A finite-volume scheme for a cross-diffusion model arising from interacting many-particle population systems

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Abstract A finite-volume scheme for a cross-diffusion model arising from the mean-field limit of an interacting particle system for multiple population species is studied. The existence of discrete solutions and a discrete entropy production inequality is proved. The proof is based on a weighted quadratic entropy that is not the sum of the entropies of the population species.

Key words: Finite volume scheme, cross-diffusion system, entropy method.

MSC (2010)**:** 35K51, 35K55, 35Q92, 65M08

1 Introduction

1.1 Presentation of the model

We consider the following cross-diffusion system:

$$\partial_t u_i + \operatorname{div}\big(-\delta\nabla u_i - u_i\nabla p_i(u)\big) = 0, \quad p_i(u) = \sum_{j=1}^n a_{ij}u_j \quad \text{in } \Omega,\ t>0, \tag{1}$$

where $i=1,\ldots,n$ with $n\geq 2$, $\Omega\subset\mathbb{R}^2$ is an open bounded polygonal domain, and $\delta>0$, $a_{ij}>0$. We impose the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \geq 0 \quad \text{in } \Omega, \quad \nabla u_i\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0,\ i=1,\ldots,n, \tag{2}$$

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where ν is the exterior unit normal vector on $\partial\Omega$. We write $u := (u_1,\ldots,u_n)$ and $u^0 := (u_1^0,\ldots,u_n^0)$. Equations (1) are derived from a weakly interacting stochastic many-particle system in the mean-field limit [7]. It can be seen as a simplification of the Shigesada-Kawasaki-Teramoto (SKT) population model [12], where the diffusion is reduced to $\delta\nabla u_i$. The two-species system was analyzed first in [3], but up to now, no analytical or numerical results are available for the n-species system. The diffusion matrix associated to (1) is neither symmetric nor positive definite but we show below that system (1) possesses an entropy structure [10] yielding gradient estimates that are the basis for the numerical analysis.

We assume that $(a_{ij}) \in \mathbb{R}^{n\times n}$ is positively stable (i.e., all eigenvalues of $A = (a_{ij})$ have positive real parts) and that the detailed-balance condition holds, i.e., there exist numbers $\pi_1,\ldots,\pi_n > 0$ such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for all } i,j = 1,\ldots,n. \tag{3}$$

Note that for the two-species model this condition is always satisfied, just set $\pi_1 = a_{21}$ and $\pi_2 = a_{12}$. Since $A_1 = \operatorname{diag}(\pi_i^{-1})$ is symmetric, positive definite and $A_2 = (\pi_i a_{ij})$ is symmetric, by [11, Prop. 6.1], the number of positive eigenvalues of $A = A_1A_2$ equals that for A_2. Thus, A_2 has only positive eigenvalues, which together with the symmetry means that A_2 is symmetric, positive definite.

Our (numerical) analysis is based on the observation that system (1) possesses an entropy structure with a weighted quadratic entropy that has not been observed before in cross-diffusion systems:

$$H[u] = \int_\Omega h(u)dx, \quad \text{where } h(u) := \frac{1}{2\delta}\sum_{i,j=1}^n \pi_i a_{ij} u_i u_j.$$

Interestingly, this entropy is not of the form $\sum_{i=1}^n h_i(u_i)$, but it mixes the species. A formal computation shows that

$$\frac{dH}{dt} + \sum_{i,j=1}^n \pi_i a_{ij}\int_\Omega \nabla u_i\cdot\nabla u_j dx + \frac{1}{\delta}\sum_{i=1}^n \pi_i \int_\Omega u_i|\nabla p_i(u)|^2 dx = 0.$$

With $\lambda > 0$ being the smallest eigenvalue of $(\pi_i a_{ij})$, we conclude the following entropy production inequality:

$$\frac{dH}{dt} + \lambda\sum_{i=1}^n \int_\Omega |\nabla u_i|^2 dx + \frac{1}{\delta}\sum_{i=1}^n \pi_i\int_\Omega u_i|\nabla p_i(u)|^2 dx \le 0.$$

Our aim is to prove this inequality for the finite-volume solutions.

