GLOBAL WEAK SOLUTIONS TO COMPRESSIBLE NAVIER-STOKES EQUATIONS FOR QUANTUM FLUIDS*

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Abstract. The global-in-time existence of weak solutions to the barotropic compressible quantum Navier-Stokes equations in a three-dimensional torus for large data is proved. The model consists of the mass conservation equation and a momentum balance equation, including a nonlinear third-order differential operator, with the quantum Bohm potential, and a density-dependent viscosity. The system has been derived by Brull and Méhats [10] from a Wigner equation using a moment method and a Chapman-Enskog expansion around the quantum equilibrium. The main idea of the existence analysis is to reformulate the quantum Navier-Stokes equations by means of a so-called effective velocity involving a density gradient, leading to a viscous quantum Euler system. The advantage of the new formulation is that there exists a new energy estimate which implies bounds on the second derivative of the particle density. The global existence of weak solutions to the viscous quantum Euler model is shown by using the Faedo-Galerkin method and weak compactness techniques. As a consequence, we deduce the existence of solutions to the quantum Navier-Stokes system if the viscosity constant is smaller than the scaled Planck constant.

Key words. Compressible Navier-Stokes equations, quantum Bohm potential, density-dependent viscosity, global existence of solutions, viscous quantum hydrodynamic equations, third-order derivative, energy estimates.

AMS subject classifications. 35G25, 35Q30, 35Q40, 35B40, 76Y05, 82D37.

1. Introduction. Quantum fluid models are used to describe, for instance, superfluids [40], quantum semiconductors [18], weakly interacting Bose gases [22], and quantum trajectories of Bohmian mechanics [46]. A hydrodynamic form of the single-state Schrödinger equation has been already found by Madelung [42]. Later, so-called quantum hydrodynamic equations have been derived by Ferry and Zhou [18] from the Bloch equation for the density matrix and by Gardner [20] from the Wigner equation by a moment method. More recently, dissipative quantum fluid models have been proposed. For instance, the moment method applied to the Wigner-Fokker-Planck equation leads to viscous quantum Euler models [21], and a Chapman-Enskog expansion in the Wigner equation leads under certain assumptions to quantum Navier-Stokes equations [10]. In this paper, we will reveal a connection between these two models by introducing an effective velocity variable, first used in capillary Korteweg-type models [5], and we will prove the global existence of weak solutions to the multidimensional initial-value problems for any finite-energy initial data.

In the following, we describe the two dissipative quantum systems studied in this paper. The barotropic quantum Navier-Stokes equations for the particle density n and the particle velocity u read as

$$n_t + \operatorname{div}(nu) = 0, \quad x \in \mathbb{T}^d,\ t > 0, \tag{1.1}$$

$$(nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) - 2\varepsilon^2 n \nabla \left(\frac{\Delta\sqrt{n}}{\sqrt{n}} \right) - nf = 2\nu \operatorname{div}(nD(u)), \tag{1.2}$$

$$n(\cdot, 0) = n_0, \quad (nu)(\cdot, 0) = n_0 u_0 \quad \text{in } \mathbb{T}^d, \tag{1.3}$$

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where $u \otimes u$ is the matrix with components $u_i u_j$, $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ is the symmetric part of the velocity gradient, and $\mathbb{T}^d$ is the d-dimensional torus ($d \le 3$). The function $p(n) = n^\gamma$ with $\gamma \ge 1$ is the pressure and f describes external forces coming, for instance, from an electric field. The physical parameters are the (scaled) Planck constant $\varepsilon > 0$ and the viscosity constant $\nu > 0$. The expression $\Delta\sqrt{n}/\sqrt{n}$ can be interpreted as a quantum potential, the so-called Bohm potential.

A nonlocal quantum Navier-Stokes system with the pressure $p(n) = n$ has been derived by Brull and Méhats by a Chapman-Enskog expansion around the quantum equilibrium of the solution to the Wigner-BGK (Bhatnagar-Gross-Krook) equation [10]. The above local system with $p(n) = n$ is obtained for nearly irrotational fluids in the $O(\varepsilon^4)$ expansion (see [31]).

The existence of global-in-time classical solutions to the one-dimensional equations (1.1)–(1.3) with a strictly positive particle density (if n_0 is strictly positive) has been shown in [31]. Up to our knowledge, there are no existence results for the multidimensional situation. In this paper, we give such a result.

In the treatment of (1.1)–(1.3), we need to overcome several mathematical difficulties. The first problem lies in the strongly nonlinear third-order differential operator and the dispersive structure of the momentum equation. In particular, as the maximum principle is not applicable, it is not clear how to obtain the positivity or nonnegativity of the particle density. In the literature, some ideas have been developed to overcome this problem. For instance, in the vanishing-viscosity method, some artificial diffusion is added to the mass equation such that the maximum principle can be applied [16]. Classically, the lower bound of the density depends on the L^∞ norm of $\operatorname{div} u$. Such a regularity cannot be expected for the system (1.1)-(1.3). Another idea is to introduce an additional pressure with negative powers of the density, which may vanish away from zero [6, 7]. This allows one to derive an L^∞ bound for $1/n$, which provides a lower bound for n. In the one-dimensional equations, strict positivity for n can be also proved [25]. For the above system, we can expect only nonnegative particle densities, which makes it necessary to define the third-order term appropriately.

The second problem is the density-dependent viscosity $\mu(n) = \nu n$ which degenerates at vacuum. In fact, most results for the Navier-Stokes equations in the literature are valid for constant viscosities $\mu(n) = \nu$ only since this allows one to derive H^1 estimates for the velocity. Recently, some works have been concerned with density-dependent viscosities in the one-dimensional equations, see e.g. [37, 45] and references therein. Multidimensional equations with $\mu(n) = \nu n$ have been examined in [4, 43]. The authors of the first paper [4] need the additional friction term $-nu|u|$, whereas the authors of [43] prove the stability of weak solutions only.

The third problem is the lack of suitable a priori estimates. Indeed, define the energy of (1.1)–(1.2) by the sum of the kinetic, internal, and quantum energies:

$$E_\varepsilon(n,u) = \int_{\mathbb{T}^d} \Big(\frac{n}{2}|u|^2 + H(n) + 2\varepsilon^2 |\nabla\sqrt{n}|^2 \Big) dx, \tag{1.4}$$

where $H(n) = n^\gamma/(\gamma-1)$ if $\gamma > 1$ and $H(n) = n(\log n - 1)$ if $\gamma = 1$. A formal computation shows that, without external forces $f = 0$,

$$\frac{dE_\varepsilon}{dt}(n,u) + \nu \int_{\mathbb{T}^d} n|D(u)|^2 dx = 0.$$

This provides an $L^\infty(0,T;H^1(\mathbb{T}^d))$ estimate for $\sqrt{n}$, but this seems to be insufficient to obtain compactness for (an approximate sequence of) $\nabla\sqrt{n}$ needed to define the quantum term in a weak or distributional sense.

Our main idea to solve these problems is to transform the quantum Navier-Stokes system by means of the so-called *effective velocity*

$$w = u + \nu\nabla \log n. \tag{1.5}$$

Then a computation (see Lemma 2.1) shows that the system (1.1)–(1.2) can be equivalently written as

$$n_t + \operatorname{div}(nw) = \nu\Delta n, \quad x \in \mathbb{T}^d,\ t > 0, \tag{1.6}$$

$$(nw)_t + \operatorname{div}(nw \otimes w) + \nabla p(n) - 2\varepsilon_0^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) - nf = \nu\Delta(nw), \tag{1.7}$$

$$n(\cdot,0) = n_0, \quad (nw)(\cdot,0) = n_0w_0 \quad \text{in } \mathbb{T}^d, \tag{1.8}$$

where $w_0 = u_0 + \nu\nabla\log n_0$ and $\varepsilon_0^2 = \varepsilon^2 - \nu^2$. The first advantage of this formulation is that it allows for an additional energy estimate if $\varepsilon > \nu$. Indeed, if $f = 0$, we compute

$$\frac{dE_{\varepsilon_0}}{dt}(n,w) + \nu\int_{\mathbb{T}^d}\big(n|\nabla w|^2 + H'(n)|\nabla n|^2 + \varepsilon_0^2 n|\nabla^2\log n|^2\big)dx = 0. \tag{1.9}$$

We show below that this provides an $L^2(0,T;H^2(\mathbb{T}^d))$ bound for $\sqrt{n}$, which allows us to find L^p gradient estimates for the current density nw. The H^2 estimate for $\sqrt{n}$ is the key of the global existence analysis. The second advantage is that we may apply the maximum principle to the parabolic equation (1.6) to deduce strict positivity of the density n if n_0 is strictly positive and the velocity w is smooth. We employ this property in an approximate version of (1.6)–(1.8), thus obtaining strict positive approximate densities. In the limit of vanishing approximation parameters, the strict positivity is lost and we obtain nonnegative densities only. We prove first the global existence of weak solutions to (1.6)–(1.8) in up to three space dimensions with general coefficients $\varepsilon_0^2 > 0$ for large data. Then, as a by-product, we deduce the global existence of solutions to the multidimensional quantum Navier-Stokes model (1.1)–(1.3) if $\varepsilon > \nu$. We notice that the case $\varepsilon = \nu$ and $d = 1$ has been treated in [31].

The viscous quantum Euler model (1.6)–(1.7) is of interest by itself. Indeed, it has been derived from a Wigner-Fokker-Planck equation by a moment method [21, 35]. The viscous terms $\nu\Delta n$ and $\nu\Delta(nu)$ arise from the moments of the Fokker-Planck collision operator. This operator also provides the momentum relaxation term $-nw/\tau$ to the right-hand side of the momentum equation, where $\tau > 0$ is the relaxation time; we have neglected it to simplify the presentation (see Remark 6.1). The system (1.6)–(1.7) without the quantum term ($\varepsilon_0 = 0$) is sometimes employed as a viscous approximation of the (one-dimensional) Euler equations in the vanishing viscosity method [28, 36]. We stress the fact that the viscous terms in the above system are of *physical* origin.

For the viscous quantum Euler system, the existence of one-dimensional solutions to the stationary problem [35] and the time-dependent problem [11, 19] has been achieved. Concerning the multidimensional transient system, there exist only local-in-time existence theorems [11, 15]. We refer to the review [13] for more details. Up to now, there exist no global existence results for the multidimensional equations.

