Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

SMALL VELOCITY AND FINITE TEMPERATURE VARIATIONS IN KINETIC RELAXATION MODELS

Kazuo Aoki

Department of Mechanical Engineering and
Science and Advanced Research Institute of Fluid Scienceand Engineering,
Graduate School of Engineering, Kyoto University, Kyoto 606-8501, Japan

Ansgar Jüngel

Institute for Analysis and Scientific Computing,
Vienna University of Technology, Wiedner Hauptstr. 8-10, 1040 Wien, Austria

Peter A. Markowich

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom

(Communicated by the associate editor name)

Abstract. A small Knuden number analysis of a kinetic equation in the diffusive scaling is performed. The collision kernel is of BGK type with a general local Gibbs state. Assuming that the flow velocity is of the order of the Knudsen number, a Hilbert expansion yields a macroscopic model with finite temperature variations, whose complexity lies in between the hydrodynamic and the energy-transport equations. Its mathematical structure is explored and macroscopic models for specific examples of the global Gibbs state are presented.

1. **Introduction.** Macroscopic models for averaged physical quantities can be derived from kinetic equations employing a moment method and a suitable closure condition. Depending on the number of moments and the assumptions on the collision operator in the kinetic equation, a hierarchy of macroscopic models is obtained, ranging from drift-diffusion, energy-transport, and hydrodynamic equations, for instance. In this paper, we assume that the collisions can be described by a BGK-type operator, named after Bhatnagar, Gross, and Krook [7, 20]. BGK models allow for a simplification of the collisional kinetic phase-space equations and have been used,

2000 *Mathematics Subject Classification.* Primary: 82C40, 35B40, 45K05; Secondary: 35M10.
Key words and phrases. Kinetic equation, diffusive limit, Gibbs state, hydrodynamic equations, energy-transport equations.

The second author acknowledges partial support from the Austrian Science Fund (FWF), grant P20214 and WK "Differential Equations", the German Science Foundation (DFG), grant JU 359/7, and the Austrian-Croatian Project of the Austrian Exchange Service (ÖAD). Part of this research was carried out during the stay of the second author at the institute of the first author; support by the exchange program of the Austrian BMWF and the Japanese JSPS is acknowledged. The work of the last author is supported by Award No. KUK-I1-007-43, funded by the King Abdullah University of Science and Technology (KAUST), and by his Royal Society Wolfson Research Merit Award.

for instance, in the Boltzmann equation for gas dynamics [9, 16], for charge transport in semiconductors [14], and for transport models of granular media [19]. BGK models are based on the assumption that large deviations from equilibrium can be described by the equilibrium distribution when certain parameters are position and time dependent. The global equilibrium is determined by the Gibbs state with constant position-space density, quasi-Fermi level, and temperature. These constants are replaced by position and time dependent functions, which are obtained from the physical conservation laws (or collisional invariants), leading to the local Gibbs state, and relaxation of the phase-space distribution to the local Gibbs state is assumed.

In ideal rarefied gas dynamics, conservation of mass, momentum, and energy holds for the particle ensemble. Then the equilibrium function (Maxwellian) has five parameters, represented by the position density, the (three-dimensional) mean velocity, and the temperature. A Hilbert expansion around the local equilibrium leads to the hydrodynamic or Euler equations of gas dynamics [16]. Rigorous results for this expansion have been proved since several decades, see e.g. [4, 5, 8]. In some physical applications, such as semiconductor theory, momentum is transferred to the crystal lattice such that only mass and energy are conserved. The Gibbs state has only two parameters, position density and temperature, and the moment method leads to the energy-transport equations, see e.g. [3, 6, 18].

In this paper, we are interested in a situation lying in between the above two cases. We suppose that mass, momentum, and energy are conserved but we allow for small velocity variations. We scale the kinetic BGK equation with general Gibbs states by using diffusive length and time scales, and we perform the formal diffusive limit. The resulting model consists of the mass conservation equation

$$\partial_t n + \mathrm{div}_x(nu_1) = 0, \tag{1}$$

where n is the position density and u_1 the (first-order) mean velocity; the momentum equation

$$\partial_t(nu_1) + \mathrm{div}_x(nu_1 \otimes u_1) + \nabla_x q - n\nabla_x V = \frac{2}{3}(ne)_0 \Delta_x u_1, \tag{2}$$

where q is the (second-order) pressure, V the external potential, and $(ne)_0$ the (zeroth-order) energy density, which is constant in space; the energy equation

$$\partial_t(ne)_0 + \frac{5}{3}(ne)_0 \mathrm{div}_x u_1 - \mathrm{div}_x(D_1\nabla_x\phi_1 + D_2\nabla_x\phi_2) = 0, \tag{3}$$

where D_1 and D_2 are diffusion coefficients, $(\phi_1, \phi_2) = (\mu/T, -1/T)$ are the entropy variables, μ is the chemical potential, and T the particle temperature. The entropy variables can be computed implicitly from $(n, (ne)_0)$ and vice versa. Moreover, it holds

$$\nabla_x(ne)_0 = D_0\nabla_x\phi_1 + D_1\nabla_x\phi_2 = 0 \tag{4}$$

for diffusion coefficients D_0 and D_1. The symmetric diffusion matrix

$$\begin{pmatrix} D_0 & D_1 \\ D_1 & D_2 \end{pmatrix}$$

is for all considered examples positive definite, thus revealing a dissipative structure which is typical for energy-transport models. In particular, we show that there exists a (negative) entropy which is nonincreasing in time (see Proposition 2). The positive definiteness of the diffusion matrix is related to the existence of an entropy functional, see [10, 14].

Both subsystems (1)-(2) and (3)-(4) are coupled through the mean velocity and the entropy variables. The form of the diffusion coefficients depends on the choice of the global Gibbs state. We specify the above model by taking Maxwell, Fermi-Dirac, and Bose-Einstein distributions.

The mass and momentum equations (1)-(2) represent the Euler equations with the viscous term $\frac{2}{3}(ne)_0\Delta_x u_1$. This term is related to the viscosity in the Navier-Stokes equations. Indeed, introducing the Lamé viscosity coefficients $\mu = \lambda = \frac{2}{3}(ne)_0$ in the Navier-Stokes viscous stress tensor $S = \mu(\nabla_x u_1 + (\nabla_x u_1)^\top) + \lambda \mathrm{div}_x u_1\, \mathbb{I}$ [13], where $\mathbb{I}$ denotes the identity matrix, we obtain $\mathrm{div}_x S = \frac{2}{3}(ne)_0\Delta_x u_1$, since $(ne)_0$ is spatially constant. In particular, the energy equation (3) is an ordinary differential equation for $(ne)_0$ (see Remark 2), which can be written as $\partial_t(ne)_0 + \frac{5}{3}g(t)(ne)_0 = 0$, where $g(t)$ is defined by

$$\mathrm{div}_x u_1 = g(t) + \frac{3}{5}(ne)_0^{-1}\mathrm{div}_x(D_1\nabla_x\phi_1 + D_2\nabla_x\phi_2). \tag{5}$$

This equation can be considered as a non-standard constraint for the pressure q in (2). In the incompressible case $\mathrm{div}_x u_1 = 0$, the pressure q can be determined by standard methods. In general, the system (1)-(4) represents a compressible situation with the “pseudo-incompressibility” condition (5). Similar constraints appear in the low-Mach number limit of some Euler equations [2, 12] (see Remark 1).