1.2 The numerical scheme

A mesh of Ω is given by a set $\mathscr{T}$ of open polygonal control volumes, a set $\mathscr{E}$ of edges, and a set $\mathscr{P}$ of points $(x_K)_{K\in\mathscr{T}}$. We assume that the mesh is admissible in the sense of Definition 9.1 in [9]. We distinguish in $\mathscr{E}$ the interior edges $\sigma = K|L$ and the exterior edges such that $\mathscr{E} = \mathscr{E}_{\rm int}\cup\mathscr{E}_{\rm ext}$. For a given control volume $K\in\mathscr{T}$, we denote by $\mathscr{E}_K$ the set of its edges. This set splits into $\mathscr{E}_K = \mathscr{E}_{{\rm int},K}\cup\mathscr{E}_{{\rm ext},K}$. For any $\sigma\in\mathscr{E}$, there exists at least one cell $K\in\mathscr{T}$ such that $\sigma\in\mathscr{E}_K$ and we denote this cell by K_σ. When σ is an interior edge, $\sigma = K|L$, K_σ can be either K or L. For all $\sigma\in\mathscr{E}$, we define $\mathrm{d}_\sigma = \mathrm{d}(x_K,x_L)$ if $\sigma = K|L\in\mathscr{E}_{\rm int}$ and $\mathrm{d}_\sigma = \mathrm{d}(x_K,\sigma)$ if $\sigma\in\mathscr{E}_{{\rm ext},K}$. Then the transmissibility coefficient is defined by $\tau_\sigma = \mathrm{m}(\sigma)/\mathrm{d}_\sigma$ for all $\sigma\in\mathscr{E}$. We assume that the mesh satisfies the following regularity constraint:

$$\exists\xi>0,\,\forall K\in\mathscr{T},\,\forall\sigma\in\mathscr{E}_K:\ \mathrm{d}(x_K,\sigma)\ge\xi\,\mathrm{d}_\sigma. \tag{4}$$

The size of the mesh is denoted by $\Delta x = \max_{K\in\mathscr{T}}\mathrm{diam}(K)$. Let $N_T\in\mathbb{N}$ be the number of time steps, $\Delta t = T/N_T$ be the time step size, and $t_k = k\Delta t$ for $k = 0,\ldots,N_T$.

Let $\mathscr{H}_{\mathscr{T}}$ be the linear space of functions $\Omega\to\mathbb{R}$ which are constant on each $K\in\mathscr{T}$. For $v\in\mathscr{H}_{\mathscr{T}}$, we introduce

$$D_{K,\sigma}v = v_{K,\sigma}-v_K,\quad D_\sigma v = |D_{K,\sigma}v|\quad\text{for all } K\in\mathscr{T},\,\sigma\in\mathscr{E}_K,$$

where $v_{K,\sigma}$ is either v_L ($\sigma = K|L$) or v_K ($\sigma\in\mathscr{E}_{{\rm ext},K}$). Finally, we define the (squared) discrete H^1 norm

$$\|v\|_{1,2,\mathscr{T}}^2 = \sum_{\sigma\in\mathscr{E}}\tau_\sigma(D_\sigma v)^2+\sum_{K\in\mathscr{T}}\mathrm{m}(K)v_K^2.$$

For all $K\in\mathscr{T}$ and $i = 1,\ldots,n$, $u_{i,K}^0$ denotes the mean value of u_i^0 over K. The finite-volume scheme for (1) reads as

$$\frac{\mathrm{m}(K)}{\Delta t}(u_{i,K}^k-u_{i,K}^{k-1})+\sum_{\sigma\in\mathscr{E}_K}\mathscr{F}_{i,K,\sigma}^k = 0,\quad i = 1,\ldots,n, \tag{5}$$

$$\mathscr{F}_{i,K,\sigma}^k = -\tau_\sigma\big(\delta D_{K,\sigma}u_i^k+u_{i,\sigma}^kD_{K,\sigma}p_i(u^k)\big)\quad\text{for all } K\in\mathscr{T},\,\sigma\in\mathscr{E}_K, \tag{6}$$

with $u^k = (u_1^k,\ldots,u_n^k)$ and $u_{i,\sigma}^k := \min\{u_{i,K}^k,u_{i,K,\sigma}^k\}$. As in [1], this definition of $u_{i,\sigma}^k$ allows us to prove the nonnegativity of $u_{i,K}^k$.

