ANALYSIS OF AN INCOMPRESSIBLE NAVIER-STOKES-MAXWELL-STEFAN SYSTEM

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Abstract. The Maxwell-Stefan equations for the molar fluxes, supplemented by the incompressible Navier-Stokes equations governing the fluid velocity dynamics, are analyzed in bounded domains with no-flux boundary conditions. The system models the dynamics of a multicomponent gaseous mixture under isothermal conditions. The global-in-time existence of bounded weak solutions to the strongly coupled model and their exponential decay to the homogeneous steady state are proved. The mathematical difficulties are due to the singular Maxwell-Stefan diffusion matrix, the cross-diffusion terms, and the different molar masses of the fluid components. The key idea of the proof is the use of a new entropy functional and entropy variables, which allows for a proof of positive lower and upper bounds of the mass densities without the use of a maximum principle.

1. Introduction

The dynamics of a multicomponent gaseous mixture can be described by the Navier-Stokes equations, which represent the balance of mass, momentum, and energy, and the Maxwell-Stefan equations, which model the diffusive transport of the components of the mixture. Applications arise, for instance, from physics (sedimentation, astrophysics), medicine (dialysis, respiratory airways), and chemistry (electrolysis, ion exchange, chemical reactors) [23]. The understanding of the analytical structure of coupled Navier-Stokes-Maxwell-Stefan systems is of great importance for an accurate modeling and efficient numerical simulation of these applications. In this paper, we make a step forward to this understanding by proving the global-in-time existence of weak solutions and their long-time behavior for Navier-Stokes-Maxwell-Stefan systems for incompressible fluids under natural assumptions.

More precisely, we consider a multicomponent fluid consisting of $N+1$ components with the mass densities ρ_i, molar masses M_i, and velocities u_i. As in [6], we prescribe a system of partial mass balances together with a common mixture momentum balance, where the

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diffusive fluxes are given by the Maxwell-Stefan relations. The partial mass balances for the molar concentrations $c_i = \rho_i/M_i$ read as

$$\partial_t c_i + \operatorname{div}(j_i + c_i u) = 0, \quad i = 1, \ldots, N+1,$$

where the barycentric velocity u and the total mass density ρ^* of the fluid are defined by $\rho^* u = \sum_{i=1}^{N+1} \rho_i u_i$ and $\rho^* = \sum_{i=1}^{N+1} \rho_i$, and the molar mass fluxes j_i are given by $j_i = c_i(u_i - u)$. By definition of j_i, it holds that $\sum_{i=1}^{N+1} M_i j_i = 0$, and therefore, one of the partial mass balances can be replaced by the continuity equation $\partial_t \rho^* + \operatorname{div}(\rho^* u) = 0$. The mixture momentum balance equations are

$$\partial_t(\rho^* u) + \operatorname{div}(\rho^* u \otimes u - S) + \nabla p = \rho^* f,$$

where p is the pressure, the force density equals $\rho^* f = \sum_{i=1}^{N+1} \rho_i f_i$, and the viscous stress tensor is $S = \nu^*(\nabla u + \nabla u^\top)$, where ν^* is the viscosity constant. In this paper, we suppose that $f_i = f$ and, as in [18], we consider the incompressible, isothermal case,

$$\rho^* = \text{const.}, \quad \operatorname{div} u = 0.$$

For simplicity, we set $\rho^* = 1$ and $\nu^* = 1$.

The above equations are closed by relating the molar mass fluxes j_i to the molar concentrations c_i by the Maxwell-Stefan equations

$$-\sum_{k=1}^{N+1} \frac{x_k j_i - x_i j_k}{D_{ik}} = c_i \nabla \mu_i - y_i \nabla p - \rho_i(f_i - f), \quad i = 1, \ldots, N+1,$$

where $x_i = c_i/c$ with $c = \sum_{k=1}^{N+1} c_i$ are the molar fractions, $y_i = \rho_i/\rho^* = \rho_i$ are the mass fractions, μ_i are the molar-based chemical potentials, and $D_{ik} = D_{ki} > 0$ for $i \neq k$ are the diffusion coefficients. Our second assumption is that the mixture of gases is ideal such that the chemical potentials can be written as $\mu_i = \ln x_i + \mu_{0i}(p)$ with $d\mu_{0i}/dp = \phi_i/c_i$, where ϕ_i is the volume fraction (see [6, Section 1.1]). Since $f_i = f$, this implies that

$$-\sum_{k=1}^{N+1} \frac{x_k j_i - x_i j_k}{D_{ik}} = \frac{c_i}{x_i} \nabla x_i + (\phi_i - y_i)\nabla p = c \nabla x_i + (\phi_i - y_i)\nabla p.$$

We assume further that the volume and mass fractions are comparable such that the contribution $(\phi_i - y_i)\nabla p$ can be neglected. The general case will be investigated in a future work. For a discussion of the above equation as well as the incompressibility condition in the context of fluid mixtures, we refer to [5, Sections 14-16]. This gives the desired closure relations

$$-\sum_{k=1}^{N+1} \frac{x_k j_i - x_i j_k}{D_{ik}} = c \nabla x_i, \quad i = 1, \ldots, N+1.$$

These relations, together with the mass balance equations, can also be derived from a system of kinetic equations with BGK-type collision operator in the Chapman-Enskog expansion [3].

Setting $J_i = M_i j_i$, the incompressible Navier-Stokes-Maxwell-Stefan system analyzed in this paper reads as

$$\partial_t \rho_i + \operatorname{div}(J_i + \rho_i u) = 0, \quad \text{in } \Omega,\ t > 0, \tag{1}$$

$$\partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p = f, \quad \operatorname{div} u = 0, \tag{2}$$

$$\nabla x_i = -\sum_{k=1}^{N+1} \frac{\rho_k J_i - \rho_i J_k}{c^2 M_i M_k D_{ik}}, \quad i = 1, \ldots, N+1, \tag{3}$$

where x_i and ρ_i are related by $x_i = \rho_i/(cM_i)$ with $c = \sum_{i=1}^{N+1} \rho_i/M_i$ and $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ is a bounded domain. Note that $\rho^* = 1$ implies that $\sum_{i=1}^{N+1} \rho_i = \sum_{i=1}^{N+1} x_i = 1$. The initial and boundary conditions are

$$\rho_i(\cdot, 0) = \rho_i^0,\ u(\cdot, 0) = u^0 \quad \text{in } \Omega, \quad \nabla \rho_i \cdot \nu = 0,\ u = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where $i = 1, \ldots, N+1$ and ν is the normal exterior unit vector on $\partial\Omega$.

Note that the fluid velocity u is solely determined by the incompressible Navier-Stokes equations (2) with the corresponding initial and boundary conditions, for given force f. The mathematical difficulties of the above system are as follows.

First, the molar mass fluxes are not explicitly given as a linear combination of the mass density gradients, which makes necessary to invert the flux-gradient relations (3). However, as the Maxwell-Stefan equations are linearly dependent, we need to invert on a subspace. In the engineering literature, this inversion is usually done in an approximate way [2]. Giovangigli [10] suggested an iterative procedure using the Perron-Frobenius theory. A general inversion result was proved by Bothe [4], again based on the Perron-Frobenius theory.

Second, because of the cross-diffusion coupling in (1) and (3), standard tools like maximum principles and regularity theory are not available. In particular, it is not clear how to prove positive lower and upper bounds for the mass densities ρ_i and even the local existence of solutions is not trivial.

Third, we need to find suitable a priori estimates for the coupled system. Difficulties arise from the facts that the molar masses M_i are generally different, which complicates the analysis, and that the velocity does not need to be bounded such that the term $\operatorname{div}(\rho_i u)$ in (1) needs to be treated carefully.

In view of these difficulties, it is not surprising that there exist only partial results on such systems in the literature. First results were concerned with the Maxwell-Stefan equations (1) and (3) with vanishing velocity $u = 0$ and equal molar masses $M = M_i$. Griepentrog [13] and later Bothe [4] derived a local existence theory; Giovangigli [11, Theorem 9.4.1] proved the global existence of solutions with initial data sufficiently close to the equilibrium state; Boudin, Grec, and Salvarani [7] investigated a particular two-component model; Jüngel and Stelzer [16] presented general global existence results; and Herberg et al. [14] prove locally well-posedness in an L^p-setting and exponential stability for the mass-based model. The Maxwell-Stefan system with given bounded velovity $u \neq 0$ was analyzed by Mucha, Pokorný, and Zatorska [20]. They imposed a special diffusion matrix which avoids the inversion problem.

Other papers were concerned with the full coupled system but in particular situations. For instance, Zatorska [24] proved the existence of weak solutions to the stationary compressible model with three fluid components and special isobaric pressures. She also proved the sequential stability of weak solutions to the two-component system on the three-dimensional torus [25]. Mucha, Pokorný, and Zatorska [19] showed a global existence result for a regularized compressible system for two components. The Navier-Stokes equations contain artificial higher-order differential operators which regularize the problem. In [17], the global existence for the incompressible Navier-Stokes-Maxwell-Stefan system was announced but not proved. For numerical approximations using a finite-volume method, we refer to [1].