Neglecting the viscous terms ($\nu = 0$), the two systems (1.1)–(1.2) and (1.6)–(1.7) reduce to the so-called quantum Euler or quantum hydrodynamic model, see, e.g. [20, 30]. First results, e.g. [32, 38, 44], have been concerned with the local existence of solutions or the global existence of near-equilibrium solutions. For the stationary problem, only the existence of "subsonic" solutions has been achieved so far [29]. Recently, the global existence of weak solutions has been shown by Antonelli and Marcati [1]. The idea of the proof is to exploit the equivalence between the quantum hydrodynamic equations (without relaxation) and the Schrödinger equation and to employ Strichartz estimates and the local smoothing property due to Vega, Constantin, and Saut. This idea cannot be used in our quantum models.

The effective velocity (1.5) has been used also in related models. First, Bresch and Desjardins employed it to derive new entropy estimates for viscous Korteweg-type and shallow-water equations [5, 6]. These models are of the type

$$n_t + \operatorname{div}(nu) = 0, \quad (nu)_t + \operatorname{div}(nu \otimes u) - nf = \operatorname{div}(S + K), \tag{1.10}$$

where $S = (\lambda \operatorname{div} u + p(n))\mathbb{I} + 2\mu D(u)$ is the viscous stress tensor, λ, μ are the viscosity coefficients, $\mathbb{I}$ is the identity matrix, and K denotes the Korteweg stress tensor. When $\operatorname{div} K = n\nabla\Delta n$, the existence of weak solutions for $\lambda = \text{const.}$, $\mu = \text{const.}$ has been shown in [14] and for $\mu = \nu n$, $\lambda = 0$ in [9]. More general Korteweg stress tensors have been considered in [2, 9, 24]. In particular, the existence of solutions to the one-dimensional problem with the term $\operatorname{div} K = n\nabla(\sigma'(n)\Delta\sigma(n))$, suggested by [5], was proved in [26].

Brenner [3] suggested the modified Navier-Stokes model

$$n_t + \operatorname{div}(nw) = 0, \quad (nu)_t + \operatorname{div}(nu \otimes w) + \nabla p(n) = \operatorname{div} S.$$

The variables u and w are interpreted as the volume and mass velocities, respectively, and they are related by the constitutive equation $u - w = \nu\nabla \log n$ with the phenomenological constant $\nu > 0$. The Brenner-Navier-Stokes system has been analyzed in [17].

The variable $nw = nu + \nu\nabla n$ was also employed in [35] to prove the existence of solutions to the one-dimensional stationary viscous quantum Euler problem with physical boundary conditions. In fact, in this case, nw is constant and it can be shown that the density n is strictly positive.

We report that new velocity variables *similar* to (1.5) have been considered too. For instance, a variable related to the effective velocity w has been employed in the analysis of the interfacial tension in the mixture of incompressible liquids [27, formula (3.6)]. Furthermore, an Euler-Korteweg model has been reformulated in [2] by using the *complex* variable $w = u + i\kappa\nabla \log n$, where $i^2 = -1$ and $\kappa = \kappa(n)$ is the capillary function. It turns out that in the new variable, the momentum equation becomes a variable-coefficient Schrödinger equation. The transformation $w = u + i\nu\nabla \log n$ can be also applied to the viscous quantum Euler model yielding Schrödinger-type equations.

Now, we state our main results.

THEOREM 1.1 (Global existence for the viscous quantum Euler model). *Let $d \le 3$, $T > 0$, ε_0, $\nu > 0$, $p(n) = n^\gamma$ with $\gamma > 3$ if $d = 3$ and $\gamma \ge 1$ if $d = 2$, $f \in L^\infty(0,T;L^\infty(\mathbb{T}^d))$, and (n_0, w_0) is such that $n_0 \ge 0$ and $E_{\varepsilon_0}(n_0, w_0)$ is finite (see (1.4) for the definition of E_{ε_0}). Then there exists a weak solution (n, w) to (1.6)–(1.8) with the regularity*

$$\sqrt{n} \in L^\infty(0,T;H^1(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d)), \quad n \ge 0 \text{ in } \mathbb{T}^d, \tag{1.11}$$

$$n \in H^1(0,T;L^2(\mathbb{T}^d)) \cap L^\infty(0,T;L^\gamma(\mathbb{T}^d)) \cap L^2(0,T;W^{1,3}(\mathbb{T}^d)), \tag{1.12}$$

$$\sqrt{n}w \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad nw \in L^2(0,T;W^{1,3/2}(\mathbb{T}^d)),$$

$$n|\nabla w| \in L^2(0,T;L^2(\mathbb{T}^d)),$$

satisfying (1.6) *pointwise and, for all smooth test functions satisfying* $\phi(\cdot,T)=0$,

$$\begin{aligned}
-\int_{\mathbb{T}^d} n_0^2 w_0\cdot\phi(\cdot,0)dx &= \int_0^T\int_{\mathbb{T}^d}\Big(n^2w\cdot\phi_t - n^2\operatorname{div}(w)w\cdot\phi - \nu(nw\otimes\nabla n):\nabla\phi \\
&\quad + nw\otimes nw:\nabla\phi + \frac{\gamma}{\gamma+1}n^{\gamma+1}\operatorname{div}\phi - 2\varepsilon_0^2\Delta\sqrt{n}\big(2\sqrt{n}\nabla n\cdot\phi + n^{3/2}\operatorname{div}\phi\big) \\
&\quad + n^2 f\cdot\phi - \nu\nabla(nw):\big(n\nabla\phi + 2\nabla n\otimes\phi\big)\Big)dxdt. \qquad (1.13)
\end{aligned}$$

The product "$A:B$" means summation over both indices of the matrices A and B. In order to control the behavior of the solutions when the particle density n vanishes, we need to define test functions for the momentum equation, which are in some sense supported on the set $\{n>0\}$. In fact, we have chosen in the weak formulation (1.13) as in [9] test functions of the form $n\phi$, where ϕ is some smooth function, in order to deal with the convection term. Indeed, the regularity $\sqrt{n}w\in L^\infty(0,T;L^2(\mathbb{T}^d))$ does not imply compactness for (an approximation of) the convection term $\sqrt{n}w\otimes\sqrt{n}w$. However, we are able to deduce gradient estimates for nw which allow us to obtain compactness for $nw\otimes nw$. This is possible thanks to the $L^2(0,T;H^2(\mathbb{T}^d))$ regularity of $\sqrt{n}$.

The existence for the quantum Navier-Stokes model is now a consequence of Theorem 1.1.

COROLLARY 1.2 (Global existence for the quantum Navier-Stokes model). *Let* $d\le 3$, $T>0$, ε, $\nu>0$ *with* $\varepsilon>\nu$, $p(n)=n^\gamma$ *with* $\gamma>3$ *if* $d=3$ *and* $\gamma\ge 1$ *if* $d=2$, $f\in L^\infty(0,T;L^\infty(\mathbb{T}^d))$, *and* (n_0,u_0) *is such that* $n_0\ge 0$ *and* $E_\varepsilon(n_0,u_0+\nu\nabla\log n_0)$ *is finite. Then there exists a weak solution* (n,u) *to* (1.1)–(1.3) *with the regularity* (1.11)–(1.12) *and*

$$\begin{aligned}
&\sqrt{n}u\in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad nu\in L^2(0,T;W^{1,3/2}(\mathbb{T}^d)),\\
&n|\nabla u|\in L^2(0,T;L^2(\mathbb{T}^d)),
\end{aligned}$$

satisfying (1.1) *pointwise and, for all smooth test functions satisfying* $\phi(\cdot,T)=0$,

$$\begin{aligned}
-\int_{\mathbb{T}^d} n_0^2u_0\cdot\phi(\cdot,0)dx &= \int_0^T\int_{\mathbb{T}^d}\Big(n^2u\cdot\phi_t - n^2\operatorname{div}(u)u\cdot\phi + nu\otimes nu:\nabla\phi \\
&\quad + \frac{\gamma}{\gamma+1}n^{\gamma+1}\operatorname{div}\phi - 2\varepsilon^2\Delta\sqrt{n}\big(2\sqrt{n}\nabla n\cdot\phi + n^{3/2}\operatorname{div}\phi\big) + n^2f\cdot\phi \\
&\quad - \nu nD(u):\big(n\nabla\phi + \nabla n\otimes\phi\big)\Big)dxdt. \qquad (1.14)
\end{aligned}$$

We explain the two (technical) restrictions $\varepsilon>\nu$ and $\gamma>3$ imposed in the above results. The condition $\varepsilon>\nu$ is necessary to obtain H^2 bounds for $\sqrt{n}$ via the viscous quantum Euler model with $\varepsilon_0^2=\varepsilon^2-\nu^2>0$. Physically, the inequality $\varepsilon>\nu$ means that the wave energy of a quantum particle with frequency ω (where ω denotes the collision frequency in the BGK model) is larger than the kinetic energy of a particle which crosses the domain in time $1/\omega$. Thus, the inequality $\varepsilon>\nu$ corresponds to an upper bound for the collision frequency. Physically this makes sense, since too many collisions "destroy" the quantum behavior of the particles.

The energy estimate (1.9) provides an $H^1(\mathbb{T}^d)$ bound for $\sqrt{n}(\cdot,t)$ and therefore an $L^3(\mathbb{T}^d)$ bound for $n(\cdot,t)$ (for $d\le 3$). Moreover, the pressure gives an $L^\gamma(\mathbb{T}^d)$ bound for $n(\cdot,t)$. This improves the $L^3(\mathbb{T}^d)$ bound only if $\gamma > 3$. In fact, this hypothesis is needed to infer an estimate for $n(\cdot,t)$ in $W^{2,p}(\mathbb{T}^d)$ with $p > 3/2$, which embeddes compactly into $W^{1,3}(\mathbb{T}^d)$. With this property at hand, for a given approximation (n_δ, n_δ) ($\delta > 0$) of (1.6)–(1.8), we infer the weak convergence (of a subsequence) of

$$\Delta\sqrt{n_\delta}\sqrt{n_\delta}\nabla n_\delta \rightharpoonup \Delta\sqrt{n}\sqrt{n}\nabla n \quad \text{in } L^1$$

as $\delta \to 0$ since $\Delta\sqrt{n_\delta}$ converges weakly in $L^2(\mathbb{T}^d)$ and $\sqrt{n_\delta}$ converges strongly in $L^6(\mathbb{T}^d)$. If $\gamma \le 3$, we can deduce compactness in $W^{1,q}(\mathbb{T}^d)$ with $q < 3$ only, which does not allow for the above convergence (see section 6 for details).

