MIXED ENTROPY ESTIMATES FOR THE POROUS-MEDIUM EQUATION WITH CONVECTION

ANSGAR JÜNGEL AND INGRID VIOLET

Abstract. In this paper, we answer the question under which conditions the porous-medium equation with convection and with periodic boundary conditions possesses gradient-type Lyapunov functionals (first-order entropies). It is shown that the weighted sum of first-order and zeroth-order entropies are Lyapunov functionals if the weight for the zeroth-order entropy is sufficiently large, depending on the strength of the convection. This provides new a priori estimates for the convective porous-medium equation. The proof is based on an extension of the algorithmic entropy construction method which is based on systematic integration by parts, formulated as a polynomial decision problem.

1. Introduction

In recent years, entropy–entropy dissipation methods proved to be very successful tools for the understanding of the structure of nonlinear diffusive equations. In particular, the methods allow for the derivation of a priori estimates and the study of the qualitative behavior of the solutions. In this paper, we apply and extend entropy methods to derive new a priori estimates for the porous-medium (and fast-diffusion) equation with convection:

$$u_t = \operatorname{div}(u^\alpha \nabla u + q u^{\beta+1}), \quad t > 0, \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{T}^d, \tag{1}$$

where $\alpha \geq -1$, $\beta > -1$, $q \in \mathbb{R}^d$, and $\mathbb{T}^d$ is the d-dimensional torus with unit volume. This equation with $\alpha > 0$ arises in a model for the flow of moisture in a porous material under the influence of gravity, where u denotes the moisture content, and it is also used to describe nonlinear heat transfer or groundwater flow [18].

The existence and uniqueness of solutions to the Cauchy problem for (1) with $\alpha > 1$ and $\beta \geq \alpha/2$ was shown by Gilding and Peletier in [11]. Díaz and Kersner [4] and Gilbert [10] developed an existence theory for more general equations of the type $u_t = a(u)_{xx} + b(u)_x$, including the functions $a(u) = u^{\alpha+1}$ and $b(u) = u^{\beta+1}$ with $\alpha \geq 0$ and $\beta > -1$. Watanabe proved the existence and uniqueness of solutions to the multi-dimensional version $u_t = \operatorname{div}(\nabla a(u) + b(u))$, including $a(u) = u^{\alpha+1}$ and $b(u) = u^{\beta+1}$ with $\beta \geq \alpha$ [19]. Several authors

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analyzed the qualitative behavior of the solutions to (1). The influence of the gravity term on the free boundary between the regions in which $u = 0$ and $u > 0$ is investigated in [9]. In [1, 3, 6, 15, 17], the long-time behavior for the solutions in the whole space was studied, and it was proved that the solutions behave, as $t \to \infty$, like certain similarity solutions. Traveling wave solutions were shown to exist in [5]. Symmetry reductions and exact solutions to (1) were derived in [8, 16]. Finally, the work [2] is concerned with a priori pointwise estimates.

In this paper, we answer the question under which conditions functionals of the type

$$E_0(t) = \int_{\mathbb{T}^d} \big(u^\gamma - \bar{u}^\gamma\big)dx, \quad \gamma > 1, \tag{2}$$

where $\bar{u} = \int_{\mathbb{T}^d} udx$, and

$$E_1(t) = \int_{\mathbb{T}^d} |\nabla u(x,t)^{\gamma/2}|^2 dx, \quad \gamma > 1, \tag{3}$$

allow for so-called entropy–entropy dissipation estimates. We call the inequality

$$\frac{dE}{dt} + P \le 0$$

an *entropy–entropy dissipation inequality* if $E(t)$ is a Lyapunov functional and if $P \ge 0$ provides estimates on the derivative of the solution u. In this situation, we refer to E as an *entropy* and to P as an *entropy dissipation* (or entropy production). An entropy of the type (3) is called to be of first order since it contains first-order derivatives of u, whereas entropy functionals of the form (2) are termed zeroth-order entropies. Entropy–entropy dissipation inequalities are very useful for the qualitative behavior of solutions such as their regularity and long-time behavior.

Entropy estimates are usually derived by suitable integrations by parts. An algorithmic entropy construction method was proposed in [12] which is based on systematic integration by parts. More precisely, all possible integrations by parts are formulated as manipulations of certain polynomials, and the task of finding entropy estimates can be reduced to prove the nonnegativity of a polynomial involving some coefficients coming from the integrations by parts. The problem to show the nonnegativity of the polynomial is formulated as a decision problem which can be solved, in principle, in an algorithmic way. This technique was applied in [12] to the one-dimensional porous-medium equation

$$u_t = (u^\alpha u_x)_x \quad \text{in } \mathbb{T}, \tag{4}$$

and it turned out that for smooth solutions and for all $\gamma > 1$, functionals (2) are zeroth-order entropies with

$$\frac{dE_0}{dt} + \frac{4\gamma(\gamma-1)}{(\alpha+\gamma)^2} \int_{\mathbb{T}^d} \big|(u^{(\alpha+\gamma)/2})_x\big|^2 dx = 0.$$

Furthermore, it is not difficult to see that the functionals (3) are first-order entropies for (4) if

$$0 < \gamma - 2\alpha < 3. \tag{5}$$

Actually, the boundary values $\gamma - 2\alpha = 0$ and $\gamma - 2\alpha = 3$ also allow for entropies, i.e. $dE_1/dt \leq 0$ but no information on the entropy dissipation is available in these cases; see [12] for details. We show in Remark 6 that there are no first-order entropies (3) for parameters outside of the interval $0 \leq \gamma - 2\alpha \leq 3$.

The entropy construction method cannot be immediately applied to the convective part of (1) since the technique was developed for homogeneous equations only. By standard integration by parts, it is easy to check that zeroth-order entropies (2) are entropies for all $\gamma > 1$ even in the presence of convection. On the other hand, it is less clear for which parameter range of γ, the first-order entropies (3) allow for entropy–entropy dissipation estimates. In this paper, we answer this question by extending the entropy construction method of [12] to the inhomogeneous equation (1). Our answer is based on two ideas.