The paper is organized as follows. In Section 2, the scaled kinetic BGK model is introduced and the formal diffusive limit is performed. The mathematical structure of the macroscopic model is examined in Section 3. Finally, examples for the global Gibbs state are considered in Section 4.

2. Formal macroscopic limit.

2.1. Scaled kinetic model.

We assume that the evolution of the particles is governed by the diffusion scaled Boltzmann-type equation

$$\varepsilon^2\partial_t f + \varepsilon\big(v\cdot\nabla_x f + \nabla_x\Phi\cdot\nabla_v f\big) = G[f] - f, \quad x, v \in \mathbb{R}^3,\ t > 0, \tag{6}$$

with initial datum $f(x,v,0) = f_I(x,v)$ for $x,\ v \in \mathbb{R}^3$. The distribution function $f(x,v,t)$ depends on the position-velocity variable (x,v) and on time $t \geq 0$. We suppose in the following that the external potential $\Phi(x,t)$ is a given function and that the Knudsen number ε is small (compared to one). The collision model is a simple BGK-type operator with the local Gibbs state $G[f]$, defined by

$$G[f] = \gamma(E_f), \quad E_f = \frac{|v-u|^2}{2T} - \frac{\mu}{T}. \tag{7}$$

Here, γ is a nonincreasing and nonnegative continuous function, and the chemical potential μ (which is the Gibbs free energy per unit mass) and the temperature T are implicitly given by the conditions

$$\int_{\mathbb{R}^3} G[f]dv = n_f := \int_{\mathbb{R}^3} f dv, \tag{8}$$

$$\int_{\mathbb{R}^3} G[f]v dv = (nu)_f := \int_{\mathbb{R}^3} f v dv, \tag{9}$$

$$\int_{\mathbb{R}^3} G[f]\frac{|v|^2}{2}dv = (ne)_f := \int_{\mathbb{R}^3} f\frac{|v|^2}{2}dv, \tag{10}$$

where n_f denotes the particle density, $(nu)_f$ the momentum density, and $(ne)_f$ the internal energy density. The mean velocity u is given by $u = (nu)_f/n_f$. The above dependency of (u, μ, T) on $(n_f, (nu)_f, (ne)_f)$ is written more explicitly as

$$\begin{pmatrix} n_f \\ (nu)_f \\ (ne)_f \end{pmatrix} = \int_{\mathbb{R}^3} \gamma\Big(\frac{|v-(nu)_f/n_f|^2}{2T(n_f,(nu)_f,(ne)_f)} - \frac{\mu(n_f,(nu)_f,(ne)_f)}{T(n_f,(nu)_f,(ne)_f)}\Big)\begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv.$$

We impose the following assumptions:

- The function γ is nonincreasing, nonnegative, and continuously differentiable on $\mathbb{R}$, the integral $\int_{\mathbb{R}} \gamma(E)E^3 dE$ exists, and $\lim_{E\to\infty} \gamma(E)E^4 = 0$.
- The potential Φ is bounded from above (such that the energy density $n_f|u_f|^2/2$ $-n_f\Phi$ is bounded from below).
- The mapping $(u, \mu, T) \mapsto (n_f, (nu)_f, (ne)_f)$ is invertible.

It should be noted that equation (6) with $G[f] = M[f]$, where $M[f]$ is the local Maxwellian based on f (see Section 4.1), is different from the standard BGK model in gas dynamics [7, 20]. In the latter model, the term $M[f] - f$ is multiplied by the local number density n_f, so that the quadratic nonlinearity in f of the original Boltzmann equation is kept at least in the so-called loss term. This difference leads to a difference in the resulting fluid-dynamic equations (see the last paragraph of Section 2.2).

We wish to derive a macroscopic model from the kinetic model (6)-(7) by performing a Hilbert expansion and a moment method. To this end, we impose two conditions on the flow velocity and the external potential in order to obtain effects on finite temperature variations:

$$u = O(\varepsilon), \quad \Phi = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{11}$$

We introduce the scaled potential $\Phi = \varepsilon^2 V$. The above assumptions have been used to derive the “ghost-effect” system for the nonlinear Boltzmann equation (see e.g. [16, Sec. 3.3] or [17]). Setting $u = 0$ and $T = 1$ in (7) and discarding conditions (9)-(10), we obtain the relaxation-time kinetic model for the mass transport. This model, together with its rigorous macroscopic limit, has been studied by Dolbeault et al. in [11]. With vanishing velocity $u = 0$ and without condition (9), the corresponding kinetic model and macroscopic limit for mass and energy transport has been considered by Aoki et al. in [3]. If the velocity u is of order one, the moment method leads to the hydrodynamic (or Euler) equations. Therefore, we expect that the moment model under assumptions (11) is of a complexity which is in between the energy-transport model (due to finite temperature variations) and the hydrodynamic model (due to the influence of the fluid velocity), see section 3 for details.

2.2. **Hilbert expansion.** We expand the distribution function $f = f_0+\varepsilon f_1+\varepsilon^2 f_2+O(\varepsilon^3)$, the moments $n_f = n_0+\varepsilon n_1+O(\varepsilon^2)$, $(ne)_f = (ne)_0+\varepsilon(ne)_1+\varepsilon^2(ne)_2+O(\varepsilon^2)$, and the energy $E_f = E_0 + O(\varepsilon)$ for $\varepsilon \to 0$. By assumption (11), we can expand $u = \varepsilon u_1 + O(\varepsilon^2)$. The formal limit $\varepsilon \to 0$ in the BGK model (6) yields

$$f_0 = G[f_0] = \gamma(E_0), \quad E_0 = \frac{|v|^2}{2T_0} - \frac{\mu_0}{T_0}. \tag{12}$$

Then, expanding $G[f] = G[f_0] + \varepsilon G_1[f_0, f_1] + \varepsilon^2 G_2[f_0, f_1, f_2] + O(\varepsilon^3)$, we find that

$$\int_{\mathbb{R}^3} G[f_0] \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = \begin{pmatrix} n_0 \\ 0 \\ (ne)_0 \end{pmatrix}, \quad \int_{\mathbb{R}^3} G_1[f_0, f_1] \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = \begin{pmatrix} n_1 \\ n_0 u_1 \\ (ne)_1 \end{pmatrix}. \tag{13}$$

Inserting the Hilbert expansion of f in the BGK model (6) and identifying the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms gives

$$f_1 = G_1[f_0, f_1] - v \cdot \nabla_x G[f_0], \quad f_2 = G_2[f_0, f_1, f_2] - \partial_t G[f_0] - v \cdot \nabla_x f_1. \tag{14}$$

Next, we multiply (6) by $\kappa(v) \in \{1, v, |v|^2/2\}$, integrate over $\mathbb{R}^3$, and identify terms of equal power of ε:

$$O(\varepsilon): \quad \mathrm{div}_x \int_{\mathbb{R}^3} G[f_0]\kappa(v) v dv = 0, \tag{15}$$

$$O(\varepsilon^2): \quad \partial_t \int_{\mathbb{R}^3} G[f_0]\kappa(v) dv + \mathrm{div}_x \int_{\mathbb{R}^3} f_1 v \kappa(v) dv = 0, \tag{16}$$

$$\begin{aligned} O(\varepsilon^3): \quad & \partial_t \int_{\mathbb{R}^3} f_1 \kappa(v) dv + \mathrm{div}_x \int_{\mathbb{R}^3} f_2 \kappa(v) v dv \\ & - \nabla_x V \cdot \int_{\mathbb{R}^3} G[f_0] \nabla_v \kappa(v) dv = 0, \end{aligned} \tag{17}$$

where $\kappa(v)v = v \otimes v$ when $\kappa(v) = v$.