The strategy of the existence proof is as follows. In section 2, we detail the reformulation of the quantum Navier-Stokes model as a viscous quantum Euler system and vice versa. The latter model is approximated in section 3 by a projection of the infinite-dimensional momentum equation onto a finite system of ordinary differential equations on a Faedo-Galerkin space with dimension N, following [16]. We need a second approximation parameter δ by adding the term $\delta(\Delta w - w)$ to the right-hand side of (1.7), which allows us to derive H^1 estimate for w. The global existence of approximate solutions follows from the energy estimates derived in section 3.2. In section 4, more a priori estimates uniform in (N,δ) are deduced. Finally, the limits $N \to \infty$ and $\delta \to 0$ are performed in sections 5 and 6, respectively.

We remark that in the literature, often additional hypotheses are needed to obtain global existence results for related equations. We mentioned above that several works are concerned with the case of constant viscosities, yielding H^1 bounds for the velocity; see, e.g. [16, 17] for Navier-Stokes equations and [14, 23] for Korteweg-type models. Nonconstant viscosity coefficients are admissible in the analysis of [6, 9, 26, 43]. Hsiao and Li [26] need the presence of the drag friction $-nu|u|$ in the momentum equation to prove the strong convergence of $\sqrt{n_\delta}w_\delta$. This convergence was obtained by Mellet and Vasseur in [43] by proving a bound in a space slightly better than $L^\infty(0,T;L^2(\mathbb{T}^d))$ (however, excluding a Faedo-Galerkin strategy). Bresch and Desjardins [7] impose conditions on the viscosity coefficients allowing for compactness results for negative powers of the particle density. The idea of multiplying the momentum equation by a power of the particle density, in order to deal with possible vacuum regions, was also employed in [9, 19].

2. Reformulation und weak formulation. We show that the quantum Navier-Stokes system (1.1)–(1.2) can be reformulated as the visous quantum Euler model (1.6)–(1.7) and we derive the weak formulation (1.13).

LEMMA 2.1. *Let (n,u) be a smooth solution to (1.1)–(1.2). Then $(n,w) = (n, u + \nu\nabla\log n)$ solves (1.6)–(1.7) with $\varepsilon_0^2 = \varepsilon^2 - \nu^2$. Conversely, if (n,w) is a smooth solution to (1.6)–(1.7), then $(n,u) = (n, w - \nu\nabla\log n)$ solves (1.1)–(1.2) with $\varepsilon^2 = \varepsilon_0^2 + \nu^2$.*

Proof. Let (n,u) be a smooth solution to (1.1)–(1.2). The mass equation transforms to

$$n_t + \operatorname{div}(nw) - \nu\Delta w = n_t + \operatorname{div}\big(n(w - \nu\nabla\log n)\big) = n_t + \operatorname{div}(nu) = 0.$$

Next, adding the elementary identities

$$\begin{aligned}
\nu(n\nabla\log n)_t &= \nu(\nabla n)_t = -\nu\nabla\mathrm{div}(nu),\\
\nu^2\mathrm{div}(n\nabla\log n\otimes\nabla\log n) &= \nu^2\Delta(n\nabla\log n) - \nu^2\mathrm{div}(n\nabla^2\log n)\\
&= \nu^2\Delta(n\nabla\log n) - 2\nu^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right),\\
\nu\mathrm{div}(n\nabla\log n\otimes u + nu\otimes\nabla\log n) &= \nu\Delta(nu) - 2\nu\mathrm{div}(nD(u)) + \nu\nabla\mathrm{div}(nu),
\end{aligned}$$

we arrive at

$$\begin{aligned}
&(nw)_t + \mathrm{div}(nw\otimes w) - \nu\Delta(nw)\\
&\quad = (nu)_t + \mathrm{div}(nu\otimes u) - 2\nu\mathrm{div}(nD(u)) - 2\nu^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right)\\
&\quad = -\nabla p(n) + nf + 2(\varepsilon^2-\nu^2)n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right).
\end{aligned}$$

Thus, (n,w) solves (1.6)–(1.7) with $\varepsilon_0^2 = \varepsilon^2-\nu^2$. ☐

LEMMA 2.2. *Let $T>0$ and let (n,w) be a (smooth) solution to* (1.6)–(1.7). *Then (n,w) solves* (1.13) *for all smooth test functions ϕ with $\phi(\cdot,T)=0$,*

Proof. Let ϕ be a smooth test function such that $\phi(\cdot,T)=0$. Multiplying (1.13) by n and integrating over $\mathbb{T}^d\times(0,T)$, we find that

$$\begin{aligned}
&-\int_{\mathbb{T}^d} n_0^2 w_0\cdot\phi(\cdot,0)dx = \int_0^T\int_{\mathbb{T}^d}(n^2w\cdot\phi)_t dx\\
&\quad = \int_0^T\int_{\mathbb{T}^d}\big(n^2w\cdot\phi_t + n(nw)_t\cdot\phi + nn_tw\cdot\phi\big)dx\\
&\quad = \int_0^T\int_{\mathbb{T}^d}\Big(n^2w\cdot\phi_t + nw\cdot\phi\big(-\nabla n\cdot w - n\mathrm{div}\,w + \nu\Delta n\big)\\
&\quad + nw\otimes nw:\nabla\phi + n(w\otimes w):(\nabla n\otimes\phi) - np'(n)\nabla n\cdot\phi + n^2f\cdot\phi\\
&\quad - 2\varepsilon_0^2\Delta\sqrt{n}\big(2\sqrt{n}\nabla n\cdot\phi + n^{3/2}\mathrm{div}\,\phi\big) - \nu\nabla(nw):\big(n\nabla\phi+\nabla n\otimes\phi\big)\Big)dx.
\end{aligned}$$

Since $n(w\cdot\phi)(w\cdot\nabla n) = n(w\otimes w):(\nabla n\otimes\phi)$, $np'(n)\nabla n = (\gamma/(\gamma+1))\nabla n^{\gamma+1}$, and

$$\nu\int_{\mathbb{T}^d} nw\cdot\phi\Delta n = -\nu\int_{\mathbb{T}^d}\big(\nabla(nw):(\nabla n\otimes\phi) + (nw\otimes\nabla n):\nabla\phi\big)dx,$$

the above formulation simplifies to (1.13). ☐

3. Faedo-Galerkin approximation. In this section, we prove the existence of solutions to approximate viscous quantum Euler equations. We proceed similarly as in [16, Chap. 7] (see [19] for the one-dimensional case).

3.1. Local existence of solutions. Let $T>0$ and let (e_k) be an orthonormal basis of $L^2(\mathbb{T}^d)$ which is also an orthogonal basis of $H^1(\mathbb{T}^d)$. Introduce the finite-dimensional space $X_N = \mathrm{span}\{e_1,\ldots,e_N\}$, $N\in\mathbb{N}$. Let $(n_0,w_0)\in C^\infty(\mathbb{T}^d)^2$ be some initial data satisfying $n_0(x)\ge\delta>0$ for $x\in\mathbb{T}^d$ for some $\delta>0$ and let the velocity $v\in C^0([0,T];X_N)$ be given. We notice that v can be written as

$$v(x,t) = \sum_{i=1}^n \lambda_i(t)e_i(x),\quad (x,t)\in\mathbb{T}^d\times[0,T],$$

for some functions $\lambda_i(t)$, and the norm of v in $C^0([0,T];X_N)$ can be formulated as

$$\|v\|_{C^0([0,T];X_N)} = \max_{t\in[0,T]} \sum_{i=1}^N |\lambda_i(t)|.$$

As a consequence, v can be bounded in $C^0([0,T];C^k(\mathbb{T}^d))$ for any $k \in \mathbb{N}$, and there exists a constant $C > 0$ depending on k such that

$$\|v\|_{C^0([0,T];C^k(\mathbb{T}^d))} \le C\|v\|_{C^0([0,T];L^2(\mathbb{T}^d))}. \tag{3.1}$$

The approximate system is defined as follows. Let $n \in C^1([0,T];C^3(\mathbb{T}^d))$ be the classical solution to

$$n_t + \operatorname{div}(nv) = \nu\Delta n, \quad n(\cdot,0) = n_0 \quad \text{in } \mathbb{T}^d \times (0,T) \tag{3.2}$$

(see, e.g. [41]). The maximum principle provides the lower and upper bounds [16, Chap. 7.3]

$$\begin{aligned}\inf_{x\in\mathbb{T}^d} n_0(x)\exp\Big(-\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{T}^d)}ds\Big) &\le n(x,t)\\ &\le \sup_{x\in\mathbb{T}^d} n_0(x)\exp\Big(\int_0^t \|\operatorname{div} v\|_{L^\infty(\mathbb{T}^d)}ds\Big) \quad \text{for } (x,t)\in\mathbb{T}^d\times[0,T].\end{aligned}$$

Since we assumed that $n_0 \ge \delta > 0$, $n(x,t)$ is strictly positive. In view of (3.1), for $\|v\|_{C^0([0,T];L^2(\mathbb{T}^d))} \le c$, there exist constants $\underline{n}(c)$ and $\overline{n}(c)$ such that

$$0 < \underline{n}(c) \le n(x,t) \le \overline{n}(c), \quad (x,t)\in\mathbb{T}^d\times[0,T].$$

We introduce the operator $S : C^0([0,T];X_N) \to C^0([0,T];C^3(\mathbb{T}^d))$ by $S(v) = n$. Since the equation for n is linear, S is Lipschitz continuous in the following sense:

$$\|S(v_1) - S(v_2)\|_{C^0([0,T];C^k(\mathbb{T}^d))} \le C(N,k)\|v_1 - v_2\|_{C^0([0,T];L^2(\mathbb{T}^d))}. \tag{3.3}$$

Next, we wish to solve the momentum equation on the space X_N. To this end, for given $n = S(v)$, we are looking for a function $w_N \in C^0([0,T];X_N)$ such that

$$\begin{aligned}-\int_{\mathbb{T}^d} n_0w_0\cdot\phi(\cdot,0)dx = \int_0^T\int_{\mathbb{T}^d}\Big(&nw_N\cdot\phi_t + n(v\otimes w_N):\nabla\phi + p(n)\operatorname{div}\phi + nf\cdot\phi\\ &- 2\varepsilon_0^2\frac{\Delta\sqrt{n}}{\sqrt{n}}\operatorname{div}(n\phi) - \nu\nabla(nw_N):\nabla\phi - \delta(\nabla w_N:\nabla\phi + w_N\cdot\phi)\Big)dxdt\end{aligned} \tag{3.4}$$

for all $\phi \in C^1([0,T];X_N)$ such that $\phi(\cdot,T) = 0$. Notice that we have added the regularization term $\delta(\Delta w_N - w_N)$. The reason is that we will apply Banach's fixed-point theorem to prove the local-in-time existence of solutions. The regularization yields the H^1 regularity of w_N needed to conclude the global existence of solutions.