The first idea is to consider, instead of the first-order entropies (3), the *mixed-order entropies*

$$E(t) = \frac{1}{2}\int_{\mathbb{T}^d} \big(|\nabla u(x,t)^{\gamma/2}|^2 + c(u(x,t)^{\gamma+2(\beta-\alpha)} - \bar{u}^{\gamma+2(\beta-\alpha)})\big)dx.$$

For a nonnegative constant c, Jensen's inequality shows that $E(t) \geq 0$. It turns out that, for suitable (α, β, γ), this functional is an entropy for (1) if the constant $c > 0$ is chosen sufficiently large. This means that the sum of an entropy and a functional, which may be *not* a Lyapunov functional, is an entropy. This fact (but with different entropies) was exploited in [7] for a population dynamics cross-diffusion model to prove the numerical convergence of semi-discrete solutions.

Our second idea concerns the treatment of the inhomogeneities in (1). In the entropy construction method of [12], the derivatives of u are identified with polynomial variables. For instance, in the one-dimensional case, the variables are $\xi_1 = u_x/u$, $\xi_2 = u_{xx}/u$ etc. Clearly, this is only possible if all powers of u are of the same order, $\alpha = \beta$. We introduce the additional variables $\eta_1 = u^{\beta-\alpha-1}u_x$ and $\eta_2 = u^{2(\beta-\alpha)-1}u_{xx}$ and work with polynomials in the variables ξ_1, ξ_2, η_1, η_2 etc. In the multi-dimensional situation, we employ the gradient, Laplacian, and Hesse matrix as "vector" variables (as in [13]). We need to show that a certain polynomial in these variables is nonnegative. Due to the inhomogeneities, the reductions of [12] do not work. We solve the problem by writing the polynomial under question as a *sum of squares*, mixing the variables ξ_i and η_j.

The originality of the paper consists of two facts. First, we prove new estimates for the porous-medium equation with convection (1) for smooth positive solutions. These estimates are the first step of a study of the qualitative behavior of the solutions. We refer to Remark 4 for some comments about an extension of the results to weak nonnegative solutions. Second, we extend the entropy construction method of [12] to an *inhomogeneous* equation. By this extension, the entropy method may be applied to a much larger class of equations than originally described in [12].

Our main results are as follows.

Theorem 1 (Multi-dimensional case)**.** *Let $d > 1$ and let u be a smooth positive solution to* (1) *with $q \in \mathbb{R}^d$. Let $A_d \in \mathbb{R}^2$ be the open domain bounded by the four line segments*

$$\alpha = \frac{1}{4}\Big(- d \pm \sqrt{d(d+8)}\Big)(\gamma - 3) \quad \text{and} \quad \alpha = \frac{1}{4d}\Big(d(\gamma-6) \pm \sqrt{d(d+8)}(\gamma-2)\Big)$$

(see Figure 1). Furthermore, let $(\alpha,\gamma) \in A_d$ and let $\beta > -1$ satisfy

$$\gamma - 2\alpha < -2\beta \quad \text{or} \quad \gamma - 2\alpha > -2\beta + 1. \tag{6}$$

Then there exist constants $\mu > 0$ (depending on α, β, γ, and d) and $c^* > 0$ (depending on α, β, γ, d, and μ) such that for all $c \geq c^*|q|^2$, it holds

$$\frac{dE}{dt} + \mu\Big(\frac{\gamma}{\alpha+\gamma}\Big)^2 \int_{\mathbb{T}^d} \big|\Delta(u^{(\alpha+\gamma)/2})\big|^2 dx \leq 0.$$

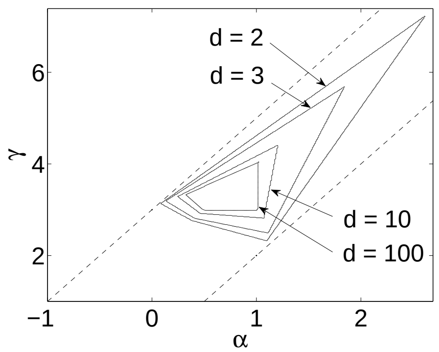


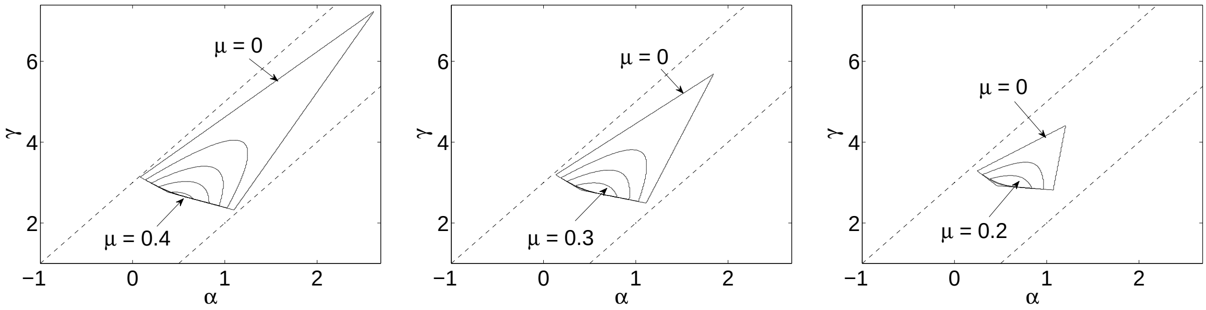
FIGURE 1. Case $d > 1$. The sets illustrate A_d in the convection-dominated case $\delta = \beta - \alpha \geq 0$. The dashed lines $\gamma - 2\alpha = 0$ and $\gamma - 2\alpha = 3$ are the boundaries of the set of admissible parameters in the one-dimensional case (see below).

The parameter μ may play an important role for the long-time behavior of the solutions since it measures the decay rate. In this paper, however, we are only concerned with a priori estimates and we do not prove any time decay of the solutions.

Depending of the choice of α, β, γ, and d, the parameters μ and c^* can be explicitly computed. Since the formulas are quite involved (see the proof below), we only illustrate the sets of all (α,γ) in the convection-dominated case $\beta \geq \alpha$ (in which (6) is satisfied) for various values of μ in Figure 2. The set of admissible (α,γ) becomes smaller if the dissipation constant μ increases. There is an upper limit for μ, i.e., for μ larger than this bound, there are no values (α,γ) for which the entropy inequality holds.

In the one-dimensional case, we can make c^* explicit.

