ANALYSIS OF A FRACTIONAL CROSS-DIFFUSION SYSTEM FOR MULTI-SPECIES POPULATIONS

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Abstract. The global in time existence of weak solutions to a cross-diffusion system with fractional diffusion in the whole space is proved. The equations describe the evolution of multi-species populations in the regime of large-distance interactions; they have been derived in the many-particle limit from moderately interacting particle systems with Lévy noise. The existence proof is based on a three-level approximation scheme, entropy and moment estimates, and a new Aubin–Lions compactness lemma in the whole space.

1. Introduction

The topic of this paper is the analysis of the following cross-diffusion system with fractional derivatives, modeling the dynamics of multi-species populations:

$$\partial_t u_i + \sigma_i(-\Delta)^\alpha u_i - \operatorname{div}\left(\sum_{j=1}^n a_{ij} u_i \nabla(-\Delta)^{(\beta-1)/2} u_j\right) = 0 \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{1}$$

$$u_i(0) = u_i^0 \quad \text{in } \mathbb{R}^d,\ i = 1, \ldots, n, \tag{2}$$

where $u_i(x,t)$ describes the population density of the ith species and $d \geq 2$ is the space dimension. The parameters are $\sigma_i \geq 0$ and $0 < \alpha, \beta < 1$. The fractional Laplacian $(-\Delta)^s$ is defined for $0 < s < 1$ as the singular integral operator

$$(-\Delta)^s u(x) = c_{d,s}\int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy, \quad \text{where } c_{d,s} = \frac{4^s\Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|},$$

for $u \in H^s(\mathbb{R}^d)$. The integral is understood as the principal value, and Γ denotes the Gamma function. The expression $\nabla(-\Delta)^{(\beta-1)/2}$ can be interpreted as a fractional partial derivative of order $\beta \in (0,1)$ and can be seen as a nonlocal gradient.

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System (1)–(2) has been derived in [7] as the many-particle limit of the following interacting particle system driven by Lévy noise:

$$dX_i^{k,N}(t) = -\sum_{j=1}^n \frac{1}{N}\sum_{\ell=1}^N a_{ij}\nabla(-\Delta)^{(\beta-1)/2}V_N(X_i^{k,N}(t) - X_j^{\ell,N}(t))dt + \sqrt{2\sigma_i}dL_i^k(t), \tag{3}$$

where $i = 1,\ldots,n$ and $k = 1,\ldots,N$, $X_i^{k,N}(t)$ is the position of the kth particle of species i at time t, V_N is a potential function, and L_i^k is a Lévy process of index $\alpha \in (0,1)$. Lévy processes include jumps and large-distance interactions instead of the short-distance interactions of Brownian motion. It was shown in [7] that if V_N converges in the sense of distributions to the delta distribution as $N \to \infty$, the empirical measures associated to (3) converge in a certain sense to limiting processes with density u_i, solving (1)–(2). The global in time existence of strong solutions is proved in [7] for sufficiently small initial data in the $H^s(\mathbb{R}^d)$ norm with $s > d/2$ in the regime $2\alpha > \beta + 1$, in which self-diffusion dominates cross-diffusion. In this paper, we prove the global in time existence of weak solutions without any smallness assumption on the initial data and for all values of α, $\beta \in (0,1)$. In particular, our proof allows for the case without self-diffusion, $\sigma_i = 0$. The key idea of our analysis is to exploit the entropy structure of (1), thus significantly extending the results of [7].

Most results on cross-diffusion systems in the literature refer to local models; see, e.g., the references in [16]. Nonlocal cross-diffusion systems have been investigated rather recently [8, 9, 12, 13, 17]. Fractional diffusion was introduced in the Keller–Segel system to model cellular population dispersal with anomalous diffusion [11]. Another application is a three-species food-chain cross-diffusion system with fractional operators [15]. In this paper, we analyze the entropy structure of fractional cross-diffusion systems for the first time.

System (1) can be seen as an extension of the local cross-diffusion system of [5], which is formally obtained from (1) by setting $\alpha = \beta = 1$. The entropy structure of the local model was investigated in [17, 18]. It turned out that such a structure holds if there exist numbers $\pi_1,\ldots,\pi_n > 0$ such that

$$\pi_i a_{ij} = \pi_j a_{ji} \quad \text{for } i,j = 1,\ldots,n,$$

and this condition is also assumed in this work. It can be interpreted as the detailed-balance condition for the Markov chain associated with (a_{ij}), and $(\pi_1,\ldots,\pi_n)$ is the invariant measure. Together with the parabolicity condition of Petrovskii [1], i.e., all eigenvalues of $(a_{ij}) \in \mathbb{R}^{n\times n}$ have a real part, this implies that the matrix $(\pi_i a_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite [6, Prop. 3]. A formal computation using a generalized Stroock–Varopoulos inequality (see Lemma 13 in the Appendix) shows that

$$\frac{d}{dt}H[u] + 4\sum_{i=1}^n \sigma_i \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}\sqrt{u_i}|^2 dx + \lambda\sum_{i=1}^n \int_{\mathbb{R}^d} |\nabla(-\Delta)^{(\beta-1)/4}u_i|^2 dx \le 0, \tag{4}$$

where $\lambda > 0$ is the smallest eigenvalue of $(\pi_i a_{ij})$ and

$$H[u] = \sum_{i=1}^{n} \pi_i \int_{\mathbb{R}^d} u_i \log u_i dx \tag{5}$$

is the entropy functional. Taking into account the mass conservation and the fractional Gagliardo–Nirenberg inequality, we obtain an estimate for u_i in $L^p(0,T;L^p(\mathbb{R}^d))$ for some $p > 2$. Together with the $L^2(0,T;H^{(\beta+1)/2}(\mathbb{R}^d))$ bound for u_i from (4), this is sufficient to handle the product $u_i \nabla(-\Delta)^{(\beta-1)/2} u_j$.

The mathematical difficulties to make these observations rigorous are of technical nature. Indeed, since the fractional integral operator is singular, we regularize the Riesz kernel $\mathcal{K}(x) = |x|^{1-\beta-d}$ of the Riesz potential $(-\Delta)^{(\beta-1)/2} v = \mathcal{K} * u$ by some kernel $\mathcal{K}^{(\varepsilon)}$ to define the approximate scheme. Unfortunately, this distroys the $L^2(0,T;H^{(\beta+1)/2}(\mathbb{R}^d))$ estimate for u_i that is needed to obtain a bound for u_i in $L^p(0,T;L^p(\mathbb{R}^d))$ for some $p > 2$ (observe that we allow for $\sigma_i = 0$). One idea to remedy this issue is to add the function

$$\kappa g_0[u_i] = \kappa\left(u_i^2 - \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{\mathbb{R}^d} u_i^2 dx \right),$$

to the equation, where $\kappa > 0$ is the second approximation parameter. This function preserves the mass conservation property and it provides an $L^1(0,T;L^1(\mathbb{R}^d))$ bound for $u_i^2(\log u_i)_+$, where $z_+ = \max\{0, z\}$ (note that (4) is derived by using formally the test function $\pi_i \log u_i$). However, in order to build an approximated solution, we need to replace g_0 by a bounded continuous mapping $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. This forces us to introduce a third level of approximation, namely

$$\kappa g_\rho[u_i](x) = \kappa u_i(x)(W_\rho * u_i)(x) - \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{\mathbb{R}^d} u_i(y)(W_\rho * u_i)(y) dy,$$

where W_ρ is a mollifier with $\rho > 0$ such that $g_\rho[u_i] \to g_0[u_i]$ a.e. in $\mathbb{R}^d$ as $\rho \to 0$. This yields, after the limit $\rho \to 0$, the desired estimate for $u_i^2(\log u_i)_+$, thus in a space slightly better than $L^2(\mathbb{R}^d)$.

The de-regularization limits $\rho \to 0$, $\varepsilon \to 0$, and $\kappa \to 0$ are based on suitable compactness lemmas. We state and prove an Aubin–Lions-type compactness result in fractional Sobolev spaces leading to strong convergence in the critical space $L^2(0,T;L^2(\mathbb{R}^d))$. Compactness in the whole space $\mathbb{R}^d$ is achieved by controlling some moments of u_i and applying the compactness result in [4, Lemma 1]; see Lemma 14 in the Appendix.

We summarize our hypotheses:

- (H1) Data: $d \geq 2$, $\sigma_i \geq 0$, $a_{ij} \geq 0$, $\alpha \in (0,1)$, and $\beta \in (0,1)$.
- (H2) Diffusion matrix: All eigenvalues of the matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ have a positive real part, and the detailed-balance condition holds, i.e., there exist $\pi_1, \ldots, \pi_n > 0$ such that $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j = 1, \ldots, n$.
- (H3) Initial data: $u^0 = (u_1^0, \ldots, u_n^0)$ satisfies $u_i^0 \in L^1(\mathbb{R}^d; (1+|x|^2)^{m/2} dx)$, $u_i^0 \log u_i^0 \in L^1(\mathbb{R}^d)$ for $i = 1, \ldots, n$, where $0 < m < \min\{1, 2\alpha\}$.

We already mentioned that Hypothesis (H2) implies that the matrix $(\pi_i a_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, and hence, its eigenvalues are real and positive. The moment assumption on u_i in Hypothesis (H3) is needed for the moment estimate of $u_i(t)$ that in turn is used to prove the compactness in $\mathbb{R}^d$.

Definition 1. *We say that $u = (u_1,\ldots,u_n) : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^n$ is a* weak solution *to* (1)–(2) *if*

$$\begin{aligned} & u_i \in L^2(0,T;H^{(\beta+1)/2}(\mathbb{R}^d)), \\ & u_i \in L^\infty(0,T;L^1(\mathbb{R}^d;(1+|x|^2)^{m/2}dx) \quad \textit{with } m>0 \textit{ as in (H3)}, \\ & \sqrt{u_i} \in L^2(0,T;H^\alpha(\mathbb{R}^d)) \quad \textit{if } \sigma_i > 0, \end{aligned}$$

equation (1) *holds in the sense of $L^q(0,T;W^{-1,q}(\mathbb{R}^d))$ for some $q>1$, and the initial condition* (2) *holds in the sense of $W^{-1,q}(\mathbb{R}^d)$.*

We show in Lemma 3 below, using the product rule for the fractional Laplacian [2, Prop. 1.5], that $u_i \in L^\infty(0,T;L^1(\Omega))$ and $\sqrt{u_i} \in L^2(0,T;H^\alpha(\mathbb{R}^d))$ imply that $(-\Delta)^{\alpha/2}u_i \in L^2(0,T;L^1(\mathbb{R}^d))$ such that the weak formulation of (1) makes sense.

Our main result is as follows.

Theorem 1 (Global existence)**.** *Let Hypotheses (H1)–(H3) hold. Then there exists a weak solution u to* (1)–(2)*, which is nonnegative, i.e. $u_i(t) \ge 0$ a.e. in $\mathbb{R}^d$, conserves the mass,*

$$\int_{\mathbb{R}^d} u_i(t)dx = \int_{\mathbb{R}^d} u_i^0 dx \quad \textit{for } t>0,\ i=1,\ldots,n,$$

and satisfies the entropy inequality,

$$\begin{aligned} (6) \qquad & \sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i(t)\log u_i(t)dx + C\sum_{i=1}^n \sigma_i \int_0^t \int_{\mathbb{R}^d} \big|(-\Delta)^{\alpha/2}\sqrt{u_i}\big|^2 dxds \\ & + \lambda \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^d} |\nabla(-\Delta)^{(\beta-1)/4}u_i|^2 dxds \le \sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i^0 \log u_i^0 dx, \quad t>0. \end{aligned}$$

The paper is organized as follows. We derive *formally* some a priori estimates in Section 2. Besides being an illustration of our strategy, the computations will be used in the subsequent sections, in particular for the limit procedure at the last approximation level. The approximate problem with three approximation levels is introduced in Section 3, and its global well-posedness is proved. In Section 4, the limit in the approximate problem is shown. Finally, we collect some technical results and prove an Aubin–Lions-type compactness lemma in the Appendix.