To solve (3.4), we follow [16, Chap. 7.3.3] and introduce the following family of operators, given a function $\rho \in L^1(\mathbb{T}^d)$ with $\rho \ge \underline{\rho} > 0$:

$$M[\rho] : X_N \to X_N^*, \quad \langle M[\rho]u, w\rangle = \int_{\mathbb{T}^d}\rho u\cdot w dx, \quad u,w\in X_N.$$

These operators are symmetric and positive definite with the smallest eigenvalue

$$\inf_{\|w\|_{L^2(\mathbb{T}^d)}=1}\langle M[\rho]w,w\rangle = \inf_{\|w\|_{L^2(\mathbb{T}^d)}=1}\int_{\mathbb{T}^d}\rho|w|^2dx \geq \inf_{x\in\mathbb{T}^d}\rho(x) \geq \underline{\rho}.$$

Hence, since X_N is finite-dimensional, the operators are invertible with

$$\|M^{-1}[\rho]\|_{L(X_N^*,X_N)} \leq \underline{\rho}^{-1},$$

where $L(X_N^*,X_N)$ is the set of bounded linear mappings from X_N^* to X_N. Moreover (see [16, Chap. 7.3.3]), M^{-1} is Lipschitz continuous in the sense

$$\|M^{-1}[\rho_1] - M^{-1}[\rho_2]\|_{L(X_N^*,X_N)} \leq C(N,\underline{\rho})\|\rho_1-\rho_2\|_{L^1(\mathbb{T}^d)} \tag{3.5}$$

for all ρ_1, $\rho_2 \in L^1(\mathbb{T}^d)$ such that ρ_1, $\rho_2 \geq \underline{\rho} > 0$.

Now, the integral equation (3.4) can be rephrased as an ordinary differential equation on the finite-dimensional space X_N:

$$\frac{d}{dt}\big(M[n(t)]w_N(t)\big) = N[v,w_N(t)], \quad t>0, \quad M[n_0]w_N(0) = M[n_0]w_0, \tag{3.6}$$

where $n = S(v)$ and

$$\begin{aligned}\langle N[v,w_N],\phi\rangle = \int_{\mathbb{T}^d}\Big(&nv\otimes w_N : \nabla\phi + p(n)\mathrm{div}\phi + nf\cdot\phi - 2\varepsilon_0^2\frac{\Delta\sqrt{n}}{\sqrt{n}}\mathrm{div}(n\phi)\\ &- \big(\nu\nabla(nw_N)+\delta\nabla w_N\big):\nabla\phi - \delta w_N\cdot\phi\Big)dx, \quad \phi\in X_N.\end{aligned}$$

The operator $N[v,\cdot]$, defined for every $t\in[0,T]$ as an operator from X_N to X_N^*, is continuous in time. Standard theory for systems of ordinary differential equations then provides the existence of a unique classical solution to (3.6), i.e., for given v, there exists a unique solution $w_N \in C^1([0,T];X_N)$ to (3.4).

Integrating (3.6) over $(0,t)$ yields the following nonlinear equation:

$$w_N(t) = M^{-1}\big[(S(w_N))(t)\big]\left(M[n_0]u_0 + \int_0^t N[w_N,w_N(s)]ds\right) \quad \text{in } X_N.$$

Taking into account the Lipschitz-type estimates (3.3) and (3.5) for S and M^{-1}, this equation can be solved by evoking the fixed-point theorem of Banach on a short time interval $[0,T']$, where $T'\leq T$, in the space $C^0([0,T'];X_N)$. In fact, we have even $w_N\in C^1([0,T'];X_N)$. Thus, there exists a unique local-in-time solution (n_N,w_N) to (3.2) and (3.4).

3.2. Global existence of solutions. In order to prove that the solution (n_N, w_N) constructed above exists on the whole time interval $[0,T]$, it is sufficient to show that (w_N) is bounded in X_N on $[0,T']$. This is achieved by employing the energy estimate.

LEMMA 3.1. *Let $T'\leq T$, and let $n_N\in C^1([0,T'];C^3(\mathbb{T}^d))$, $w_N\in C^1([0,T'];X_N)$ be a local-in-time solution to* (3.2) *and* (3.4) *with $n=n_N$ and $v=w_N$. Then*

$$\begin{aligned}\frac{dE_{\varepsilon_0}}{dt}(n_N,w_N) &+ \nu\int_{\mathbb{T}^d}\big(n_N|\nabla w_N|^2 + H''(n_N)|\nabla n_N|^2 + \varepsilon_0^2 n_N|\nabla^2\log n_N|^2\big)dx\\ &+ \delta\int_{\mathbb{T}^d}\big(|\nabla w_N|^2+|w_N|^2\big)dx\\ &\leq \frac{\nu}{2}\int_{\mathbb{T}^d}n_N|w_N|^2dx + \frac{1}{2\nu}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\|n_0\|_{L^1(\mathbb{T}^d)},\end{aligned}$$

where the energy E_{ε_0} is defined in (1.4).

Proof. First, we multiply (3.2) by $H'(n_N)-|w_N|^2/2-2\varepsilon_0^2\Delta\sqrt{n_N}/\sqrt{n_N}$, integrate over $\mathbb{T}^d$, and integrate by parts:

$$\begin{aligned}0=\int_{\mathbb{T}^d}\Big(&\partial_t H(n_N)-\frac{1}{2}|w_N|^2\partial_t n_N+2\varepsilon_0^2\partial_t|\nabla\sqrt{n_N}|^2-n_NH''(n_N)\nabla n_N\cdot w_N\\&+n_Nw_N\cdot\nabla w_N\cdot w_N-2\varepsilon_0^2\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\operatorname{div}(n_Nw_N)+\nu H''(n_N)|\nabla n_N|^2\\&-\nu\nabla n_N\cdot\nabla w_N\cdot w_N+2\nu\varepsilon_0^2\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\Delta n_N\Big)dx.\end{aligned}$$

Then, using the test function $w_N\in C^1([0,T];X_N)$ in (3.4), with $v=w_N$ and $n=n_N:=S(v)=S(w_N)$, and integrating by parts leads to

$$\begin{aligned}0=\int_{\mathbb{T}^d}\Big(&|w_N|^2\partial_t n_N+\frac{1}{2}n_N\partial_t|w_N|^2-n_Nw_N\otimes w_N:\nabla w_N\\&+p'(n_N)\nabla n_N\cdot w_N+2\varepsilon_0^2\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\operatorname{div}(n_Nw_N)-n_Nf\cdot w_N\\&+\nu\nabla n_N\cdot\nabla w_N\cdot w_N+\nu n_N|\nabla w_N|^2+\delta|\nabla w_N|^2+\delta|w_N|^2\Big)dx.\end{aligned}$$

Adding both equations gives, since $n_NH''(n_N)=p'(n_N)$ and $w_N\cdot\nabla w_N\cdot w_N=w_N\otimes w_N:\nabla w_N$,

$$\begin{aligned}0=\int_{\mathbb{T}^d}\Big(&\partial_t\Big(\frac{n_N}{2}|w_N|^2+H(n_N)+2\varepsilon_0^2|\nabla\sqrt{n_N}|^2\Big)-n_Nf\cdot w_N\\&+2\nu\varepsilon_0^2\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\Delta n_N+\nu H''(n_N)|\nabla n_N|^2+\nu n_N|\nabla w_N|^2+\delta|\nabla w_N|^2+\delta|w_N|^2\Big)dx.\end{aligned}$$

The identity $2n_N\nabla(\Delta\sqrt{n_N}/\sqrt{n_N})=\operatorname{div}(n_N\nabla^2\log n_N)$ yields

$$\begin{aligned}\int_{\mathbb{T}^d}\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\Delta n_Ndx&=-\int_{\mathbb{T}^d}n_N\nabla\log n_N\cdot\nabla\left(\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\right)dx\\&=-\frac{1}{2}\int_{\mathbb{T}^d}\nabla\log n_N\cdot\operatorname{div}(n_N\nabla^2\log n_N)dx=\frac{1}{2}\int_{\mathbb{T}^d}n_N|\nabla^2\log n_N|^2dx.\end{aligned}\tag{3.7}$$

Hence,

$$\begin{aligned}\frac{dE_{\varepsilon_0}}{dt}(n_N,w_N)&+\nu\int_{\mathbb{T}^d}\big(n_N|\nabla w_N|^2+H''(n_N)|\nabla n_N|^2+\varepsilon_0^2n_N|\nabla^2\log n_N|^2\big)dx\\&+\delta\int_{\mathbb{T}^d}\big(|\nabla w_N|^2+|w_N|^2\big)dx=\int_{\mathbb{T}^d}n_Nf\cdot w_Ndx.\end{aligned}$$

Finally, the right-hand side is estimated by

$$\int_{\mathbb{T}^d}n_Nf\cdot w_Ndx\le\frac{\nu}{2}\int_{\mathbb{T}^d}n_N|w_N|^2dx+\frac{1}{2\nu}\|f\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\|n_0\|_{L^1(\mathbb{T}^d)},$$

since n conserves mass, $\|n_N\|_{L^\infty(0,T';L^1(\mathbb{T}^d))}=\|n_0\|_{L^1(\mathbb{T}^d)}$ for $0\le t\le T'$. ◻