Theorem 2 (One-dimensional case)**.** *Let $d = 1$ and let u be a smooth positive solution to* (1) *with $q \in \mathbb{R}$. Let $\alpha \geq -1$ and $\gamma > 1$ be such that $0 < \gamma - 2\alpha < 3$, and let $\beta > -1$ satisfy*



(a) $d = 2$ (b) $d = 3$ (c) $d = 10$

FIGURE 2. Case $d > 1$. Sets of admissible (α, γ) for $\mu = 0, 0.1, 0.2, \ldots$ in the convection-dominated case. The dashed lines $\gamma - 2\alpha = 0$ and $\gamma - 2\alpha = 3$ are the boundaries of the set of admissible parameters in the one-dimensional case (see below).

(6). *Finally, choose* $c \geq c^*|q|^2$, *where*

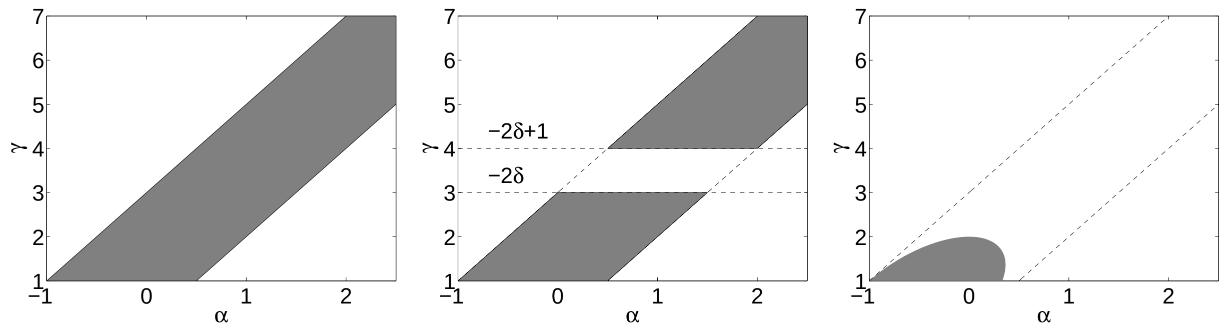
$$c^* = \frac{9\beta^2(\beta+1)^2\gamma^2}{-16(\gamma+2\beta-2\alpha)(\gamma+2\beta-2\alpha-1)(\gamma-2\alpha)(\gamma-2\alpha-3)} > 0.$$

Then there exists $0 < \mu \leq 1$ *such that*

$$\frac{dE}{dt} + \mu\Big(\frac{\gamma}{\alpha+\gamma}\Big)^2 \int_{\mathbb{T}} \big|\big(u^{(\alpha+\gamma)/2}\big)_{xx}\big|^2 dx \leq 0. \tag{7}$$

The region of admissible (α, γ) is illustrated in Figure 3 (a) and (b) for cases in which the difference $\delta = \beta - \alpha$ is given; $\delta > 0$ corresponds to the convection-dominated case. The choice of μ can also be made precise; see Proposition 5 for details. For instance, Figure 3 (c) illustrates the region of admissible (α, γ) for which $\mu = 1$. For (α, γ) outside of this region, it holds $\mu < 1$.

The paper is organized as follows. In the next section, we explain the systematic integration by parts and derive the polynomial formulation. Section 3 is concerned with multi-dimensional entropy estimates, whereas the one-dimensional situation is considered in Section 4.



(a) $\delta = \beta - \alpha \geq 0$ (b) $\delta = \beta - \alpha < 0$ (c) $\mu = 1$

FIGURE 3. Case $d = 1$. (a)–(b) The gray region illustrates admissible pairs (α, γ) with $0 < \gamma - 2\alpha < 3$ for given $\delta = \beta - \alpha$. (c) The gray region illustrates pairs (α, γ) for which $\mu = 1$ can be chosen.

2. Formulation as a polynomial decision problem

Let u be a smooth positive solution to (1). To simplify the notation, we set $\delta = \beta - \alpha$. A computation shows that the derivative of E can be written as

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2}\int_{\mathbb{T}^d} \big(-\gamma u^{\gamma/2-1}\Delta(u^{\gamma/2}) + c(\gamma+2\delta)u^{\gamma+2\delta-1}\big)\operatorname{div}(u^\alpha \nabla u + q u^{\beta+1})dx \\ &= -\frac{\gamma^2}{4}\int_{\mathbb{T}^d} u^{\alpha+\gamma}\Big[\Big(\frac{\Delta u}{u}\Big)^2 + \Big(\alpha + \frac{\gamma}{2} - 1\Big)\Big|\frac{\nabla u}{u}\Big|^2\frac{\Delta u}{u} + \alpha\Big(\frac{\gamma}{2}-1\Big)\Big|\frac{\nabla u}{u}\Big|^4 \\ &\quad + (\beta+1)u^\delta q\cdot\frac{\nabla u}{u}\frac{\Delta u}{u} + (\beta+1)\Big(\frac{\gamma}{2}-1\Big)u^\delta q\cdot\frac{\nabla u}{u}\Big|\frac{\nabla u}{u}\Big|^2 \\ &\quad - \frac{2c}{\gamma^2}(\gamma+2\delta)u^{2\delta}\frac{\Delta u}{u} - \frac{2c}{\gamma^2}\alpha(\gamma+2\delta)\Big|u^\delta\frac{\nabla u}{u}\Big|^2 - \frac{2c}{\gamma^2}(\beta+1)(\gamma+2\delta)u^{3\delta}q\cdot\frac{\nabla u}{u}\Big]dx. \end{aligned} \tag{8}$$

Notice that the last term vanishes since, in view of the periodic boundary conditions,

$$(\alpha+\gamma+3\delta)\int_{\mathbb{T}^d} u^{\alpha+\gamma+3\delta}q\cdot\frac{\nabla u}{u}dx = \int_{\mathbb{T}^d}\operatorname{div}\big(u^{\alpha+\gamma+3\delta}q\big)dx = 0.$$

The entropy dissipation can be written as

$$\begin{aligned} J &= \Big(\frac{\gamma}{\alpha+\gamma}\Big)^2\int_{\mathbb{T}^d}\big(\Delta u^{(\alpha+\gamma)/2}\big)^2dx \\ &= \frac{\gamma^2}{4}\int_{\mathbb{T}^d} u^{\alpha+\gamma}\Big[\Big(\frac{\Delta u}{u}\Big)^2 + \frac{1}{4}(\alpha+\gamma-2)^2\Big|\frac{\nabla u}{u}\Big|^4 + (\alpha+\gamma-2)\Big|\frac{\nabla u}{u}\Big|^2\frac{\Delta u}{u}\Big]dx. \end{aligned} \tag{9}$$