1.3 Main result

The main result of this work is the existence of nonnegative solutions to scheme (5)-(6), which preserve the entropy production inequality.

Theorem 1 (Existence of discrete solutions). *Assume that* $u^0 \in L^2(\Omega)^n$ *with* $u_i^0 \geq 0$, $\delta > 0$, $a_{ij} > 0$, (a_{ij}) *is positively stable, and* (3) *holds. Then there exists a solution* $(u_K^k)_{K\in\mathscr{T},k=0,\ldots,N_T}$ *with* $u_K^k = (u_{1,K}^k,\ldots,u_{n,K}^k)$ *to scheme* (5)-(6) *satisfying* $u_{i,K}^k \geq 0$ *for all* $K \in \mathscr{T}$, $i = 1,\ldots,n$, *and* $k = 0,\ldots,N_T$. *Moreover, the following discrete entropy production inequality holds:*

$$\sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^k) + \Delta t\lambda \sum_{i=1}^n \sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma u_i^k)^2 + \frac{\Delta t}{\delta}\sum_{i=1}^n \sum_{\sigma\in\mathscr{E}} \tau_\sigma \pi_i u_{i,\sigma}^k (D_\sigma p_i(u^k))^2 \leq \sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^{k-1}), \quad (7)$$

where λ *denotes the smallest eigenvalue of* $(\pi_i a_{ij})$.

We expect that the detailed-balance condition (3) can be replaced by a weak cross-diffusion condition as in [6]. The positive stability of (a_{ij}) implies the parabolicity of (1) in the sense of Petrovskii. Indeed, $(\pi_i a_{ij})$ and $\mathrm{diag}(u_i/\pi_i)$ are symmetric, positive definite matrices for $u \in (0,\infty)^n$. Thus, its product $(u_i a_{ij})$ has only positive eigenvalues [4, Theorem 7] which proves the claim. The assumption that the diffusion coefficient δ is the same for all species is a simplification needed to conclude that $h(u)$ is coercive, $h(u) \geq (\lambda/2)|u|^2$ for $u \in \mathbb{R}^n$. It can be removed by exploiting the Shannon entropy to show first that u_i is nonnegative, but this requires more technical effort which will be detailed in a future work.

2 Proof of Theorem 1

We proceed by induction. For $k = 0$, we have $u_i^0 \geq 0$ by assumption. Assume that there exists a solution u^{k-1} for some $k \in \{2,\ldots,N_T\}$ such that $u_i^{k-1} \geq 0$ in Ω, $i = 1,\ldots,n$. The construction of a solution u^k is split in several steps.

Step 1: Definition of a linearized problem. Let $R > 0$, we set

$$Z_R := \big\{ w = (w_1,\ldots,w_n) \in (\mathscr{H}_\mathscr{T})^n \;:\; \|w_i\|_{1,2,\mathscr{T}} < R \quad \text{for } i = 1,\ldots,n \big\},$$

and let $\varepsilon > 0$ be given. We define the mapping $F_\varepsilon : Z_R \to \mathbb{R}^{\theta n}$ by $F_\varepsilon(w) = w^\varepsilon$, with $\theta = \#\mathscr{T}$, where $w^\varepsilon = (w_1^\varepsilon,\ldots,w_n^\varepsilon)$ is the solution to the linear problem