Notation. The space $W^{s,p}(\mathbb{R}^d)$ with $s>0$ and $1\le p\le\infty$ is the usual fractional Sobolev space; we set $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$. We write $\|\cdot\|_p$ for the norm in $L^p(\mathbb{R}^d)$, $1\le p\le\infty$, and we define for $m>0$ the space

$$L^1(\mathbb{R}^d;(1+|x|^2)^{m/2}dx) = \left\{ v \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} v(x)(1+|x|^2)^{m/2}dx < \infty \right\}.$$

The characteristic function on a set $B \subset \mathbb{R}^d$ is written as 1_B. Finally, we denote by $C > 0$ a generic constant whose value may change from line to line.

2. A priori estimates

In this section, we derive *formally* some a priori estimates. First, we provide a proof for the entropy inequality (4).

Lemma 2. *Let u be a smooth solution to* (1)–(2). *Then*

$$\frac{d}{dt}H[u] + 4\sum_{i=1}^n \sigma_i \int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}\sqrt{u_i}|^2 dx + \lambda \sum_{i=1}^n \int_{\mathbb{R}^d} |\nabla(-\Delta)^{(\beta-1)/4}u_i|^2 dx \le 0, \tag{7}$$

where the entropy $H[u]$ is defined in (5) *and $\lambda > 0$ is the smallest eigenvalue of $(\pi_i a_{ij}) \in \mathbb{R}^{n\times n}$.*

Proof. Using $\log u_i$ *formally* as a test function in (1) yields

$$\frac{d}{dt}H[u] = -\sum_{i=1}^n \sigma_i \int_{\mathbb{R}^d} \log u_i (-\Delta)^\alpha u_i dx - \sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathbb{R}^d} \nabla u_i \cdot \nabla(-\Delta)^{(\beta-1)/2}u_j dx. \tag{8}$$

We integrate by parts in the last integral and use the positive definiteness of the matrix $(\pi_i a_{ij})$ to obtain

$$\begin{aligned}\sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathbb{R}^d} \nabla u_i \cdot \nabla(-\Delta)^{(\beta-1)/2}u_j dx &= \sum_{i,j=1}^n \pi_i a_{ij} \int_{\mathbb{R}^d} \nabla(-\Delta)^{(\beta-1)/4}u_i \cdot \nabla(-\Delta)^{(\beta-1)/4}u_j dx \\ &\ge \lambda \sum_{i=1}^n \int_{\mathbb{R}^d} |\nabla(-\Delta)^{(\beta-1)/4}u_i|^2 dx.\end{aligned}$$

We apply the generalized Stroock–Varopolous inequality in Lemma 13 (see Appendix A) to the first integral on the right-hand side of (8) to conclude the proof. ☐

The mass conservation and entropy inequality (7) yield the following bounds for $i = 1, \ldots, n$:

$$\|u_i\|_{L^\infty(0,\infty;L^1(\mathbb{R}^d))} + \sigma_i\|\sqrt{u_i}\|_{L^2(0,T;H^\alpha(\mathbb{R}^d))} + \lambda\|u_i\|_{L^2(0,T;H^{(\beta+1)/2}(\mathbb{R}^d))} \le C. \tag{9}$$

We derive further a priori estimates from the entropy inequality and the Gagliardo–Nirenberg inequality.

Lemma 3. *Let u be a smooth solution to* (1)–(2). *Then there exists a constant $C > 0$, not depending on u, such that for $i = 1, \ldots, n$,*

$$\|\nabla(-\Delta)^{(\beta-1)/2}u_i\|_{L^2(0,T;L^{2d/(d+\beta-1)}(\mathbb{R}^d))} + \sqrt{\sigma_i}\|(-\Delta)^{\alpha/2}u_i\|_{L^2(0,T;L^1(\mathbb{R}^d))} \le C. \tag{10}$$

Moreover, there exist $p^*, q^* > 1$ such that for $i, j = 1, \ldots, n$,

$$\|u_i\nabla(-\Delta)^{(\beta-1)/2}u_j\|_{L^{q^*}(0,T;L^{p^*}(\mathbb{R}^d))} \le C. \tag{11}$$

Proof. We claim that

$$\|u_i\|_{L^q(0,T;L^p(\mathbb{R}^d))} \le C, \quad \text{where } q \ge 2 \text{ and } \frac{1}{p} + \frac{1}{q}\left(1 + \frac{\beta+1}{d}\right) = 1. \tag{12}$$

By interpolation, it holds for $1 < p, q < \infty$ and $0 < \theta < 1$ with $1/p = \theta/r + (1-\theta)$ that

$$\|u_i\|^q_{L^q(0,T;L^p(\mathbb{R}^d))} = \int_0^T \|u_i\|_p^q dt \le \int_0^T \|u_i\|_r^{\theta q}\|u_i\|_1^{(1-\theta)q} dt.$$

Taking into account estimate (9) and the fractional Sobolev embedding $H^{(\beta+1)/2}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ for $r = 2d/(d-\beta-1)$ [10, Theorem 6.5] and choosing $\theta = 2/q \le 1$, we find that

$$\|u_i\|^q_{L^q(0,T;L^p(\mathbb{R}^d))} \le C\|u_i\|^{(1-\theta)q}_{L^\infty(0,T;L^1(\mathbb{R}^d))} \int_0^T \|u_i\|^{\theta q}_{H^{(\beta+1)/2}(\mathbb{R}^d)} dt \le C.$$

Inserting $\theta = 2/q$ and $1/r = 1/2 - (\beta+1)/(2d)$, we have

$$\frac{1}{p} = \frac{\theta}{r} + (1-\theta) = \frac{2}{q}\left(\frac{1}{2} - \frac{\beta+1}{2d}\right) + 1 - \frac{2}{q} = 1 - \frac{1}{q}\left(1 + \frac{\beta+1}{d}\right),$$

which proves the claim. Choosing $q = p$ in (12) yields

$$\|u_i\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C \quad \text{for } p = 2 + \frac{\beta+1}{d} > 2.$$

It follows from the Hardy–Littlewood–Sobolev inequality (Lemma 12 with $s = (1-\beta)/4$) that

$$\|\nabla(-\Delta)^{(\beta-1)/2}u_j\|_{2d/(d+\beta-1)} \le C\|\nabla(-\Delta)^{(\beta-1)/4}u_j\|_2$$

and therefore, because of (9),

$$\|\nabla(-\Delta)^{(\beta-1)/2}u_j\|_{L^2(0,T;L^{2d/(d+\beta-1)}(\mathbb{R}^d))} \le C. \tag{13}$$

Since $2d/(d+\beta-1) > 2$, the product $u_i\nabla(-\Delta)^{(\beta-1)/2}u_j$ is bounded in some $L^{q^*}(0,T;L^{p^*}(\mathbb{R}^d))$ for suitable $q^*, p^* > 1$.

It remains to derive the bound for $(-\Delta)^{\alpha/2}u_i$. By the product rule for the fractional Laplacian [2, Prop. 1.5],

$$(-\Delta)^{\alpha/2}u_i(x) = 2\sqrt{u_i}(x)(-\Delta)^{\alpha/2}\sqrt{u_i}(x) - c_{d,\alpha/2}\int_{\mathbb{R}^d} \frac{(\sqrt{u_i(x)} - \sqrt{u_i(y)})^2}{|x-y|^{d+\alpha}} dy.$$

We take the $L^1(\mathbb{R}^d)$ norm and use the Cauchy–Schwarz inequality to find that

$$\begin{aligned}\|(-\Delta)^{\alpha/2}u_i\|_1 &\le 2\|\sqrt{u_i}\|_2\|(-\Delta)^{\alpha/2}\sqrt{u_i}\|_2 + c_{d,\alpha/2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\sqrt{u_i(x)} - \sqrt{u_i(y)})^2}{|x-y|^{d+\alpha}} dxdy \\ &= 2\|u_i\|_1^{1/2}\|(-\Delta)^{\alpha/2}\sqrt{u_i}\|_2 + 2\int_{\mathbb{R}^d} \sqrt{u_i}(-\Delta)^{\alpha/2}\sqrt{u_i}dx \\ &\le 4\|u_i\|_1^{1/2}\|(-\Delta)^{\alpha/2}\sqrt{u_i}\|_2.\end{aligned}$$

After taking the square and integrating over time, we obtain

$$\sigma_i\|(-\Delta)^{\alpha/2}u_i\|^2_{L^2(0,T;L^1(\mathbb{R}^d))} \le 4\sigma_i\|u_i\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\|(-\Delta)^{\alpha/2}\sqrt{u_i}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \le C.$$

This ends the proof. □

Next, we derive some moment bounds for u_i.

Lemma 4. *Let u be a smooth solution to* (1)–(2) *and $0 < m < \min\{1, 2\alpha\}$. Then there exists a constant $C > 0$, independent of u, such that*

$$\sup_{0<t<T} \int_{\mathbb{R}^d} u_i(x,t)(1+|x|^2)^{m/2} dx \le C(T), \quad i = 1, \ldots, n.$$

Proof. We use *formally* the test function $(1+|x|^2)^{m/2}$ with $0 < m < \min\{1, 2\alpha\}$ in (1):

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} u_i(1+|x|^2)^{m/2}dx &= -\sigma_i \int_{\mathbb{R}^d} u_i(-\Delta)^\alpha(1+|x|^2)^{m/2}dx \\ &\quad - \sum_{j=1}^n a_{ij}\int_{\mathbb{R}^d} u_i\nabla(1+|x|^2)^{m/2}\cdot\nabla(-\Delta)^{(\beta-1)/2}u_j dx. \end{aligned} \tag{14}$$

To estimate the first term on the right-hand side, we claim that there exists $C > 0$ such that

$$|(-\Delta)^\alpha(1+|x|^2)^{m/2}| \le C(1+|x|^2)^{m/2} \quad \text{for all } x \in \mathbb{R}^d. \tag{15}$$

Indeed, we infer from [10, Lemma 3.2] that

$$\begin{aligned} -(-\Delta)^\alpha(1+|x|)^{m/2} &= \frac{c_{d,\alpha}}{2}\int_{\mathbb{R}^d} \frac{(1+|x+y|^2)^{m/2}+(1+|x-y|^2)^{m/2}-2(1+|x|^2)^{m/2}}{|y|^{d+2\alpha}}dy \\ &=: I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{c_{d,\alpha}}{2}\int_{\{|y|>1\}} \frac{(1+|x+y|^2)^{m/2}+(1+|x-y|^2)^{m/2}-2(1+|x|^2)^{m/2}}{|y|^{d+2\alpha}}dy, \\ I_2 &= \frac{c_{d,\alpha}}{2}\int_{\{|y|\le 1\}} \frac{(1+|x+y|^2)^{m/2}+(1+|x-y|^2)^{m/2}-2(1+|x|^2)^{m/2}}{|y|^{d+2\alpha}}dy. \end{aligned}$$

The triangle inequality implies that

$$|I_1| \le C\int_{\{|y|>1\}} \frac{|x|^m+|y|^m}{|y|^{d+2\alpha}}dy \le C(1+|x|^2)^{m/2},$$

since the integrability is ensured if $m-d-2\alpha < -d$ or, equivalently, $m < 2\alpha$. The function $\Phi_x(y) := (1+|x+y|^2)^{m/2}+(1+|x-y|^2)^{m/2}-2(1+|x|^2)^{m/2}$ satisfies $\Phi_y(0) = |\nabla_y\Phi_y(0)| = 0$ and

$$|D_y^2\Phi_x(y)| \le C(1+|x+y|^2)^{m/2-1} + C(1+|x-y|^2)^{m/2-1}, \quad x, y \in \mathbb{R}^d,$$

and this expression is bounded for all $x, y \in \mathbb{R}^d$. We infer from Taylor's theorem that $|\Phi_x(y)| = \frac{1}{2}|D_y^2\Phi_x(\theta y)||y|^2 \le C|y|^2$ for $y \in \mathbb{R}^d$, where $\theta \in [0,1]$ is a suitable number. Therefore, $|I_2| \le C\int_{\{|y|\le 1\}}|y|^{2-d-2\alpha}dy \le C \le C(1+|x|^2)^{m/2}$, since $\alpha < 1$. This shows the claim.