We wish to show that $-dE/dt-\mu J$ is nonnegative for some $\mu\geq 0$ by employing systematic integration by parts. To this end, we employ the following integration-by-parts formulas:

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}^d}\operatorname{div}\Big(u^{\alpha+\gamma+2\delta}\frac{\nabla u}{u}\Big)dx = \int_{\mathbb{T}^d}u^{\alpha+\gamma}\Big(u^{2\delta}\frac{\Delta u}{u}+(\alpha+\gamma+2\delta-1)\Big|u^\delta\frac{\nabla u}{u}\Big|^2\Big)dx,\\
I_2 &= \int_{\mathbb{T}^d}\operatorname{div}\Big(u^{\alpha+\gamma+\delta}q\Big|\frac{\nabla u}{u}\Big|^2\Big)dx\\
&= \int_{\mathbb{T}^d}u^{\alpha+\gamma}\Big(2u^\delta q^\top\frac{\nabla^2 u}{u}\frac{\nabla u}{u}+(\alpha+\gamma+\delta-2)u^\delta q\cdot\frac{\nabla u}{u}\Big|\frac{\nabla u}{u}\Big|^2\Big)dx,\\
I_3 &= \int_{\mathbb{T}^d}\operatorname{div}\Big(u^{\alpha+\gamma+\delta}q\cdot\frac{\nabla u}{u}\frac{\nabla u}{u}\Big)dx\\
&= \int_{\mathbb{T}^d}u^{\alpha+\gamma}\Big(u^\delta q\cdot\frac{\nabla u}{u}\frac{\Delta u}{u}+u^\delta q^\top\frac{\nabla^2 u}{u}\frac{\nabla u}{u}+(\alpha+\gamma+\delta-2)u^\delta q\cdot\frac{\nabla u}{u}\Big|\frac{\nabla u}{u}\Big|^2\Big)dx,\\
I_4 &= \int_{\mathbb{T}^d}\operatorname{div}\Big(u^{\alpha+\gamma}\Big|\frac{\nabla u}{u}\Big|^2\frac{\nabla u}{u}\Big)dx\\
&= \int_{\mathbb{T}^d}u^{\alpha+\gamma}\Big(\Big|\frac{\nabla u}{u}\Big|^2\frac{\Delta u}{u}+(\alpha+\gamma-3)\Big|\frac{\nabla u}{u}\Big|^4+2\frac{\nabla u^\top}{u}\frac{\nabla^2 u}{u}\frac{\nabla u}{u}\Big)dx,\\
I_5 &= \int_{\mathbb{T}^d}\operatorname{div}\Big(u^{\alpha+\gamma}\Big(\frac{\nabla^2 u}{u}-\frac{\Delta u}{u}\mathbb{I}\Big)\frac{\nabla u}{u}\Big)dx\\
&= \int_{\mathbb{T}^d}u^{\alpha+\gamma}\Big((\alpha+\gamma-3)\Big(\frac{\nabla u^\top}{u}\frac{\nabla^2 u}{u}\frac{\nabla u}{u}-\Big|\frac{\nabla u}{u}\Big|^2\frac{\Delta u}{u}\Big)+\Big\|\frac{\nabla^2 u}{u}\Big\|^2-\Big(\frac{\Delta u}{u}\Big)^2\Big)dx,
\end{aligned}
$$

where $\mathbb{I}$ is the identity matrix, $\nabla^2 u$ the Hessian of u, and $\|\nabla^2 u\|$ the Euclidian norm of the Hessian. Clearly, in view of the periodic boundary conditions, $I_j=0$ for all $j=1,\ldots,5$.

The idea of systematic integration by parts is to formulate the integrands of the above integrals as polynomials by translating the derivatives to polynomial variables:

$$
\xi_G=\frac{\nabla u}{u},\quad \xi_L=\frac{\Delta u}{u},\quad \xi_H=\frac{\nabla^2 u}{u},\quad \eta_G=u^\delta\frac{\nabla u}{u},\quad \eta_L=u^{2\delta}\frac{\Delta u}{u}.
$$

The new idea here is to introduce variables with mixed orders in powers of u. Then the integrand of $-dE/dt$ (up to the factor $\gamma^2 u^{\alpha+\gamma}/4$) translates to

$$
\begin{aligned}
S_0(\zeta) &= \xi_L^2+\Big(\alpha+\frac{\gamma}{2}-1\Big)|\xi_G|^2\xi_L+\alpha\Big(\frac{\gamma}{2}-1\Big)|\xi_G|^4+(\beta+1)q\cdot\eta_G\xi_L\\
&\quad+(\beta+1)\Big(\frac{\gamma}{2}-1\Big)q\cdot\eta_G|\xi_G|^2-\frac{2c}{\gamma^2}(\gamma+2\delta)\eta_L-\frac{2c}{\gamma^2}\alpha(\gamma+2\delta)|\eta_G|^2,
\end{aligned}
\tag{10}
$$

where $\zeta=(\xi_G,\xi_L,\xi_H,\eta_G,\eta_L)$; the integrand of the entropy dissipation (9) translates to

$$
D(\zeta)=\xi_L^2+\frac14(\alpha+\gamma-2)^2|\xi_G|^4+(\alpha+\gamma-2)|\xi_G|^2\xi_L;
$$

and the integrands of I_j become (up to the factor $\gamma^2 u^{\alpha+\gamma}/4$)

$$\begin{aligned}
T_1(\zeta) &= \eta_L + (\alpha+\gamma+2\delta-1)|\eta_G|^2, \\
T_2(\zeta) &= 2q^\top \xi_H \eta_G + (\alpha+\gamma+\delta-2) q\cdot\eta_G |\xi_G|^2, \\
T_3(\zeta) &= q\cdot\eta_G\xi_L + q^\top\xi_H\eta_G + (\alpha+\gamma+\delta-2) q\cdot\eta_G|\xi_G|^2, \\
T_4(\zeta) &= |\xi_G|^2\xi_L + (\alpha+\gamma-3)|\xi_G|^4 + 2\xi_G^\top\xi_H\xi_G, \\
T_5(\zeta) &= (\alpha+\gamma-3)\xi_G^\top\xi_H\xi_G - (\alpha+\gamma-3)|\xi_G|^2\xi_L + \|\xi_H\|^2 - \xi_L^2.
\end{aligned}$$

