^1(\Omega))$, $v_{0,i} = \varphi_i^L(u^0) \in L^1(\Omega)$, $w_i = \sum_{j=1}^n \partial_j\varphi_i^L(u) f_j(u) \in L^1(Q_T)$, $\mu_i = -\mu_i^L \in \mathcal{M}(\overline{\Omega}\times[0,T))$, $q_i = 0$, and

$$z_i = -\sum_{j=1}^{n} \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^{n} A_{j\ell}(u)\nabla u_\ell - u_j b_j\bigg) \in L^2(Q_T;\mathbb{R}^n).$$

Then we obtain from (36) that for all $\xi \in C^\infty([0,\infty)^n)$ with $\xi' \in C_0^\infty([0,\infty);\mathbb{R}^n)$ and for all $\phi \in C_0^\infty(\overline{\Omega}\times[0,T))$,

$$\begin{aligned}
\bigg| &-\int_0^T\int_\Omega \xi(\varphi^L(u))\partial_t\phi dxdt - \int_\Omega \xi(\varphi^L(u^0))\phi(\cdot,0)dx \\
&+ \int_0^T\int_\Omega \sum_{i,k=1}^{n} \partial_i\partial_k\xi(\varphi^L(u)) \sum_{j=1}^{n} \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^{n} A_{j\ell}(u)\nabla u_\ell - u_j b_j\bigg) \\
&\quad \times \sum_{m=1}^{n} \partial_m\varphi_k^L(u)\nabla u_m\phi dxdt \qquad (46)\\
&+ \int_0^T\int_\Omega \sum_{i=1}^{n} \partial_i\xi(\varphi^L(u)) \sum_{j=1}^{n} \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^{n} A_{j\ell}(u)\nabla u_\ell - u_j b_j\bigg)\cdot\nabla\phi dxdt \\
&- \int_0^T\int_\Omega \sum_{i=1}^{n} \partial_i\xi(\varphi^L(u)) \sum_{j=1}^{n} \partial_j\varphi_i^L(u) f_j(u)\phi dxdt\bigg| \\
&\le C(\Omega)\|\phi\|_{L^\infty(Q_T)} \sup_{v\in[0,\infty)^n} |\xi'(v)| \sum_{i=1}^{n} |\mu_i^L|(\overline{\Omega}\times[0,T)).
\end{aligned}$$

We wish to perform the limit $L \to \infty$ in (46). We deduce from the mean value theorem and (L3) that

$$\|\xi(\varphi^L(u)) - \xi(u)\|_{L^1(Q_T)} \le \sup_{v\in[0,\infty)^n} |\xi'(v)|\, \|\varphi^L(u) - u\|_{L^1(Q_T)} \le C\|\varphi^L(u) - u\|_{L^1(Q_T)}.$$

By definition of φ_i^L, $\varphi_i^L(u)$ converges pointwise to u_i as $L \to \infty$. Together with the linear bound for φ_i^L from (L1), $0 \le \varphi_i^L(u) \le u_i + 2\sum_{k=1}^n u_k \in L^1(Q_T)$, which gives a uniform bound, we can apply Lebesgues dominated convergence theorem to conclude that $\varphi_i^L(u) \to u_i$ strongly in $L^1(Q_T)$ and consequently, $\xi(\varphi^L(u)) \to \xi(u)$ strongly in $L^1(Q_T)$. Similarly, $\xi(\varphi^L(u^0)) \to \xi(u^0)$ strongly in $L^1(Q_T)$. Therefore, the first two integrals in (46) converge:

$$\begin{aligned} -\int_0^T\int_\Omega \xi(\varphi^L(u))\partial_t\phi dxdt &\to -\int_0^T\int_\Omega \xi(u)\partial_t\phi dxdt, \\ -\int_\Omega \xi(\varphi^L(u^0))\phi(\cdot,0)dx &\to -\int_\Omega \xi(u^0)\phi(\cdot,0)dx. \end{aligned}$$

Next, consider the last integral on the left-hand side of (46). Let $L_0 > 0$ be such that $\operatorname{supp}\xi' \subset [0, L_0/n)^n$ and let $L > L_0$. We distinguish the cases $\sum_{i=1}^n u_i \ge L_0$ and $\sum_{i=1}^n u_i < L_0$. In the former case, it follows from (L8) that $\sum_{i=1}^n \varphi_i^L(u) \ge L_0$ or $\varphi^L(u) \notin [0, L_0/n)^n$ and, in particular, $\varphi^L(u) \notin \operatorname{supp}\xi'$. Hence, $\partial_i\xi(\varphi^L(u)) = 0$ and $\partial_i\partial_k\xi(\varphi^L(u)) = 0$. In the latter case, we deduce from (L2) that $\varphi^L(u) = u$ and consequently $\partial_j\varphi_i^L(u) = \delta_{ij}$. Furthermore, we have

$$\{u \in \operatorname{supp}\xi'\} \subset \left\{u \in \left[0, \frac{L_0}{n}\right)^n\right\} \subset \left\{\sum_{i=1}^n u_i < L_0\right\}.$$

This allows us to reformulate the last term on the left-hand side of (46):

$$\begin{aligned} \int_0^T\int_\Omega &\sum_{i=1}^n \partial_i\xi(\varphi^L(u)) \sum_{j=1}^n \partial_j\varphi_i^L(u) f_j(u)\phi dxdt \\ &= \int_0^T\int_\Omega \sum_{i=1}^n \chi_{\{\sum_{i=1}^n u_i \ge L_0\}} \partial_i\xi(\varphi^L(u)) \sum_{j=1}^n \partial_j\varphi_i^L(u) f_j(u)\phi dxdt \\ &\quad + \int_0^T\int_\Omega \sum_{i=1}^n \chi_{\{\sum_{i=1}^n u_i < L_0\}} \partial_i\xi(\varphi^L(u)) \sum_{j=1}^n \partial_j\varphi_i^L(u) f_j(u)\phi dxdt \\ &= \int_0^T\int_\Omega \sum_{i=1}^n \chi_{\{\sum_{i=1}^n u_i < L_0\}} \partial_i\xi(u) f_i(u)\phi dxdt \\ &= \int_0^T\int_\Omega \sum_{i=1}^n \partial_i\xi(u) f_i(u)\phi dxdt, \end{aligned}$$

and this expression does not depend on L. In a similar way, we compute the third and fourth integrals on the left-hand side of (46):

$$\begin{aligned}\int_0^T \int_\Omega \sum_{i,k=1}^n \partial_i\partial_k\xi(\varphi^L(u)) \sum_{j=1}^n \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg) \\ \times \sum_{m=1}^n \partial_m\varphi_k^L(u)\nabla u_m\phi dxdt \\ = \int_0^T \int_\Omega \sum_{i,k=1}^n \partial_i\partial_k\xi(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla u_k\phi dxdt,\end{aligned}$$

$$\begin{aligned}\int_0^T \int_\Omega \sum_{i=1}^n \partial_i\xi(\varphi^L(u)) \sum_{j=1}^n \partial_j\varphi_i^L(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla\phi dxdt \\ = \int_0^T \int_\Omega \sum_{i=1}^n \partial_i\xi(u)\bigg(\sum_{\ell=1}^n A_{j\ell}(u)\nabla u_\ell - u_jb_j\bigg)\cdot\nabla\phi dxdt.\end{aligned}$$

Finally, because of (37), the right-hand side of (46) vanishes in the limit $L \to \infty$. Therefore, passing to the limit $L \to \infty$ in (46), we see that (12) holds. This concludes the proof. $\square$

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