After this article was completed, we learned of the paper [18] in which the global existence of solutions to the coupled system (1)-(4) is proved. The main idea in [18] is to modify the flux-gradient system equivalently in such a way that it becomes invertible. This yields a particular structure of the diffusion coefficients a_{ij} in $J_i = \sum_{j=1}^{N+1} a_{ij}\nabla x_j$, allowing for the use of the classical maximum principle.

In this paper, we present a global existence result for the full system (1)-(4), allowing for different molar masses M_i, using a different approach than in [18], namely an entropy-dissipation method. We remove the last component of the fluid by setting $\rho_{N+1} = 1-\sum_{i=1}^N \rho_i$, thus obtaining a directly invertible flux-gradient system [4]. By "symmetrizing" the diffusion system via an introduction of so-called entropy variables, we solve an equivalent diffusion system whose solution defines nonnegative densities ρ_i satisfying $\sum_{i=1}^N \rho_i \le 1$ *without the use of a maximum principle*. This idea was first employed in [16] for the Maxwell-Stefan system with $u = 0$. Compared to [16], we have to overcome some additional difficulties detailed below. In contrast to [18], we are able to prove the exponential decay of the weak solutions to equilibrium by using the entropy-dissipation method.

In order to state our main results, we introduce the following spaces (see [21, Chapter I]). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega \in C^{1,1}$ and let

$$\begin{aligned} \mathcal{H} &= \{u \in L^2(\Omega;\mathbb{R}^d) : \operatorname{div} u = 0,\ u\cdot\nu|_{\partial\Omega} = 0\}, \\ \mathcal{V} &= \{u \in H_0^1(\Omega;\mathbb{R}^d) : \operatorname{div} u = 0\}, \quad \mathcal{V}_2 = \mathcal{V}\cap H^2(\Omega;\mathbb{R}^d), \\ \widetilde{H}^2(\Omega;\mathbb{R}^N) &= \{q \in H^2(\Omega;\mathbb{R}^N) : \nabla q\cdot\nu|_{\partial\Omega} = 0\}. \end{aligned} \tag{5}$$

We define similarly the space $\widetilde{H}^2(\Omega)$. We recall that functions $u \in L^2(\Omega;\mathbb{R}^d)$ with $\operatorname{div} u \in L^2(\Omega)$ satisfy $u\cdot\nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ such that the space $\mathcal{H}$ is well defined [21, Theorem I.1.2].

Theorem 1 (Global existence)**.** *Let $d = 1,2,3$, $T > 0$, and $D_{ij} = D_{ji} > 0$ for $i,j = 1,\ldots,N+1$, $i \ne j$. Suppose that $f \in L^2(0,T;\mathcal{V}')$, $u^0 \in \mathcal{H}$, and let $\rho_1^0,\ldots,\rho_{N+1}^0 \in L^1(\Omega)$ be nonnegative functions which satisfy $\sum_{i=1}^{N+1}\rho_i^0 = 1$ and $h(\rho^0) < +\infty$, where $\rho^0 = (\rho_1^0,\ldots,\rho_N^0)$ and h is defined in* (7) *below. Then there exists a global weak solution $(u,\rho_1,\ldots,\rho_{N+1})$ to*

(1)-(4) *(in the sense of* (27)-(28) *below) such that* $\rho_i \geq 0$, $\sum_{j=1}^{N+1} \rho_i = 1$ *in* $\Omega \times (0,T)$, *and*

$$u \in L^\infty(0,T;\mathcal{H}) \cap L^2(0,T;\mathcal{V}), \quad \partial_t u \in L^2(0,T;\mathcal{V}_2'),$$
$$\rho_i \in L^2(0,T;H^1(\Omega)), \quad \partial_t \rho_i \in L^2(0,T;\widetilde{H}^2(\Omega)'), \quad i = 1,\dots,N+1.$$

Remark 2. For a given force f, the velocity u is solely determined by the Navier-Stokes equations (2) with initial and boundary conditions. In the analysis, we may first solve (2), then plug the velocity u into the continuity equation (1), and hence solve (1) and (3). Therefore, Theorem 1 also applies to any velocity field which enjoys the regularity for the weak solutions of the incompressible Navier-Stokes equations.

We stress the fact that although the diffusion coefficients D_{ij} are constant, the diffusion matrix of the inverted Maxwell-Stefan system (see (6) below) depends on the mass densities in a nonlinear way. Note that the same existence result holds when we allow for reaction terms in (1) which are locally Lipschitz continuous and quasi-positive; see [4, 16].

The key ideas of the proof are as follows. First, we write (3) more compactly as $\nabla x = A(\rho)J$, where $x = (x_1,\dots,x_{N+1})$, $\rho = (\rho_1,\dots,\rho_{N+1})$, $J = (J_1,\dots,J_{N+1})$, and $A(\rho)$ is a matrix. Using the Perron-Frobenius theory, Bothe [4] proved that $A(\rho)$ can be inverted on its image. As in [16], it turns out that it is more convenient to work with the system in N components by eliminating the last equation in (1). We set $x' = (x_1,\dots,x_N)$ and similarly for the other vectors. Then, inverting $\nabla x' = -A_0(\rho)J'$ (Lemma 4), (1) becomes

$$\partial_t \rho' + (u\cdot\nabla)\rho' - \operatorname{div}(A_0(\rho)^{-1}\nabla x') = 0. \tag{6}$$

This equation can be analyzed by exploiting its entropy structure. Indeed, we associate to this system the entropy density (or, more precisely, Gibbs free energy)

$$h(\rho') = c\sum_{i=1}^{N+1} x_i(\ln x_i - 1) + c, \tag{7}$$

where $\rho_{N+1} = 1 - \sum_{i=1}^N \rho_i$ is interpreted as a function of the other mass densities. We "symmetrize" (6) by introducing the entropy variables

$$w_i = \frac{\partial h}{\partial \rho_i} = \frac{\ln x_i}{M_i} - \frac{\ln x_{N+1}}{M_{N+1}}, \quad i = 1,\dots,N, \tag{8}$$

and set $w = (w_1,\dots,w_N)$. The second equality in (8) is shown in Lemma 5 below. Denoting by $D^2h(\rho')$ the Hessian of h with respect to ρ', (6) is equivalent to

$$\partial_t \rho' + (u\cdot\nabla)\rho' - \operatorname{div}(B(w)\nabla w) = 0, \tag{9}$$

where $B(w) = A_0^{-1}(\rho')(D^2h)^{-1}(\rho')$ is symmetric and positive definite (Lemma 9). This formulation reveals the parabolic structure of the equations. The mass density vector ρ' is interpreted as a function of w. If all molar masses are equal, $M_i = M$, this function can be written as $\rho_i(w) = \exp(Mw_i)(1 + \sum_{j=1}^N \exp(Mw_j))^{-1}$ [16], showing that

$$0 < \rho' < 1 \quad \text{and} \quad \sum_{i=1}^N \rho_i < 1. \tag{10}$$

This formulation is no longer possible if the molar masses are different. In this situation, ρ' is implicitly given as a function of w; there is no explicit formula anymore. However, we are able to show that the mapping $\rho' \mapsto w$, defined by (8) and $x_i = \rho_i/(cM_i)$, can be inverted and that (10) still holds (Corollary 8). We emphasize that this property is obtained without the use of a maximum principle. The idea is to prove the existence of weak solutions w to (9) and to conclude (10) directly from the inverted relation $\rho = \rho(w)$.

The entropy $H(\rho') = \int_\Omega h(\rho')dz$ provides suitable a priori estimates. Indeed, using w as a test function in (9), a computation (see Lemma 12 and the proof of Theorem 1) shows the entropy-dissipation inequality

$$\frac{dH}{dt} = -\int_\Omega \nabla w : B(w)\nabla w dz \leq -C_B \sum_{i=1}^{N+1} \int_\Omega |\nabla\sqrt{x_i}|^2 dz, \tag{11}$$

where the constant $C_B > 0$ only depends on the diffusion coefficients D_{ij} and the molar masses M_i and the double point ":" signifies summation over both matrix indices. The key point is that the integral $\int_\Omega((u\cdot\nabla)\rho')\cdot w dz$ in (9) vanishes (Lemma 11). This property allows us to "separate" the solution's behavior of the Maxwell-Stefan system and the given velocity of Navier-Stokes equations. It yields H^1 estimates for $\sqrt{x_i}$ from which we conclude H^1 bounds for ρ_i (Lemma 15). We note that a diffusion inequality which directly implies the above entropy-dissipation inequality was first established in [12, Section 4].

The proof of Theorem 1 is based on a semi-discretization in time of both the Navier-Stokes equations (2) and Maxwell-Stefan equations (9) with time step $\tau > 0$, together with a regularization using the operator $\varepsilon(\Delta^2 w + w)$ in (9), which guarantees the coercivity in w. The existence of a solution to the approximate problem is shown by means of the Leray-Schauder fixed-point theorem. The discrete analogon of the entropy-dissipation inequality (11) provides bounds uniform in the approximation parameters τ and ε. By weak compactness and the Aubin lemma, this allows us to perform the limit $(\tau, \varepsilon) \to 0$.