The inequality $-dE/dt - \mu J$ is nonnegative if there are constants $c_1,\ldots,c_5 \in \mathbb{R}$ such that $S_1(\zeta) = (S_0 - \mu D + c_1 T_1 + \cdots + c_5 T_5)(\zeta) \geq 0$ for all ζ. We wish to eliminate some terms, in particular the first-order expressions in $q\cdot\eta_G$, η_L, and ξ_H. (We retain the third-order expression $\eta_G|\xi_G|^2$ since this term can be estimated by $|\eta_G|^2$ and $|\xi_G|^4$. The expression $q\cdot\eta_G\xi_L$ could be estimated by $|\eta_G|^2$ and ξ_L^2 but we prefer to eliminate this term.) We choose

$$\begin{aligned}
c_1 &= 2c(\gamma+2\delta)/\gamma^2 && \text{to eliminate } \eta_L, \\
c_3 &= -(\beta+1) && \text{to eliminate } q\cdot\eta_G\xi_L, \\
c_2 &= -c_3/2 = (\beta+1)/2 && \text{to eliminate } q^\top\xi_H\eta_G, \\
c_4 &= -(\alpha+\gamma-3)c_5/2 && \text{to eliminate } \xi_G^\top\xi_H\xi_G.
\end{aligned}$$

This gives the polynomial

$$S_1(\zeta) = (1-\kappa-\mu)\xi_L^2 + a_2|\xi_G|^2\xi_L + a_3|\xi_G|^4 + a_4 q\cdot\eta_G|\xi_G|^2 + a_5|\eta_G|^2 + \kappa\|\xi_H\|^2,$$

where we have set $\kappa = c_5$ and

$$\begin{aligned}
a_2 &= \alpha + \frac{\gamma}{2} - 1 - \frac{3}{2}(\alpha+\gamma-3)\kappa - \mu(\alpha+\gamma-2), \\
a_3 &= \alpha\Big(\frac{\gamma}{2}-1\Big) - \frac{1}{2}(\alpha+\gamma-3)^2\kappa - \frac{\mu}{4}(\alpha+\gamma-2)^2, \\
a_4 &= -\frac{1}{2}\beta(\beta+1), \quad a_5 = \frac{2c}{\gamma^2}(\gamma+2\delta)(\gamma+2\delta-1).
\end{aligned}$$

There are two difficulties to prove the nonnegativity of S. First, due to the zeroth-order entropy, the homogeneity in the order of the derivatives is broken which makes it impossible to employ the techniques of [12] directly. Second, it is not clear how to estimate the norm of the Hessian variable ξ_H. For the second problem, an easy choice would be to choose $\kappa \geq 0$ and to neglect the term $\kappa\|\xi_H\|^2$ since it is nonnegative. However, by this choice, we neglect a possibly useful positive contribution. We employ the following elementary estimate:

$$\|\xi_H\|^2 = \Big\|\frac{\nabla^2 u}{u}\Big\|^2 \geq \frac{1}{d}\Big(\frac{\Delta u}{u}\Big)^2 = \frac{1}{d}\xi_L^2.$$

Then we can write, if κ is nonnegative,

$$(1-\kappa)\xi_L^2 + \kappa\|\xi_H\|^2 \geq \Big(1 - \frac{d-1}{d}\kappa\Big)\xi_L^2$$

and obtain

$$S_1(\zeta) \geq S(\zeta) := a_1\xi_L^2 + a_2|\xi_G|^2\xi_L + a_3|\xi_G|^4 + a_4 q\cdot\eta_G|\xi_G|^2 + a_5|\eta_G|^2, \tag{11}$$

where

$$a_1 = 1 - \frac{d-1}{d}\kappa - \mu.$$

The advantage of this approach is that we obtain an optimal result in the one-dimensional case (the inequality becomes an equality). If $d\to\infty$, we essentially neglect the contribution of $\|\xi_H\|^2$. In principle, this could be improved by employing the following refined estimate between the Hessian and the Laplacian [13, Lemma 2.1]:

$$\|\nabla^2 u\|^2 \geq \frac{1}{d}(\Delta u)^2 + \frac{d}{d-1}\Big(\frac{1}{|\nabla u|^2}(\nabla u)^\top\nabla^2 u\nabla u - \frac{1}{d}(\Delta u)\Big)^2.$$

In order to make full use of this estimate, as in [13], we need to employ the variables ξ_G, ξ_L/d, and ξ_M, defined by $(\xi_L/d+\xi_M)|\xi_G|^2 = \xi_G^\top\xi_H\xi_G$. However, with these variables, we are not able to prove the nonnegativity of the above polynomial. The difficulty comes from the different orders of the powers of u. The disadvantage of our approach is that the estimate is only sharp in the one-dimensional case and that the estimates will depend on the space dimension.

The problem to deal with the inhomogeneity is solved by a formulation of the polynomial as a sum of squares which mixes the orders of derivatives. We suggest to write the polynomial S in (11) as

$$\begin{aligned} S(\zeta) &= a_1\Big(\xi_L + \frac{a_2}{2a_1}|\xi_G|^2\Big)^2 + \frac{a_5}{|q|^2}\Big(q\cdot\eta_G + \frac{a_4|q|^2}{2a_5}|\xi_G|^2\Big)^2 \\ &\quad + \Big(a_3 - \frac{a_2^2}{4a_1} - \frac{a_4^2|q|^2}{4a_5}\Big)|\xi_G|^4 + a_5\Big(|\eta_G|^2 - \Big|\frac{q}{|q|}\cdot\eta_G\Big|^2\Big). \end{aligned} \tag{12}$$