We estimate the last term in (14). Choosing $p = 2d/(d-\beta+1)$ in (12), we find that

$$\|u_i\|_{L^q(0,T;L^{2d/(d-\beta+1)}(\mathbb{R}^d))} \le C, \quad \text{where } q = \frac{2(d+\beta+1)}{d+\beta-1} > 2.$$

Because of (12) and (13), the product $u_i\nabla(-\Delta)^{(\beta-1)/2}u_j$ can be estimated according to

$$\|u_i\nabla(-\Delta)^{(\beta-1)/2}u_j\|_{L^r(0,T;L^1(\mathbb{R}^d))} \le C \quad \text{for some } r > 1.$$

Taking into account that $\nabla(1+|x|^2)^{m/2}$ is bounded in $\mathbb{R}^d$ if $m < 1$, we obtain

$$-\sum_{j=1}^n a_{ij}\int_0^t\int_{\mathbb{R}^d} u_i\nabla(1+|x|^2)^{m/2}\cdot\nabla(-\Delta)^{(\beta-1)/2}u_j dxd\tau \le C.$$

Summarizing, we conclude from (14) that

$$\int_{\mathbb{R}^d} u_i(t)(1+|x|^2)^{m/2}dx \le \int_{\mathbb{R}^d} u_i^0(1+|x|^2)^{m/2}dx + C\int_0^t\int_{\mathbb{R}^d} u_i(1+|x|^2)^{m/2}dxds + C$$

for some $C > 0$, which shows the result after applying Gronwall's lemma. □

Lemma 5. *Let u be a smooth solution to* (1)–(2)*. Then there exist constants $C > 0$ and $p > 1$, independent of u, such that*

$$\|\partial_t u_i\|_{L^p(0,T;W^{-1,p}(\mathbb{R}^d))} \le C.$$

Proof. It follows from estimates (10) and (11) that there exists $p > 1$ such that

$$\begin{aligned}\|\partial_t u_i\|_{L^p(0,T;W^{-1,p}(\mathbb{R}^d))} &\le \sigma_i\|(-\Delta)^{\alpha/2}u_i\|_{L^p(0,T;L^p(\mathbb{R}^d))} \\ &\quad + \sum_{j=1}^n a_{ij}\|u_i\nabla(-\Delta)^{(\beta-1)/2}u_j\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C,\end{aligned}$$

which finishes the proof. □

3. Approximate scheme

We approximate equation (1) by introducing three approximation levels. First, we regularize the Riesz potential. Noting that $(-\Delta)^{(\beta-1)/2}u = \mathcal{K}_{(1-\beta)/2} * u$, where $\mathcal{K}_{(1-\beta)/2}(x) = |x|^{1-\beta-d}$ for $x \in \mathbb{R}^d$ is the kernel of the Riesz potential, we define the approximation $\mathcal{K}_s^{(\varepsilon)}$ of $\mathcal{K}_s$ by

$$\begin{aligned}&\mathcal{K}_s^{(\varepsilon)} := \widetilde{\mathcal{K}}_{s/2}^{(\varepsilon)} * \widetilde{\mathcal{K}}_{s/2}^{(\varepsilon)}, \quad \text{where } \widetilde{\mathcal{K}}_{s/2}^{(\varepsilon)} \in C_0^2(\mathbb{R}^d), \\ &0 \le \widetilde{\mathcal{K}}_{s/2}^{(\varepsilon)} \le \widetilde{\mathcal{K}}_{s/2}^{(\varepsilon')} \le \mathcal{K}_{s/2} \quad \text{in } \mathbb{R}^d \text{ for } 0 < \varepsilon' < \varepsilon, \\ &\widetilde{\mathcal{K}}_{s/2}^{(\varepsilon)}(x) = \mathcal{K}_{s/2}(x) \quad \text{for } \varepsilon \le |x| \le 1/\varepsilon.\end{aligned} \tag{16}$$

Since $\int_{\mathbb{R}^d} u\mathcal{K}_s^{(\varepsilon)}dx$ generally does not preserve the nonnegativity for $u \ge 0$, we define $\mathcal{K}_s^{(\varepsilon)}$ as a "convolution square" to guarantee this property. Second, we introduce the mollifier

$$W_\rho(x) := \rho^{-d}W_1(x/\rho) \quad \text{for } x \in \mathbb{R}^d, \quad \text{where}$$

$$W_1 \in C_0^0(\mathbb{R}^d), \quad W_1 \geq 0 \text{ in } \mathbb{R}^d, \quad \|W_1\|_1 = 1,$$

that satisfies $W_\rho * u \to u$ a.e. in $\mathbb{R}^d$, and the mapping $g_\rho : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \cap L_0^1(\mathbb{R}^d)$,

$$g_\rho[u](x) := u(x)(W_\rho * u)(x) - \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{\mathbb{R}^d} u(y)(W_\rho * u)(y)dy, \quad u \in L^2(\mathbb{R}^d),$$

where $L_0^1(\mathbb{R}^d)$ is the space of $L^1(\mathbb{R}^d)$ functions with vanishing average. This mapping satisfies the following properties:

$$\|g_\rho[u]\|_1 \leq 2\|u\|_2^2, \quad \|g_\rho[u]\|_2 \leq C(\rho)\|u\|_2^2, \tag{17}$$

$$\|g_\rho[u] - g_\rho[v]\|_2 \leq C(\rho)\|u+v\|_2\|u-v\|_2, \tag{18}$$

where $u, v \in L^2(\mathbb{R}^d)$. These inequalities follow from the Young convolution inequality,

$$\begin{aligned} \|u(W_\rho * u)\|_1 &\leq \|u\|_2\|W_\rho * u\|_2 \leq \|u\|_2^2\|W_\rho\|_1 = \|u\|_2^2, \\ \|u(W_\rho * u)\|_2 &\leq \|u\|_2\|W_\rho * u\|_\infty \leq \|u\|_2^2\|W_\rho\|_2 \leq C(\rho)\|u\|_2^2. \end{aligned}$$

As explained in the introduction, the function g_ρ is needed to obtain an $L^2 \log L^2$ estimate, which is used to obtain strong convergence of the sequence of approximate solutions in $L^2(\mathbb{R}^d)$. Furthermore, we add a Laplacian to (1). This leads to the approximate problem

$$\begin{aligned} \partial_t u_i^{(\rho,\varepsilon,\kappa)} - \kappa\Delta u_i^{(\rho,\varepsilon,\kappa)} + \sigma_i(-\Delta)^\alpha u_i^{(\rho,\varepsilon,\kappa)} + \kappa g_\rho[u_i^{(\rho,\varepsilon,\kappa)}] \\ = \operatorname{div}\bigg(\sum_{j=1}^n a_{ij}(u_i^{(\rho,\varepsilon,\kappa)})_+ \nabla\mathcal{K}_{(1-\beta)/2}^{(\varepsilon)} * u_j^{(\rho,\varepsilon,\kappa)}\bigg), \end{aligned} \tag{19}$$

$$u_i^{(\rho,\varepsilon,\kappa)}(0) = u_i^0 \quad \text{in } \mathbb{R}^d,\ i = 1, \ldots, n, \tag{20}$$

where $z_+ = \max\{0, z\}$ denotes the positive part of $z \in \mathbb{R}$.

3.1. **Local well-posedness of the approximate problem.** We prove the existence of a local solution to (19)–(20) by applying Banach's fixed-point theorem. To this end, we introduce for $R > 2\|u_0\|_2$ and $T > 0$ the space

$$X_{R,T} := \big\{v \in C^0([0,T]; L^2(\mathbb{R}^d)) : \|v_i\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \leq R,\ i = 1, \ldots, n\big\}$$

and the fixed-point mapping $F : X_{R,T} \to X_{R,T}$, $F(v) = u$, where $u = (u_1, \ldots, u_n)$ is the unique solution to the linear problem

$$\partial_t u_i - \kappa\Delta u_i + \sigma_i(-\Delta)^\alpha u_i = -\kappa g_\rho[v_i] + \operatorname{div}\bigg(\sum_{j=1}^n a_{ij}(v_i)_+ \nabla\mathcal{K}_{(1-\beta)/2}^{(\varepsilon)} * v_j\bigg), \tag{21}$$

$$u_i(0) = u_i^0 \quad \text{in } \mathbb{R}^d,\ i = 1, \ldots, n.$$

Since the kernel is regularized, this problem has a unique solution $u = (u_1, \ldots, u_n)$ with $u_i \in L^2(0,T; H^1(\mathbb{R}^d))$, $\partial_t u_i \in L^2(0,T; H^{-1}(\mathbb{R}^d))$, implying that $u_i \in C^0([0,T]; L^2(\Omega))$.

We show that the mapping F is well defined. We use the test function u_i in the weak formulation of (21) and take into account (17):

$$
\begin{aligned}
\frac{1}{2}\|u_i(t)\|_2^2 &+ \kappa\int_0^t\|\nabla u_i\|_2^2 ds + \sigma_i\int_0^t\|(-\Delta)^{\alpha/2}u_i\|_2^2 ds \\
&\le \frac{1}{2}\|u_i^0\|_2^2 + C(\kappa,\rho)\int_0^t\|v_i\|_2^2\|u_i\|_2 ds \\
&\quad + C\sum_{j=1}^n\int_0^t\|(v_i)_+\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*v_j\|_2\|\nabla u_i\|_2 ds.
\end{aligned}
$$

We apply the Young (convolution) inequality to obtain for $0<t<T$,

$$
\begin{aligned}
\|u_i(t)\|_2^2 &+ \kappa\int_0^t\|\nabla u_i\|_2^2 ds + \sigma_i\int_0^t\|(-\Delta)^{\alpha/2}u_i\|_2^2 ds \\
&\le 2\|u_i^0\|_{L^2(\mathbb{R}^d)}^2 + C(\kappa,\rho)\int_0^t\|v_i\|_2^4 ds \\
&\quad + C(\kappa)\sum_{j=1}^n\int_0^t\|(v_i)_+\|_2^2\|\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}\|_2^2\|v_j\|_2^2 ds \\
&\le 2\|u_i^0\|_{L^2(\mathbb{R}^d)}^2 + C(\varepsilon,\kappa,\rho)T\sum_{j=1}^n\|v_j\|^4_{L^\infty(0,T;L^2(\mathbb{R}^d))}.
\end{aligned}
$$

Therefore, since $\|u_i^0\|_2 < R/2$, if $T>0$ is sufficiently small, we infer that $u\in X_{R,T}$, proving the well-posedness of F.