System (1)-(3) admits the homogeneous steady state $\bar\rho_i^0 = \text{meas}(\Omega)^{-1}\|\rho_i^0\|_{L^1(\Omega)}$ or $\bar x_i^0 = \bar\rho_i^0/(\bar c^0 M_i)$, where $\bar c^0 = \sum_{i=1}^{N+1} \bar\rho_i^0/M_i$. We prove that the solution to (1)-(3), constructed in Theorem 1, converges exponentially fast to this stationary state. For this, we introduce the relative entropy

$$H^*(\rho) = \sum_{i=1}^{N+1} \int_\Omega c x_i \ln \frac{x_i}{\bar x_i^0} dz. \tag{12}$$

Theorem 3 (Exponential decay)**.** *Let the assumptions of Theorem 1 hold. We assume that there exists $0 < \eta < 1$ such that $\rho_i^0 \geq \eta$ for $i = 1, \ldots, N+1$. Let (u, ρ) be the weak solution, whose existence is guaranteed by Theorem 1. Then there exist constants $C > 0$, only depending on ρ_i^0 and M_i, and $\lambda > 0$, only depending on Ω and M_i, such that for all $t > 0$ and $i = 1, \ldots, N+1$,*

$$\|x_i(\cdot, t) - \bar x_i^0\|_{L^1(\Omega)} \leq Ce^{-\lambda t}\sqrt{H^*(\rho^0)},$$

where $x_i = \rho_i/(cM_i)$ with $c = \sum_{i=1}^{N+1} \rho_i/M_i$ and $\bar x_i^0 = \rho_i^0/(\bar c^0 M_i)$ with $\bar c^0 = \sum_{i=1}^{N+1} \bar\rho_i^0/M_i$.

The proof is based on the entropy-dissipation inequality (11) by relating the entropy dissipation with the entropy via the logarithmic Sobolev inequality [15, Remark 3.7]. Similarly as in [16], the difficulty of the proof is that the approximate solution does not conserve the L^1 norm because of the presence of the regularizing ε-terms. The estimations of these terms make the proof rather technical.

Compared to our previous work [16], the main novelty in this paper is the treatment of the molar masses M_i, which may be not equal. Therefore, we need to distinguish between the mass densities ρ_i and the molar fractions x_i, which makes necessary to derive some additional estimates. In particular, the proof of the positive definiteness of the Hessian of h, which implies the positive definiteness of $B(w)$, is rather involved (see Lemma 9).

The paper is organized as follows. In Section 2, we prove some auxiliary results needed for the main proofs. In particular, we show properties of the relations between w, ρ, and x and of the matrices D^2h and $B(w)$. The proofs of Theorems 1 and 3 are presented in Sections 3 and 4, respectively.

2. Preparations

In this section, we show some auxiliary results which are used in the proofs of the main theorems.

2.1. **Equivalent formulation of (1) and (3).** We recall the notation $\rho = (\rho', \rho_{N+1})$, $\rho' = (\rho_1, \ldots, \rho_N)$ and similarly for x and J, defined by $x_i = \rho_i/(cM_i)$ and $J_i = M_i j_i$ $(i = 1, \ldots, N+1)$. The matrix $\nabla\rho$ consists of the elements $\partial\rho_i/\partial z_j$ $(1 \le i \le N+1$, $1 \le j \le d)$, and we define similarly ∇x and ∇w. Then we can formulate (1) and (3) more compactly as

$$\partial_t\rho + (u\cdot\nabla)\rho + \operatorname{div} J = 0, \quad \nabla x = AJ, \tag{13}$$

where the $(N+1)\times(N+1)$ matrix $A = A(\rho) = (A_{ij})$ is defined by

$$\begin{aligned} A_{ij} &= d_{ij}\rho_i \quad \text{if } i \ne j,\ i,j = 1,\ldots,N+1, \\ A_{ij} &= -\sum_{k=1,\,k\ne i}^{N+1} d_{ik}\rho_k \quad \text{if } i = j = 1,\ldots,N+1, \end{aligned} \tag{14}$$

and $d_{ij} = 1/(c^2M_iM_jD_{ij})$. It is shown in [16, Section 2] that the system of $N+1$ equations $\nabla x = AJ$ can be reduced to the first N components, leading to

$$\partial_t\rho' + (u\cdot\nabla)\rho' + \operatorname{div} J' = 0, \quad \nabla x' = A_0J', \tag{15}$$

where the $N\times N$ matrix $A_0 = A_0(\rho') = (A^0_{ij})$ is given by

$$\begin{aligned} A^0_{ij} &= -(d_{ij} - d_{i,N+1})\rho_i \quad \text{if } i \ne j,\ i,j = 1,\ldots,N, \\ A^0_{ij} &= \sum_{k=1,\,k\ne i}^{N} (d_{ik} - d_{i,N+1})\rho_k + d_{i,N+1} \quad \text{if } i = j = 1,\ldots,N. \end{aligned} \tag{16}$$

Lemma 4. *The matrix A_0 is invertible and the elements of its inverse A_0^{-1} are uniformly bounded in $\rho_1, \ldots, \rho_N \in [0,1]$.*

Proof. The definition $c = \sum_{i=1}^{N+1} \rho_i/M_i$ and the property $0 \le \rho_i \le 1$ imply that

$$\left(\max_{1\le i\le N+1} M_i\right)^{-1} \le c \le \left(\min_{1\le i\le N+1} M_i\right)^{-1}. \tag{17}$$

Hence, the coefficients $d_{ij} = 1/(c^2 M_i M_j D_{ij})$ are bounded uniformly in $\rho_k \in [0,1]$. Therefore, the proof of Lemma 2.3 in [16] applies, proving the result. □

2.2. **Entropy variables.** We recall the relations $x_i = \rho_i/(cM_i)$, $c = \sum_{i=1}^{N+1} \rho_i/M_i$, and $\sum_{i=1}^{N+1} \rho_i = 1$. Since

$$x_{N+1} = \frac{\rho_{N+1}}{cM_{N+1}} = \frac{1}{cM_{N+1}}\left(1 - \sum_{i=1}^N \rho_i\right), \quad c = \sum_{i=1}^N \frac{\rho_i}{M_i} + \frac{1}{M_{N+1}}\left(1 - \sum_{i=1}^N \rho_i\right),$$

we may interpret the entropy density (7) as a function of $\rho' = (\rho_1, \ldots, \rho_N)$, which gives

$$\begin{aligned} h(\rho') &= c\sum_{i=1}^N x_i(\ln x_i - 1) + cx_{N+1}(\ln x_{N+1} - 1) + c \\ &= \sum_{i=1}^N \frac{\rho_i}{M_i}\left(\ln\frac{\rho_i}{M_i} - 1\right) + \frac{\rho_{N+1}}{M_{N+1}}\left(\ln\frac{\rho_{N+1}}{M_{N+1}} - 1\right) - c(\ln c - 1), \end{aligned} \tag{18}$$

First, we prove that the entropy variables can be written as in (8).

Lemma 5. *The entropy variables $w_i = \partial h(\rho')/\partial\rho_i$ are given by*

$$w_i = \frac{\ln x_i}{M_i} - \frac{\ln x_{N+1}}{M_{N+1}}, \quad i = 1, \ldots, N. \tag{19}$$

Proof. The proof is just a computation. Indeed, we infer from

$$\frac{\partial c}{\partial\rho_i} = \frac{\partial}{\partial\rho_i}\left(\sum_{k=1}^N \frac{\rho_k}{M_k} + \frac{1}{M_{N+1}}\left(1 - \sum_{k=1}^N \rho_k\right)\right) = \frac{1}{M_i} - \frac{1}{M_{N+1}}$$

for $i = 1, \ldots, N$ that

$$\begin{aligned} \frac{\partial h}{\partial\rho_i}(\rho') &= \frac{1}{M_i}\ln\frac{\rho_i}{M_i} - \frac{1}{M_{N+1}}\ln\frac{\rho_{N+1}}{M_{N+1}} - \left(\frac{1}{M_i} - \frac{1}{M_{N+1}}\right)\ln c \\ &= \frac{1}{M_i}\ln\frac{\rho_i}{cM_i} - \frac{1}{M_{N+1}}\ln\frac{\rho_{N+1}}{cM_{N+1}}, \end{aligned}$$

and since $\rho_i/(cM_i) = x_i$, the conclusion follows. □

We claim that we can invert the mapping $x' \mapsto w$, defined by (19).

Lemma 6. *Let $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$ be given. Then there exists a unique $(x_1, \ldots, x_N) \in (0,1)^N$ satisfying $\sum_{i=1}^N x_i < 1$ such that (19) holds with $x_{N+1} = 1 - \sum_{i=1}^N x_i$. In particular, the mapping $\mathbb{R}^N \to (0,1)^N$, $x'(w) = (x_1, \ldots, x_N)$, is bounded.*

Proof. Introduce the function $f(s) = \sum_{i=1}^N (1-s)^{M_i/M_{N+1}} \exp(M_i w_i)$ for $s \in [0,1]$. Then f is strictly decreasing in $[0,1]$ and $0 = f(1) < f(s) < f(0) = \sum_{i=1}^N \exp(M_i w_i)$ for $s \in (0,1)$. By continuity, there exists a unique fixed point $s_0 \in (0,1)$, $f(s_0) = s_0$. Defining $x_i = (1-s_0)^{M_i/M_{N+1}} \exp(M_i w_i)$ for $i = 1, \ldots, N$, we infer that $x_i > 0$ and $\sum_{i=1}^N x_i = f(s_0) = s_0 < 1$. Hence, in view of $x_{N+1} = 1 - s_0$, (19) holds. □

Given ρ, we can define $x_i = \rho_i/(cM_i)$, where $c = \sum_{i=1}^{N+1} \rho_i/M_i$. The following lemma ensures that this mapping is invertible.