We stress the fact that this formulation is nontrivial since we mix the variables η_G and $|\xi_G|^2$ in the second sum. The last summand is nonnegative since the norm of $q\cdot\eta_G/|q|$ is never larger than the norm of η_G. The polynomial S is nonnegative if $\kappa\geq 0$, $a_1>0$, $a_5>0$, and $4a_1a_3a_5 - a_2^2a_5 - a_1a_4^2|q|^2 \geq 0$. The first three conditions are equivalent to

$$0\leq\mu\leq 1,\quad 0\leq\kappa<\frac{d}{d-1}(1-\mu)\ (\kappa\in\mathbb{R}\text{ if } d=1),\quad c(\gamma+2\delta)(\gamma+2\delta-1)>0, \tag{13}$$

and the last condition can be written as

$$\begin{aligned} 0 &\leq 4d\gamma^2(4a_1a_3a_5 - a_2^2a_5 - a_1a_4^2|q|^2) \\ &= -\beta^2(\beta+1)^2\gamma^2 d\Big(1-\frac{d-1}{d}\kappa-\mu\Big)|q|^2 - 2c(\gamma+2\delta)(\gamma+2\delta-1)(b_1\kappa^2+b_2\kappa+b_3), \end{aligned} \tag{14}$$

where

$$
\begin{aligned}
b_1 &= (\alpha+\gamma-3)^2(d+8), \\
b_2 &= -4(d-\mu)\alpha^2 + 2(3d+4\mu-4)\alpha\gamma + 2(d+2\mu)\gamma^2 + 4(d\mu-4d-4\mu+4)\alpha \\
&\quad + 2(2d\mu-9d-8\mu)\gamma + 4(-4d\mu+9d+4\mu), \\
b_3 &= d\big(4(1-\mu)\alpha^2 - 4(1-\mu)\alpha\gamma + \gamma^2 + 8(1-\mu)\alpha - 4\gamma + 4\big).
\end{aligned}
$$

Condition (14) can be satisfied for sufficiently large values of c, if there exists $0 \le \kappa < d(1-\mu)/(d-1)$ such that $b_1\kappa^2 + b_2\kappa + b_3 < 0$. Indeed, we have to choose

$$
c \ge c^*|q|^2 = \frac{\beta^2(\beta+1)^2\gamma^2(d-(d-1)\kappa-d\mu)}{2c(\gamma+2\delta)(\gamma+2\delta-1)(-b_1\kappa^2-b_2\kappa-b_3)}|q|^2.
$$

It is convenient to distinguish in the following the cases $d > 1$ and $d = 1$.

3. The multi-dimensional case

We need to analyze the polynomial $p(\kappa) = b_1\kappa^2 + b_2\kappa + b_3$ in (14) and to find $\kappa \in [0, \widehat{\kappa})$ such that $p(\kappa) < 0$, where $\widehat{\kappa} = d(1-\mu)/(d-1)$.

First, we notice that b_1 and b_3 are nonnegative. This is clear for b_1. When we consider b_3 as a function of (α,γ), its minimum is attained at $(0,2)$ and $b_3(0,2) = 0$. Therefore, $b_3(\alpha,\gamma) \ge 0$ for all (α,γ). The polynomial p has real roots if $b_2^2 - 4b_1b_3 \ge 0$. If b_2 is positive, the roots are negative (since $b_2^2 - 4b_1b_3 \le b_2^2$), and we cannot find $\kappa \ge 0$ such that $p(\kappa) < 0$. If $b_2 = 0$, the polynomial has no roots. Therefore, it must hold $b_2 < 0$. In this case, the roots are positive. The value $p(\widehat{\kappa})$ can be written as the square of a polynomial in α and γ and hence, it is nonnegative. This shows that either the roots of p are larger than all values of the interval $[0, \widehat{\kappa})$ or they lie in that interval. In the former case, there cannot exist $\kappa \in [0, \widehat{\kappa})$ such that $p(\kappa) < 0$. In order to guarantee the latter case, we have to require that $p'(\widehat{\kappa}) > 0$.

Summarizing, we can find $\kappa \in [0, \widehat{\kappa})$ such that $p(\kappa) < 0$ if the following conditions are satisfied:

$$
b_2^2 - 4b_1b_3 > 0, \quad b_2 < 0, \quad p'(\widehat{\kappa}) = 2b_1\widehat{\kappa} + b_2 > 0. \tag{15}
$$

Let $A_{d,\mu}$ be the set of all (α,γ) satisfying (15). Notice that (α,γ) with $\alpha+\gamma-3 = 0$ cannot be an element of any $A_{d,\mu}$ since otherwise, $b_1 = (\alpha+\gamma-3)^2(d+8) = 0$ implies that $p'(\widehat{\kappa}) = b_2 > 0$, contradicting $b_2 < 0$. The set $A_{d,\mu}$ may be empty. We claim that it is not empty if $\mu = 0$. Indeed, in this situation, the discriminant factorizes,

$$
\begin{aligned}
0 < b_2^2 - 4b_1b_3 = -32\big(2\alpha^2 + d\alpha\gamma - d\gamma^2 - 3d\alpha + 6d\gamma - 9d\big)& \\
\times \big(2d\alpha^2 + d\alpha\gamma - \gamma^2 - 6d\alpha + (4-d)\gamma + 4d - 4\big)&.
\end{aligned} \tag{16}
$$

The two factors are quadratic polynomials in (α,γ). The corresponding discriminants vanish if and only if

$$
\alpha = \frac14\Big(-d \pm \sqrt{d(d+8)}\Big)(\gamma-3) \quad \text{or} \quad \alpha = \frac{1}{4d}\Big(-d(\gamma-6) \pm \sqrt{d(d+8)}(\gamma-2)\Big).
$$

It can be seen that the boundary of the set of all (α,γ) satisfying all three conditions in (15) consists of four straight line segments (see Figure 1).

We have shown the following result.

Theorem 3. *Let $d>1$ and let u be a smooth positive solution to* (1) *with $q\in\mathbb{R}^d$. Then there exists $\mu^*\in(0,1]$ such that for all $0\le\mu<\mu^*$, there exists a non-empty set $A_{d,\mu}\subset\mathbb{R}^2$ of admissible parameters. Let $(\alpha,\gamma)\in A_{d,\mu}$ and $\beta>-1$ satisfy* (6). *Then there exists $c^*>0$ such that for all $c\ge c^*|q|^2$, it holds*

$$\frac{dE}{dt}+\mu\Big(\frac{\gamma}{\alpha+\gamma}\Big)^2\int_{\mathbb{T}^d}\big|\Delta(u^{(\alpha+\gamma)/2})\big|^2dx\le 0.$$

From this result, Theorem 1 follows. Indeed, let $(\alpha,\gamma)\in A_d=A_{d,0}$. Then, by a continuity argument, there exists $\mu>0$ such that $(\alpha,\gamma)\in A_{d,\mu}$, and the above theorem applies.