4. A priori estimates. Let $(n_N, w_N) \in C^1([0,T];C^3(\mathbb{T}^d)) \times C^1([0,T];X_N)$ be a solution to the approximate system (3.2) and (3.4). We infer from the energy estimate of Lemma 3.1 and Gronwall's lemma the uniform bounds

$$\|\sqrt{n_N}\|_{L^\infty(0,T;H^1(\mathbb{T}^d))} \le C, \tag{4.1}$$

$$\|n_N\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^d))} \le C, \tag{4.2}$$

$$\|\sqrt{n_N}w_N\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \|\sqrt{n_N}\nabla w_N\|_{L^2(0,T;L^2(\mathbb{T}^d))} \le C, \tag{4.3}$$

$$\sqrt{\delta}\|w_N\|_{L^2(0,T;H^1(\mathbb{T}^d))} \le C, \tag{4.4}$$

where the constant $C > 0$ is here and in the following a generic constant which is independent of N and δ. The $L^\infty(0,T;H^1(\mathbb{T}^d))$ estimate for $\sqrt{n_N}$ gives immediately an $L^\infty(0,T;L^3(\mathbb{T}^d))$ bound for n_N, since $H^1(\mathbb{T}^d)$ embeddes continuously into $L^6(\mathbb{T}^d)$ for $d \le 3$. Thus, the estimate (4.2) improves this bound only if $\gamma > 3$. In the case $d = 2$, $H^1(\mathbb{T}^d)$ embeddes continuously into $L^\alpha(\mathbb{T}^d)$ for any $\alpha < \infty$ and hence, (n_N) is bounded in $L^\infty(0,T;L^p(\mathbb{T}^d))$ for any $\gamma \ge 1$. In the following, we assume that $\gamma > 3$ if $d = 3$ and $\gamma \ge 1$ if $d = 2$.

We recall the Gagliardo-Nirenberg inequality (see p. 1034 in [47]).

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^d$ ($d \ge 1$) be a bounded open set with $\partial\Omega \in C^{0,1}$, $m \in \mathbb{N}$, $1 \le 3p, q, r \le \infty$. Then there exists a constant $C > 0$ such that for all $u \in W^{m,p}(\Omega) \cap L^q(\Omega)$,*

$$\|D^\alpha\|_{L^r(\Omega)} \le C\|u\|^\theta_{W^{m,p}(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)},$$

where $0 \le |\alpha| \le m-1$, $\theta = |\alpha|/m$, and $|\alpha| - d/r = \theta(m - d/p) - (1-\theta)d/p$. If $m - |\alpha| - d/p \notin \mathbb{N}_0$, then $\theta \in [|\alpha|/m, 1]$ is allowed.

The energy inequality of Lemma 3.1 allows us to conclude some estimates.

LEMMA 4.2 (Estimates for $\sqrt{n_N}$ and $\sqrt[4]{n_N}$). *The following uniform estimate holds for some constant $C > 0$ which is independent of N and δ:*

$$\|\sqrt{n_N}\|_{L^2(0,T;H^2(\mathbb{T}^d))} + \|\sqrt[4]{n_N}\|_{L^4(0,T;W^{1,4}(\mathbb{T}^d))} \le C. \tag{4.5}$$

Proof. The lemma follows from the energy estimate in Lemma 3.1, the inequality

$$\int_{\mathbb{T}^d} n_N|\nabla^2 \log n_N|^2 dx \ge \kappa_d \int_{\mathbb{T}^d} |\nabla^2\sqrt{n_N}|^2 dx, \tag{4.6}$$

with $\kappa_2 = 7/8$ and $\kappa_3 = 11/15$, which is shown in [34], and the inequality

$$\int_{\mathbb{T}^d} n_N|\nabla^2 \log n_N|^2 dx \ge \kappa \int_{\mathbb{T}^d} |\nabla \sqrt[4]{n_N}|^4 dx, \quad \kappa > 0,$$

which is proved in the appendix. ◻

We are able to deduce more regularity from the H^2 bound for $\sqrt{n_N}$.

LEMMA 4.3 (Space regularity for n_N and $n_N w_N$). *The following uniform estimates hold for some constant $C > 0$ not depending on N and δ:*

$$\|n_N w_N\|_{L^2(0,T;W^{1,3/2}(\mathbb{T}^d))} \le C, \tag{4.7}$$

$$\|n_N\|_{L^2(0,T;W^{2,p}(\mathbb{T}^d))} \le C, \tag{4.8}$$

$$\|n_N\|_{L^{4\gamma/3+1}(0,T;L^{4\gamma/3+1}(\mathbb{T}^d))} \le C, \tag{4.9}$$

where $p = 2\gamma/(\gamma+1)$ *if* $d = 3$ *and* $p < 2$ *if* $d = 2$.

We remark that for $\gamma > 3$ it holds $p > 3/2$ and hence, the embedding $W^{2,p}(\mathbb{T}^d) \hookrightarrow W^{1,3}(\mathbb{T}^d)$ is compact.

Proof. Since $d \le 3$, the space $H^2(\mathbb{T}^d)$ embeddes continuously into $L^\infty(\mathbb{T}^d)$, showing that $(\sqrt{n_N})$ is bounded in $L^2(0,T;L^\infty(\mathbb{T}^d))$ (see (4.5)). Thus, in view of (4.3), $n_N w_N = \sqrt{n_N}\sqrt{n_N}w_N$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^d))$. By (4.1) and (4.5), $(\nabla\sqrt{n_N})$ is bounded in $L^2(0,T;L^6(\mathbb{T}^d))$ and $(\sqrt{n_N})$ is bounded in $L^\infty(0,T;L^6(\mathbb{T}^d))$. This, together with (4.3), implies that

$$\nabla(n_N w_N) = 2\nabla\sqrt{n_N} \otimes (\sqrt{n_N}w_N) + \sqrt{n_N}\nabla w_N\sqrt{n_N}$$

is uniformly bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$, proving the first claim.

For the second claim, we observe first that, by the Gagliardo-Nirenberg inequality (see Lemma 4.1), with $p = 2\gamma/(\gamma+1)$ and $\theta = 1/2$,

$$\begin{aligned}\|\nabla\sqrt{n_N}\|^4_{L^4(0,T;L^{2p}(\mathbb{T}^d))} &\le C\int_0^T \|\sqrt{n_N}\|^{4\theta}_{H^2(\mathbb{T}^d)}\|\sqrt{n_N}\|^{4(1-\theta)}_{L^{2\gamma}(\mathbb{T}^d)}dt \\ &\le C\|\sqrt{n_N}\|^{4(1-\theta)}_{L^\infty(0,T;L^{2\gamma}(\mathbb{T}^d))}\int_0^T \|\sqrt{n_N}\|^2_{H^2(\mathbb{T}^d)}dt \le C.\end{aligned}$$

Thus, $(\sqrt{n_N})$ is bounded in $L^4(0,T;W^{1,2p}(\mathbb{T}^d))$. Notice that in the case $d = 3$, $\gamma > 3$ implies that $2p > 3$ which gives a uniform bound for $\sqrt{n_N}$ in $L^4(0,T;L^\infty(\mathbb{T}^d))$. If $d = 2$, (n_N) is bounded in $L^\infty(0,T;H^1(\mathbb{T}^d)) \hookrightarrow L^\infty(0,T;L^\alpha(\mathbb{T}^d))$ for all $\alpha < \infty$. Then we may replace in the above estimate 2γ by α, obtaining an $L^4(0,T;W^{1,2p}(\mathbb{T}^d))$ bound for all $p < 2$. Hence, in the two-dimensional case, all $\gamma \ge 1$ are admissible. The estimate on $\nabla\sqrt{n_N}$ in $L^4(0,T;L^{2p}(\mathbb{T}^d))$ shows that

$$\nabla^2 n_N = 2\big(\sqrt{n_N}\nabla^2\sqrt{n_N} + \nabla\sqrt{n_N}\otimes\nabla\sqrt{n_N}\big)$$

is bounded in $L^2(0,T;L^p(\mathbb{T}^d))$ which proves the second claim.

Finally, the Gagliardo-Nirenberg inequality, with $\theta = 3/(4\gamma+3)$ and $q = 2(4\gamma+3)/3$,

$$\begin{aligned}\|\sqrt{n_N}\|^q_{L^q(0,T;L^q(\mathbb{T}^d))} &\le C\int_0^T \|\sqrt{n_N}\|^{q\theta}_{H^2(\mathbb{T}^d)}\|\sqrt{n_N}\|^{q(1-\theta)}_{L^{2\gamma}(\mathbb{T}^d)}dt \\ &\le C\|n_\delta\|^{q(1-\theta)}_{L^\infty(0,T;L^\gamma(\mathbb{T}^d))}\int_0^T \|\sqrt{n_N}\|^2_{H^2(\mathbb{T}^d)}dt \le C,\end{aligned}$$

shows that n_N is bounded in $L^{q/2}(0,T;L^{q/2}(\mathbb{T}^d))$. This finishes the proof. ◻

Lemma 4.4 (Time regularity for n_N and $n_N w_N$). *The following uniform estimates hold for* $s > d/2+1$*:*

$$\|\partial_t n_N\|_{L^2(0,T;L^{3/2}(\mathbb{T}^d))} \le C, \tag{4.10}$$

$$\|\partial_t(n_N w_N)\|_{L^{4/3}(0,T;(H^s(\mathbb{T}^d))^*)} \le C. \tag{4.11}$$

Proof. By (4.7) and (4.8), we find that $\partial_t n_N = -\mathrm{div}(n_N w_N) + \nu\Delta n_N$ is uniformly bounded in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$, achieving the first claim.