Step 1: expansion of the stress tensor. Defining the stress tensor P by

$$P = \int_{\mathbb{R}^3} G[f](v-u) \otimes (v-u) dv,$$

we can expand, employing (13) and $nu \otimes u = n_0 u_1 \otimes u_1 + O(\varepsilon^2)$,

$$\int_{\mathbb{R}^3} G[f] v \otimes v dv = P + nu \otimes u = P_0 + \varepsilon P_1 + \varepsilon^2 (P_2 + n_0 u_1 \otimes u_1) + O(\varepsilon^3), \tag{18}$$

where

$$\begin{aligned} P_0 &= \frac{2}{3}(ne)_0 \mathbb{I} = \int_{\mathbb{R}^3} G[f_0] v \otimes v dv, \\ P_1 &= \int G_1[f_0, f_1] v \otimes v dv, \\ P_2 &= \int_{\mathbb{R}^3} G_2[f_0, f_1, f_2] v \otimes v dv - n_0 u_1 \otimes u_1, \end{aligned}$$

and $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{3\times 3}$. In fact, these tensors may be identified with scalars since, using (8)-(10),

$$\begin{aligned} P &= \frac{1}{3} \int_{\mathbb{R}^3} G[f] |v-u|^2 dv \, \mathbb{I} = \frac{1}{3} \int_{\mathbb{R}^3} G[f](|v|^2 - 2u \cdot v + |u|^2) dv \, \mathbb{I} \\ &= \frac{1}{3}(2ne - n|u|^2) \mathbb{I}. \end{aligned}$$

The expansion $P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + O(\varepsilon^3)$ then gives

$$P_0 = \frac{2}{3}(ne)_0 \, \mathbb{I}, \quad P_1 = \frac{2}{3}(ne)_1 \, \mathbb{I}, \quad P_2 = \frac{1}{3}\big(2(ne)_2 - n_0|u_1|^2\big) \, \mathbb{I}.$$

In particular, we may write $P_1 = p_1 \, \mathbb{I}$ and $P_2 = p_2 \mathbb{I}$, where

$$p_1 = \frac{2}{3}(ne)_1, \quad p_2 = \frac{1}{3}\big(2(ne)_2 - n_0|u_1|^2\big).$$

Equation (15) with $\kappa(v) = v$ implies that

$$0 = \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v \otimes v dv = \frac{1}{3}\nabla_x \int_{\mathbb{R}^3} G[f_0]|v|^2 dv = \frac{2}{3}\nabla_x (ne)_0$$

and hence $\operatorname{div}_x P_0 = \frac{2}{3}\nabla_x (ne)_0 = 0$. Notice that (15) is automatically satisfied for $\kappa(v) = 1$ and $\kappa(v) = |v|^2/2$ since $G[f_0]$ is an even function in v.

Step 2: moments of f_1 and f_2. We compute, using (14), (13), and $\operatorname{div}_x P_0 = 0$,

$$\int_{\mathbb{R}^3} f_1 v dv = \int_{\mathbb{R}^3} G_1[f_0, f_1] v dv - \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v \otimes v dv = n_0 u_1. \tag{19}$$

Next, by (18),

$$\begin{aligned}\int_{\mathbb{R}^3} f_1 v \otimes v dv &= \int_{\mathbb{R}^3} G_1[f_0, f_1] v \otimes v dv - \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v \otimes v \otimes v dv \\ &= \int_{\mathbb{R}^3} G_1[f_0, f_1] v \otimes v dv = P_1,\end{aligned} \tag{20}$$

since $G[f_0] v \otimes v \otimes v$ is an odd function in v and thus, its integral vanishes. A computation shows that

$$\begin{aligned}0 &= \int_{\mathbb{R}^3} G[f](v_i - u_i)(v_j - u_j)(v_k - u_k) dv \\ &= \int_{\mathbb{R}^3} G[f] v_i v_j v_k dv - \big(P_{ij} u_k + P_{ik} u_j + P_{jk} u_i\big) + 2n u_i u_j u_k\end{aligned}$$

for $i, j, k = 1, 2, 3$. Expanding the right-hand side and identifying the terms of order $O(\varepsilon)$ gives

$$\int_{\mathbb{R}^3} G_1[f_0, f_1] v_i v_j v_k dv = P_{0,ij} u_{1,k} + P_{0,ik} u_{1,j} + P_{0,jk} u_{1,i}.$$

This allows us to compute the third-order moment of f_1:

$$\begin{aligned}\int_{\mathbb{R}^3} f_1 v_i v_j v_k dv &= \int_{\mathbb{R}^3} G_1[f_0, f_1] v_i v_j v_k dv - \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v_i v_j v_k v dv \\ &= P_{0,ij} u_{1,k} + P_{0,ik} u_{1,j} + P_{0,jk} u_{1,i} - \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v_i v_j v_k v dv \\ &= \frac{2}{3}(ne)_0 \big(\delta_{ij} u_{1,k} + \delta_{ik} u_{1,j} + \delta_{jk} u_{1,i}\big) - \operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v_i v_j v_k v dv.\end{aligned}$$

Differentiating the last integral, we obtain from (12)

$$\begin{aligned}\operatorname{div}_x \int_{\mathbb{R}^3} G[f_0] v_i v_j v_k v dv &= \int_{\mathbb{R}^3} \gamma'(E_0) \left(-\frac{|v|^2}{2} \nabla_x \Big(-\frac{1}{T_0} \Big) - \nabla_x \Big(\frac{\mu_0}{T_0} \Big) \right) \cdot v v_i v_j v_k dv \\ &= C_1^{ijk} \cdot \nabla_x \phi_1 + C_2^{ijk} \cdot \nabla_x \phi_2,\end{aligned}$$

where $(\phi_1, \phi_2) = (\mu_0/T_0, -1/T_0)$ are the entropy variables and the coefficients $C_m = (C_m^{ijk\ell}) \in \mathbb{R}^{3\times3\times3\times3}$, $C_m^{ijk} = (C_m^{ijk\ell}) \in \mathbb{R}^3$, $m = 0, 1$, are defined by

$$C_1^{ijk\ell} = -\int_{\mathbb{R}^3} \gamma'(E_0) v_i v_j v_k v_\ell dv, \quad C_2^{ijk\ell} = -\int_{\mathbb{R}^3} \gamma'(E_0) v_i v_j v_k v_\ell \frac{|v|^2}{2} dv. \tag{21}$$

We conclude that for $i,j,k,\ell=1,2,3$

$$\int_{\mathbb{R}^3} f_1 v_i v_j v_k dv = \frac{2}{3}(ne)_0\big(\delta_{ij}u_{1,k}+\delta_{ik}u_{1,j}+\delta_{jk}u_{1,i}\big) - \big(C_1^{ijk}\cdot\nabla_x\phi_1 + C_2^{ijk}\cdot\nabla_x\phi_2\big). \tag{22}$$