Remark 4. The above proof is valid for smooth positive solutions only. It is possible to extend the results, however, under stronger conditions on the parameters, to weak nonnegative solutions. To this end, consider the nondegenerate equation

$$u_t=\operatorname{div}((u^\alpha+\varepsilon)\nabla u+qu^{\beta+1}),\quad t>0,\quad u(x,0)=u_0(x)+\varepsilon,\quad x\in\mathbb{T}^d,$$

where $\varepsilon>0$. If u_0 is smooth, there exists a smooth positive solution u_ε to this problem. Then we can proceed similarly as above, but we have now two diffusion coefficients, one with $\alpha>0$ ($u^\alpha\nabla u$) and one with $\alpha=0$ ($\varepsilon\nabla u$). For instance, in the one-dimensional case, we need to satisfy both $0<\gamma-2\alpha<3$, $(\gamma+2\beta-2\alpha)(\gamma+2\beta-2\alpha-1)>0$ *and* $0<\gamma<3$, $(\gamma+2\beta)(\gamma+2\beta-1)>0$. Since u_ε can be shown to converge in some gradient norm, as $\varepsilon\to0$, to the solution to (1), the above entropy estimates are valid also for weak nonnegative solutions, under stronger assumptions on the parameters.

4. The one-dimensional case

In the one-dimensional case $d=1$, the problem simplifies. First, we remark that the shift polynomial T_5 vanishes since $\xi_H=\xi_L$. Thus, we do not need to take into account the coefficient c_5, and we have $\kappa=c_4\in\mathbb{R}$. Then we have to show the nonnegativity of the polynomial

$$S(\zeta)=a_1\xi_L^2+a_2\xi_G^2\xi_L+a_3\xi_G^4+a_4q\eta_G\xi_G^2+a_5\eta_G^2,$$

where

$$\begin{aligned}
a_1&=1-\mu,\quad a_2=\alpha+\frac{\gamma}{2}-1+3\kappa-\mu(\alpha+\gamma-2),\\
a_3&=\alpha\Big(\frac{\gamma}{2}-1\Big)+(\alpha+\gamma-3)\kappa-\frac{\mu}{4}(\alpha+\gamma-2)^2,\\
a_4&=-\frac12\beta(\beta+1),\quad a_5=\frac{2c}{\gamma^2}(\gamma+2\delta)(\gamma+2\delta-1).
\end{aligned}$$

This polynomial is written similarly as above as a sum of squares,

$$S(\zeta)=a_1\Big(\xi_L+\frac{a_2}{2a_1}\xi_G^2\Big)^2+\frac{a_5}{q^2}\Big(q\eta_G+\frac{a_4q^2}{2a_5}\xi_G^2\Big)^2+\Big(a_3-\frac{a_2^2}{4a_1}-\frac{a_4^2q^2}{4a_5}\Big)\xi_G^4.$$

This expression is nonnegative if $a_1 > 0$, $a_5 > 0$, and $4a_1a_3a_5 - a_2^2a_5 - a_1a_4^2q^2 \geq 0$. These inequalities are equivalent to $\mu < 1$, $c(\gamma+2\delta)(\gamma+2\delta-1) > 0$, and

$$0 \leq 4\gamma^2(4a_1a_3a_5 - a_2^2a_5 - a_1a_4^2q^2) = d_1\kappa^2 + d_2\kappa + d_3,$$

where

$$\begin{aligned}
d_1 &= -72c(\gamma+2\delta)(\gamma+2\delta-1), \\
d_2 &= 8c(\gamma+2\delta)(\gamma+2\delta-1)\big(2(\mu-1)\alpha + (1+2\mu)\gamma - 6\big), \\
d_3 &= -\beta^2(\beta+1)^2\gamma^2(1-\mu)q^2 \\
&\quad - 2c(\gamma+2\delta)(\gamma+2\delta-1)\big(4(1-\mu)\alpha^2 + \gamma^2 - 4(1-\mu)\alpha\gamma + 8(1-\mu)\alpha - 4\gamma + 4\big).
\end{aligned}$$

The inequality holds true for some $\kappa \in \mathbb{R}$ if and only if the discriminant $d_2^2 - 4d_1d_3$ is nonnegative,

$$0 \leq d_2^2 - 4d_1d_3 = 64c(\gamma+2\delta)(\gamma+2\delta-1)(\alpha+\gamma-3)^2\big(e_1\mu^2 + e_2\mu + e_3\big),$$

where

$$\begin{aligned}
e_1 &= 4c(\gamma+2\delta)(\gamma+2\delta-1)(\alpha+\gamma)^2 > 0, \\
e_2 &= \frac{9}{2}\beta^2(\beta+1)^2\gamma^2q^2 + 4c(\gamma+2\delta)(\gamma+2\delta-1)\big(7\alpha^2 + \gamma^2 - 10\alpha\gamma + 12\alpha - 6\gamma\big), \\
e_3 &= -\frac{9}{2}\beta^2(\beta+1)^2\gamma^2q^2 - 8c(\gamma+2\delta)(\gamma+2\delta-1)(\gamma-2\alpha)(\gamma-2\alpha-3).
\end{aligned}$$

If we choose (α,γ) such that $(\gamma-2\alpha)(\gamma-2\alpha-3) < 0$ and $c > 0$ such that

$$c \geq \frac{9\beta^2(\beta+1)^2\gamma^2q^2}{-16(\gamma+2\delta)(\gamma+2\delta-1)(\gamma-2\alpha)(\gamma-2\alpha-3)} > 0, \tag{17}$$

then $e_3 > 0$ and there exists $0 < \mu \leq 1$ satisfying $e_1\mu^2 + e_2\mu + e_3 \geq 0$, i.e., the discriminant $d_2^2 - 4d_1d_3$ becomes nonnegative with such a choice of μ. This shows Theorem 2.