The sequence $(n_N w_N \otimes w_N)$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$; hence, $(\mathrm{div}(n_N w_N \otimes w_N))$ is bounded in $L^\infty(0,T;(W^{1,\infty}(\mathbb{T}^d))^*)$ and, because of the continuous embedding of $H^s(\mathbb{T}^d)$ into $W^{1,\infty}(\mathbb{T}^d)$ for $s > d/2+1$, also in $L^\infty(0,T;\,(H^s(\mathbb{T}^d))^*)$. The estimate

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} n_N\nabla\left(\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\right)\cdot\phi dxdt &= -\int_0^T\int_{\mathbb{T}^d}\Delta\sqrt{n_N}\big(2\nabla\sqrt{n_N}\cdot\phi+\sqrt{n_N}\operatorname{div}\phi\big)dxdt\\ &\le \|\Delta\sqrt{n_N}\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big(2\|\sqrt{n_N}\|_{L^4(0,T;W^{1,3}(\mathbb{T}^d))}\|\phi\|_{L^4(0,T;L^6(\mathbb{T}^d))}\\ &\quad+\|\sqrt{n_N}\|_{L^\infty(0,T;L^6(\mathbb{T}^d))}\|\phi\|_{L^2(0,T;W^{1,3}(\mathbb{T}^d))}\big)\\ &\le C\|\phi\|_{L^4(0,T;W^{1,3}(\mathbb{T}^d))}\end{aligned}$$

for all $\phi \in L^4(0,T;W^{1,3}(\mathbb{T}^d))$ proves that $n_N\Delta\sqrt{n_N}/\sqrt{n_N}$ is uniformly bounded in $L^{4/3}(0,T;(W^{1,3}(\mathbb{T}^d))^*) \hookrightarrow L^{4/3}(0,T;(H^s(\mathbb{T}^d))^*)$. In view of (4.9), (n_N^γ) is bounded in $L^{4/3}(0,T;L^{4/3}(\mathbb{T}^d)) \hookrightarrow L^{4/3}(0,T;(H^s(\mathbb{T}^d))^*)$. Furthermore, by (4.7), $\Delta(n_Nw_N)$ is uniformly bounded in $L^2(0,T;(W^{1,3}(\mathbb{T}^d))^*)$ and, by (4.4), $(\delta\Delta w_N)$ is bounded in $L^2(0,T;(H^1(\mathbb{T}^d))^*)$. Therefore,

$$\begin{aligned}(n_Nw_N)_t = -\operatorname{div}(n_Nw_N\otimes w_N) - \nabla(n_N^\gamma) + 2\varepsilon_0^2 n_N\nabla\left(\frac{\Delta\sqrt{n_N}}{\sqrt{n_N}}\right) + n_Nf\\ + \nu\Delta(n_Nw_N) + \delta\Delta w_N\end{aligned}$$

is uniformly bounded in $L^{4/3}(0,T;(H^s(\mathbb{T}^d))^*)$. □

The $L^4(0,T;W^{1,4}(\mathbb{T}^d))$ bound (4.5) on $\sqrt[4]{n_N}$ provides a uniform estimate for $\partial_t\sqrt{n_N}$.

LEMMA 4.5 (Time regularity for $\sqrt{n_N}$). *The following estimate holds:*

$$\|\partial_t\sqrt{n_N}\|_{L^2(0,T;(H^1(\mathbb{T}^d))^*)} \le C. \tag{4.12}$$

Proof. Dividing the mass equation by $\sqrt{n_N}$ gives

$$\begin{aligned}\partial_t\sqrt{n_N} &= -\nabla\sqrt{n_N}\cdot w_N - \frac{1}{2}\sqrt{n_N}\operatorname{div} w_N + \nu\big(\Delta\sqrt{n_N} + 4|\nabla\sqrt[4]{n_N}|^2\big)\\ &= -\operatorname{div}(\sqrt{n_N}w_N) + \frac{1}{2}\sqrt{n_N}\operatorname{div} w_N + \nu\big(\Delta\sqrt{n_N} + 4|\nabla\sqrt[4]{n_N}|^2\big).\end{aligned}$$

The first term on the right-hand side is bounded in $L^2(0,T;(H^1(\mathbb{T}^d))^*)$, by (4.3). The remaining terms are uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^d))$, see (4.3) and (4.5). □

5. The limit $N\to\infty$**.** We perform first the limit $N\to\infty$, $\delta > 0$ being fixed. The limit $\delta\to 0$ is carried out in section 6. We consider both limits separately since the weak formulation (1.13) for the continuous viscous quantum Euler model is different from its approximation (3.2) and (3.4).

We conclude from the Aubin lemma, taking into account the regularity (4.8) and (4.10) for n_N, the regularity (4.5) and (4.12) for $\sqrt{n_N}$, and the regularity (4.7) and (4.11) for n_Nw_N, that there exist subsequences of (n_N), $(\sqrt{n_N})$, and (n_Nw_N), which are not relabeled, such that, for some functions n and j, as $N\to\infty$,

$$\begin{aligned} n_N &\to n && \text{strongly in } L^2(0,T;L^\infty(\mathbb{T}^d)),\\ \sqrt{n_N} &\rightharpoonup \sqrt{n} && \text{weakly in } L^2(0,T;H^2(\mathbb{T}^d)),\\ \sqrt{n_N} &\to \sqrt{n} && \text{strongly in } L^2(0,T;H^1(\mathbb{T}^d)),\\ n_Nw_N &\to j && \text{strongly in } L^2(0,T;L^2(\mathbb{T}^d)).\end{aligned}$$

Here we have used that the embeddings $W^{2,p}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ ($p > 3/2$), $H^2(\mathbb{T}^d) \hookrightarrow H^1(\mathbb{T}^d)$, and $W^{1,3/2}(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$ are compact. The estimate (4.4) on w_N provides further the existence of a subsequence (not relabeled) such that, as $N \to \infty$,

$$w_N \rightharpoonup w \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T}^d)).$$

Then, since $(n_N w_N)$ converges weakly to nw in $L^1(0,T;L^6(\mathbb{T}^d))$, we infer that $j = nw$.

We are now in the position to let $N \to \infty$ in the approximate system (3.2) and (3.4) with $n = n_N$ and $v = w_N$. Clearly, the limit $N \to \infty$ shows immediately that n solves

$$n_t + \operatorname{div}(nw) = \nu \Delta n \quad \text{in } \mathbb{T}^d \times (0,T).$$

Next, we consider the weak formulation (3.4) term by term. The strong convergence of $(n_N w_N)$ in $L^2(0,T;L^2(\mathbb{T}^d))$ and the weak convergence of (w_N) in $L^2(0,T;L^6(\mathbb{T}^d))$ leads to

$$n_N w_N \otimes w_N \rightharpoonup nw \otimes w \quad \text{weakly in } L^1(0,T;L^{3/2}(\mathbb{T}^d)).$$

Furthermore, in view of (4.7) (up to a subsequence),

$$\nabla(n_N w_N) \rightharpoonup \nabla(nw) \quad \text{weakly in } L^2(0,T;L^{3/2}(\mathbb{T}^d)).$$

The $L^\infty(0,T;L^\gamma(\mathbb{T}^d))$ bound for (n_N) shows that $n_N^\gamma \rightharpoonup z$ weakly* in $L^\infty(0,T; L^1(\mathbb{T}^d))$ for some function z and, since $n_N^\gamma \to n^\gamma$ a.e., $z = n^\gamma$. Finally, the above convergence results show that the limit $N \to \infty$ of

$$\int_{\mathbb{T}^d} \frac{\Delta\sqrt{n_N}}{\sqrt{n_N}} \operatorname{div}(n_N\phi)dx = \int_{\mathbb{T}^d} \Delta\sqrt{n_N}\big(2\nabla\sqrt{n_N}\cdot\phi + \sqrt{n_N}\operatorname{div}\phi\big)dx$$

equals, for sufficiently smooth test functions,

$$\int_{\mathbb{T}^d} \Delta\sqrt{n}\big(2\nabla\sqrt{n}\cdot\phi + \sqrt{n}\operatorname{div}\phi\big)dx.$$

We have shown that (n, nw) solves $n_t + \operatorname{div}(nw) = \nu\Delta n$ pointwise in $\mathbb{T}^d \times (0,T)$ and, for all test functions ϕ such that the integrals are defined,

$$\begin{aligned} -\int_{\mathbb{T}^d} n_0 w_0 \cdot \phi(\cdot,0)dx &= \int_0^T \int_{\mathbb{T}^d} \Big(nw\cdot\phi_t + nw\otimes w : \nabla\phi + p(n)\operatorname{div}\phi + nf\cdot\phi \qquad (5.1) \\ &\quad - 2\varepsilon_0^2 \Delta\sqrt{n}(2\nabla\sqrt{n}\cdot\phi + \sqrt{n}\operatorname{div}\phi) - \big(\nu\nabla(nw) + \delta\nabla w\big) : \nabla\phi - \delta w\cdot\phi\Big)dxdt. \end{aligned}$$

6. The limit $\delta \to 0$. Let (n_δ, w_δ) be a solution to (3.2) and (5.1), with the regularity proved in the previous section. By employing the test function $n_\delta\phi$ in (5.1) (which is possible as long as the integrals are well defined), we obtain, according to Lemma 2.2,

$$\begin{aligned} -\int_{\mathbb{T}^d} n_0^2 w_0\cdot\phi(\cdot,0)dx &= \int_0^T\int_{\mathbb{T}^d}\Big(n_\delta^2 w_\delta\cdot\phi_t - n_\delta^2\operatorname{div}(w_\delta)w_\delta\cdot\phi - \nu(n_\delta w_\delta\otimes\nabla n_\delta) : \nabla\phi \\ &\quad + n_\delta w_\delta \otimes n_\delta w_\delta : \nabla\phi + \frac{\gamma}{\gamma+1} n_\delta^{\gamma+1}\operatorname{div}\phi + n_\delta^2 f\cdot\phi \\ &\quad - 2\varepsilon_0^2\Delta\sqrt{n_\delta}\big(2\sqrt{n}\nabla n\cdot\phi + n_\delta^{3/2}\operatorname{div}\phi\big) - \nu\nabla(n_\delta w_\delta) : \big(n_\delta\nabla\phi + 2\nabla n_\delta\otimes\phi\big) \\ &\quad - \delta\nabla w_\delta : \big(n_\delta\nabla\phi + \nabla n_\delta\otimes\phi\big) - \delta n_\delta w_\delta\cdot\phi\Big)dxdt. \end{aligned} \tag{6.1}$$

The Aubin lemma and the regularity results from section 4 allow us to extract subsequences (not relabeled) such that as $\delta \to 0$, for some functions n and j,

$$n_\delta \to n \quad \text{strongly in } L^2(0,T;W^{1,p}(\mathbb{T}^d)),\ 3 < p < 6\gamma/(\gamma+3), \tag{6.2}$$

$$n_\delta w_\delta \to j \quad \text{strongly in } L^2(0,T;L^q(\mathbb{T}^d)),\ 1 \le q < 3, \tag{6.3}$$

$$\sqrt{n_\delta} \to \sqrt{n} \quad \text{strongly in } L^\infty(0,T;L^r(\mathbb{T}^d)),\ 1 \le r < 6. \tag{6.4}$$

Estimate (4.3) and Fatou's lemma yield

$$\int_{\mathbb{T}^d} \liminf_{\delta\to 0} \frac{|n_\delta w_\delta|^2}{n_\delta} dx < \infty.$$

This implies that $j = 0$ in $\{n = 0\}$. Then, when we define the limit velocity $w := j/n$ in $\{n \neq 0\}$ and $w := 0$ in $\{n = 0\}$, we have $j = nw$. By (4.3), there exists a subsequence (not relabeled) such that

$$\sqrt{n_\delta} w_\delta \rightharpoonup g \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^d)) \tag{6.5}$$

for some function g. Hence, since $\sqrt{n_\delta}$ converges strongly to $\sqrt{n}$ in $L^2(0,T;L^\infty(\mathbb{T}^d))$, we infer that $n_\delta w_\delta = \sqrt{n_\delta}(\sqrt{n_\delta}w_\delta)$ converges weakly to $\sqrt{n}g$ in $L^2(0,T;L^2(\mathbb{T}^d))$ and $\sqrt{n}g = nw = j$. In particular, $g = j/\sqrt{n}$ in $\{n \neq 0\}$.