In particular, after setting $i=j$ and summing over $i=1,2,3$,

$$\int_{\mathbb{R}^3} f_1\frac{|v|^2}{2}v_k dv = \frac{5}{3}(ne)_0 u_{1,k} - \big(D_1^k\cdot\nabla_x\phi_1 + D_2^k\cdot\nabla_x\phi_2\big), \tag{23}$$

where the coefficients are defined as

$$D_1^k = \frac{1}{2}\sum_{i=1}^3 C_1^{iik}, \quad D_2^k = \frac{1}{2}\sum_{i=1}^3 C_2^{iik}. \tag{24}$$

Finally, we compute the second-order moment of f_2:

$$\begin{aligned}\int_{\mathbb{R}^3} f_2 v\otimes v dv &= \int_{\mathbb{R}^3} G_2[f_0,f_1,f_2]v\otimes v dv - \partial_t\int_{\mathbb{R}^3} G[f_0]v\otimes v dv \\ &\quad - \operatorname{div}_x\int_{\mathbb{R}^3} f_1 v\otimes v\otimes v dv \\ &= P_2 + n_0u_1\otimes u_1 - \partial_t P_0 - \operatorname{div}_x\int_{\mathbb{R}^3} f_1 v\otimes v\otimes v dv,\end{aligned}$$

where we have used (18). We differentiate (22) to calculate the last integral:

$$\sum_{k=1}^3\frac{\partial}{\partial x_k}\int_{\mathbb{R}^3} f_1 v_i v_j v_k dv = \frac{2}{3}(ne)_0\Big(\operatorname{div}_x u_1\delta_{ij} + \frac{\partial u_{1,j}}{\partial x_i} + \frac{\partial u_{1,i}}{\partial x_j}\Big) - \operatorname{div}_x\big(C_1^{ij}\nabla_x\phi_1 + C_2^{ij}\nabla_x\phi_2\big).$$

Because of $P_2 = p_2\,\mathbb{I}$, this gives

$$\int_{\mathbb{R}^3} f_2 v\otimes v dv = p_2\,\mathbb{I} + n_0u_1\otimes u_1 - \partial_t P_0 - \frac{2}{3}(ne)_0\big(\operatorname{div}_x u_1\mathbb{I} + \nabla_x u_1 + (\nabla_x u_1)^\top\big) + \operatorname{div}_x\big(C_1\nabla_x\phi_1 + C_2\nabla_x\phi_2\big). \tag{25}$$

We recall that $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{3\times 3}$.

Step 3: computation of the moments and diffusion coefficients. Passing to spherical coordinates, the moments and the diffusion coefficients can be simplified. In fact, we compute

$$n_0 = 4\pi\int_0^\infty \gamma\Big(\frac{r^2}{2T_0} - \frac{\mu_0}{T_0}\Big)r^2 dr = 4\pi T_0^{3/2} g_1\Big(\frac{\mu_0}{T_0}\Big), \tag{26}$$

$$(ne)_0 = 4\pi\int_0^\infty \gamma\Big(\frac{r^2}{2T_0} - \frac{\mu_0}{T_0}\Big)\frac{r^4}{2} dr = 2\pi T_0^{5/2} g_2\Big(\frac{\mu_0}{T_0}\Big), \tag{27}$$

where

$$g_i(z) = \int_0^\infty \gamma\Big(\frac{r^2}{2} - z\Big)r^{2i}dr = \int_0^\infty \gamma(y-z)(2y)^{i-1/2}dy, \quad i\geq 1.$$

A computation shows that the diffusion coefficients (21) and (24) can be written as

$$C_1^{ijk\ell} = \frac{4\pi}{15} T_0^{7/2} g_3'\Big(\frac{\mu_0}{T_0}\Big)\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big), \quad D_1^{k\ell} = \frac{2\pi}{3} T_0^{7/2} g_3'\Big(\frac{\mu_0}{T_0}\Big)\delta_{k\ell},$$
$$C_2^{ijk\ell} = \frac{2\pi}{15} T_0^{9/2} g_4'\Big(\frac{\mu_0}{T_0}\Big)\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big), \quad D_2^{k\ell} = \frac{\pi}{3} T_0^{9/2} g_4'\Big(\frac{\mu_0}{T_0}\Big)\delta_{k\ell},$$

where g_i' is the derivative of g_i. Integrating by parts and using the assumptions on γ, we find that $g_{i+1}'(z) = (2i+1)g_i(z)$, $i \le 3$. Then

$$C_1^{ijk\ell} = \frac{4\pi}{3} T_0^{7/2} g_2\Big(\frac{\mu_0}{T_0}\Big)\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big), \quad D_1^{k\ell} = \frac{10\pi}{3} T_0^{7/2} g_2\Big(\frac{\mu_0}{T_0}\Big)\delta_{k\ell}, \tag{28}$$

$$C_2^{ijk\ell} = \frac{14\pi}{15} T_0^{9/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big), \quad D_2^{k\ell} = \frac{7\pi}{3} T_0^{9/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\delta_{k\ell}. \tag{29}$$

Step 4: computation of the moment equations. Equation (16) with $\kappa(v) = 1$ becomes, using (19),

$$\partial_t n_0 + \mathrm{div}_x(n_0 u_1) = 0.$$

Employing (20) and (13), equation (16) with $\kappa(v) = v$ can be written as

$$\nabla_x p_1 = \mathrm{div}_x P_1 = 0.$$

Furthermore, because of (23), equation (16) with $\kappa(v) = |v|^2/2$ equals

$$\partial_t (ne)_0 + \frac{5}{3}(ne)_0 \mathrm{div}_x u_1 - \mathrm{div}_x\big(D_1\nabla_x\phi_1 + D_2\nabla_x\phi_2\big) = 0.$$

In view of (28) and (29), we can write

$$\begin{aligned} D_1\nabla_x\phi_1 + D_2\nabla_x\phi_2 &= \frac{2\pi}{3}\Big(5T_0^{7/2} g_2\Big(\frac{\mu_0}{T_0}\Big)\nabla_x\Big(\frac{\mu_0}{T_0}\Big) + \frac{7}{2}T_0^{9/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\frac{\nabla_x T_0}{T_0^2}\Big) \\ &= \frac{2\pi}{3}\nabla_x\Big(T_0^{7/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\Big), \end{aligned}$$

so that the energy equation becomes

$$\partial_t (ne)_0 + \frac{5}{3}(ne)_0 \mathrm{div}_x u_1 - \frac{2\pi}{3}\Delta_x\Big(T_0^{7/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\Big) = 0. \tag{30}$$

Since $(ne)_0$ is space-independent, we may write this equation as

$$\partial_t (ne)_0 + \frac{5}{3}(ne)_0 \mathrm{div}_x\left(u_1 - \frac{1}{5}\nabla_x\Big(T_0 \frac{g_3(\mu_0/T_0)}{g_2(\mu_0/T_0)}\Big)\right) = 0,$$

using (27). As a consequence, the divergence term has to be space-independent too, and the above equation is an ordinary differential equation for $(ne)_0$, showing that $(ne)_0(t)$ is positive for all time if $(ne)_0$ is positive initially.