Next, we show that F is a contraction on $X_{R,T}$. Let $v,v'\in X_{R,T}$ and set $u=F(v)$, $u'=F(v')$. The test function u_i-u_i' in the weak formulation of

$$
\begin{aligned}
&\partial_t(u_i-u_i') - \kappa\Delta(u_i-u_i') + \sigma_i(-\Delta)^\alpha(u_i-u_i') + \kappa(g_\rho[v_i]-g_\rho[v_i']) \\
&\quad = \operatorname{div}\left(\sum_{j=1}^n a_{ij}\big[\big((v_i)_+-(v_i')_+\big)\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*v_j + (v_i')_+\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*(v_j-v_j')\big]\right)
\end{aligned}
$$

leads, after similar computations as before and using (18), for $0<t<T$, to

$$
\begin{aligned}
&\|(u_i-u_i')(t)\|_2^2 + \kappa\int_0^t\|\nabla(u_i-u_i')\|_2^2 ds + \sigma_i\int_0^t\|(-\Delta)^{\alpha/2}(u_i-u_i')\|_2^2 ds \\
&\quad \le C(\rho)\int_0^t\|v_i+v_i'\|_2^2\|v_i-v_i'\|_2^2 ds + C(\kappa)\sum_{j=1}^n\int_0^t\big(\|(v_i)_+-(v_i')_+\|_2^2\|\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*v_j\|_\infty^2 \\
&\qquad + \|(v_i')_+\|_2^2\|\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*(v_j-v_j')\|_\infty^2\big) ds \\
&\quad \le C(\varepsilon,\kappa,\rho)\sum_{j,k=1}^n\int_0^t\|v_j-v_j'\|_2^2\big(\|v_k\|_2^2+\|v_k'\|_2^2\big) ds
\end{aligned}
$$

$$\leq C(\varepsilon,\kappa,\rho,R)T\sum_{j=1}^n\|v_j-v_j'\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))}.$$

Hence, if $T>0$ is sufficiently small, F is a contraction on $X_{R,T}$. We conclude from Banach's fixed-point theorem that there exists $T^*>0$ and a unique fixed point $u\in X_{R,T^*}$ of F, i.e. a unique solution $u^{(\varepsilon,\kappa,\rho)}\in L^2(0,T^*;H^1(\mathbb{R}^d))$ with $\partial_t u_i^{(\varepsilon,\kappa,\rho)}\in L^2(0,T^*;H^{-1}(\mathbb{R}^d))$ to (19)–(20).

3.2. **Uniform bounds and global well-posedness.** We show that the solution $u=u^{(\varepsilon,\kappa,\rho)}\in C^0([0,T^*];L^2(\mathbb{R}^d))$, derived in the previous subsection, is actually global in time. First, we prove that $u_i(t)\geq 0$ for $t\in[0,T^*]$. We use the test function $(u_i)_-=\min\{0,u_i\}$ as a test function in the weak formulation of (19):

$$\begin{aligned}
\frac12\int_{\mathbb{R}^d}(u_i)_-^2(t)dx &+ \kappa\int_0^t\int_{\mathbb{R}^d}|\nabla(u_i)_-|^2dxds = -\sigma_i\int_0^t\int_{\mathbb{R}^d}(u_i)_-(-\Delta)^\alpha u_i dxds\\
&- \kappa\int_0^t\int_{\mathbb{R}^d}(u_i)_-g_\rho[u_i]dxds-\sum_{j=1}^n a_{ij}\int_0^t\int_{\mathbb{R}^d}(u_i)_+(\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*u_j)\cdot\nabla(u_i)_-dxds\\
&=: I_3+I_4+I_5.
\end{aligned}$$

Since $(u_i)_+\nabla(u_i)_-=(u_i)_+1_{\{u_i<0\}}\nabla u_i=0$, we have $I_5=0$. Moreover, by a symmetry argument (also see [3, Lemma 7.4]),

$$\begin{aligned}
I_3 &= -\frac{\sigma_i c_{d,\alpha}}{2}\int_{\mathbb{R}^d}\frac{[(u_i(x))_- - (u_i(y))_-](u_i(x)-u_i(y))}{|x-y|^{d+2\alpha}}dxdy\leq 0,\quad\text{and}\\
I_4 &\leq -\int_0^t\int_{\mathbb{R}^d}u_i(W_\rho*u_i)(u_i)_-dxds = -\int_0^t\int_{\mathbb{R}^d}(u_i)_-^2(W_\rho*u_i)dxds\\
&\leq \int_0^t\|W_\rho*u_i\|_\infty\|(u_i)_-\|_2^2ds\leq C(\rho)\int_0^t\|u_i\|_2\|(u_i)_-\|_2^2ds.
\end{aligned}$$

We conclude that for $0<t<T^*$,

$$\|(u_i)_-^2(t)\|_2^2\leq C(\rho)\int_0^t\|u_i\|_2\|(u_i)_-\|_2^2ds.$$

Since $t\mapsto\|u_i(t)\|_2$ is continuous $[0,T^*]$, we can apply the Gronwall lemma to conclude that $(u_i)_-(t)=0$ and hence $u_i(t)\geq 0$ for $t\in[0,T^*]$.

Now, we show the conservation of mass.

Lemma 6 (Conservation of mass)**.** *Let $u=u^{(\varepsilon,\kappa,\rho)}$ be a weak solution to* (19)–(20) *on* $[0,T^*]$. *Then $\|u_i(t)\|_1=\|u_i^0\|_1$ for any $t\in[0,T^*]$.*

Proof. Let $R\geq 1$, $\gamma>d$ and introduce the cutoff function $\psi_R:\mathbb{R}^d\to[0,\infty)$ by

$$\psi_R(x)=\psi_1(x/R),\quad \psi_1(x)=(1+|x|^2)^{-\gamma/2}\quad\text{for }x\in\mathbb{R}^d.$$

The following estimates hold:

$$|\nabla\psi_R(x)|\leq CR^{-1}\psi_R(x),\quad |\Delta\psi_R(x)|\leq CR^{-2}\psi_R(x)\quad\text{for }x\in\mathbb{R}^d. \tag{22}$$

We claim that

$$-(-\Delta)^\alpha\psi_R(x) \le CR^{-2\alpha}\psi_R(x) \text{ for } x\in\mathbb{R}^d, \quad \lim_{R\to\infty}\|(-\Delta)^\alpha\psi_R\|_\infty = 0. \tag{23}$$

It is sufficient to prove the first statement for $R=1$, thanks to a scaling argument, while the proof for $R=1$ is similar to that one for (15). The second statement in (23) follows from $(-\Delta)^\alpha\psi_R(x) = R^{-2\alpha}((-\Delta)^\alpha\psi_1)(x/R)$ and the property $(-\Delta)^\alpha\psi_1\in L^\infty(\mathbb{R}^d)$.

Since $\psi_R\in H^1(\mathbb{R}^d)$ for $\gamma>d$, we can use ψ_R as a test function in the weak formulation of (19):

$$\begin{aligned}\int_{\mathbb{R}^d} u_i(t)\psi_R dx - \int_{\mathbb{R}^d} u_i^0\psi_R dx &= \kappa\int_0^t\int_{\mathbb{R}^d} u_i\Delta\psi_R dxds - \kappa\int_0^t\int_{\mathbb{R}^d} g_\rho[u_i]\psi_R dxds \\ &\quad - \sigma_i\int_0^t\int_{\mathbb{R}^d} u_i(-\Delta)^\alpha\psi_R dxds - \sum_{j=1}^n a_{ij}\int_0^t\int_{\mathbb{R}^d} u_i\nabla\psi_R\cdot\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j dxds.\end{aligned} \tag{24}$$

We deduce from (22) that

$$\begin{aligned}\int_{\mathbb{R}^d} u_i(t)\psi_R dx - \int_{\mathbb{R}^d} u_i^0\psi_R dx &\le 2\kappa\int_0^t\|u_i\|_2^2 ds + CR^{-2}\int_0^t\int_{\mathbb{R}^d} u_i\psi_R dxds \\ &\quad + CR^{-1}\sum_{j=1}^n\int_0^t\|u_i\|_2\|\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j\|_2 ds \\ &\le 2\kappa\int_0^t\|u_i\|_2^2 ds + C\int_0^t\int_{\mathbb{R}^d} u_i\psi_R dxds + C\sum_{j=1}^n\int_0^t\|u_i\|_2\|u_j\|_2 ds \\ &\le C\sum_{j=1}^n\int_0^t\|u_j\|_2^2 ds + C\int_0^t\int_{\mathbb{R}^d} u_i\psi_R dxds.\end{aligned}$$

Summing this inequality over $i=1,\ldots,n$, observing that $u_i\in C^0([0,T^*];L^2(\mathbb{R}^d))$, and applying Gronwall's lemma shows that

$$\sup_{0<t<T^*}\int_{\mathbb{R}^d} u_i(t)\psi_R dx \le C(T^*).$$

The monotone convergence theorem allows us to perform the limit $R\to\infty$ leading to

$$\sup_{0<t<T^*}\int_{\mathbb{R}^d} u_i(t) dx \le C(T^*).$$

At this point, because of (22), (23), and the fact that $\int_{\mathbb{R}^d} g_\rho[u_i]dx = 0$, the limit $R\to\infty$ in (24) gives the conservation of mass:

$$\int_{\mathbb{R}^d} u_i(t)dx - \int_{\mathbb{R}^d} u_i^0 dx = 0 \quad \text{for } t\in[0,T^*],$$

finishing the proof. □

The next step is the proof of a bound for u_i in $C^0([0,T^*];L^2(\mathbb{R}^d))$, which allows us to extend the local solution globally.

Lemma 7 ($L^2(\mathbb{R}^d)$ estimate)**.** *Let $u = u^{(\varepsilon,\kappa,\rho)}$ be a weak solution to (19)–(20) on $[0,T^*]$. Then*

$$\|u_i\|_{L^\infty(0,T^*;L^2(\mathbb{R}^d))} + \sqrt{\kappa}\|\nabla u_i\|_{L^2(0,T^*;L^2(\mathbb{R}^d))} \le C(\varepsilon,T^*).$$

Proof. We use u_i as a test function in (19) and estimate in a similar way as before:

$$\begin{aligned}
&\frac{1}{2}\|u_i\|_2^2 - \frac{1}{2}\|u_i^0\|_2^2 + \kappa\int_0^t \|\nabla u_i\|_2^2 ds + \sigma_i\int_0^t \|(-\Delta)^{\alpha/2}u_i\|_2^2 ds \\
&\quad = -\kappa\int_0^t\int_{\mathbb{R}^d} g_\rho[u_i]u_i dxds - \sum_{j=1}^n a_{ij}\int_0^t\int_{\mathbb{R}^d} u_i\nabla u_i\cdot\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j dxds \\
&\quad \le -\kappa\int_0^t\int_{\mathbb{R}^d} g_\rho[u_i]u_i dxds + \frac{1}{2}\sum_{j=1}^n a_{ij}\int_0^t\int_{\mathbb{R}^d} u_i^2\Delta\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j dxds \\
&\quad \le C(\varepsilon)\sum_{j=1}^n \|u_j\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\int_0^t \|u_i\|_2^2 ds.
\end{aligned}$$

Then mass conservation and Gronwall's lemma yield the conclusion. □

We deduce from Lemma 7 that the solution u to (19)–(20) exists for all $t \ge 0$.

4. Limit in the approximate problem

We first derive some estimates uniform in $(\varepsilon,\kappa,\rho)$ and perform then the limits $\rho\to 0$, $\varepsilon\to 0$, and $\kappa\to 0$ in this order.

4.1. Uniform estimates.

A uniform bound for a moment of $u_i = u_i^{(\varepsilon,\kappa,\rho)}$ can be derived in a similar way as in Lemma 4. To make the proof rigorous, we may proceed as in the proof of the conservation of mass in Section 3.2 by testing (19) with $(1+|\cdot|^2)^{m/2}\psi_R$. This leads to the estimate

$$\sup_{0<t<T}\int_{\mathbb{R}^d}(1+|x|^2)^{m/2}u_i(t)dx \le C(\varepsilon,u^0,T), \quad \text{where } 0<m<\min\{1,2\alpha\}. \tag{25}$$

The following lemma states the entropy inequality for the approximate problem.