Now, we are able to pass to the limit $\delta \to 0$ in the weak formulation (6.1) term by term. The strong convergences (6.2) and (6.3) imply that

$$n_\delta^2 w_\delta \to n^2 w \quad \text{strongly in } L^1(0,T;L^q(\mathbb{T}^d)),\ q < 3,$$

$$n_\delta w_\delta \otimes \nabla n_\delta \to nw \otimes \nabla n \quad \text{strongly in } L^1(0,T;L^{3/2}(\mathbb{T}^d)).$$

The strong convergence of $n_\delta w_\delta$ immediately gives

$$n_\delta w_\delta \otimes n_\delta w_\delta \to nw \otimes nw \quad \text{strongly in } L^1(0,T;L^{q/2}(\mathbb{T}^d)),\ q < 3.$$

Furthermore, we have

$$\nabla n_\delta \to \nabla n \quad \text{strongly in } L^2(0,T;L^p(\mathbb{T}^d)) \text{ by (6.2)}, p > 3,$$

$$\sqrt{n_\delta} \to \sqrt{n} \quad \text{strongly in } L^\infty(0,T;L^r(\mathbb{T}^d)) \text{ by (6.4) with } r = 2p/(p-2),$$

$$\Delta\sqrt{n_\delta} \rightharpoonup \Delta\sqrt{n} \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)) \text{ by (4.5)}.$$

It holds $r < 6$ since we have $p > 3$. This implies that

$$\Delta\sqrt{n_\delta}\sqrt{n_\delta}\nabla n_\delta \rightharpoonup \Delta\sqrt{n}\sqrt{n}\nabla n \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

Here, we need the assumption $\gamma > 3$ if $d = 3$ which allows us to obtain compactness of (n_δ) in $W^{1,p}(\mathbb{T}^d)$ with $p > 3$. This assumption is also needed in the following argument: Since $\nabla(n_\delta w_\delta)$ converges weakly in $L^2(0,T;L^{3/2}(\mathbb{T}^d))$ (see (4.7)) and and ∇n_δ converges strongly in $L^2(0,T;L^3(\mathbb{T}^d))$ (see (6.2)), we obtain

$$\nabla(n_\delta w_\delta)\cdot\nabla n_\delta \rightharpoonup \nabla(nw)\cdot\nabla n \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

The almost everywhere convergence of n_δ and the $L^{4\gamma/3+1}(0,T;L^{4\gamma/3+1}(\mathbb{T}^d))$ bound on n_δ (see (4.9)), together with the fact that $4\gamma/3+1 > \gamma+1$, proves that

$$n_\delta^{\gamma+1} \to n^{\gamma+1} \quad \text{strongly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

Using the estimate (4.4) for $\sqrt{\delta}w_\delta$, we obtain further, for smooth test functions,

$$
\begin{aligned}
\delta\int_{\mathbb{T}^d}&\nabla w_\delta : (n_\delta\nabla\phi + \nabla n_\delta\otimes\phi)dx \\
&\le \sqrt{\delta}\big\|\sqrt{\delta}\nabla w_\delta\big\|_{L^2(0,T;L^2(\mathbb{T}^d))}\big(\|n_\delta\|_{L^2(0,T;L^\infty(\mathbb{T}^d))}\|\phi\|_{L^\infty(0,T;H^1(\mathbb{T}^d))} \\
&\quad + \|n_\delta\|_{L^2(0,T;W^{1,3}(\mathbb{T}^d))}\|\phi\|_{L^\infty(0,T;L^6(\mathbb{T}^d))}\big) \to 0, \\
\delta\int_{\mathbb{T}^d}&n_\delta w_\delta\cdot\phi dx \le \delta\|n_\delta w_\delta\|_{L^2(0,T;L^3(\mathbb{T}^d))}\|\phi\|_{L^2(0,T;L^{3/2}(\mathbb{T}^d))} \to 0 \quad \text{as } \delta\to 0.
\end{aligned}
$$

It remains to show the convergence of $n_\delta^2\mathrm{div}(w_\delta)w_\delta$. To this end, we proceed similarly as in [9] and introduce the functions $G_\alpha\in C^\infty([0,\infty))$, $\alpha>0$, satisfying $G_\alpha(x)=1$ for $x\ge 2\alpha$, $G_\alpha(x)=0$ for $x\le\alpha$, and $0\le G_\alpha\le 1$. Then we can estimate the low-density part of $n_\delta^2\mathrm{div}(w_\delta)w_\delta$ by

$$
\begin{aligned}
\big\|(1-&G_\alpha(n_\delta))n_\delta^2\mathrm{div}(w_\delta)w_\delta\big\|_{L^1(0,T;L^1(\mathbb{T}^d))} \\
&\le \|(1-G_\alpha(n_\delta))\sqrt{n_\delta}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\|\sqrt{n_\delta}\mathrm{div}\, w_\delta\|_{L^2(0,T;L^2(\mathbb{T}^d))} \\
&\quad\times\|n_\delta w_\delta\|_{L^2(0,T;L^2(\mathbb{T}^d))} \\
&\le C\|(1-G_\alpha(n_\delta))\sqrt{n_\delta}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \le C\sqrt{\alpha},
\end{aligned}
\tag{6.6}
$$

where $C>0$ is independent of δ and α. We write

$$
G_\alpha(n_\delta)n_\delta\mathrm{div}\, w_\delta = \mathrm{div}(G_\alpha(n_\delta)n_\delta w_\delta) - n_\delta w_\delta\otimes\nabla n_\delta\Big(G_\alpha'(n_\delta)+\frac{G_\alpha(n_\delta)}{n_\delta}\Big). \tag{6.7}
$$

As $\delta\to 0$, the first term on the right-hand side converges strongly to $\mathrm{div}(G_\alpha(n)nw)$ in $L^1(0,T;(H^1(\mathbb{T}^d))^*)$ since $G_\alpha(n_\delta)$ converges strongly to $G_\alpha(n)$ in $L^p(0,T;L^p(\mathbb{T}^d))$ for any $p<\infty$ and $n_\delta w_\delta$ converges strongly to nw in $L^2(0,T;L^q(\mathbb{T}^d))$ for any $q<3$. In view of (6.4) and (6.5), we infer the weak* convergence $n_\delta w_\delta\rightharpoonup\sqrt{n}g=nw$ in $L^\infty(0,T;L^{2r/(r+2)}(\mathbb{T}^d))$ for all $r<6$. Thus, because of (6.2),

$$
n_\delta w_\delta\otimes\nabla n_\delta \rightharpoonup nw\otimes\nabla n \quad \text{weakly in } L^2(0,T;L^\theta(\mathbb{T}^d)),
$$

where $\theta=2pr/(2p+2r+pr)$. It is possible to choose $3<p\le 6\gamma/(\gamma+3)$ and $r<6$ such that $\theta>1$. Then, together with the strong convergence of $G_\alpha'(n_\delta)+G_\alpha(n_\delta)/n_\delta$ to $G_\alpha'(n)+G_\alpha(n)/n$ in $L^p(0,T;L^p(\mathbb{T}^d))$ for any $p<\infty$, the limit $\delta\to 0$ in (6.7) yields the identity

$$
G_\alpha(n)n\mathrm{div}\, w = \mathrm{div}(G_\alpha(n)nw) - nw\otimes\nabla n\Big(G_\alpha'(n)+\frac{G_\alpha(n)}{n}\Big)
$$

in $L^1(0,T;(H^2(\mathbb{T}^d))^*)$. Since $G_\alpha(n_\delta)n_\delta\mathrm{div}\, w_\delta$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$, we conclude that

$$
G_\alpha(n_\delta)n_\delta\mathrm{div}\, w_\delta \rightharpoonup G_\alpha(n)n\mathrm{div}\, w \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T}^d)).
$$

Moreover, in view of the strong convergence of $n_\delta w_\delta$ to nw in $L^2(0,T;L^q(\mathbb{T}^d))$ for all $q<3$, we infer that

$$
G_\alpha(n_\delta)n_\delta\mathrm{div}(w_\delta)n_\delta w_\delta \rightharpoonup G_\alpha(n)n^2\mathrm{div}(w)w \quad \text{weakly in } L^1(0,T;L^{q/2}(\mathbb{T}^d)).
$$

We write, for $\phi \in L^\infty(0,T;L^\infty(\mathbb{T}^d))$,

$$\begin{aligned}
\int_{\mathbb{T}^d} &\big(n_\delta^2 \mathrm{div}(w_\delta)w_\delta - n^2\mathrm{div}(w)w\big)\cdot\phi dx \qquad (6.8)\\
&= \int_{\mathbb{T}^d}\big(G_\alpha(n_\delta)n_\delta^2\mathrm{div}(w_\delta)w_\delta - G_\alpha(n)n^2\mathrm{div}(w)w\big)\cdot\phi dx\\
&\quad + \int_{\mathbb{T}^d}\big(G_\alpha(n) - G_\alpha(n_\delta)\big)n^2\mathrm{div}(w)w\cdot\phi dx\\
&\quad + \int_{\mathbb{T}^d}(1-G_\alpha(n_\delta))\big(n_\delta^2\mathrm{div}(w_\delta)w_\delta - n^2\mathrm{div}(w)w\big)\cdot\phi dx.
\end{aligned}$$