Next, using $\mathrm{div}_x\partial_t P_0 = 0$, (17) with $\kappa(v) = v$ reads as

$$\begin{aligned} \partial_t(n_0u_1) &+ \mathrm{div}_x(P_2 + n_0 u_1\otimes u_1) + \mathrm{div}_x\mathrm{div}_x(C_1\nabla\phi_1 + C_2\nabla\phi_2) - n_0\nabla_x V \\ &= \frac{2}{3}(ne)_0\big(\Delta_x u_1 + 2\nabla_x \mathrm{div}_x u_1\big). \end{aligned}$$

Since

$$C_1^{ijk\ell}\frac{\partial\phi_1}{\partial x_\ell} + C_2^{ijk\ell}\frac{\partial\phi_2}{\partial x_\ell} = \frac{4\pi}{15}\frac{\partial}{\partial x_\ell}\Big(T_0^{7/2} g_3\Big(\frac{\mu_0}{T_0}\Big)\Big)\big(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\big),$$

we infer that

$$\mathrm{div}_x\mathrm{div}_x(C_1\nabla\phi_1 + C_2\nabla\phi_2) = \frac{4\pi}{5}\nabla_x\Delta_x\Big(T_0^{7/2}g_3\Big(\frac{\mu_0}{T_0}\Big)\Big). \tag{31}$$

Differentiating the energy equation (30) with respect to the spatial variable, it follows that

$$5(ne)_0\nabla_x\mathrm{div}_x u_1 = 2\pi\nabla_x\Delta_x\big(T_0^{7/2}g_3(\phi_1)\big).$$

Hence, we conclude that

$$\mathrm{div}_x\mathrm{div}_x(C_1\nabla\phi_1 + C_2\nabla\phi_2) = 2(ne)_0\nabla_x\mathrm{div}_x u_1,$$

and the momentum equation simplifies to

$$\partial_t(n_0u_1) + \mathrm{div}_x(n_0u_1\otimes u_1) + \nabla_x q - n_0\nabla_x V = \frac{2}{3}(ne)_0\Delta_x u_1,$$

where we have used the relation $P_2 = p_2\,\mathbb{I}$ from step 1 and the definition

$$q = p_2 + \frac{2}{3}(ne)_0\mathrm{div}_x u_1. \tag{32}$$

If (15)-(17) were satisfied with $\kappa(v) = 1, v, |v|^2/2$, the Hilbert expansion of f would be solvable up to third order. Since we require that (17) is satisfied with $\kappa(v) = v$ only, we obtain solvability up to second order.

Collecting the above results, we have shown the following theorem.

Theorem 2.1. *Let* (11) *and the assumptions on page 4 hold. Then the Hilbert expansion of f is solvable up to second order if the functions u_1, μ_0, T_0, p_1, and q solve the system of equations*

$$\nabla_x(ne)_0 = 0, \quad \nabla_x p_1 = 0, \quad \partial_t n_0 + \mathrm{div}_x(n_0u_1) = 0, \tag{33}$$

$$\partial_t(ne)_0 + \frac{5}{3}(ne)_0\mathrm{div}_x\left(u_1 - \frac{1}{5}\nabla_x\Big(T_0\frac{g_3(\mu_0/T_0)}{g_2(\mu_0/T_0)}\Big)\right) = 0, \tag{34}$$

$$\partial_t(n_0u_1) + \mathrm{div}_x(n_0u_1\otimes u_1) + \nabla_x q - n_0\nabla_x V = \frac{2}{3}(ne)_0\Delta_x u_1, \tag{35}$$

where the particle density n_0 and the energy density $(ne)_0$ depend on (μ_0, T_0) (or, equivalently, (ϕ_1, ϕ_2)),

$$n_0 = 4\pi T_0^{3/2}g_1\Big(\frac{\mu_0}{T_0}\Big), \quad (ne)_0 = 2\pi T_0^{5/2}g_2\Big(\frac{\mu_0}{T_0}\Big),$$

and g_i is defined by

$$g_i(z) = \int_0^\infty \gamma\Big(\frac{r^2}{2} - z\Big)r^{2i}dr = \int_0^\infty \gamma(y-z)(2y)^{i-1/2}dy, \quad i = 1,2,3.$$

We observe that the mapping $(\phi_1,\phi_2)\mapsto(n_0,(ne)_0)$ is invertible since

$$\det\frac{\partial(n_0,(ne)_0)}{\partial(\phi_1,\phi_2)} = \Big(\int_{\mathbb{R}^3}\gamma'(E_0)\frac{|v|^2}{2}dv\Big)^2 - \int_{\mathbb{R}^3}\gamma'(E_0)dv\int_{\mathbb{R}^3}\gamma'(E_0)\frac{|v|^4}{4}dv$$

is negative, by the Cauchy-Schwarz inequality (and the corresponding condition for equality). The last equation in (33) expresses mass conservation, equation (34) expresses energy conservation, and (35) is the momentum balance equation. Compared to the ghost-effect equations based on the Boltzmann equation (or the standard BGK model) in [16, p. 117], the divergence of the thermal stress in (35) does not appear. In fact, in our model, the divergence of the thermal stress is given by (31), and we employ as in [16] the energy equation (30) to replace it by a first-order term in the velocity, which is hidden in the new pressure q, see (32). The absence of the divergence of the thermal stress in (35) should be attributed to the fact that the structure of our kinetic equation (6) is slightly different from that of the standard BGK model (see Section 2.1).

3. **Mathematical structure of the model.** From a mathematical point of view, the model equations (33)-(35) can be interpreted as follows. The mass and momentum equations

$$\partial_t n_0 + \mathrm{div}_x(n_0 u_1) = 0, \tag{36}$$

$$\partial_t(n_0 u_1) + \mathrm{div}_x(n_0 u_1 \otimes u_1) + \nabla_x q - n_0 \nabla_x V = \frac{2}{3}(ne)_0 \Delta_x u_1 \tag{37}$$

are the compressible Navier-Stokes equations, where $\frac{2}{3}(ne)_0$ plays the role of a space-independent viscosity and the pressure q is – similarly to the incompressible case – the Lagrange multiplier of the scalar constraint (5). The following proposition gives an energy estimate for the above system.

Proposition 1. *Let n_0, $(ne)_0 = (ne)_0(t)$, and $V = V(x)$ be given and let (u_1, q) be a (smooth) solution to (36)-(37) such that $|u_1(x,t)|$ decays sufficiently fast to zero as $|x| \to \infty$ uniformly in t. Then*

$$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{1}{2}n_0|u_1|^2 - n_0 V\Big)dx + \frac{2}{3}(ne)_0\int_{\mathbb{R}^3}|\nabla_x u_1|^2 dx = \int_{\mathbb{R}^3} q\,\mathrm{div}_x u_1 dx. \tag{38}$$

The term $q\mathrm{div}_x u_1$ in (38) expresses the energy change due to the work of compression. In the incompressible case, it vanishes, and the energy becomes nonincreasing in time.