$$\varepsilon \left(\sum_{\sigma\in\mathscr{E}_K} \tau_\sigma D_{K,\sigma}(w_i^\varepsilon) + \mathrm{m}(K) w_{i,K}^\varepsilon \right) = -\left(\frac{\mathrm{m}(K)}{\Delta t}(u_{i,K} - u_{i,K}^{k-1}) + \sum_{\sigma\in\mathscr{E}_K} \mathscr{F}_{i,K,\sigma}^+ \right), \quad (8)$$

for $K \in \mathscr{T}$, $i = 1,\ldots,n$, and $\mathscr{F}_{i,K,\sigma}^+$ is defined in (6) with $u_{i,\sigma}$ replaced by $\overline{u}_{i,\sigma} = \min\{u_{i,K}^+, u_{i,K,\sigma}^+\}$, where $z^+ = \max\{0,z\}$. Here, $u_{i,K}$ is a function of $w_{1,K},\ldots,w_{n,K}$, defined by the entropy variables

$$w_{i,K} = \frac{\pi_i}{\delta} p_i(u_K) = \sum_{j=1}^{n} \frac{\pi_i a_{ij}}{\delta} u_j \quad \text{for all } K \in \mathscr{T},\ i = 1,\ldots,n. \tag{9}$$

This is a linear system with the invertible coefficient matrix $(\pi_i a_{ij}/\delta)$, and so, the function $u_K = u(w_K)$ is well-defined. The existence of a unique solution w_i^ε to the linear scheme (8)-(9) is now a consequence of [9, Lemma 3.2].

Step 2: Continuity of F_ε. We fix $i \in \{1,\ldots,n\}$. Multiplying (8) by $w_{i,K}^\varepsilon$ and summing over $K \in \mathscr{T}$, we obtain, after discrete integration by parts,

$$\varepsilon \|w_i^\varepsilon\|_{1,2,\mathscr{T}}^2 = -\sum_{K\in\mathscr{T}} \frac{\mathrm{m}(K)}{\Delta t}(u_{i,K} - u_{i,k}^{k-1}) w_{i,K}^\varepsilon + \sum_{\substack{\sigma\in\mathscr{E}_{\mathrm{int}}\\ \sigma=K|L}} \mathscr{F}_{i,K,\sigma}^{+} D_{K,\sigma} w_i^\varepsilon =: J_1 + J_2.$$

By the Cauchy-Schwarz inequality and the definition of $\mathscr{F}_{i,K,\sigma}^{+}$, we find that

$$|J_1| \le \frac{1}{\Delta t}\left(\sum_{K\in\mathscr{T}} \mathrm{m}(K)(u_{i,K} - u_{i,K}^{k-1})^2\right)^{1/2} \left(\sum_{K\in\mathscr{T}} \mathrm{m}(K)(w_{i,K}^\varepsilon)^2\right)^{1/2}$$
$$|J_2| \le \left(\sum_{\sigma\in\mathscr{E}} \tau_\sigma \big(\delta D_\sigma u_i + \bar{u}_{i,\sigma} D_\sigma p_i(u)\big)^2\right)^{1/2} \left(\sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma w_i^\varepsilon)^2\right)^{1/2}.$$

Hence, since u_i is a linear combination of $(w_1,\ldots,w_n) \in Z_R$, there exists a constant $C(R) > 0$ which is independent of w^ε such that $|J_1| + |J_2| \le C(R)\|w_i^\varepsilon\|_{1,2,\mathscr{T}}$. Inserting these estimations, it follows that $\varepsilon\|w_i^\varepsilon\|_{1,2,\mathscr{T}} \le C(R)$.

We turn to the proof of the continuity of F_ε. Let $(w^m)_{m\in\mathbb{N}} \subset Z_R$ be such that $w^m \to w$ as $m \to \infty$. The previous estimate shows that $w^{\varepsilon,m} := F_\varepsilon(w^m)$ is bounded uniformly in $m \in \mathbb{N}$. Thus, there exists a subsequence of $(w^{\varepsilon,m})$, which is not relabeled, such that $w^{\varepsilon,m} \to w^\varepsilon$ as $m \to \infty$. Passing to the limit $m \to \infty$ in scheme (8)-(9) and taking into account the continuity of the nonlinear functions, we see that w_i^ε is a solution to (8)-(9) for $i = 1,\ldots,n$ and $w^\varepsilon = F_\varepsilon(w)$. Because of the uniqueness of the limit function, the whole sequence converges, which proves the continuity.