Lemma 7. *Let $x' = (x_1, \ldots, x_N) \in (0,1)^N$ and $x_{N+1} = 1 - \sum_{i=1}^N x_i > 0$ be given and define for $i = 1, \ldots, N+1$,*

$$\rho_i(x') = \rho_i = cM_i x_i, \quad \text{where } c = \frac{1}{\sum_{k=1}^{N+1} M_k x_k}.$$

Then $(\rho_1, \ldots, \rho_N) \in (0,1)^N$ is the unique vector satisfying $\rho_{N+1} = 1 - \sum_{i=1}^N \rho_i > 0$, $x_i = \rho_i/(cM_i)$ for $i = 1, \ldots, N+1$, and $c = \sum_{k=1}^{N+1} \rho_k/M_k$.

The proof follows immediately from $\sum_{k=1}^{N+1} \rho_k/M_k = \sum_{k=1}^{N+1} cx_k = c$, and the fact that $\rho_i/M_i x_i = \sum_{k=1}^{N+1} \rho_k/M_k$ for $i = 1, 2, ..., N$ have unique solutions $\rho_i = M_i x_i / \sum_{k=1}^{N+1} M_k x_k$ for $i = 1, 2, ..., N$ by applying Cramer's rule.

Combining Lemmas 6 and 7, we infer the following result.

Corollary 8. *Let $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$ be given. Then there exists a unique vector $(\rho_1, \ldots, \rho_N) \in (0,1)^N$ satisfying $\sum_{i=1}^N \rho_i < 1$ such that (19) holds for $\rho_{N+1} = 1 - \sum_{i=1}^N \rho_i$ and $x_i = \rho_i/(cM_i)$ with $c = \sum_{i=1}^{N+1} \rho_i/M_i$. Moreover, the mapping $\mathbb{R}^N \to (0,1)^N$, $\rho'(w) = (\rho_1, \ldots, \rho_N)$, is bounded.*

2.3. **Hessian of the entropy density.** We prove some properties of the Hessian $(H_{ij}) = (\partial^2 h(\rho')/\partial\rho_i \partial\rho_j)_{1 \le i,j \le N} = (\partial w_i/\partial \rho_j)_{1 \le i,j \le N}$ and the matrix $(G_{ij}) = (\partial w_i/\partial x_j)_{1 \le i,j \le N}$. Differentiating (18) gives

$$H_{ij} = \frac{\delta_{ij}}{M_i \rho_i} + \frac{1}{M_{N+1}\rho_{N+1}} - \frac{1}{c}\left(\frac{1}{M_i} - \frac{1}{M_{N+1}}\right)\left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right), \quad i, j = 1, \ldots, N,$$

where δ_{ij} denotes the Kronecker delta.

Lemma 9. *The matrix (H_{ij}) is symmetric and positive definite for all $\rho_1, \ldots, \rho_N > 0$ satisfying $\sum_{i=1}^N \rho_i < 1$.*

Proof. We claim that the principal minors $\det H_k$ of (H_{ij}) satisfy

$$\det H_k > \frac{2}{cM_{N+1}\prod_{\ell=1}^k M_\ell}\left(\sum_{i,j=1,\, i<j}^{k} \frac{1}{\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k \rho_\ell} + \sum_{j=1}^{k} \frac{1}{\prod_{\ell=1,\,\ell\neq j}^k \rho_\ell}\right) > 0 \tag{20}$$

for $k = 1, \ldots, N$. Then the positive definiteness of (H_{ij}) follows from Sylvester's criterion. It remains to prove (20). Since each column of H_k can be written for $j = 1, 2, ..., k$, as the difference

$$\begin{pmatrix} \delta_{1j}(M_1\rho_1)^{-1} + (M_{N+1}\rho_{N+1})^{-1} \\ \vdots \\ \delta_{kj}(M_k\rho_k)^{-1} + (M_{N+1}\rho_{N+1})^{-1} \end{pmatrix} - \frac{1}{c}\left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right)\begin{pmatrix} M_1^{-1} - M_{N+1}^{-1} \\ \vdots \\ M_k^{-1} - M_{N+1}^{-1} \end{pmatrix},$$

a calculation shows that

$$\begin{aligned} \det H_k &= \frac{1}{\prod_{\ell=1}^k M_\ell\rho_\ell}\left(\sum_{j=1}^k \frac{M_j\rho_j}{M_{N+1}\rho_{N+1}} + 1\right) \\ &\quad - \frac{1}{c}\sum_{j=1}^k\left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right)\frac{1}{\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} \\ &\quad \times\left(\sum_{i=1,\,i\neq j}^k \frac{M_i\rho_i}{M_{N+1}\rho_{N+1}}\left(\frac{1}{M_j} - \frac{1}{M_i}\right) + \left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right)\right). \end{aligned}$$

Multiplying this expression by c and rearranging the terms, we find that

$$\begin{aligned} c\det H_k &= \left(\sum_{j=1}^k \frac{c}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} + \frac{c}{\prod_{\ell=1}^k M_\ell\rho_\ell}\right) \\ &\quad - \sum_{j=1}^k\left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right)^2\frac{1}{\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} \\ &\quad - \sum_{j=1}^k\left(\frac{1}{M_j} - \frac{1}{M_{N+1}}\right)\sum_{i=1,\,i\neq j}^k\left(\frac{1}{M_j} - \frac{1}{M_i}\right)\frac{M_i\rho_i}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Recalling that $c = \sum_{\ell=1}^{N+1}\rho_\ell/M_\ell$, we can estimate as follows:

$$\begin{aligned} I_1 &> \sum_{j=1}^k \frac{\sum_{i=1,\,i\neq j}^k \rho_i/M_i + \rho_{N+1}/M_{N+1}}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} + \frac{\sum_{j=1}^k \rho_j/M_j}{\prod_{\ell=1}^k M_\ell\rho_\ell} \\ &= \sum_{j=1}^k\left(\sum_{i=1,\,i\neq j}^k \frac{1}{M_i^2 M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k M_\ell\rho_\ell} + \frac{1}{M_{N+1}^2\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell}\right) \\ &\quad + \sum_{j=1}^k \frac{1}{M_j^2\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} \\ &= \sum_{i,j=1,\,i<j}^k \frac{M_i^{-2} + M_j^{-2}}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k M_\ell\rho_\ell} + \sum_{j=1}^k \frac{M_j^{-2} + M_{N+1}^{-2}}{\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell}. \end{aligned}$$

Using $\sum_{i,j} b_{ij}(a_j - a)(a_j - a_i) = \sum_{i<j} b_{ij}(a_j - a_i)^2$ for numbers a, $a_i \in \mathbb{R}$ and $b_{ij} = b_{ji} \in \mathbb{R}$, the last term I_3 can be formulated as

$$I_3 = -\sum_{j=1}^k \sum_{i=1,\, i\neq j}^k \frac{(M_j^{-1} - M_{N+1}^{-1})(M_j^{-1} - M_i^{-1})}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k M_\ell\rho_\ell} = -\sum_{i,j=1,\, i<j}^k \frac{(M_i^{-1} - M_j^{-1})^2}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k M_\ell\rho_\ell}.$$

Therefore, we infer that

$$\begin{aligned} c\det H_k &> \sum_{i,j=1,\, i<j}^k \frac{2M_i^{-1}M_j^{-1}}{M_{N+1}\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k M_\ell\rho_\ell} + \sum_{j=1}^k \frac{2M_j^{-1}M_{N+1}^{-1}}{\prod_{\ell=1,\,\ell\neq j}^k M_\ell\rho_\ell} \\ &= \frac{2}{M_{N+1}\prod_{\ell=1}^k M_\ell}\left(\sum_{i,j=1,\, i<j}^k \frac{1}{\rho_{N+1}\prod_{\ell=1,\,\ell\neq i,j}^k \rho_\ell} + \sum_{j=1}^k \frac{1}{\prod_{\ell=1,\,\ell\neq j}^k \rho_\ell}\right), \end{aligned}$$

and (20) follows. □

The coefficients $G_{ij} = \partial w_i/\partial x_j$ are given by

$$G_{ij} = \frac{1}{M_{N+1}x_{N+1}} + \frac{\delta_{ij}}{M_i x_i} = c\left(\frac{1}{\rho_{N+1}} + \frac{\delta_{ij}}{\rho_i}\right), \quad i,j = 1,\ldots,N, \tag{21}$$

since $x_i = \rho_i/(cM_i)$. We recall that $w(\rho')$ is computed in (19).