Proof. We calculate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{1}{2}n_0|u_1|^2 - n_0 V\Big)dx &= \int_{\mathbb{R}^3}\Big(-\frac{1}{2}|u_1|^2\partial_t n_0 + \partial_t(n_0 u_1)\cdot u_1 - \partial_t n_0 V\Big)dx \\ &= \int_{\mathbb{R}^3}\Big(\frac{1}{2}\mathrm{div}_x(n_0 u_1)|u_1|^2 + \nabla_x u_1 : (n_0 u_1 \otimes u_1)\Big)dx \\ &\quad + \int_{\mathbb{R}^3}\big(\mathrm{div}_x(n_0 u_1)V + n_0 \nabla_x V \cdot u_1\big)dx \\ &\quad + \int_{\mathbb{R}^3}\Big(q\,\mathrm{div}_x u_1 - \frac{2}{3}(ne)_0|\nabla_x u_1|^2\Big)dx.\end{aligned}$$

An integration by parts in the first two integrals shows that both integrals vanish. Hence, the conclusion follows. □

Remark 1. As mentioned in the introduction, the Euler system (36)-(37) may be supplemented by the equation $\mathrm{div}_x u_1 = h(x,t)$, where $h(x,t)$ depends on $(ne)_0$, μ_0, and T_0 (see (5)). This equation may be interpreted as a non-standard (scalar) constraint for the (scalar) Lagrange multiplier q in the momentum equation (37). Such constraints appear in the low-Mach number limit in the Euler equations and are referred to as pseudo-incompressibility conditions [2, 12]. It is not surprising that we obtain a similar constraint since in the low-Mach number limit, the time scale is $O(1/\varepsilon)$ and the velocity is assumed to be of the order $O(\varepsilon)$ which are exactly our scaling assumptions. The numerical approximation of pseudo-incompressible equations is described in [2]; the analytical treatment, however, seems to be not clear. □

The Euler equations (36)-(37) are coupled to the energy equations

$$\nabla_x(ne)_0 = 0, \quad \partial_t(ne)_0 + \frac{5}{3}(ne)_0 \mathrm{div}_x u_1 - \mathrm{div}_x(D_1\nabla_x\phi_1 + D_2\nabla_x\phi_2) = 0. \tag{39}$$

We recall that the diffusion coefficients D_i are defined in (28)-(29). We claim that this system has an energy-transport-type structure. To explain this, we differentiate the energy density, employing (27):

$$\frac{2}{3}\nabla_x(ne)_0 = \frac{4\pi}{3}T_0^{5/2}g_2'(\phi_1)\nabla_x\phi_1 + \frac{10\pi}{3}T_0^{7/2}g_2(\phi_1)\nabla_x\phi_2 = D_0\nabla_x\phi_1 + D_1\nabla_x\phi_2,$$

where

$$D_0 = 4\pi T_0^{5/2}g_1(\phi_1), \quad D_1 = \frac{10\pi}{3}T_0^{7/2}g_2(\phi_1).$$

Then we can write

$$\partial_t\begin{pmatrix}0\\(ne)_0\end{pmatrix} - \mathrm{div}_x\left(\mathcal{D}\nabla_x\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\right) + \frac{5}{3}(ne)_0\mathrm{div}_x\begin{pmatrix}0\\u_1\end{pmatrix} = 0, \tag{40}$$

where the diffusion matrix $\mathcal{D}$ is defined by

$$\mathcal{D} = \begin{pmatrix}D_0 & D_1\\ D_1 & D_2\end{pmatrix} = \frac{\pi}{3}T_0^{5/2}\begin{pmatrix}12g_1(\phi_1) & 10T_0g_2(\phi_1)\\ 10T_0g_2(\phi_1) & 7T_0^2g_3(\phi_1)\end{pmatrix}. \tag{41}$$

This matrix is symmetric and we will show in section 4 that it is positive definite in all considered examples. Another formulation of the above system is obtained by adding the mass equation to the first component of (40):

$$\partial_t\begin{pmatrix}n_0\\(ne)_0\end{pmatrix} - \mathrm{div}_x\left(\mathcal{D}\nabla_x\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\right) + \mathrm{div}_x\begin{pmatrix}n_0u_1\\\frac{5}{3}(ne)_0u_1\end{pmatrix} = 0. \tag{42}$$

The entropic structure of this system (in the sense of [10, 14]) can be understood by computing the time derivative of the (negative) relative entropy density

$$h(x,t) = m\cdot(\phi-\phi_{\mathrm{eq}}) + \int_{\mathbb{R}^3}\gamma^*\Big(-\phi_1 - \frac{|v|^2}{2}\phi_2\Big)(x,t)dv - \int_{\mathbb{R}^3}\gamma^*\Big(\frac{|v|^2}{2}\Big)dv,$$

where $m = (n_0,(ne)_0)$ are the moments, $\phi = (\phi_1,\phi_2) = (\mu_0/T_0, -1/T_0)$ are the entropy variables, $\phi_{\mathrm{eq}} = (0,-1)$ is the equilibrium value, and γ^* is a primitive of γ.

Proposition 2. *Let u_1 and $V = V(x)$ be given and let (ϕ_1,ϕ_2) (or $(n_0,(ne)_0)$) be a (smooth) solution to (42) satisfying $\phi(x,t)-\phi_{\mathrm{eq}}\to 0$ as $|x|\to\infty$ uniformly in t. Then*

$$\frac{d}{dt}\int_{\mathbb{R}^3}h(x,t)dx + \int_{\mathbb{R}^3}\sum_{k=1}^3\Big(\frac{\partial\phi}{\partial x_k}\Big)^\top\mathcal{D}\Big(\frac{\partial\phi}{\partial x_k}\Big)dx = 0.$$

The second term on the left-hand side can be interpreted as the entropy dissipation; it is nonnegative if the diffusion matrix $\mathcal{D}$ is positive (semi-) definite. In this situation, the (negative) entropy is nonincreasing in time.

Proof. We observe that

$$\frac{\partial h}{\partial\phi_1} = \frac{\partial m}{\partial\phi_1}\cdot(\phi-\phi_{\mathrm{eq}}) + n_0 - \int_{\mathbb{R}^3}\gamma\Big(-\phi_1 - \frac{|v|^2}{2}\phi_2\Big)dv = \frac{\partial m}{\partial\phi_1}\cdot(\phi-\phi_{\mathrm{eq}}),$$

$$\frac{\partial h}{\partial\phi_2} = \frac{\partial m}{\partial\phi_2}\cdot(\phi-\phi_{\mathrm{eq}}) + (ne)_0 - \int_{\mathbb{R}^3}\gamma\Big(-\phi_1 - \frac{|v|^2}{2}\phi_2\Big)\frac{|v|^2}{2}dv = \frac{\partial m}{\partial\phi_2}\cdot(\phi-\phi_{\mathrm{eq}}),$$

and therefore

$$\partial_t m \cdot (\phi - \phi_{\rm eq}) = \sum_{i=1}^2 \frac{\partial m}{\partial \phi_i} \cdot (\phi - \phi_{\rm eq})\partial_t \phi_i = \sum_{i=1}^2 \frac{\partial h}{\partial \phi_i}\partial_t \phi_i = \frac{\partial h}{\partial t}.$$