Step 3: Existence of a fixed point. We claim that the map F_ε admits a fixed point. We use a topological degree argument [8], i.e., we prove that $\deg(I - F_\varepsilon, Z_R, 0) = 1$, where deg is the Brouwer topological degree. Since deg is invariant by homotopy, it is sufficient to prove that any solution $(w^\varepsilon,\rho) \in \overline{Z}_R \times [0,1]$ to the fixed-point equation $w^\varepsilon = \rho F_\varepsilon(w^\varepsilon)$ satisfies $(w^\varepsilon,\rho) \notin \partial Z_R \times [0,1]$ for sufficiently large values of $R > 0$. Let (w^ε,ρ) be a fixed point and $\rho \ne 0$, the case $\rho = 0$ being clear. Then w_i^ε/ρ solves

$$\varepsilon\left(\sum_{\sigma\in\mathscr{E}_K} \tau_\sigma D_{K,\sigma}(w_i^\varepsilon) + \mathrm{m}(K) w_{i,K}^\varepsilon\right) = -\rho\left(\frac{\mathrm{m}(K)}{\Delta t}(u_{i,K}^\varepsilon - u_{i,K}^{k-1}) + \sum_{\sigma\in\mathscr{E}_K} \mathscr{F}_{i,K,\sigma}^{+,\varepsilon}\right), \tag{10}$$

for all $K \in \mathscr{T}$, $i = 1,\ldots,n$, and $\mathscr{F}_{i,K,\sigma}^{+,\varepsilon}$ is defined as in (6) with u replaced by u^ε. The following discrete entropy production inequality is the key argument.

Lemma 1 (Discrete entropy production inequality). *Let the assumptions of Theorem 1 hold. Then, for any* $\rho \in (0,1]$ *and* $\varepsilon > 0$,

$$\rho \sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^\varepsilon) + \varepsilon\Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathscr{T}}^2 + \rho\Delta t\lambda \sum_{i=1}^n \sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma u_i^\varepsilon)^2 + \rho\frac{\Delta t}{\delta}\sum_{i=1}^n\sum_{\sigma\in\mathscr{E}} \tau_\sigma \pi_i \overline{u}_{i,\sigma}^\varepsilon (D_\sigma p_i(u^\varepsilon))^2 \le \rho \sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^{k-1}), \qquad (11)$$

with $\lambda > 0$ *being the smallest eigenvalue of* $(\pi_i a_{ij})$ *and obvious notations for* $\overline{u}_{i,\sigma}^\varepsilon$.

Proof. We multiply (10) by $\Delta t w_{i,K}^\varepsilon$ and sum over i and $K \in \mathscr{T}$. This gives, after discrete integration by parts, $\varepsilon\Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathscr{T}}^2 + J_3 + J_4 + J_5 = 0$, where

$$J_3 = \rho \sum_{i=1}^n \sum_{K\in\mathscr{T}} \mathrm{m}(K)(u_{i,K}^\varepsilon - u_{i,K}^{k-1})w_{i,K}^\varepsilon,$$

$$J_4 = -\rho\Delta t \sum_{i=1}^n \sum_{\substack{\sigma\in\mathscr{E}_{\rm int}\\ \sigma=K|L}} \tau_\sigma \delta D_{K,\sigma} u_i^\varepsilon w_{i,K}^\varepsilon,$$

$$J_5 = \rho\Delta t \sum_{i=1}^n \sum_{\substack{\sigma\in\mathscr{E}_{\rm int}\\ \sigma=K|L}} \tau_\sigma \overline{u}_{i,\sigma}^\varepsilon D_{K,\sigma} p_i(u^\varepsilon) D_{K,\sigma} w_{i,K}^\varepsilon.$$