Lemma 10. *It holds for all $\rho_1,\ldots,\rho_N > 0$ satisfying $\rho_{N+1} = 1 - \sum_{i=1}^N \rho_i > 0$:*

- (i) *The matrix $G(\rho') = (G_{ij})$ and its inverse $G^{-1}(\rho')$ are positive definite.*
- (ii) $\nabla w(\rho') = G(\rho')\nabla x'(\rho')$.
- (iii) *The elements of the $N \times N$ matrix $d\rho'/dx' = (\partial\rho_i/\partial x_k)$ are bounded by a constant which depends only on the molar masses M_i.*
- (iv) *The $N \times N$ matrix $B(\rho') = A_0^{-1}(\rho')G^{-1}(\rho')$ is symmetric, positive definite, and its elements are uniformly bounded.*

Proof. (i) The explicit expression (21) shows that $G(\rho')$ is symmetric. Since all principal minors $\det G_k$ of $G(\rho')$,

$$\det G_k = \frac{\sum_{i=1}^k M_i x_i + M_{N+1}x_{N+1}}{(\prod_{i=1}^k M_i x_i)M_{N+1}x_{N+1}}, \quad k = 1,\ldots,N,$$

are positive, Sylvester's criterion implies that $G(\rho')$ is positive definite. Consequently, also $G^{-1}(\rho')$ is positive definite.

(ii) We infer from (19) that

$$\nabla w_i = \frac{\nabla x_i}{M_i x_i} + \sum_{j=1}^N \frac{\nabla x_j}{M_{N+1}x_{N+1}} = \sum_{j=1}^N G_{ij}\nabla x_j, \quad i = 1,2,\ldots,N,$$

and hence $\nabla w = G(\rho')\nabla x'$.

(iii) By Lemma 7, it follows that

$$\frac{\partial\rho_i}{\partial x_k} = cM_i\delta_{ik} - c^2 M_i x_i(M_k - M_{N+1}), \quad i,k = 1,\ldots,N, \tag{22}$$

where $c = 1/\sum_{j=1}^{N+1} M_j x_j$. The claim follows from the inequalities $0 < x_i < 1$ and the bounds (17).

(iv) We set $G(\rho') = cK(\rho')$, where the elements K_{ij} of $K(\rho')$ are given by $K_{ij} = 1/\rho_{N+1} + \delta_{ij}/\rho_i$ for $i,j = 1,\ldots,N$. In view of part (i) of the proof, the matrix $K(\rho')$ is symmetric and positive definite, hence invertible. Then, by Lemma 2.4 in [16], $A_0^{-1}(\rho')K^{-1}(\rho')$ is symmetric and positive definite and its elements are uniformly bounded. Consequently, the same holds for $B(\rho') = c^{-1}A_0^{-1}(\rho')K^{-1}(\rho')$. This ends the proof. ☐

From Lemma 10 follows that

$$A_0^{-1}(\rho')\nabla x'(\rho') = \big(A_0^{-1}(\rho')G^{-1}(\rho')\big)(G(\rho')\nabla x'(\rho')) = B(\rho')\nabla w(\rho'). \tag{23}$$

We have shown at the end of Section 2.2 that ρ' can be interpreted as a function of w. Therefore, setting $B(w) := B(\rho'(w))$, (6) can be written as

$$\partial_t \rho'(w) + (u\cdot\nabla)\rho'(w) - \operatorname{div}(B(w)\nabla w) = 0. \tag{24}$$

The boundary conditions are given by

$$\nabla w_i \cdot \nu = 0 \quad \text{on } \partial\Omega,\ t>0, \quad i = 1,\ldots,N, \tag{25}$$

since $\nabla\rho_j\cdot\nu = 0$ on $\partial\Omega$ for all j implies that

$$\nabla x_i \cdot \nu = \nabla\frac{\rho_i}{cM_i}\cdot\nu = \frac{\nabla\rho_i\cdot\nu}{cM_i} - \sum_{j=1}^{N+1}\frac{\rho_i\nabla\rho_j\cdot\nu}{c^2 M_i M_j} = 0$$

and thus $\nabla w_i\cdot\nu = (G(\rho')\nabla x)_i\cdot\nu = 0$ on $\partial\Omega$.

2.4. **Some estimates.** We show two results which are needed in the proof of the existence theorem.

Lemma 11. *Let $u\in\mathcal{V}$ and $w\in H^1(\Omega)$. Then*

$$\int_\Omega ((u\cdot\nabla)\rho'(w))\cdot w\,dz = 0.$$

Proof. Using $\operatorname{div} u = 0$, the characterization (19) of w_i, and $\rho_i/M_i = cx_i$, we obtain after an integration by parts,

$$\begin{aligned}\int_\Omega ((u\cdot\nabla)\rho'(w))\cdot w\,dz &= \sum_{i=1}^N\int_\Omega (u\cdot\nabla\rho_i(w))w_i\,dz = -\sum_{i=1}^N\int_\Omega (u\cdot\nabla w_i)\rho_i(w)\,dz \\ &= -\sum_{i=1}^N\int_\Omega \rho_i(w)u\cdot\left(\frac{\nabla x_i}{M_i x_i} - \frac{\nabla x_{N+1}}{M_{N+1}x_{N+1}}\right)dz \\ &= -\sum_{i=1}^N\int_\Omega cu\cdot\nabla x_i\,dz + \int_\Omega\sum_{i=1}^N \rho_i\frac{u\cdot\nabla x_{N+1}}{M_{N+1}x_{N+1}}dz.\end{aligned}$$

Because of $\sum_{i=1}^N \rho_i = 1-\rho_{N+1}$ and $\rho_{N+1}/(M_{N+1}x_{N+1}) = c$, the last integral equals

$$\int_\Omega \frac{1-\rho_{N+1}}{M_{N+1}x_{N+1}} u\cdot\nabla x_{N+1}dz = \frac{1}{M_{N+1}}\int_\Omega u\cdot\nabla(\ln x_{N+1})dz - \int_\Omega cu\cdot\nabla x_{N+1}dz$$
$$= \sum_{i=1}^N\int_\Omega cu\cdot\nabla x_i dz,$$

where we integrated by parts and used $\operatorname{div} u = 0$ and $x_{N+1} = 1-\sum_{i=1}^N x_i$. This shows the lemma. ☐

In the following, we employ the notation $f(x) = (f(x_1),\ldots,f(x_{N+1}))$ for vectors $x = (x_1,\ldots,x_{N+1})$ and arbitrary functions f.

Lemma 12. *Let $w\in H^1(\Omega)$. Then there exists a constant $C_B>0$, only depending on the coefficients D_{ij} and M_i such that*

$$\int_\Omega \nabla w : B(w)\nabla w dz \ge C_B\int_\Omega |\nabla\sqrt{x}|^2 dz.$$

Proof. We follow the proof of Lemma 3.2 in [16]. In contrast to that proof, we have to take into account the different molar masses M_i which complicates the analysis. First, we claim that

$$\nabla w : B(w)\nabla w = \nabla s : (-\widetilde{A})^{-1}\nabla x,$$

where $s = (\ln x_1/M_1,\ldots,\ln x_{N+1}/M_{N+1})$ and $\widetilde{A} = A|_{\mathrm{im}(A)}$. To prove this claim, we set $r' = (r_1,\ldots,r_N)^\top = B(w)\nabla w\in\mathbb{R}^{N\times d}$ and $r_{N+1} = -\sum_{i=1}^N r_i\in\mathbb{R}^d$. Then, by (19),

$$\nabla w : B(w)\nabla w = \sum_{i=1}^N\left(\frac{\nabla\ln x_i}{M_i} - \frac{\nabla\ln x_{N+1}}{M_{N+1}}\right)\cdot r_i = \sum_{i=1}^{N+1}\frac{\nabla\ln x_i}{M_i}\cdot r_i = \nabla s : r, \tag{26}$$

where $r = (r', r_{N+1})^\top$. By (23), $\nabla x' = A_0 r'$, and the definitions (14) and (16) of A and A_0, respectively, we obtain for $i = 1,\ldots,N$,

$$\nabla x_i = \sum_{j=1,\,j\ne i}^N (d_{ij}-d_{i,N+1})(\rho_j r_i^\top - \rho_i r_j^\top) + d_{i,N+1}r_i^\top = (-Ar)_i = (-\widetilde{A}r)_i,$$

since $\mathrm{im}(A) = (\mathrm{span}(1,\ldots,1))^\perp$ and each column of r is an element of $\mathrm{im}(A)$. Moreover, each column of $\widetilde{A}r$ is also an element of $\mathrm{im}(A)$, so that

$$(-\widetilde{A}r)_{N+1} = -\sum_{i=1}^N(-\widetilde{A}r)_i = -\sum_{i=1}^N\nabla x_i = \nabla x_{N+1}.$$

Therefore, $\nabla x = -\widetilde{A}r$. It is shown in [16, Lemma 2.2] that $\widetilde{A}$ is invertible. Thus, $r = (-\widetilde{A})^{-1}\nabla x$, and inserting this expression into (26) proves the claim.