Then, multiplying (42) by $\phi - \phi_{\rm eq}$ and integrating over $\mathbb{R}^3$, we find that

$$\int_{\mathbb{R}^3} \partial_t m \cdot (\phi - \phi_{\rm eq})dx + \int_{\mathbb{R}^3} (\nabla\phi)^\top \mathcal{D}(\nabla\phi)dx = \int_{\mathbb{R}^3} \Big(n_0 u_1 \cdot \nabla\phi_1 + \frac{5}{3}(ne)_0 u_1 \cdot \nabla\phi_2 \Big) dx.$$

With the definition of the diffusion coefficients, it follows that $D_0 = n_0 T_0$ and $D_1 = \frac{5}{3}T_0(ne)_0$. Thus, $\frac{2}{3}\nabla_x(ne)_0 = D_0\nabla_x\phi_1 + D_1\nabla_x\phi_2 = T_0(n_0\nabla_x\phi_1 + \frac{5}{3}(ne)_0\nabla_x\phi_2)$,

$$\partial_t \int_{\mathbb{R}^3} h dx + \int_{\mathbb{R}^3} (\nabla\phi)^\top \mathcal{D}(\nabla\phi)dx = \frac{2}{3}\int_{\mathbb{R}^3} \frac{u_1}{T_0} \cdot \nabla_x(ne)_0 dx,$$

and the conclusion follows since $\nabla_x(ne)_0 = 0$. □

Remark 2. The energy equations (39) can be reformulated in a different way than explained above. In fact, the first equation in (39) implies that $(ne)_0$ is a function of time only. Then the expression $g(t) = \mathrm{div}_x(u_1 - \nabla_x(T_0 g_3/(5g_2)))$ in the second equation (see (34)) is independent of x, becoming just the ordinary differential equation $\partial_t(ne)_0 + \frac{5}{3}g(t)(ne)_0 = 0$ for $t > 0$. □

4. **Examples.** We consider the Maxwellian, Fermi-Dirac, and Bose-Einstein distributions.

4.1. **Maxwellian distribution.** We specify (33)-(35) for the Maxwellian distribution $\gamma(E) = \exp(-E)$. Then

$$g_1(z) = \frac{1}{2}\sqrt{2\pi}e^z, \quad g_2(z) = \frac{3}{2}\sqrt{2\pi}e^z, \quad g_3(z) = \frac{15}{2}\sqrt{2\pi}e^z,$$

the moments (26) and (27) are calculated as

$$n_0 = (2\pi T_0)^{3/2} e^{\mu/T_0}, \quad (ne)_0 = \frac{3}{2}(2\pi)^{3/2} T_0^{5/2} e^{\mu_0/T_0} = \frac{3}{2}n_0 T_0,$$

and the term in the energy equation, involving g_3, equals

$$\frac{2\pi}{3}\Delta_x\big(T_0^{7/2} g_3(\mu_0/T_0)\big) = \frac{5}{2}(2\pi)^{3/2}\Delta_x\big(T_0^{7/2}e^{\mu_0/T_0}\big) = \frac{5}{2}\Delta_x(n_0 T_0^2).$$

We summarize the equations. System (33)-(35) can be written in the variables n_0, T_0, u_1, p_1, and q as

$$\begin{aligned} &\nabla_x(n_0 T_0) = 0, \quad \nabla_x p_1 = 0, \quad \partial_t n_0 + \mathrm{div}_x(n_0 u_1) = 0, \\ &\frac{3}{2}\partial_t(n_0 T_0) + \frac{5}{2}n_0 T_0 \mathrm{div}_x(u_1 - \nabla_x T_0) = 0, \\ &\partial_t(n_0 u_1) + \mathrm{div}_x(n_0 u_1 \otimes u_1) + \nabla_x q - n_0 \nabla_x V = n_0 T_0 \Delta_x u_1. \end{aligned}$$

The diffusion matrix (41)

$$\mathcal{D} = n_0 T_0 \begin{pmatrix} 1 & \frac{5}{2}T_0 \\ \frac{5}{2}T_0 & \frac{35}{4}T_0^2 \end{pmatrix}$$

is symmetric and positive definite. Furthermore, the (relative) entropy becomes

$$\int_{\mathbb{R}^3} h dx = \int_{\mathbb{R}^3} \Big(n_0 \log \frac{n_0}{T_0^{3/2}} - \frac{5}{2} n_0 + \frac{3}{2} n_0 T_0 + (2\pi)^{3/2} \Big) dx,$$

which is the usual expression in fluid dynamics.

4.2. **Fermi-Dirac distribution.** Let $\gamma(E) = 1/(1+e^E)$. We compute the functions g_i:

$$g_1(z) = \sqrt{\frac{\pi}{2}} F_{1/2}(z), \quad g_2(z) = 3\sqrt{\frac{\pi}{2}} F_{3/2}(z), \quad g_3(z) = 15\sqrt{\frac{\pi}{2}} F_{5/2}(z),$$

where F_a is the Fermi integral with index a, defined by

$$F_a(z) = \frac{1}{\Gamma(a+1)} \int_0^\infty \frac{y^a dy}{1+e^{y-z}}, \quad z \in \mathbb{R}, \ a > -1, \tag{43}$$

and Γ is the Gamma function satisfying $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(a+1) = a\Gamma(a)$. Then the particle and energy densities are

$$\begin{aligned} n_0 &= (2\pi T_0)^{3/2} F_{1/2}\Big(\frac{\mu_0}{T_0}\Big), \\ (ne)_0 &= \frac{3}{2}(2\pi)^{3/2} T_0^{5/2} F_{3/2}\Big(\frac{\mu_0}{T_0}\Big) = \frac{3}{2} n_0 T_0 \frac{F_{3/2}(\mu_0/T_0)}{F_{1/2}(\mu_0/T_0)}, \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{2\pi}{3} \Delta_x \big(T_0^{7/2} g_3(\mu_0/T_0) \big) &= \frac{5}{2} (2\pi)^{3/2} \Delta_x \Big(T_0^{7/2} F_{5/2}(\mu_0/T_0) \Big) \\ &= \frac{5}{2} \Delta_x \Big(n_0 T_0^2 \frac{F_{5/2}(\mu_0/T_0)}{F_{1/2}(\mu_0/T_0)} \Big), \end{aligned}$$

equations (33)-(35) equal

$$\begin{aligned} &\nabla_x \Big(n_0 T_0 \frac{F_{3/2}(\mu_0/T_0)}{F_{1/2}(\mu_0/T_0)} \Big) = 0, \quad \nabla_x p_1 = 0, \quad \partial_t n_0 + \operatorname{div}_x(n_0 u_1) = 0, \\ &\frac{3}{2} \partial_t \Big(n_0 T_0 \frac{F_{3/2}(\mu_0/T_0)}{F_{1/2}(\mu_0/T_0)} \Big) \\ &\qquad + \frac{5}{2} n_0 T_0 \frac{F_{3/2}(\mu_0/T_0)}{F_{1/2}(\mu_0/T_0)} \operatorname{div}_x \Big(u_1 - \nabla_x \Big(T_0 \frac{F_{5/2}(\mu_0/T_0)}{F_{3/2}(\mu_0/T_0)} \Big) \Big) = 0, \\ &\partial_t(n_0 u_1) + \operatorname{div}_x(n_0 u_1 \otimes u_1) + \nabla_x p_1 - n_0 \nabla_x V = n_0 T_0 \Delta_x u_1. \end{aligned}$$