Now we discuss how large we can choose μ. For this, we observe that the discriminant of the quadratic polynomial in μ can be written as

$$e_2^2 - 4e_1e_3 = \frac{9}{4}\big(3\beta^2(\beta+1)^2\gamma^2q^2 + 8c(\gamma+2\delta)(\gamma+2\delta-1)P(\alpha,\gamma)\big)^2,$$

where

$$P(\alpha,\gamma) = 3\alpha^2 + \gamma^2 - 2\alpha\gamma + 4\alpha - 2\gamma.$$

We define

$$Z = \frac{3\beta^2(\beta+1)^2\gamma^2q^2}{8c(\gamma+2\delta)(\gamma+2\delta-1)} > 0.$$

We distinguish three cases: $P(\alpha,\gamma) \leq -Z$, $-Z < P(\alpha,\gamma) < 0$, and $P(\alpha,\gamma) \geq 0$. Let first $P(\alpha,\gamma) \leq -Z$. Then

$$\sqrt{e_2^2 - 4e_1e_3} = -\frac{3}{2}\big(3\beta^2(\beta+1)^2\gamma^2q^2 + 8c(\gamma+2\delta)(\gamma+2\delta-1)P(\alpha,\gamma)\big),$$

and the smallest root μ_1 of $e_1\mu^2 + e_2\mu + e_3 = 0$ equals

$$\mu_1 = \frac{1}{2e_1}\Big(- e_2 - \sqrt{e_2^2 - 4e_1e_3}\Big) = 1.$$

Thus, we can choose $\mu = 1$. Next, let $-Z < P(\alpha,\gamma) < 0$. Then

$$\sqrt{e_2^2 - 4e_1e_3} = \frac{3}{2}\big(3\beta^2(\beta+1)^2\gamma q^2 + 8c(\gamma+2\delta)(\gamma+2\delta-1)P(\alpha,\gamma)\big),$$

and

$$\mu_1 = -\frac{1}{(\alpha+\gamma)^2}\Big(\frac{9\beta^2(\beta+1)^2\gamma^2q^2}{8c(\gamma+2\delta)(\gamma+2\delta-1)} + 2(\gamma-2\alpha)(\gamma-2\alpha-3)\Big). \tag{18}$$

Under condition (17), it holds $\mu_1 > 0$. Furthermore, we have $\mu_1 < 1$ since this is equivalent to

$$3Z = \frac{9\beta^2(\beta+1)^2\gamma q^2}{8c(\gamma+2\delta)(\gamma+2\delta-1)} > -(\alpha+\gamma)^2 - 2(\gamma-2\alpha)(\gamma-2\alpha-3) = -3P(\alpha,\gamma), \tag{19}$$

which is true since $P(\alpha,\gamma) > -Z$. Thus, we have to choose $\mu \le \mu_1 < 1$. Finally, let $P(\alpha,\gamma) \ge 0$. Then μ_1 is given by (18) and it holds $\mu_1 \ge 1$ such that we can choose $\mu = 1$.

Proposition 5. *Define* $P(\alpha,\gamma) = 3\alpha^2 + \gamma^2 - 2\alpha\gamma + 4\alpha - 2\gamma$ *and*

$$c_1 = \frac{3\beta^2(\beta+1)^2\gamma^2}{-8(\gamma+2(\beta-\alpha))(\gamma+2(\beta-\alpha)-1)P(\alpha,\gamma)}q^2.$$

Then if $P(\alpha,\gamma) < 0$ *and* $c > c_1$, *we can choose* $\mu = 1$ *in* (7), *otherwise* $\mu \le \mu_1 < 1$, *where* μ_1 *is defined in* (18).

We remark that c_1 is larger than the constant c^* in Theorem 2 since, by (19), the inequality $c_1 > c^*$ is equivalent to

$$(\alpha+\gamma)^2 = 3P(\alpha,\gamma) - 2(2\alpha-\gamma)(2\alpha-\gamma+3) > 0.$$

Remark 6. For the one-dimensional porous-medium equation without convection, we can prove that if $E_1(t) = \int_{\mathbb{T}} |u_x(x,t)^{\gamma/2}|^2 dx$ is an entropy then $0 \le \gamma - 2\alpha \le 3$ must be satisfied. This shows the optimality of the entropy parameter range for the convectionless equation. The idea of the proof is to employ a periodic regularisation of the initial datum $u_0(x) = |x|^{3/(\alpha+\gamma)}$. Indeed, using u_0 *formally* in the entropy dissipation (see (8))

$$D[u] = -\frac{dE_1}{dt} = \frac{\gamma^2}{4}\int_{\mathbb{T}} u^{\alpha+\gamma}\Big(\Big(\frac{u_{xx}}{u}\Big)^2 + \Big(\alpha+\frac{\gamma}{2}-1\Big)\Big(\frac{u_x}{u}\Big)^2\frac{u_{xx}}{u} + \alpha\Big(\frac{\gamma}{2}-1\Big)\Big(\frac{u_x}{u}\Big)^4\Big)dx,$$

a computation shows that

$$D[u_0] = -\frac{\gamma^2}{4}\frac{9}{2(\alpha+\gamma)^4}(\gamma-2\alpha)(\gamma-2\alpha-3)\int_{\mathbb{T}} |x|^{-1}dx.$$

Suppose that $(\gamma-2\alpha)(\gamma-2\alpha-3) > 0$. Ignoring the fact that $|x|^{-1}$ is not integrable at $x = 0$, we conclude that, starting with the initial datum u_0, the functional $E_1(t)$ is increasing for small times $t \ge 0$ and cannot be an entropy. Thus, it must hold $(\gamma-2\alpha)(\gamma-2\alpha-3) \le 0$ which is equivalent to $0 \le \gamma - 2\alpha \le 3$.

Now, the problem of the nonintegrability of u_0 can be overcome, see [14]. The argument of [14] was extended in [12, Thm. 19] and the following result was proved: If the polynomial $S_0(\zeta)$ corresponding to the entropy dissipation $-dE_1/dt$ is negative at $\xi^* = (\xi_1, \xi_2) = (1, 1-3(\alpha+\gamma))$ then E_1 cannot be an entropy. We apply this result to our situation, suppose $(\gamma-2\alpha)(\gamma-2\alpha-3) > 0$ and calculate (see (10))

$$S_0(\xi^*) = \xi_2^2 + \Big(\alpha + \frac{\gamma}{2} - 1\Big)\xi_1^2\xi_2 + \alpha\Big(\frac{\gamma}{2} - 1\Big)\xi_1^4 = -\frac{1}{18}(\gamma-2\alpha)(\gamma-2\alpha-3) < 0.$$

Thus, for such (α,γ), E_1 cannot be an entropy.

Unfortunately, it seems to be difficult to extend the above argument to the convective porous-medium equation since the proof of [14] (and [12]) uses the homogeneity of the underlying differential equation.

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Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstr. 8-10, 1040 Wien, Austria

E-mail address: juengel@anum.tuwien.ac.at

Laboratoire Paul Painlevé, CNRS–UMR 8524, Université Lille 1, Cité scientifique, 59655 Villeneuve d'Ascq, France

E-mail address: ingrid.violet@math.univ-lille1.fr