Next, we introduce the symmetric matrix $\widetilde{A}_S = P^{-1/2}\widetilde{A}P^{1/2}$, where

$$P^{1/2} = \mathrm{diag}((M_1x_1)^{1/2},\ldots,(M_{N+1}x_{N+1})^{1/2}).$$

Then $(-\widetilde{A}_S)^{-1} = P^{-1/2}(-\widetilde{A})^{-1}P^{1/2}$. Arguing similarly as in [16, Lemma 2.2], we find that $(-\widetilde{A}_S)^{-1}$ is a self-adjoint endomorphism whose smallest eigenvalue is bounded from below by some positive constant, say $C_0 > 0$, which depends only on (D_{ij}). This gives

$$\begin{aligned}
\nabla w : B(w)\nabla w &= \nabla s : (-\widetilde{A})^{-1}\nabla x \\
&= 4\nabla\sqrt{x} : \operatorname{diag}\big(M_1^{-1}x_1^{-1/2},\ldots,M_{N+1}^{-1}x_{N+1}^{-1/2}\big)(-\widetilde{A})^{-1}\operatorname{diag}\big(x_1^{1/2},\ldots,x_{N+1}^{1/2}\big)\nabla\sqrt{x} \\
&= 4\nabla\sqrt{x} : \Big(\operatorname{diag}\big(M_1^{-1}x_1^{-1/2},\ldots,M_{N+1}^{-1}x_{N+1}^{-1/2}\big)P^{1/2}\Big)\left(P^{-1/2}(-\widetilde{A})^{-1}P^{1/2}\right) \\
&\quad \times \Big(P^{-1/2}\operatorname{diag}\big(x_1^{1/2},\ldots,x_{N+1}^{1/2}\big)\Big)\nabla\sqrt{x} \\
&= 4\nabla\sqrt{x} : \operatorname{diag}\big(M_1^{-1/2},\ldots,M_{N+1}^{-1/2}\big)(-\widetilde{A}_S)^{-1}\operatorname{diag}\big(M_1^{-1/2},\ldots,M_{N+1}^{-1/2}\big)\nabla\sqrt{x} \\
&\geq C_0\big|\operatorname{diag}\big(M_1^{-1/2},\ldots,M_{N+1}^{-1/2}\big)\nabla\sqrt{x}\big|^2 \\
&\geq C_B|\nabla\sqrt{x}|^2,
\end{aligned}$$

where $C_B = C_0(\max_{1\leq i\leq N+1} M_i)^{-1/2}$. □

3. Proof of Theorem 1

We say that (u,ρ) is a weak solution to (1)-(4) if for any $v \in C_0^\infty(\Omega\times[0,T);\mathbb{R}^d)$ with $\operatorname{div} v = 0$,

$$\begin{aligned}
-\int_0^T\int_\Omega u\cdot\partial_t v dz\,dt + \int_0^T\int_\Omega ((u\cdot\nabla)u)\cdot v dz\,dt + \int_0^T\int_\Omega \nabla u : \nabla v dz\,dt \\
= \int_0^T \langle f, v\rangle dt + \int_\Omega u^0\cdot v(\cdot,0)dz, \qquad (27)
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$; and if for any $q \in C_0^\infty(\overline{\Omega}\times[0,T);\mathbb{R}^N)$ with $\nabla q\cdot\nu|_{\partial\Omega} = 0$,

$$\begin{aligned}
-\int_0^T\int_\Omega \rho'\cdot\partial_t q dz\,dt + \int_0^T\int_\Omega \nabla q : A_0^{-1}(\rho')\nabla x'(\rho')dz\,dt + \int_0^T\int_\Omega ((u\cdot\nabla)\rho')\cdot q dz\,dt \\
= \int_\Omega (\rho^0)'\cdot q(\cdot,0)dz. \qquad (28)
\end{aligned}$$

The proof of Theorem 1 is divided into several steps.

First, by standard theory of the incompressible Navier-Stokes equations [21], there exists a divergence-free weak solution to (27), satisfying

$$\|u\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq C(u^0, f), \qquad (29)$$

and enjoying the regularity stated in Theorem 1. Thus, in the following, we only need to solve (28).

3.1. **Approximate problem.** Let $M \in \mathbb{N}$ and set $\tau = T/M$. Let $k \in \{1, \dots, M\}$. Given $w^{k-1} \in L^\infty(\Omega;\mathbb{R}^N)$, we solve a regularized approximate problem for (1) and (3): For any $q \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$:

$$\int_\Omega \frac{\rho'(w^k) - \rho'(w^{k-1})}{\tau} \cdot q dz + \int_\Omega \nabla q : A_0^{-1}(\rho'(w^k))\nabla x'(\rho'(w^k)) dz \\ + \int_\Omega ((u\cdot\nabla)\rho'(w^k)) \cdot q dz + \varepsilon \int_\Omega (\Delta w^k \cdot \Delta q + w^k \cdot q) dz = 0, \tag{30}$$

where $\rho'(w^k)$ is defined in Corollary 8. Because of (23), equation (30) is equivalent to

$$\int_\Omega \frac{\rho'(w^k) - \rho'(w^{k-1})}{\tau} \cdot q dz + \int_\Omega \nabla q : B(w^k)\nabla w^k dz \\ + \int_\Omega ((u\cdot\nabla)\rho'(w^k)) \cdot q dz + \varepsilon \int_\Omega (\Delta w^k \cdot \Delta q^k + w^k \cdot q) dz = 0, \tag{31}$$

Define for $0 < \eta < 1$ the space of bounded, strictly positive functions

$$Y_\eta = \Big\{ q = (q_1, \dots, q_N) \in L^\infty(\Omega;\mathbb{R}^N) : q_i \geq \eta \text{ for } i = 1, \dots .N,\ q_{N+1} = 1 - \sum_{i=1}^N q_i \geq \eta \Big\}.$$

Lemma 13. *Let $\eta^{k-1} \in (0,1)$ and $\rho^{k-1} \in Y_{\eta^{k-1}}$ with $\rho^{k-1} = \rho'(w^{k-1})$. Then there exist $\eta^k \in (0,1)$ and $w^k \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ which solves (30) satisfying $\rho'(w^k) \in Y_{\eta^k}$.*

Proof. Step 1. Let $\bar{w} \in L^\infty(\Omega;\mathbb{R}^N)$. Let $\sigma \in [0,1]$. We prove that there exists a unique $w \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ to

$$a_2(w,q) = F_2(q) \quad \text{for } q \in \widetilde{H}^2(\Omega;\mathbb{R}^N), \tag{32}$$

where for w, $q \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$,

$$a_2(w,q) = \varepsilon \int_\Omega (\Delta w \cdot \Delta q + w \cdot q) dz + \int_\Omega \nabla q : B(\bar{w})\nabla w dz$$
$$F_2(q) = -\frac{\sigma}{\tau} \int_\Omega (\rho'(\bar{w}) - \rho^{k-1}) \cdot q dz + \sigma \int_\Omega ((u\cdot\nabla)q) \cdot \rho'(\bar{w}) dz.$$

We infer from Lemma 10 (iv) that $a_2(\cdot,\cdot)$ is a bounded bilinear form on $\widetilde{H}^2(\Omega;\mathbb{R}^N)$, and from the positive definiteness of $B(\bar{w})$ (see also Lemma 10 (iv)) follows that

$$a_2(w,w) \geq \varepsilon \int_\Omega (|\Delta w|^2 + |w|^2) dz \geq C\|w\|^2_{H^2(\Omega)}.$$

Since $\rho'(\bar{w})$ is a bounded function, by Corollary 8, we infer that F_2 is bounded on $\widetilde{H}^2(\Omega;\mathbb{R}^N)$. Then the Lax-Milgram lemma provides the existence of a unique solution $w \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ to (32).

Step 2. This defines the fixed-point mapping $S : L^\infty(\Omega;\mathbb{R}^N) \times [0,1] \to L^\infty(\Omega;\mathbb{R}^N)$, $S(\bar{w},\sigma) = w$, where w solves (32). By construction, $S(\bar{w}, 0) = 0$ for all $w \in L^\infty(\Omega;\mathbb{R}^N)$. Since the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact, standard arguments show that S is continuous and compact. It remains to prove that there exists a constant $C > 0$ such that $\|w\|_{L^\infty(\Omega)} \le C$ for all $(w,\sigma) \in L^\infty(\Omega;\mathbb{R}^N) \times [0,1]$ satisfying $w = S(w,\sigma)$.