Lemma 8 (Entropy inequality for the approximate problem)**.** *Let $u = u^{(\varepsilon,\kappa,\rho)}$ be a weak solution to (19)–(20). Then there exists a constant $C>0$ that is independent of $(\varepsilon,\kappa,\rho)$ such that for $t>0$,*

$$\begin{aligned}
&\sum_{i=1}^n \pi_i\int_{\mathbb{R}^d} u_i(t)\log u_i(t)dx + 4\kappa\sum_{i=1}^n \pi_i\int_0^t\int_{\mathbb{R}^d}|\nabla\sqrt{u_i}|^2 dxds \\
&\quad + C\sum_{i=1}^n \sigma_i\int_0^t\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/2}\sqrt{u_i}|^2 dxds + \lambda\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i|^2 dxds \\
&\quad + \kappa\sum_{i=1}^n \pi_i\int_0^t\int_{\mathbb{R}^d} u_i(\log u_i)_+ W_\rho * u_i dxds
\end{aligned} \tag{26}$$

$$\le \sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i^0 \log u_i^0 dx + \kappa C t + \kappa C \int_0^t \int_{\mathbb{R}^d} u_i^2 dx ds,$$

recalling that $\lambda > 0$ *is the smallest eigenvalue of* $(\pi_i a_{ij}) \in \mathbb{R}^{n\times n}$*.*

Proof. The usual idea to derive the entropy estimate is to use $\pi_i \log u_i$ as a test function in the weak formulation of (19). Since this function is not an element of $L^2(0,T;H^1(\mathbb{R}^d))$, we need to regularize. Instead, we use $\pi_i(\log(u_i+\eta) - \log\eta) \in L^2(0,T;H^1(\mathbb{R}^d))$ with $0<\eta<1$ as a test function. Thanks to mass conservation, we have

$$\begin{aligned}\langle \partial_t u_i, \log(u_i+\eta) - \log\eta\rangle &= \frac{d}{dt}\int_{\mathbb{R}^d}\big((u_i+\eta)\log(u_i+\eta) - \eta\log\eta - (1+\log\eta)u_i\big)dx \\ &= \frac{d}{dt}\int_{\mathbb{R}^d}\big((u_i+\eta)\log(u_i+\eta) - \eta\log\eta\big)dx.\end{aligned}$$

Setting $H_\eta[u] = \sum_{i=1}^n \pi_i \int_{\mathbb{R}^d}((u_i+\eta)\log(u_i+\eta) - \eta\log\eta)dx$, we infer from the weak formulation of (19), after summing over $i=1,\ldots,n$, that

$$H_\eta[u(t)] - H_\eta[u^0] + 4\kappa\sum_{i=1}^n \pi_i \int_0^t\int_{\mathbb{R}^d} |\nabla\sqrt{u_i+\eta}|^2 dx ds =: I_6 + I_7 + I_8, \quad \text{where}$$

$$I_6 = -\sum_{i=1}^n \sigma_i\pi_i \int_0^t\int_{\mathbb{R}^d} \log(u_i+\eta)(-\Delta)^\alpha u_i dx ds,$$

$$I_7 = -\kappa\sum_{i=1}^n \pi_i \int_0^t\int_{\mathbb{R}^d} g_\rho[u_i]\log(u_i+\eta) dx ds,$$

$$I_8 = -\sum_{i,j=1}^n \pi_i a_{ij}\int_0^t\int_{\mathbb{R}^d} \frac{u_i}{u_i+\eta}\nabla u_i\cdot\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j dx ds.$$

We use the generalized Stroock–Varopoulos inequality (Lemma 13) to estimate I_6:

$$I_6 \le -C\sum_{i=1}^n \int_0^t\int_{\mathbb{R}^d} |(-\Delta)^{\alpha/2}\sqrt{u_i+\eta}|^2 dx ds.$$

The definition of $g_\rho[u_i]$ yields

$$\begin{aligned} I_7 &= -\kappa\sum_{i=1}^n \pi_i\int_0^t\int_{\mathbb{R}^d} u_i\log(u_i+\eta) W_\rho * u_i dx ds \\ &\quad + \kappa\sum_{i=1}^n \pi_i\int_0^t\bigg(\int_{\mathbb{R}^d}\log(u_i+\eta)\frac{e^{-|x|^2}}{\pi^{d/2}}dx\bigg)\bigg(\int_{\mathbb{R}^d} u_i W_\rho * u_i dy\bigg)ds \\ &=: I_{71} + I_{72}.\end{aligned}$$

Since the logarithm is increasing, we find that

$$I_{71} \le -\kappa\sum_{i=1}^n \pi_i\int_0^t\int_{\mathbb{R}^d} u_i\log u_i W_\rho * u_i dx ds$$

$$\le -\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}u_i(\log u_i)_+W_\rho*u_idxds+\kappa\sum_{i=1}^n\pi_i\int_0^t\|u_i(\log u_i)_-\|_2\|u_i\|_2ds$$
$$\le -\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}u_i(\log u_i)_+W_\rho*u_idxds+\kappa C\sum_{i=1}^n\int_0^t\|u_i\|_1^{1/2}\|u_i\|_2ds,$$

where we used the inequality $u_i^2(\log u_i)_-^2\le u_i$ in the last step. The inequality $\log(u_i+\eta)\le C(1+u_i)$ and mass conservation imply that

$$I_{72}\le\kappa\sum_{i=1}^n\int_0^t\bigg(C+C\int_{\mathbb{R}^d}u_i\frac{e^{-|x|^2}}{\pi^{d/2}}dx\bigg)\bigg(\int_{\mathbb{R}^d}u_iW_\rho*u_idy\bigg)ds\le\kappa C\int_0^t\|u_i\|_2^2ds.$$

We infer that

$$I_7\le\kappa Ct+\kappa C\int_0^t\|u_i\|_2^2ds-\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}u_i(\log u_i)_+W_\rho*u_idxds.$$

Finally, by the definition of $\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}$, the positive definiteness of the matrix (π_ia_{ij}), and integration by parts,

$$\begin{aligned}I_8&=-\sum_{i,j=1}^n\pi_ia_{ij}\int_0^t\int_{\mathbb{R}^d}(\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i)\cdot(\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_j)dxds\\&\quad+\sum_{i,j=1}^n\pi_ia_{ij}\int_0^t\int_{\mathbb{R}^d}\frac{\eta}{u_i+\eta}\nabla u_i\cdot\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*u_jdxds\\&\le-\lambda\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i|^2dxds+I_{81}(\eta),\end{aligned}$$

where

$$I_{81}(\eta)=-\sum_{i,j=1}^n\pi_ia_{ij}\int_0^t\int_{\mathbb{R}^d}\eta(\log(u_i+\eta)-\log\eta)\Delta\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*u_jdxds.$$

We summarize the previous estimates:

$$\begin{aligned}&H_\eta[u(t)]-H_\eta[u^0]+4\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}|\nabla\sqrt{u_i+\eta}|^2dxds\\&\quad+C\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/2}\sqrt{u_i+\eta}|^2dxds+\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}u_i(\log u_i)_+W_\rho*u_idxds\\&\quad+\lambda\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i|^2dxds\le\kappa Ct+\kappa C\int_0^t\|u_i\|_2^2ds+I_{81}(\eta).\end{aligned}\tag{27}$$

Before performing the limit $\eta\to 0$, we estimate the error term $I_{81}(\eta)$:

$$
\begin{aligned}
I_{81}(\eta) &\le C\sum_{i,j=1}^n \|\Delta\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*u_j\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))}\int_0^t\int_{\mathbb{R}^d}\eta(\log(u_i+\eta)-\log\eta)dxds \\
&\le C(\varepsilon)\sum_{i,j=1}^n \|u_j\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\int_0^t\int_{\mathbb{R}^d}\eta(\log(u_i+\eta)-\log\eta)dxds.
\end{aligned}
$$

By mass conservation, the first factor is bounded, while the second one tends to zero as $\varepsilon\to 0$. Indeed, it holds that $\eta(\log(u_i+\eta)-\log\eta)\to 0$ a.e. in $\mathbb{R}^d\times(0,T)$ as $\eta\to 0$ and $0\le\eta(\log(u_i+\eta)-\log\eta)\le u_i\in L^\infty(0,T;L^1(\mathbb{R}^d))$, and therefore, we can apply the dominated convergence theorem leading to $I_{81}(\eta)\to 0$ as $\eta\to 0$.

At this point, we can take the limit $\eta\to 0$ in (27) by applying dominated convergence, Fatou's lemma, and the weak lower semicontinuity of the $L^2(\mathbb{R}^d)$ norm to conclude the proof. □

We deduce from the upper bound for $u_i\log u_i$, mass conservation, and the moment bound that $u_i\log u_i$ is bounded in $L^1(\mathbb{R}^d)$, as stated in the following lemma.

Lemma 9. *Let $u=u^{(\varepsilon,\kappa,\rho)}$ be a weak solution to (19)–(20). Then for any $T>0$,*

$$
\|u_i\log u_i\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}\le C.
$$

Proof. The proof is similar to that one in [14, Section 2]. In fact, the result holds for any function $0\le v\in L^\infty(0,T;L^1(\mathbb{R}^d))$ satisfying

$$
\sup_{0<t<T}\int_{\mathbb{R}^d}v(t)\big(\log v(t)+(1+|x|^2)^{m/2}\big)dx\le C(T),
$$

where $m>0$. We show that $\sup_{0<t<T}\|v(t)\log v(t)\|_1\le C(T)$. For this, we write

$$
\begin{aligned}
\int_{\mathbb{R}^d}|v\log v|dx &= -\int_{\{v<1\}}v\log v dx+\int_{\{v\ge 1\}}v\log vdx \\
&= -2\int_{\{v<1\}}v\log vdx+\int_{\mathbb{R}^d}v\log vdx\le -2\int_{\{v<1\}}v\log vdx+C.
\end{aligned}
$$

We use the Cauchy–Schwarz inequality to the integral on the right-hand side:

$$
\begin{aligned}
-\int_{\{v<1\}}v\log vdx &= \int_{\{v<1\}}v^{(1-\delta)/2}v^{(1+\delta)/2}\log\frac{1}{v}dx \\
&\le \bigg(\int_{\{v<1\}}v^{1-\delta}dx\bigg)^{1/2}\bigg(\int_{\{v<1\}}v\Big(v^{\delta/2}\log\frac{1}{v}\Big)^2dx\bigg)^{1/2},
\end{aligned}
$$

where $\delta\in(0,1)$. The function $(0,1)\to\mathbb{R}$, $s\mapsto s^{\delta/2}\log(1/s)$, is bounded by a constant $C(\delta)$. Therefore, taking into account mass conservation for v and the Hölder inequality,

$$
-\int_{\{v<1\}}v\log vdx\le C(\delta)\bigg(\int_{\{v<1\}}v^{1-\delta}dx\bigg)^{1/2}
$$

$$
\begin{aligned}
&= C(\delta)\bigg(\int_{\{v<1\}}(1+|x|^2)^{m(1-\delta)/2}v(x)^{1-\delta}(1+|x|^2)^{-m(1-\delta)/2}dx\bigg)^{1/2} \\
&\le C(\delta)\bigg(\int_{\{v<1\}}(1+|x|^2)^{m/2}v(x)dx\bigg)^{(1-\delta)/2}\bigg(\int_{\{v<1\}}(1+|x|^2)^{-m(1-\delta)/(2\delta)}dx\bigg)^{\delta/2}.
\end{aligned}
$$

The moment estimate shows that the first integral is bounded, while the second one is finite if $m(1-\delta)/(2\delta) > d$ or $\delta < m/(m+2d)$. This proves the claim. $\square$

We deduce from the previous lemmas the following estimates.