For J_3, we use the convexity of h for its estimation; for J_4, we take into account the symmetry of τ_σ with respect to $\sigma = K|L$, definition (9) of w_i^ε and the positive definiteness of $(\pi_i a_{ij})$; and for J_5, we employ definition (9) of w_i^ε:

$$J_3 \ge \rho \sum_{K\in\mathscr{T}} \mathrm{m}(K)\big(h(u_K^\varepsilon) - h(u_K^{k-1})\big),$$

$$J_4 = \rho\Delta t \sum_{i,j=1}^n \sum_{\substack{\sigma\in\mathscr{E}_{\rm int}\\ \sigma=K|L}} \tau_\sigma \pi_i a_{ij} D_{K,\sigma} u_i^\varepsilon D_{K,\sigma} u_j^\varepsilon \ge \rho\Delta t\lambda \sum_{i=1}^n \sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma u_i^\varepsilon)^2,$$

$$J_5 = \rho\frac{\Delta t}{\delta}\sum_{i=1}^n \sum_{\sigma\in\mathscr{E}} \tau_\sigma \pi_i \overline{u}_{i,\sigma}^\varepsilon (D_\sigma p_i(u^\varepsilon))^2.$$

Putting all the estimations together completes the proof. □

We proceed with the topological degree argument. Lemma 1 implies that

$$\varepsilon\Delta t \sum_{i=1}^n \|w_i^\varepsilon\|_{1,2,\mathscr{T}}^2 \le \rho \sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^{k-1}) \le \sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^{k-1}).$$

Then, if we define $R := (\varepsilon\Delta t)^{-1/2}(\sum_{K\in\mathscr{T}} \mathrm{m}(K)h(u_K^{k-1}))^{1/2} + 1$, we conclude that $w^\varepsilon \notin \partial Z_R$ and $\deg(I - F_\varepsilon, Z_R, 0) = 1$. Thus, F_ε admits a fixed point

Step 4: Limit $\varepsilon \to 0$. Recall that $h(u_K) \ge (\lambda/2)|u_K|^2$ (note that $u_{i,K} \in \mathbb{R}$ at this point). Thus, by Lemma 1, there exists a constant $C > 0$ depending only on the mesh but not on ε such that for all $K \in \mathscr{T}$ and $i = 1, \dots, n$,

$$|u_{i,K}^{\varepsilon}| \le C(\lambda) \left(\sum_{K \in \mathscr{T}} \mathrm{m}(K) h(u_K^{k-1}) \right)^{1/2}.$$

Thus, up to a subsequence, for $i = 1, \dots, n$ and for all $K \in \mathscr{T}$, we infer the existence of $u_{i,K} \in \mathbb{R}$ such that $u_{i,K}^{\varepsilon} \to u_{i,K}$ as $\varepsilon \to 0$. We deduce from (11) that there exists a subsequence (not relabeled) such that $\varepsilon w_{i,K}^{\varepsilon} \to 0$ for any $K \in \mathscr{T}$ and $i = 1, \dots, n$. Hence, the limit $\varepsilon \to 0$ in (10) yields the existence of a solution to (8) with $\varepsilon = 0$.

Let $i \in \{1, \dots, n\}$ and $K \in \mathscr{T}$ such that $u_{i,K} = \min_{L \in \mathscr{T}} u_{i,L}$. We multiply (8) with $\varepsilon = 0$ by $\Delta t u_{i,K}^{-}$ with $z^{-} = \min\{0, z\}$ and use the induction hypothesis:

$$\begin{aligned} \mathrm{m}(K)(u_{i,K}^{-})^2 - \Delta t \sum_{\sigma \in \mathscr{E}_K} \tau_\sigma (\delta + a_{ii} \overline{u}_{i,\sigma}) D_{K,\sigma}(u_i) u_{i,K}^{-} & \\ - \Delta t \sum_{j \ne i} \sum_{\sigma \in \mathscr{E}_K} \tau_\sigma a_{ij} \overline{u}_{i,\sigma} D_{K,\sigma}(u_j) u_{i,K}^{-} &= 0. \end{aligned}$$