Let $w \in L^\infty(\Omega;\mathbb{R}^N)$ be such a fixed point. Then it solves (32) with $\bar w$ replaced by w. Taking $w \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ as a test function, it follows from Lemma 11 that

$$\frac{\sigma}{\tau}\int_\Omega (\rho'(w) - \rho'(w^{k-1}))\cdot w dz + \int_\Omega \nabla w : B(w)\nabla w dz + \varepsilon\int_\Omega (|\Delta w|^2 + |w|^2)dz = 0.$$

By Lemma 9, the entropy density h, defined in (7), is convex. This implies that $h(\rho'(w)) - h(\rho^{k-1}) \le (dh/d\rho')\cdot(\rho'(w) - \rho^{k-1}) = w\cdot(\rho'(w) - \rho^{k-1})$ (see Lemma 5). We infer from the positive definiteness of $B(w)$ (see Lemma 10 (iv)) that

$$\sigma\int_\Omega h(\rho'(w))dz + \varepsilon\tau\int_\Omega (|\Delta w|^2 + |w|^2)dz \le \sigma\int_\Omega h(\rho^{k-1})dz.$$

This yields the desired uniform H^2 bound and hence uniform L^∞ bound for w. By the Leray-Schauder fixed-point theorem, there exists a solution $w \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ to (31). According to Corollary 8, we can define $\rho_1(w),\ldots,\rho_N(w) > 0$ satisfying $\rho_{N+1}(w) := 1 - \sum_{i=1}\rho_i(w) > 0$, and we set $\eta^k = \min_{1\le i\le N+1}\operatorname{ess\,inf}_\Omega \rho_i(w) > 0$. Then, by construction, $\rho'(w) \in Y_{\eta^k}$. ☐

3.2. **Uniform estimates.** Let $\rho^0 = (\rho_1^0,\ldots,\rho_{N+1}^0)$ satisfing $\rho_i^0 \ge 0$ for $i = 1,\ldots,N+1$ and $\sum_{i=1}^{N+1}\rho_i^0 = 1$. Let $0 < \eta^0 \le 1/(2(N+1))$ and define

$$\rho_i^{\eta^0} = \frac{\rho_i^0 + 2\eta^0}{1 + 2\eta^0(N+1)}, \quad i = 1,\ldots,N+1.$$

Then $\rho_i^{\eta^0} \ge \eta^0$ for all $i = 1,\ldots,N+1$ and $\sum_{i=1}^{N+1}\rho_i^{\eta^0} = 1$. Finally, let $w^0 \in L^\infty(\Omega;\mathbb{R}^N)$ be defined by (19). Applying Lemma 13 iteratively, we obtain a sequence of approximate solutions $w^k \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ to (30) such that $\rho'(w^k) \in Y_{\eta^k}$, where $\eta^k \in (0,1)$. For the following, we set $\rho^k = \rho'(w^k)$ for $k \ge 0$, slightly abusing our notation.

Lemma 14. *For any $1 \le k \le M$ and sufficiently small $\eta^0 > 0$, it holds that*

(33)
$$\int_\Omega h(\rho^k)dz + C_B\tau\sum_{j=1}^k \|\nabla\sqrt{x(\rho^j)}\|_{L^2(\Omega)}^2 + \varepsilon\tau\sum_{j=1}^k\int_\Omega (|\Delta w^j|^2 + |w^j|^2)dz \le \int_\Omega h(\rho^0)dz + 1,$$

where $\sqrt{x(\rho^j)} = (\sqrt{x_1(\rho^j)},\ldots,\sqrt{x_{N+1}(\rho^j)})$, $x_i(\rho^j) = \rho_i^j/(cM_i)$ for $i = 1,\ldots,N+1$, $c = \sum_{k=1}^{N+1}\rho_k^j/M_k$, and $C_B > 0$ is obtained from Lemma 12.

Proof. Lemma 12 and Step 3 of the proof of Lemma 13 imply after summation over $j = 1,\ldots,k$ that

$$\int_\Omega h(\rho^k)dz + C_B\tau\sum_{j=1}^k \|\sqrt{x(\rho^j)}\|_{L^2(\Omega)}^2 + \varepsilon\tau\sum_{j=1}^k\int_\Omega (|\Delta w^j|^2 + |w^j|^2)dz \le \int_\Omega h(\rho^{\eta^0})dz.$$

By dominated convergence,

$$\lim_{\eta^0\to 0}\int_\Omega h(\rho^{\eta^0})dz = \int_\Omega h(\rho^0)dz,$$

and hence, for sufficiently small $\eta^0 > 0$,

$$\int_\Omega h(\rho^{\eta^0})dz \le \int_\Omega h(\rho^0)dz + 1.$$

This proves (33). □

Lemma 15. *It holds that*

$$\tau\sum_{k=1}^M \|\nabla x(\rho^k)\|_{L^2(\Omega)}^2 + \tau\sum_{k=1}^M \|\nabla\rho^k\|_{L^2(\Omega)}^2 + \tau\sum_{k=1}^M \left\|\frac{\rho^k-\rho^{k-1}}{\tau}\right\|_{\widetilde{H}^2(\Omega)'}^2 \le C(u^0,\rho^0,f), \tag{34}$$

Proof. Since $x_i(\rho^k) = \rho_i^k/(cM_i)$ with $c = \sum_{k=1}^{N+1}\rho_k/M_k$ is bounded by one, we find that

$$\|\nabla x(\rho^k)\|_{L^2(\Omega)} \le 2\|\sqrt{x(\rho^k)}\|_{L^2(\Omega)}\|\nabla\sqrt{x(\rho^k)}\|_{L^2(\Omega)} \le 2\|\nabla\sqrt{x(\rho^k)}\|_{L^2(\Omega)}.$$

Thus, by (33),

$$\tau\sum_{k=1}^M \|\nabla x(\rho^k)\|_{L^2(\Omega)}^2 \le C(\rho^0).$$

Then it follows from Lemma 10 (iii) that $|\nabla\rho^k| \le C|\nabla x'(\rho^k)|$. Hence,

$$\tau\sum_{k=1}^M \|\nabla\rho^k\|_{L^2(\Omega)}^2 \le C\tau\sum_{k=1}^M \|\nabla x'(\rho^k)\|_{L^2(\Omega)}^2 \le C(\rho^0).$$

We deduce from (30), the boundedness of the elements of $A_0^{-1}(\rho^k)$ (see Lemma 4), and the uniform estimate for u in L^2 (see (29)) that for $q \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}
&\left|\frac{1}{\tau}\int_\Omega(\rho^k-\rho^{k-1})\cdot q dz\right| \\
&\le \|A_0^{-1}(\rho^k)\|_{L^\infty(\Omega)}\|\nabla x'(\rho^k)\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)} \\
&\quad + \|u\|_{L^2(\Omega)}\|\nabla\rho^k\|_{L^2(\Omega)}\|q\|_{L^\infty(\Omega)} + \varepsilon(\|\Delta w^k\|_{L^2(\Omega)} + \|w^k\|_{L^2(\Omega)})\|q\|_{L^2(\Omega)} \\
&\le C(u^0,f)\big(\|\nabla x'(\rho^k)\|_{L^2(\Omega)} + \|\nabla\rho^k\|_{L^2(\Omega)} + \varepsilon\|w^k\|_{H^2(\Omega)}\big)\|q\|_{H^2(\Omega)}.
\end{aligned}$$

Taking into account the above uniform estimates for $\nabla x'(\rho^k)$ and $\nabla\rho^k$ in L^2 and the estimate (33) for $\sqrt{\varepsilon}w^k$ in H^2, it follows that

$$\begin{aligned}
\tau\sum_{k=1}^M \left\|\frac{\rho^k-\rho^{k-1}}{\tau}\right\|_{\widetilde{H}^2(\Omega)'}^2 &\le C(u^0,f)\tau\sum_{k=1}^M\Big(\|\nabla x'(\rho^k)\|_{L^2(\Omega)}^2 + \|\nabla\rho^k\|_{L^2(\Omega)}^2 + \varepsilon^2\|w^k\|_{H^2(\Omega)}^2\Big) \\
&\le C(u^0,\rho^0,f).
\end{aligned}$$

This ends the proof. □

3.3. **Proof of Theorem 1.** Define the piecewise constant function $\rho^{(\tau)}(x,t)=\rho(x)$ and the difference quotient

$$\partial_t^\tau \rho^{(\tau)}(x,t)=\frac{\rho^k(x)-\rho^{k-1}(x)}{\tau}$$

for $x\in\Omega$, $(k-1)\tau<t\le k\tau$, $k=1,\ldots,M$. Similarly, we define $f^{(\tau)}$ and $w^{(\tau)}$. Lemmas 14 and 15 imply immediately the following uniform estimates:

$$\|x'(\rho^{(\tau)})\|_{L^\infty(0,T;L^\infty(\Omega))}+\|x'(\rho^{(\tau)})\|_{L^2(0,T;H^1(\Omega))}\le C, \tag{35}$$

$$\|\rho^{(\tau)}\|_{L^\infty(0,T;L^\infty(\Omega))}+\|\rho^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}+\|\partial_t^\tau\rho^{(\tau)}\|_{L^2(0,T;\widetilde{H}^2(\Omega)')}\le C, \tag{36}$$

$$\sqrt{\varepsilon}\|w^{(\tau)}\|_{L^2(0,T;H^2(\Omega))}\le C. \tag{37}$$

The weak formulation (30) can be written for any $q\in C_0^\infty(\overline{\Omega}\times[0,T);\mathbb{R}^N)$ with $\nabla q\cdot\nu|_{\partial\Omega}=0$ as follows:

$$\int_0^T\int_\Omega \partial_t^\tau\rho^{(\tau)}\cdot q dz\,dt+\int_0^T\int_\Omega \nabla q:A_0^{-1}(\rho^{(\tau)})\nabla x'(\rho^{(\tau)})dz\,dt+\int_0^T\int_\Omega((u\cdot\nabla)\rho^{(\tau)})\cdot q dz\,dt$$

$$=-\varepsilon\int_0^T\int_\Omega(\Delta w^{(\tau)}\cdot\Delta q+w^{(\tau)}\cdot q)dz\,dt. \tag{38}$$

Estimates (36) for $(\rho^{(\tau)})$ allow us to apply Aubin-Lions's lemma in the version of [9] which yields the existence of subsequences of $(\rho^{(\tau)})$ (not relabeled) such that, as $(\varepsilon,\tau)\to 0$,

$$\rho^{(\tau)}\to\rho' \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Furthermore, the strong convergence of $(\rho^{(\tau)})$ and the boundedness of the elements of A_0^{-1} and x' yield $A_0^{-1}(\rho^{(\tau)})\to A_0^{-1}(\rho')$, $x'(\rho^{(\tau)})\to x'(\rho')$ strongly in $L^p(0,T;L^p(\Omega))$ for any $p<\infty$. Together with the weak convergence (again up to a subsequence) of $(\nabla x'(\rho^{(\tau)}))$, we infer that

$$\nabla x'(\rho^{(\tau)})\rightharpoonup\nabla x'(\rho) \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

Finally, we note that $\varepsilon w^{(\tau)}\to 0$ strongly in $L^2(0,T;H^2(\Omega))$ as $(\varepsilon,\tau)\to 0$. These convergences are sufficient to pass to the limit $(\varepsilon,\tau)\to 0$ in (38) yielding a global solution ρ' to (28). In view of the a priori estimates uniform in η^0 and the finiteness of the initial entropy, we can perform the limit $\eta^0\to 0$ and hence conclude the existence result for general initial data. The theorem is proved.