Lemma 10 (Uniform estimates)**.** *Let $u = u^{(\varepsilon,\kappa,\rho)}$ be a weak solution to* (19)–(20)*. Then there exist constants $q > 1$ and $C(\varepsilon, T) > 0$, which is independent of (κ, ρ), such that for $t > 0$,*

$$\sqrt{\kappa}\|\sqrt{u_i}\|_{L^2(0,T;H^1(\mathbb{R}^d))} + \sqrt{\sigma_i}\|\sqrt{u_i}\|_{L^2(0,T;H^\alpha(\mathbb{R}^d))} \le C(\varepsilon,T), \tag{28}$$

$$\|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i\|_{L^2(0,T;L^2(\mathbb{R}^d))} + \kappa\|u_i(\log u_i)_+ W_\rho * u_i\|_{L^1(0,T;L^1(\mathbb{R}^d))} \le C(\varepsilon,T), \tag{29}$$

$$\|u_i\log u_i\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \|\partial_t u_i\|_{L^q(0,T;W^{-1,q}(\mathbb{R}^d))} \le C(\varepsilon,T). \tag{30}$$

Proof. Estimates (28) and (29) follow from Lemmas 6 and 8. The first estimate in (30) is proved in Lemma 9. It remains to prove the second estimate in (30).

Let $p > \max\{d/(1-\alpha), 2d/(1-\beta)\} > 2$ with $1/p+1/q = 1$ and use $\phi \in C^0([0,T];C_0^\infty(\mathbb{R}^d))$ as a test function in the weak formulation of (19):

$$
\begin{aligned}
&\int_0^T\langle\partial_t u_i,\phi\rangle dt =: I_9 + \cdots + I_{12}, \quad \text{where} \qquad (31)\\
&I_9 = -\kappa\int_0^T\int_{\mathbb{R}^d}\nabla u_i\cdot\nabla\phi dxdt, \\
&I_{10} = -\sigma_i\int_0^T\int_{\mathbb{R}^d}(-\Delta)^{\alpha/2}u_i(-\Delta)^{\alpha/2}\phi dxdt, \\
&I_{11} = -\kappa\int_0^T\int_{\mathbb{R}^d}g_\rho[u_i]\phi dxdt, \\
&I_{12} = -\sum_{j=1}^n a_{ij}\int_0^T\int_{\mathbb{R}^d}u_i\nabla\phi\cdot\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * (\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_j)\big)dxdt.
\end{aligned}
$$

We estimate the integrals $I_9, \ldots, I_{12}$. First, by Lemma 7 with $T^* = T$, it holds that $\sqrt{\kappa}\|\nabla u_i\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C(\varepsilon,T)$. We infer from (28) that $\sqrt{\kappa}\nabla u_i = 2\sqrt{\kappa}\sqrt{u_i}\nabla\sqrt{u_i}$ is bounded in $L^1(\mathbb{R}^d)$, i.e. $\sqrt{\kappa}\|\nabla u_i\|_{L^1(0,T;L^1(\mathbb{R}^d))} \le C(\varepsilon,T)$. Hence, since $q < 2$, it follows by interpolation that $\sqrt{\kappa}\|\nabla u_i\|_{L^q(0,T;L^q(\mathbb{R}^d))} \le C(\varepsilon,T)$. We deduce that

$$|I_9| \le \kappa\|\nabla u_i\|_{L^q(0,T;L^q(\mathbb{R}^d))}\|\nabla\phi\|_{L^p(0,T;L^p(\mathbb{R}^d))} \le C\|\phi\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}.$$

We can prove, using the generalized Stroock–Varopoulos inequality (Lemma 13) in a similar way as in Lemma 3, that $\sqrt{\sigma_i}\|(-\Delta)^{\alpha/2}u_i\|_{L^2(0,T;L^1(\mathbb{R}^d))} \le C$. Therefore, since $p > d/(1-\alpha)$,

$$|I_{10}| \le \sigma_i\|(-\Delta)^{\alpha/2}u_i\|_{L^2(0,T;L^1(\mathbb{R}^d))}\|(-\Delta)^{\alpha/2}\phi\|_{L^2(0,T;L^\infty(\mathbb{R}^d))} \le C\|\phi\|_{L^2(0,T;W^{1,p}(\mathbb{R}^d))}.$$

It follows from property (17) of $g_\rho[u_i]$, the $L^\infty(0,T;L^2(\mathbb{R}^d))$ estimate of u_i in Lemma 7, and the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ that

$$|I_{11}| \le \int_0^T \|g_\rho[u_i]\|_1\|\phi\|_\infty dt \le C\int_0^T \|u_i\|_2^2\|\phi\|_{W^{1,p}(\mathbb{R}^d)}dt \le C(\varepsilon,T)\|\phi\|_{L^p(0,T;W^{1,p}(\mathbb{R}^d))}.$$

Finally, the Hardy–Littlewood–Sobolev inequality (Lemma 12 with $r = 2d/(d+1-\beta)$) and the Hölder inequality with $1/q_2 + 1/p = 1/r$ lead to

$$\begin{aligned}|I_{12}| &\le C(\varepsilon)\int_0^T \|u_i\nabla\phi\|_r\|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_j\|_2 dt \\ &\le C(\varepsilon)\int_0^T \|u_i\|_{q_2}\|\nabla\phi\|_p\|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_j\|_2 dt \le C(\varepsilon,T)\|\nabla\phi\|_{L^p(0,T;L^p(\mathbb{R}^d))},\end{aligned}$$

where we used Lemma 7, mass conservation, the fact that $q_1 \in [1,2]$, and estimate (29) in the last step. Putting together the estimates for $I_9,\ldots,I_{12}$, we conclude the proof from (31) for $\phi \in L^p(0,T;W^{1,p}(\mathbb{R}^d))$ with $p > \max\{d/(1-\alpha), 2d/(1-\beta)\}$. □

4.2. **Limit $\rho \to 0$.** We conclude from Lemma 14 in the Appendix that

$$V := \left\{v \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d}(1+|x|^2)^{m/2}|v(x)|dx < \infty\right\}$$

is compactly embedded into $L^2(\mathbb{R}^d)$. Moreover, the embedding $L^2(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$ is continuous for any $s > 0$. The uniform $L^2(0,T;H^1(\mathbb{R}^d))$ bound in Lemma 7 and the moment bound (25) show that $(u^{(\varepsilon,\kappa,\rho)})$ is bounded in $L^2(0,T;V)$, while, by estimate (30), $(\partial_t u_i^{(\varepsilon,\kappa,\rho)})$ is bounded in $L^1(0,T;H^{-s}(\mathbb{R}^d))$. It follows from the Aubin–Lions lemma that there exists a subsequence, which is not relabeled, such that, as $\rho \to 0$,

$$u_i^{(\varepsilon,\kappa,\rho)} \to u_i \quad \text{strongly in } L^2(0,T;L^2(\mathbb{R}^d)).$$

Since $(u_i^{(\varepsilon,\kappa,\rho)})$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$ by Lemma 7, the Gagliardo–Nirenberg inequality provides a uniform bound in $L^{2+4/d}(0,T;L^{2+4/d}(\mathbb{R}^d))$. Hence, there exists $2 < p < 2+4/d$ such that

$$u_i^{(\varepsilon,\kappa,\rho)} \to u_i \quad \text{strongly in } L^p(0,T;L^p(\mathbb{R}^d)). \tag{32}$$

Given the uniform bounds in Lemma 10, it is quite standard to perform the limit $\rho \to 0$ in (19). We consider here only the term that explicitly depends on ρ, namely

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} g_\rho[u_i^{(\varepsilon,\kappa,\rho)}]\phi dxdt &= \int_0^T\int_{\mathbb{R}^d} u_i^{(\varepsilon,\kappa,\rho)}(W_\rho * u_i^{(\varepsilon,\kappa,\rho)})\phi dxdt \\ &\quad - \int_0^T\left(\int_{\mathbb{R}^d} u_i^{(\varepsilon,\kappa,\rho)}(W_\rho * u_i^{(\varepsilon,\kappa,\rho)})dx\right)\left(\int_{\mathbb{R}^d}\frac{e^{-|x|^2}}{\pi^{d/2}}\phi dx\right)dt\end{aligned}$$

for test functions $\phi \in L^2(0,T;L^\infty(\mathbb{R}^d))$. Since $\|W_\rho\|_1 = 1$ and $(u_i^{(\varepsilon,\kappa,\rho)})$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d))$, we have

$$W_\rho * u_i^{(\varepsilon,\kappa,\rho)} \rightharpoonup^* u_i \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{R}^d)).$$

This implies that, for suitable test functions,

$$\int_0^T \int_{\mathbb{R}^d} g_\rho[u_i^{(\varepsilon,\kappa,\rho)}]\phi dxdt \to \int_0^T \int_{\mathbb{R}^d} g_0[u_i]\phi dxdt \quad \text{as } \varepsilon \to 0,$$

where

$$g_0[v](x) := v(x)^2 - \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{\mathbb{R}^d} v(y)^2 dy, \quad v \in L^2(\mathbb{R}^d).$$

We have proved that the limit $u_i^{(\varepsilon,\kappa)} := u_i$ is a solution to

$$\begin{aligned} &\partial_t u_i^{(\varepsilon,\kappa)} - \kappa\Delta u_i^{(\varepsilon,\kappa)} + \sigma_i(-\Delta)^\alpha u_i^{(\varepsilon,\kappa)} + \kappa g_0[u_i^{(\varepsilon,\kappa)}] \\ &\qquad = \operatorname{div}\left(\sum_{j=1}^n a_{ij} u_i^{(\varepsilon,\kappa)} \nabla \mathcal{K}^{(\varepsilon)}_{(1-\beta)/2} * u_j^{(\varepsilon,\kappa)}\right) \quad \text{in } \mathbb{R}^d,\ t>0, \\ &u_i^{(\varepsilon,\kappa)}(\cdot,0) = u_i^0 \quad \text{in } \mathbb{R}^d,\ i=1,\ldots,n. \end{aligned} \tag{33}$$

The strong convergence (32), Fatou's lemma, and the weak lower semicontinuity of the $L^2(\mathbb{R}^d)$ norm allow us to take the limit $\rho \to 0$ in the approximate entropy inequality (26), leading to

$$\begin{aligned} &\sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i^{(\varepsilon,\kappa)}(t)\log u_i^{(\varepsilon,\kappa)}(t)dx + 4\kappa \sum_{i=1}^n \pi_i \int_0^t \int_{\mathbb{R}^d} \left|\nabla\sqrt{u_i^{(\varepsilon,\kappa)}}\right|^2 dxds \\ &\qquad + C\sum_{i=1}^n \sigma_i \int_0^t \int_{\mathbb{R}^d} \left|(-\Delta)^{\alpha/2}\sqrt{u_i^{(\varepsilon,\kappa)}}\right|^2 dxds + \lambda \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^d} |\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)}|^2 dxds \\ &\qquad + \kappa \sum_{i=1}^n \pi_i \int_0^t \int_{\mathbb{R}^d} (u_i^{(\varepsilon,\kappa)})^2 (\log u_i^{(\varepsilon,\kappa)})_+ dxds \\ &\quad \le \sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i^0 \log u_i^0 dx + \kappa Ct + \kappa C \int_0^t \int_{\mathbb{R}^d} (u_i^{(\varepsilon,\kappa)})^2 dxds. \end{aligned}$$

The last integral on the right-hand side can be controlled by the last integral on the left-hand side. Therefore,

$$\begin{aligned} &\sum_{i=1}^n \pi_i \int_{\mathbb{R}^d} u_i^{(\varepsilon,\kappa)}(t)\log u_i^{(\varepsilon,\kappa)}(t)dx + 4\kappa \sum_{i=1}^n \pi_i \int_0^t \int_{\mathbb{R}^d} \left|\nabla\sqrt{u_i^{(\varepsilon,\kappa)}}\right|^2 dxds \\ &\qquad + C\sum_{i=1}^n \sigma_i \int_0^t \int_{\mathbb{R}^d} \left|(-\Delta)^{\alpha/2}\sqrt{u_i^{(\varepsilon,\kappa)}}\right|^2 dxds + \lambda \sum_{i=1}^n \int_0^t \int_{\mathbb{R}^d} |\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} u_i^{(\varepsilon,\kappa)}|^2 dxds \end{aligned} \tag{34}$$

$$+\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}(u_i^{(\varepsilon,\kappa)})^2(\log u_i^{(\varepsilon,\kappa)})_+dxds \le \sum_{i=1}^n\pi_i\int_{\mathbb{R}^d}u_i^0\log u_i^0dx + \kappa C(t+1).$$

This shows that the uniform bounds in Lemma 10 also hold for $u^{(\varepsilon,\kappa)}$ with constants independent of ε.