The second term is nonpositive since $\overline{u}_{i,\sigma} \ge 0$ and $D_{K,\sigma}(u_i) \ge 0$, by the choice of K. The last term vanishes since $\overline{u}_{i,\sigma} u_{i,K}^{-} = u_{i,K}^{+} u_{i,K}^{-} = 0$, by the definition of $\overline{u}_{i,\sigma}$. This shows that $u_{i,L} \ge u_{i,K} \ge 0$ for all $L \in \mathscr{T}$ and $i = 1, \dots, n$. Passing to the limit $\varepsilon \to 0$ in (11) yields inequality (7), which completes the proof of Theorem 1.

3 Convergence analysis and perspectives

In this section, we sketch the proof of the convergence of the scheme and possible extensions of the method presented in this paper.

- Let us give the main features of the proof of convergence. First, thanks to the a priori estimates given by (7) and assumption (4), we prove the existence of a constant $C > 0$ independent of Δx and Δt such that for all $i = 1, \dots, n$ and $\phi \in C_0^{\infty}(\Omega \times (0,T))$,

$$\sum_{k=1}^{N_T} \sum_{K \in \mathscr{T}} \mathrm{m}(K)(u_{i,K}^{k} - u_{i,K}^{k-1}) \phi(x_K, t_k) \le C \|\nabla \phi\|_{L^{\infty}(\Omega \times (0,T))}. \tag{12}$$

  Next, we consider a sequence of admissible meshes $(\mathscr{T}_\eta, \Delta t_\eta)_{\eta > 0}$ of $\Omega \times (0,T)$, indexed by the size $\eta = \{\Delta x, \Delta t\}$, satisfying (4) uniformly in η. For any $\eta > 0$, we denote by $u_\eta = (u_{1,\eta}, \dots, u_{n,\eta})$ the piecewise constant (in time and space) finite-volume solution constructed in Theorem 1. We deduce, thanks to [2, Theorem 3.9] and (12), that there exist nonnegative functions $u_1, \dots, u_n$ such that, up to a subsequence,

$$u_{i,\eta} \to u_i \quad \text{a.e. in } \Omega \times (0,T) \text{ as } \eta \to 0, \quad i = 1, \dots, n.$$

Moreover, we conclude from (7) that $u_{i,\eta} \in L^\infty(0,T;L^2(\Omega)) \subset L^2(\Omega\times(0,T))$ uniformly in η for $i=1,\ldots,n$. Hence, $(u_{i,\eta})$ is equi-integrable in $L^2(\Omega\times(0,T))$. Thus, applying the Vitali convergence theorem, we deduce that, up to a subsequence, $u_{i,\eta}\to u_i$ strongly in $L^2(\Omega\times(0,T))$ as $\eta\to 0$, $i=1,\ldots,n$. The discrete entropy production inequality yields a uniform bound of the discrete gradient ∇^η of $u_{i,\eta}$ in $L^2(\Omega\times(0,T))$; see [5] for a definition of ∇^η. It follows from [5, Lemma 4.4] that, up to a subsequence,

$$\nabla^\eta u_{i,\eta} \rightharpoonup \nabla u_i \quad \text{weakly in } L^2(\Omega\times(0,T)) \text{ as } \eta\to 0,\ i=1,\ldots,n.$$

Finally, following the method developed in [5], we prove that the limit function $u=(u_1,\ldots,u_n)$ is a weak solution to (1)-(2).

- We already mentioned that system (1) can be interpreted as a simplification of the SKT model. In a future work, we will analyze a structure-preserving finite-volume approximation of the full SKT model. Such a discretization was analyzed in [1], but only for positive definite diffusion matrices associated to (1). We will extend the analysis of [1] without this assumption.

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