For fixed $\alpha > 0$, the first integral converges to zero as $\delta \to 0$. Furthermore, the last integral can be estimated by $C\sqrt{\alpha}$ uniformly in δ (see (6.6)). For the second integral, we recall that $G_\alpha(n_\delta) \to G_\alpha(n)$ strongly in $L^p(0,T;L^p(\mathbb{T}^d))$ for all $p < \infty$. Furthermore, by the Gagliardo-Nirenberg inequality, the bounds of nw in $L^2(0,T;W^{1,3/2}(\mathbb{T}^d))$ and $L^\infty(0,T;L^{3/2}(\mathbb{T}^d))$ imply that $nw \in L^{5/2}(0,T;L^{5/2}(\mathbb{T}^d))$. Thus, since $\sqrt{n}\mathrm{div}\, w \in L^2(0,T;L^2(\mathbb{T}^d))$ and $\sqrt{n} \in L^q(0,T;L^q(\mathbb{T}^d))$ with $q = 8\gamma/3+2$ (see (4.9)),

$$n^2\mathrm{div}(w)w = \sqrt{n}(\sqrt{n}\mathrm{div}\, w)nw \in L^r(0,T;L^r(\mathbb{T}^d)), \quad r = \frac{18\gamma+21}{20\gamma+15} > 1.$$

As a consequence, the second integral converges to zero as $\delta \to 0$. Thus, in the limit $\delta \to 0$, (6.8) can be made arbitrarily small and hence,

$$n_\delta^2\mathrm{div}(w_\delta)w_\delta \rightharpoonup n^2\mathrm{div}(w)w \quad \text{weakly in } L^1(0,T;L^1(\mathbb{T}^d)).$$

We have proved that (n,w) solves (1.6) and (1.13) for smooth initial data. Let (n_0,w_0) be some finite-energy initial data, i.e. $n_0 \ge 0$ and $E_{\varepsilon_0}(n_0,w_0) < \infty$, and let (n_0^δ, w_0^δ) be smooth approximations satisfying $n_0^\delta \ge \delta > 0$ in $\mathbb{T}^d$ and $\sqrt{n_0^\delta} \to \sqrt{n_0}$ strongly in $H^1(\mathbb{T}^d)$, $\sqrt{n_0^\delta}w_0^\delta \to \sqrt{n_0}w_0$ strongly in $L^2(\mathbb{T}^d)$ as $\delta \to 0$. In particular, $\sqrt{n_0^\delta} \to \sqrt{n_0}$ strongly in $L^6(\mathbb{T}^d)$ and therefore, $n_0^\delta w_0^\delta \to n_0 w_0$ strongly in $L^{3/2}(\mathbb{T}^d)$. By the above proof, there exists a weak solution (n_δ, w_δ) to (1.6)-(1.8) with initial data (n_0^δ, w_0^δ) satisfying all the above bounds. In particular, $(n_\delta, n_\delta w_\delta)$ converges strongly in some spaces to (n, nw) as $\delta \to 0$ and there exist uniform bounds for (n_δ) in $H^1(0,T;L^{3/2}(\mathbb{T}^d))$ and for $(n_\delta w_\delta)$ in $W^{1,4/3}(0,T;(H^s(\mathbb{T}^d))^*)$. Thus, up to subsequences, as $\delta \to 0$,

$$\begin{aligned}
n_0^\delta = n_\delta(\cdot,0) &\rightharpoonup n(\cdot,0) \quad \text{weakly in } L^{3/2}(\mathbb{T}^d),\\
n_0^\delta w_0^\delta = (n_\delta w_\delta)(\cdot,0) &\rightharpoonup (nw)(\cdot,0) \quad \text{weakly in } (H^s(\mathbb{T}^d))^*.
\end{aligned}$$

This shows that $n(\cdot,0) = n_0$ and $(nw)(\cdot,0) = n_0 w_0$ in the sense of distributions. We conclude the proof of Theorem 1.1. Corollary 1.2 follows from this theorem after setting $\varepsilon_0^2 = \varepsilon^2 - \nu^2 > 0$ and $u = w - \nu\nabla\log n$.

Remark 6.1 (Momentum relaxation term). The above proof also works when we include the relaxation term $-nu/\tau$ to the right-hand side of (1.2). In the viscous quantum Euler model, this term becomes $-(nw-\nu\nabla n)/\tau$. The existence proof for the approximate system in section 3 does not change, see e.g. [19] for the one-dimensional situation. Now, the convergence results of this section imply that $n_\delta w_\delta - \nu\nabla n_\delta$ converges strongly to $nw - \nu\nabla n$ in $L^2(0,T;L^2(\mathbb{T}^d))$.

Remark 6.2 (Positivity of the particle density). We have shown that the particle density is nonnegative. This does not exclude vacuum regions $\{n = 0\}$. Notice that the maximum principle can be applied to (1.6) only if the velocity is regular, e.g. $\operatorname{div} w \in L^\infty$. In the literature, there are only few results concerning *positive* densities in fluid models. For instance, in the Brenner-Navier-Stokes model with constant viscosity, Feireisl and Vasseur [17] proved that the density is positive except on a set of Lebesgue measure zero. Furthermore, in [35] it is shown that the solution of the one-dimensional stationary viscous quantum Euler model admits strictly positive particle densities.

Remark 6.3 (Boundary conditions). Without the third-order quantum term, it is possible to treat Dirichlet or no-slip boundary conditions for the velocity u [16]. Moreover, according to [8], energy estimates for the new energy $E_{\varepsilon_0}(n, w)$ can be derived if the boundary condition $\nabla n \times \vec{\nu} = 0$ is imposed, where $\vec{\nu}$ denotes the exterior unit normal on the boundary. However, the situation is less clear concerning the choice of boundary conditions for the particle density in quantum fluid models. In fact, many authors impose periodic boundary conditions [6, 9, 11, 19, 26, 38], insulating boundary conditions [13], or they consider the whole-space problem [39]. Boundary conditions satisfying the Shapiro-Lopatinskii criterion have been examined in [12]. Furthermore, in [32, 35] Dirichlet-type conditions have been employed in the analyzed, but only for the (simpler) one-dimensional equations.

Appendix. We prove the following result which is used in Lemma 4.2: Let u be a smooth positive function on $\mathbb{T}^d$ ($d \geq 1$). Then

$$\int_{\mathbb{T}^d} u^2 |\nabla^2 \log u|^2 dx \geq \frac{16(4d-1)}{(d+2)^2} \int_{\mathbb{T}^d} |\nabla \sqrt{u}|^4 dx.$$

Proof. The proof is inspired by the extension of the entropy construction method introduced in [34]. The main idea is to formalize the integrations by parts. The case $d = 1$ is a consequence of the results of [33]; therefore we assume that $d > 1$. To simplify the computations, we introduce as in [34] the functions

$$\theta = \frac{|\nabla u|}{u}, \quad \lambda = \frac{1}{d}\frac{\Delta u}{u}, \quad (\lambda + \mu)\theta^2 = \frac{1}{u^3} \nabla u^\top \nabla^2 u \nabla u,$$

and $\rho > 0$ by

$$|\nabla^2 u|^2 = \Big(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2\Big)u^2.$$

It is shown in [34] that ρ is well defined. A computation shows that

$$\begin{aligned} J &= \int_{\mathbb{T}^d} u^2 \Big(\frac{|\nabla^2 u|^2}{u^2} - 2\frac{1}{u^3}\nabla^\top \nabla^2 u \nabla u + \frac{|\nabla u|^4}{u^4}\Big) dx \\ &= \int_{\mathbb{T}^d} u^2 \Big(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2 - 2(\lambda + \mu)\theta^2 + \theta^4 \Big) dx. \end{aligned}$$

This integral is compared to

$$K = 16 \int_{\mathbb{T}^d} |\nabla \sqrt{u}|^4 dx = \int_{\mathbb{T}^d} u^2 \theta^4 dx,$$

i.e., we wish to determine a constant $c_0 > 0$ such that $J - c_0 K \geq 0$ for all (positive smooth) functions u. We perform integration by parts in $J - c_0 K$ by adding a linear combination of the "dummy" integrals

$$J_1 = \int_{\mathbb{T}^d} \operatorname{div}\big((\nabla^2 u - \Delta u \mathbb{I})\nabla u\big)dx = \int_{\mathbb{T}^d} u^2\Big(- d(d-1)\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2\Big)dx = 0,$$
$$J_2 = \int_{\mathbb{T}^d} \operatorname{div}(u^{-1}|\nabla u|^2\nabla u)dx = \int_{\mathbb{T}^d} u^2\big((d+2)\lambda\theta^2 + 2\mu\theta^2 - \theta^4\big)dx = 0,$$

where $\mathbb{I}$ is the unit matrix in $\mathbb{R}^{d\times d}$. The integrals vanish in view of the periodic boundary conditions. The goal is to find constants $c_0 > 0$, $c_1 \in \mathbb{R}$, and $c_2 \in \mathbb{R}$ such that $I := J - c_0K = J - c_0K + c_1J_1 + c_2J_2 \geq 0$. We obtain

$$I = \int_{\mathbb{T}^d} u^2\Big(d(1-c_1(d-1))\lambda^2 + \frac{d}{d-1}(1+c_1)\mu^2 + c_1\rho^2 + (-2+c_2(d+2))\lambda\theta^2 + 2(c_2-1)\mu\theta^2 + (1-c_0-c_2)\theta^4\Big)dx.$$

The choice $c_1 = 1/(d-1) > 0$ and $c_2 = 2/(d+2)$ eliminates the terms involving λ and leads to

$$I \geq \int_{\mathbb{T}^d} u^2\big(a_1\mu^2 + 2a_2\mu\theta^2 + a_3\theta^4\big)dx,$$

where $a_1 = d^2/(d-1)^2$, $a_2 = -d/(d+2)$, and $a_3 = d/(d+2) - c_0$. This integral is nonnegative if the integrand is nonnegative pointwise. This is the case if and only if $a_1 > 0$ and $a_1a_3 - a_2^2 \geq 0$ which is equivalent to $c_0 \leq (4d-1)/(d+2)^2$. $\square$

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