Lemma 11. *The solution $u_i := u_i^{(\varepsilon,\kappa)}$ constructed above satisfies the following uniform estimates with a constant $C(T)>0$ that is independent of ε and κ:*

$$\sqrt{\kappa}\|\sqrt{u_i}\|_{L^2(0,T;H^1(\mathbb{R}^d))} + \sqrt{\sigma_i}\|\sqrt{u_i}\|_{L^2(0,T;H^\alpha(\mathbb{R}^d))} \le C(T), \tag{35}$$

$$\|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i\|_{L^2(0,T;L^2(\mathbb{R}^d))} + \kappa\|u_i^2(\log u_i)_+\|_{L^1(0,T;L^1(\mathbb{R}^d))} \le C(T), \tag{36}$$

$$\sqrt{\sigma_i}\|(-\Delta)^{\alpha/2}u_i\|_{L^2(0,T;L^1(\mathbb{R}^d))} + \|\partial_t u_i\|_{L^q(0,T;W^{-1,q}(\mathbb{R}^d))} \le C(T) \tag{37}$$

$$\|u_i\log u_i\|_{L^\infty(0,T;L^1(\mathbb{R}^d))} + \sup_{0<t<T}\int_{\mathbb{R}^d}|x|^m u_i(t)dx \le C(T), \tag{38}$$

where $q>1$.

Proof. Estimates (35)–(36) follow from (34). The first estimate in (37) is a consequence of the $L^\infty(0,T;L^1(\mathbb{R}^d))$ bound for u_i and the $L^2(0,T;L^2(\mathbb{R}^d))$ norm for $(-\Delta)^{\alpha/2}\sqrt{u_i}$; see the proof of (10). The second estimate in (37) is shown as in Lemma 10, now using the ε-independent entropy estimates. The moment estimate for u_i can be proved as in Lemma 4. Compared to (25), we are able to derive a uniform bound independent of ε. This is possible since we have an ε-independent $L^2(\mathbb{R}^d)$ bound for u_i after having performed the limit $\rho\to 0$. This bound is not available for $u_i^{(\varepsilon,\kappa,\rho)}$, since its $L^2(\mathbb{R}^d)$ estimate depends on ε; see Lemma 7. The critical term becomes, using the Hardy–Littlewood–Sobolev inequality and a cutoff function ψ_R,

$$\begin{aligned}&\bigg|\sum_{j=1}^n a_{ij}\int_0^T\int_{\mathbb{R}^d}u_i\nabla(1+|x|^2)^{m/2}\cdot\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*(\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_j)\big)\psi_R dxdt\bigg| \\ &\quad\le C\int_0^T\|u_i\|_2\|\nabla(1+|x|^2)^{m/2}\|_\infty\|\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_j\|_2dt \le C.\end{aligned}$$

Then, proceeding as in the proofs of Lemmas 4 and 6 (to handle the cutoff), we obtain the moment estimate for u_i. This estimate, together with the upper bound for $\int_{\mathbb{R}^d}u_i\log u_idx$ from (34) and the mass conservation property, imply the $L^\infty(0,T;L^1(\mathbb{R}^d))$ bound for $u_i\log u_i$, by proceeding as in the proof of Lemma 9. □

4.3. **Limit $\varepsilon\to 0$.** The uniform bounds of Lemma 11 allow us to apply the compactness result of Aubin–Lions-type in Lemma 15 below to conclude that there exists a subsequence (not relabeled) such that

$$u_i^{(\varepsilon,\kappa)} \to u_i^{(\kappa)} \quad \text{strongly in } L^2(0,T;L^2(\mathbb{R}^d)) \text{ as } \varepsilon\to 0.$$

We wish to perform the limit $\varepsilon \to 0$ in (33). The only nontrivial term is that one on the right-hand side of (33). We first notice that, by Lemma 11,

$$\nabla \widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)} \rightharpoonup \xi_i \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^d)) \tag{39}$$

for some $\xi_i \in L^2(0,T;L^2(\mathbb{R}^d))$, $i=1,\ldots,n$. To identify ξ_i, we consider

$$\int_0^T\int_{\mathbb{R}^d} \phi\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)}\big)dxdt = \int_0^T\int_{\mathbb{R}^d} u_i^{(\varepsilon,\kappa)}\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * \phi\big)dxdt, \tag{40}$$

and we wish to pass to the limit $\varepsilon \to 0$ on the right-hand side. For this, we remark that it follows from the definition of $\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}$ that

$$\begin{aligned} \|\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * \phi\|_{L^2(0,T;L^{2d/(d-1+\beta)}(\mathbb{R}^d))} &\le \|\mathcal{K}_{(1-\beta)/4} * |\phi|\|_{L^2(0,T;L^{2d/(d-1+\beta)}(\mathbb{R}^d))} \\ &\le C\|\phi\|_{L^2(0,T;L^2(\mathbb{R}^d))}. \end{aligned} \tag{41}$$

It holds that $0 \le \widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} \nearrow \mathcal{K}_{(1-\beta)/4}$ a.e. in $\mathbb{R}^d$ and $\mathcal{K}_{(1-\beta)/4}$ (the kernel of the Riesz potential) is integrable in the unit ball $B_1(0)$, while its square $\mathcal{K}^2_{(1-\beta)/4}$ is integrable in $\mathbb{R}^d \setminus B_1(0)$. Hence, we infer from Young's convolution inequality and the monotone convergence theorem that

$$\begin{aligned} &\big\|\big[\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} - \mathcal{K}_{(1-\beta)/4}\big)1_{B_1(0)}\big] * \phi\big\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\ &\qquad \le \big\|\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} - \mathcal{K}_{(1-\beta)/4}\big)1_{B_1(0)}\big\|_1 \|\phi\|_{L^2(0,T;L^2(\mathbb{R}^d))} \to 0, \\ &\big\|\big[\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} - \mathcal{K}_{(1-\beta)/4}\big)1_{\mathbb{R}^d\setminus B_1(0)}\big] * \phi\big\|_{L^2(0,T;L^2(\mathbb{R}^d))} \\ &\qquad \le \big\|\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} - \mathcal{K}_{(1-\beta)/4}\big)1_{\mathbb{R}^d\setminus B_1(0)}\big\|_2 \|\phi\|_{L^2(0,T;L^1(\mathbb{R}^d))} \to 0, \end{aligned}$$

such that for $\phi \in L^2(0,T;L^2(\mathbb{R}^d)\cap L^1(\mathbb{R}^d))$,

$$\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * \phi \to \mathcal{K}_{(1-\beta)/4} * \phi \quad \text{strongly in } L^2(0,T;L^2(\mathbb{R}^d)). \tag{42}$$

Thus, we deduce from (40) and the strong convergence of $(u_i^{(\varepsilon,\kappa)})$ in $L^2(0,T;L^2(\mathbb{R}^d))$ that

$$\begin{aligned} \int_0^T\int_{\mathbb{R}^d} \phi\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)}\big)dxdt &\to \int_0^T\int_{\mathbb{R}^d} u_i^{(\kappa)}\mathcal{K}_{(1-\beta)/4} * \phi dxdt \\ &= \int_0^T\int_{\mathbb{R}^d} \phi\big(\mathcal{K}_{(1-\beta)/4} * u_i^{(\kappa)}\big)dxdt, \end{aligned}$$

which means that

$$\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)} \rightharpoonup \mathcal{K}_{(1-\beta)/4} * u_i^{(\kappa)} \quad \text{weakly in } L^2(0,T;(L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d))').$$

Hence, we can identify the limit ξ_i in (39), leading to the convergence

$$\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4} * u_i^{(\varepsilon,\kappa)} \rightharpoonup \nabla\mathcal{K}_{(1-\beta)/4} * u_i^{(\kappa)} \quad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^d)). \tag{43}$$

We claim that a similar weak limit holds for $\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}$ instead of $\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}$. To this end, we use definition (16) of $\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}$ and convergences (42) and (43):

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^d}\phi\nabla\mathcal{K}^{(\varepsilon)}_{(1-\beta)/2}*u_i^{(\varepsilon,\kappa)}dxdt &= \int_0^T\int_{\mathbb{R}^d}\phi\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i^{(\varepsilon,\kappa)}dxdt \\
&= \int_0^T\int_{\mathbb{R}^d}\big(\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*u_i^{(\varepsilon,\kappa)}\big)\big(\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/4}*\phi\big)dxdt \\
&\to \int_0^T\int_{\mathbb{R}^d}\big(\nabla\mathcal{K}_{(1-\beta)/4}*u_i^{(\kappa)}\big)\big(\mathcal{K}_{(1-\beta)/4}*\phi\big)dxdt \\
&= \int_0^T\int_{\mathbb{R}^d}\phi\mathcal{K}_{(1-\beta)/4}*\nabla\mathcal{K}_{(1-\beta)/4}*u_i^{(\kappa)}dxdt \\
&= \int_0^T\int_{\mathbb{R}^d}\phi\nabla\mathcal{K}_{(1-\beta)/2}*u_i^{(\kappa)}dxdt
\end{aligned}$$

for any $\phi\in L^2(0,T;L^2(\mathbb{R}^d))$, where we used the representation $\mathcal{K}_s*f=(-\Delta)^{-s}f$. We infer that

$$\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/2}*u_i^{(\varepsilon,\kappa)}\rightharpoonup\nabla\mathcal{K}_{(1-\beta)/2}*u_i^{(\kappa)}\quad\text{weakly in } L^2(0,T;L^2(\mathbb{R}^d)).$$

Together with the strong $L^2(\mathbb{R}^d)$ convergence of $(u_i^{(\varepsilon,\kappa)})$, it follows that

$$u_i^{(\varepsilon,\kappa)}\nabla\widetilde{\mathcal{K}}^{(\varepsilon)}_{(1-\beta)/2}*u_j^{(\varepsilon,\kappa)}\rightharpoonup u_i^{(\kappa)}\nabla\mathcal{K}_{(1-\beta)/2}*u_j^{(\kappa)}\quad\text{weakly in } L^1(0,T;L^1(\mathbb{R}^d)).$$

These convergences allow us to perform the limit $\varepsilon\to 0$ in (33) to conclude that $u^{(\kappa)}$ solves

$$\begin{aligned}
&\partial_t u_i^{(\kappa)}-\kappa\Delta u_i^{(\kappa)}+\sigma(-\Delta)^\alpha u_i^{(\kappa)}+\kappa g_0[u_i^{(\kappa)}] \\
&\qquad=\operatorname{div}\Big(\sum_{j=1}^n a_{ij}u_i^{(\kappa)}\nabla(-\Delta)^{(\beta-1)/2}u_j^{(\kappa)}\Big)\quad\text{in }\mathbb{R}^d,\ t>0, \\
&u_i^{(\kappa)}(\cdot,0)=u_i^0\quad\text{in }\mathbb{R}^d,\ i=1,\ldots,n.
\end{aligned}\tag{44}$$

The limit $\varepsilon\to 0$ in the entropy inequality (34) leads to

$$\begin{aligned}
&\sum_{i=1}^n\pi_i\int_{\mathbb{R}^d}u_i^{(\kappa)}(t)\log u_i^{(\kappa)}(t)dx+4\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}\Big|\nabla\sqrt{u_i^{(\kappa)}}\Big|^2dxds \\
&+C\sum_{i=1}^n\sigma_i\int_0^t\int_{\mathbb{R}^d}\Big|(-\Delta)^{\alpha/2}\sqrt{u_i^{(\kappa)}}\Big|^2dxds+\lambda\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}|\nabla(-\Delta)^{(\beta-1)/4}u_i^{(\kappa)}|^2dxds \\
&+\kappa\sum_{i=1}^n\pi_i\int_0^t\int_{\mathbb{R}^d}(u_i^{(\kappa)})^2(\log u_i^{(\kappa)})_+dxds\le\sum_{i=1}^n\pi_i\int_{\mathbb{R}^d}u_i^0\log u_i^0dx+\kappa C(t+1)
\end{aligned}\tag{45}$$

for $t>0$. Estimates (35)–(38) also hold for $u^{(\kappa)}$ with the exception that the first bound in (36) is replaced by

$$\|\nabla(-\Delta)^{(\beta-1)/4}u_i^{(\kappa)}\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C, \quad i=1,\ldots,n. \tag{46}$$

4.4. **Limit $\kappa\to 0$.** We deduce from (12) with $q=2(d+\beta+1)/d$ and the κ-uniform bounds for $u_i^{(\kappa)}$ that $(u_i^{(\kappa)})$ is bounded in $L^{2(d+\beta+1)/d}(0,T;L^2(\mathbb{R}^d))$. Together with estimate (46), we conclude that $(u_i^{(\kappa)})$ is bounded in $L^2(0,T;H^{(\beta+1)/2}(\mathbb{R}^d))$.