The diffusion matrix (41) becomes

$$\mathcal{D} = (2\pi)^{3/2} T_0^{5/2} \begin{pmatrix} F_{1/2}(\phi_1) & \frac{5}{2} T_0 F_{3/2}(\phi_1) \\ \frac{5}{2} T_0 F_{3/2}(\phi_1) & \frac{35}{4} T_0^2 F_{5/2}(\phi_1) \end{pmatrix}.$$

This matrix is symmetric and positive definite since

$$\det \mathcal{D} = \frac{5}{4} T_0^2 \big(7 F_{1/2}(\phi_1) F_{5/2}(\phi_1) - 5 F_{3/2}(\phi_1)^2 \big) > 0,$$

which is a consequence of following general property of Fermi integrals (see [1, Lemma 4.2] for a proof):

$$(a+2) F_{a-1}(z) F_{a+1}(z) - (a+1) F_a(z)^2 > 0, \quad z \in \mathbb{R}, \ a > 0.$$

4.3. **Bose-Einstein distribution.** The Bose-Einstein case $\gamma(E) = 1/(e^E - 1)$ is similar to the Fermi-Dirac case. Indeed, introducing the polylogarithms (or Bose-Einstein integrals [15])

$$\mathrm{Li}_{a+1}(e^z) = \frac{1}{\Gamma(a+1)} \int_0^\infty \frac{y^a dy}{e^{y-z} - 1}, \quad z > 0, \ a > 0,$$

we obtain

$$n_0 = (2\pi T_0)^{3/2} \mathrm{Li}_{3/2}(e^{\mu_0/T_0}),$$
$$(ne)_0 = (2\pi)^{3/2} T_0^{5/2} \mathrm{Li}_{5/2}(e^{\mu_0/T_0}) = \frac{3}{2} n_0 T_0 \frac{\mathrm{Li}_{5/2}(e^{\mu_0/T_0})}{\mathrm{Li}_{3/2}(e^{\mu_0/T_0})}.$$

Then, after a similar computation as in the previous subsection, the model equations (33)-(35) are written as

$$\nabla_x \Big(n_0 T_0 \frac{\mathrm{Li}_{5/2}(e^{\mu_0/T_0})}{\mathrm{Li}_{3/2}(e^{\mu_0/T_0})} \Big) = 0, \quad \nabla_x p_1 = 0, \quad \partial_t n_0 + \mathrm{div}_x(n_0 u_1) = 0,$$
$$\frac{3}{2} \partial_t \Big(n_0 T_0 \frac{\mathrm{Li}_{5/2}(e^{\mu_0/T_0})}{\mathrm{Li}_{3/2}(e^{\mu_0/T_0})} \Big) + \frac{5}{2} n_0 T_0 \frac{\mathrm{Li}_{5/2}(e^{\mu_0/T_0})}{\mathrm{Li}_{3/2}(e^{\mu_0/T_0})} \mathrm{div}_x \left(u_1 - \nabla_x \Big(T_0 \frac{\mathrm{Li}_{7/2}(e^{\mu_0/T_0})}{\mathrm{Li}_{5/2}(e^{\mu_0/T_0})} \Big) \right) = 0,$$
$$\partial_t(n_0 u_1) - \mathrm{div}_x(n_0 u_1 \otimes u_1) + \nabla_x q - n_0 \nabla_x V = n_0 T_0 \Delta_x u_1,$$

and the diffusion matrix $\mathcal{D}$ reads as

$$\mathcal{D} = (2\pi)^{3/2} T_0^{5/2} \begin{pmatrix} \mathrm{Li}_{3/2}(e^{\mu_0/T_0}) & \frac{5}{6} T_0 \mathrm{Li}_{5/2}(e^{\mu_0/T_0}) \\ \frac{5}{6} T_0 \mathrm{Li}_{5/2}(e^{\mu_0/T_0}) & \frac{7}{12} T_0^2 \mathrm{Li}_{7/2}(e^{\mu_0/T_0}) \end{pmatrix}.$$

REFERENCES

[1] G. Albinus. A thermodynamically motivated formulation of the energy model of semiconductor devices. *Z. Angw. Math. Mech.* 76 (1996), 289-292.
[2] A. Almgren, J. Bell, C. Rendleman, and M. Zingale. Low Mach number modeling of type Ia supernovae. I. Hydrodynamics. *Astrophys. J.* 637 (2006), 922-936.
[3] K. Aoki, P. Markowich, and S. Takata. Kinetic relaxation models for energy transport. *J. Stat. Phys.* 127 (2007), 287-312.
[4] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Statist. Phys.* 63 (1991), 323-344.
[5] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. *Comm. Pure Appl. Math.* 46 (1993), 667-753.
[6] N. Ben Abdallah and P. Degond. On a hierarchy of macroscopic models for semiconductors. *J. Math. Phys.* 37 (1996), 3308-3333.
[7] P. Bhatnagar, E. Gross, and M. Krook. A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems. *Phys. Rev.* 94 (1954), 511-525.
[8] R. Caflisch. The fluid dynamic limit of the nonlinear Boltzmann equation. *Comm. Pure Appl. Math.* 33 (1980), 651-666.
[9] C. Cercignani. *The Boltzmann equation and Its Applications.* Springer, Berlin, 1988.
[10] P. Degond, S. Génieys, and A. Jüngel. Symmetrization and entropy inequality for general diffusion equations. *C. R. Acad. Sci. Paris, Sér. I Math.* 325 (1997), 963-968.
[11] J. Dolbeault, P. Markowich, D. Ölz, and C. Schmeiser. Nonlinear diffusion as limit of kinetic equations with relaxation collision kernels. *Arch Ration. Mech. Anal.* 186 (2007), 133-158.
[12] D. Durran. Improving the anelastic approximation. *J. Atmos. Sci.* 46 (1989), 1453-1461.
[13] E. Feireisl. *Dynamics of Viscous Compressible Fluids.* Oxford University Press, Oxford, 2004.

[14] A. Jüngel. *Transport Equations for Semiconductors*. Lecture Notes in Physics 773. Springer, Berlin, 2009.
[15] J. Robinson. Note on the Bose-Einstein integral functions. *Phys. Rev.* 83 (1951), 678-679.
[16] Y. Sone. *Molecular Gas Dynamics. Theory, Techniques and Applications*. Birkhäuser, Basel, 2006.
[17] Y. Sone, K. Aoki, S. Takata, H. Sugimoto, and A. Bobylev. Inappropriateness of the heat-conduction equation for description of a temperature field of a stationary gas in the continuum limit: Examination by asymptotic analysis and numerical computation of the Boltzmann equation. *Phys. Fluids* 8 (1995), 628-638.
[18] R. Stratton. Diffusion of hot and cold electrons in semiconductor barriers. *Phys. Rev.* 126 (1962), 2002-2014.
[19] C. Villani. Mathematics of granular materials. *J. Stat. Phys.* 124 (2006), 781-822.
[20] P. Welander. On the temperature jump in a rarefied gas. *Ark. Fys.* 7 (1954), 507-553.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: aoki@aero.mbox.media.kyoto-u.ac.jp
E-mail address: juengel@anum.tuwien.ac.at
E-mail address: p.a.markowich@damtp.cam.ac.uk