We claim that the embedding $H^{(\beta+1)/2}(\mathbb{R}^d)\cap L^1(\mathbb{R}^d;|x|dx)\hookrightarrow L^2(\mathbb{R}^d)$ is compact. This claim follows from [10, Corollary 7.2], applied to balls (which are extension domains due to [10, Theorem 5.4]), [4, Lemma 1], and a Cantor diagonal argument. In view of the moment estimate for $u_i^{(\kappa)}$ and the $L^q(0,T;W^{-1,q}(\mathbb{R}^d))$ bound for $\partial_t u_i^{(\kappa)}$, the Aubin–Lions lemma yields, up to a subsequence, the convergence

$$u_i^{(\kappa)}\to u_i \quad \text{strongly in } L^2(0,T;L^2(\mathbb{R}^d)) \text{ as } \kappa\to 0.$$

All the terms in (44) have been already estimated in Section 2 except those depending explicitly on κ, in particular

$$\kappa g_0[u_i^{(\kappa)}](x) = \kappa u_i^{(\kappa)}(x)^2 - \kappa\frac{e^{-|x|^2}}{\pi^{d/2}}\int_{\mathbb{R}^d} u_i^{(\kappa)}(y)^2dy.$$

The strong convergence of $u_i^{(\kappa)}$ in $L^2(\mathbb{R}^d))$ implies that

$$\kappa g_0[u_i^{(\kappa)}]\to 0 \quad \text{strongly in } L^1(0,T;L^1(\mathbb{R}^d)).$$

Therefore, we can perform the limit $\kappa\to 0$ in (44) to infer that u is a weak solution to (1). The entropy inequality (45) for $u^{(\kappa)}$ and Fatou's lemma yield in the limit $\varepsilon\to 0$ the entropy inequality (6) for u. This finishes the proof of Theorem 1.

Appendix A. Auxiliary results

We collect some results from fractional calculus used in this paper. The following lemma can be found in [20, Chap. V, Sect. 1.2].

Lemma 12 (Hardy–Littlewood–Sobolev inequality)**.** *Let $0<s<1$ and $1<p<\infty$. Then there exists a constant $C>0$ such that for all $u\in L^p(\mathbb{R}^d)$,*

$$\|(-\Delta)^{-s}u\|_q \le C\|u\|_p, \quad \text{where } \frac{1}{p}=\frac{1}{q}+\frac{2s}{d}.$$

The following Stroock–Varopoulos-type inequality is known even for general functions; see, e.g., [19, Lemma 7.2].

Lemma 13 (Generalized Stroock–Varopoulos inequality)**.** *Let $u\in H^s(\mathbb{R}^d)$ be such that $u\ge 0$ in $\mathbb{R}^d$ and $\log(u)(-\Delta)^s u\in L^1(\mathbb{R}^d)$, where $0<s<1$. Then*

$$\int_{\mathbb{R}^d}\log(u)(-\Delta)^s u dx \ge 4\int_{\mathbb{R}^d}|(-\Delta)^{s/2}\sqrt{u}|^2dx.$$

Proof. A symmetry argument shows that

$$\int_{\mathbb{R}^d}\log(u)(-\Delta)^s u dx = \frac{c_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(\log u(x)-\log u(y))}{|x-y|^{d+2s}}dxdy.$$

Elementary computations yield

$$\begin{aligned}&(u(x)-u(y))(\log u(x)-\log u(y))\\ &\quad = 4\big(\sqrt{u(x)}-\sqrt{u(y)}\big)^2\frac{\sqrt{u(x)}+\sqrt{u(y)}}{2(\sqrt{u(x)}-\sqrt{u(y)})}\big(\log\sqrt{u(x)}-\log\sqrt{u(y)}\big).\end{aligned}$$

We claim that

$$\frac{\sqrt{u(x)}+\sqrt{u(y)}}{2(\sqrt{u(x)}-\sqrt{u(y)})}\big(\log\sqrt{u(x)}-\log\sqrt{u(y)}\big)\geq 1.$$

Notice that the above relation holds in the limit $u(x)\to u(y)$ as $x\to y$. Therefore, we can assume without loss of generality that $u(x)>u(y)$. Defining $Z=\log\sqrt{u(x)}-\log\sqrt{u(y)}>0$, the previous inequality is equivalent to

$$\frac{e^Z+1}{e^Z-1}\frac{Z}{2}\geq 1 \quad\Longleftrightarrow\quad e^Z+1\geq\frac{e^Z-1}{Z/2}.$$

Multiplying both sides of the inequality by $e^{-Z/2}/2$ yields the elementary relation $\sinh(Z/2)\leq(Z/2)\cosh(Z/2)$, which is true.

This gives, using the same symmetry argument as before,

$$\begin{aligned}\int_{\mathbb{R}^d}\log u(-\Delta u)^s u dx &\geq 2c_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\sqrt{u(x)}-\sqrt{u(y)})^2}{|x-y|^{d+2s}}dxdy\\ &= 4\int_{\mathbb{R}^d}\sqrt{u}(-\Delta)^s\sqrt{u}dx = 4\int_{\mathbb{R}^d}|(-\Delta)^{s/2}\sqrt{u}|^2dx,\end{aligned}$$

finishing the proof. □

The following lemma can be proved exactly as in [4, Lemma 1].

Lemma 14 (Compactness)**.** *Let $d\geq 2$, $1\leq p<d$, $m>0$, and $0<r\leq p$. Then the space*

$$\Big\{v\in W^{1,p}(\mathbb{R}^d):\int_{\mathbb{R}^d}(1+|x|^2)^{m/2}|v(x)|^r dx<\infty\Big\}$$

is compactly embedded into $L^q(\mathbb{R}^d)$ for any $\max\{1,m\}\leq q<dp/(d-p)$.

The previous lemma allows us to prove a compactness result in $\mathbb{R}^d$ of Aubin–Lions type.

Lemma 15. *Let $d\geq 1$, $T>0$, $m>0$, $s\geq 0$, and let (u_ε) be a family of nonnegative functions satisfying*

$$\begin{aligned}\|\sqrt{u_\varepsilon}\|_{L^2(0,T;H^1(\mathbb{R}^d))}+\|\partial_t u_\varepsilon\|_{L^1(0,T;H^{-s}(\mathbb{R}^d))}&\leq C,\\ \|u_\varepsilon^2(\log u_\varepsilon)_+\|_{L^1(0,T;L^1(\mathbb{R}^d))}+\|(1+|\cdot|^2)^{m/2}u_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^d))}&\leq C\end{aligned}$$

for some $C > 0$ independent of $\varepsilon > 0$. Then, up to a subsequence,

$$u_\varepsilon \to u \quad \textit{strongly in } L^2(0,T;L^2(\mathbb{R}^d)) \textit{ as } \varepsilon \to 0.$$

Proof. The bounds for (u_ε) imply that $\nabla u_\varepsilon = 2\sqrt{u_\varepsilon}\nabla\sqrt{u_\varepsilon}$ is bounded in $L^2(0,T;L^1(\mathbb{R}^d))$ and consequently, (u_ε) is bounded in $L^2(0,T;W^{1,1}(\mathbb{R}^d))$. By Lemma 14, $V := \{v \in W^{1,1}(\mathbb{R}^d) : \int_{\mathbb{R}^d}(1+|x|^2)^{m/2}v(x)dx < \infty\}$ is compactly embedded into $L^q(\mathbb{R}^d)$ for any $1 \le q < d/(d-1)$.

If $s \ge d/2$, the embedding $H^s(\mathbb{R}^d) \hookrightarrow L^{q'}(\mathbb{R}^d)$ for $q' = q/(q-1) > d$ implies that $L^q(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$ is continuous. Thus, we can apply the standard Aubin–Lions lemma with the spaces $V \hookrightarrow L^q(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$. If $s < d/2$, it holds that $H^{d/2}(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$ and $H^{-s}(\mathbb{R}^d) \hookrightarrow H^{-d/2}(\mathbb{R}^d)$ and consequently, $(\partial_t u_\varepsilon)$ is bounded in $L^1(0,T;H^{-d/2}(\mathbb{R}^d))$. In any case, the Aubin–Lions lemma can be applied with $V \hookrightarrow L^q(\mathbb{R}^d) \hookrightarrow H^{-\max\{d/2,s\}}(\mathbb{R}^d)$. Thus, there exists a subsequence of (u_ε), which is not relabeled, such that $u_\varepsilon \to u$ strongly in $L^2(0,T;L^q(\mathbb{R}^d))$ as $\varepsilon \to 0$.

It remains to show that this convergence holds in $L^2(0,T;L^2(\mathbb{R}^d))$. To this end, we observe that there exists $C > 0$ such that $f(z) := z^2\log(1+z^2) \le C(1+z^2(\log z^2)_+)$ for $z \in \mathbb{R}$, and for any $1 < q < 2$ and $\delta > 0$, there exists $C(\delta) > 0$ such that $z^2 \le \delta f(z) + C(\delta)|z|^q$ for $s \in \mathbb{R}$. Since f is even, increasing on $[0,\infty)$, and convex, this gives with $z = (u_\varepsilon - u)/2$ and for any $\delta > 0$,

$$\begin{aligned} \frac14\|u_\varepsilon - u\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} &\le \delta\int_0^T\int_{\mathbb{R}^d} f\left(\frac{u_\varepsilon - u}{2}\right)dxdt + C(\delta)\|u_\varepsilon - u\|^q_{L^q(0,T;L^q(\mathbb{R}^d))} \\ &\le \delta\int_0^T\int_{\mathbb{R}^d} f\left(\frac{u_\varepsilon + u}{2}\right)dxdt + C(\delta)\|u_\varepsilon - u\|^q_{L^q(0,T;L^q(\mathbb{R}^d))} \\ &\le \frac{\delta}{2}\int_0^T\int_{\mathbb{R}^d}(f(u_\varepsilon) + f(u))dxdt + C(\delta)\|u_\varepsilon - u\|^q_{L^q(0,T;L^q(\mathbb{R}^d))}. \end{aligned}$$

We apply the limes superior $\varepsilon \to 0$ to both sides and use the strong convergence of (u_ε):

$$\limsup_{\varepsilon\to0}\|u_\varepsilon - u\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \le 2\delta\limsup_{\varepsilon\to0}\int_0^T\int_{\mathbb{R}^d}(f(u_\varepsilon)+f(u))dxdt \le \delta C.$$

Since $\delta > 0$ is arbitrary, the conclusion follows. □

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