4. Proof of Theorem 3

Let w^k be a solution to (31). First, we prove L^1 bounds for $\rho_i^k=\rho_i(w^k)$ and $c^k=\sum_{i=1}^{N+1}\rho_i^k/M_k$.

Lemma 16 (Uniform L^1 norms for ρ^k)**.** *There exist constants $\gamma_0>0$, depending on ρ^0, and $\varepsilon_0>0$ such that for all $0<\gamma<\min\{1,\gamma_0\}$ and $0<\varepsilon<\varepsilon_0$,*

$$\big|\|\rho_i^k\|_{L^1(\Omega)}-\|\rho_i^0\|_{L^1(\Omega)}\big|\le\gamma\|\rho_i^0\|_{L^1(\Omega)},\quad i=1,\ldots,N, \tag{39}$$

$$\big| \|\rho_{N+1}^k\|_{L^1(\Omega)} - \|\rho_{N+1}^0\|_{L^1(\Omega)} \big| \le \gamma \sum_{I=1}^N \|\rho_i^0\|_{L^1(\Omega)}. \tag{40}$$

Furthermore, $\|\rho_{N+1}^k\|_{L^1(\Omega)} \ge \frac{1}{2}\|\rho_{N+1}^0\|_{L^1(\Omega)} > 0$.

Proof. The proof is similar to the proof of Lemma 4.1 in [16]. The main difference is that the entropy differs from that of [16] which makes some changes necessary. We recall that $\tau = T/M$ with $T > 0$ and $M \in \mathbb{N}$. Using the test function $q = e_i$ in (31), where e_i is the ith unit vector of $\mathbb{R}^N$, and observing that

$$\int_\Omega ((u\cdot\nabla)\rho'(w^k))\cdot e_i dz = -\int_\Omega \operatorname{div}(u^k)\rho_i(w^k)dz = 0,$$

we have

$$\int_\Omega \rho_i^k dz = \int_\Omega \rho_i^{k-1} - \varepsilon\tau\int_\Omega w_i^k dz, \quad i = 1,\ldots,N.$$

Solving this recursion, we deduce that

$$\int_\Omega \rho_i^k dz = \int_\Omega \rho_i^0 dz - \varepsilon\tau\sum_{j=1}^k \int_\Omega w_i^j dz, \quad i = 1,\ldots,N. \tag{41}$$

Thus, we need to bound the L^1 norm of w_i^j. Recalling that $H(\rho^k) = \int_\Omega h(\rho'(w^k))dz$, we infer from Step 3 of the proof of Lemma 13 that

$$H(\rho^k) + \varepsilon\tau\int_\Omega |w_i^k|^2 dz \le H(\rho^{k-1})$$

or, solving the recursion,

$$H(\rho^k) + \varepsilon\tau\sum_{j=1}^k \int_\Omega |w_i^j|^2 dz \le H(\rho^0). \tag{42}$$

It follows from the definition of the entropy and estimate (17) that the entropy can be bounded from below:

$$H(\rho^k) = \int_\Omega c^k \sum_{j=1}^{N+1}\big(x_i^k(\ln x_i^k - 1) + 1\big) - N\int_\Omega c^k dz \ge -C_1 := -N\text{meas}(\Omega)M_*^{-1},$$

where $c^k = \sum_{i=1}^{N+1}\rho_i^k/M_i$, $x_i^k = \rho_i^k/(c^kM_i)$, and $M_* = \min_{1\le i\le N+1} M_i$. Therefore, (42) implies that

$$\varepsilon\tau\sum_{j=1}^k\int_\Omega |w_i^j|^2 dz \le H(\rho^0) - H(\rho^k) \le H(\rho^0) + C_1.$$

The L^1 norm of w_i^k can be estimated by its L^2 norm by applying the Cauchy-Schwarz inequality:

$$\varepsilon\tau\sum_{j=1}^k\int_\Omega |w_i^j|dz \le \varepsilon\tau\sqrt{\text{meas}(\Omega)}\sum_{j=1}^k \|w_i^j\|_{L^2(\Omega)} \le \varepsilon\tau\sqrt{k\text{meas}(\Omega)}\left(\sum_{j=1}^k \|w_i^j\|_{L^2(\Omega)}^2\right)^{1/2}$$

$$= \sqrt{\varepsilon\tau k\mathrm{meas}(\Omega)}\bigg(\varepsilon\tau\sum_{j=1}^k\|w_i^j\|_{L^2(\Omega)}^2\bigg)^{1/2} \le \sqrt{\varepsilon T\mathrm{meas}(\Omega)(H(\rho^0)+C_1)},$$

where we used $\tau k \le T$. We conclude from (41) that

$$\big|\|\rho_i^k\|_{L^1(\Omega)} - \|\rho_i^0\|_{L^1(\Omega)}\big| \le \sqrt{\varepsilon T\mathrm{meas}(\Omega)(H(\rho^0)+C_1)}.$$

Given $0<\gamma<1$, let $\varepsilon>0$ satisfy

$$\sqrt{\varepsilon} \le \frac{\gamma\min_{1\le j\le N}\|\rho_j^0\|_{L^1(\Omega)}}{\sqrt{T\mathrm{meas}(\Omega)(H(\rho^0)+C_1)}}. \tag{43}$$

This proves (39).

For $i=N+1$, we estimate

$$\begin{aligned}\big|\|\rho_{N+1}^k\|_{L^1(\Omega)} - \|\rho_{N+1}^0\|_{L^1(\Omega)}\big| &= \bigg|\int_\Omega\bigg(1-\sum_{i=1}^N\rho_i^k\bigg)dz - \int_\Omega\bigg(1-\sum_{i=1}^N\rho_i^0\bigg)dz\bigg| \\ &\le \sum_{i=1}^N\big|\|\rho_i^k\|_{L^1(\Omega)} - \|\rho_i^0\|_{L^1(\Omega)}\big| \le \gamma\sum_{i=1}^N\|\rho_i^0\|_{L^1(\Omega)}\end{aligned}$$

which proves (40). From this estimate follows that

$$\|\rho_{N+1}^k\|_{L^1(\Omega)} \ge \|\rho_{N+1}^0\|_{L^1(\Omega)} - \gamma\sum_{i=1}^N\|\rho_i^0\|_{L^1(\Omega)}.$$

Hence, defining

$$\gamma_0 = \frac{\|\rho_{N+1}^0\|_{L^1(\Omega)}}{2\sum_{i=1}^N\|\rho_i^0\|_{L^1(\Omega)}} \tag{44}$$

and choosing $0<\gamma<\min\{1,\gamma_0\}$, we deduce that $\|\rho_{N+1}^k\|_{L^1(\Omega)} \ge \frac{1}{2}\|\rho_{N+1}^0\|_{L^1(\Omega)}$. □

Lemma 17 (Uniform L^1 norms for c^k)**.** *With γ as in Lemma 16, it holds that*

$$\big|\|c^k\|_{L^1(\Omega)} - \|c^0\|_{L^1(\Omega)}\big| \le M_0\gamma\|c^0\|_{L^1(\Omega)},$$

where $M_0 = \max_{1\le i\le N}|1-M_i/M_{N+1}|$.

Proof. We employ the definitions $c^k = \sum_{i=1}^{N+1}\rho_i^k/M_i$ and $\sum_{i=1}^{N+1}\rho_i^k = 1$ and the estimate (39) to obtain

$$\begin{aligned}\big|\|c^k\|_{L^1(\Omega)} - \|c^0\|_{L^1(\Omega)}\big| &= \bigg|\sum_{i=1}^{N+1}\frac{1}{M_i}\int_\Omega(\rho_i^k-\rho_i^0)dz\bigg| \\ &= \bigg|\sum_{i=1}^N\bigg(\frac{1}{M_i}-\frac{1}{M_{N+1}}\bigg)\int_\Omega(\rho_i^k-\rho_i^0)dz\bigg| \le M_0\sum_{i=1}^N\frac{1}{M_i}\big|\|\rho_i^k\|_{L^1(\Omega)} - \|\rho_i^0\|_{L^1(\Omega)}\big|\end{aligned}$$

$$\leq M_0\gamma \sum_{i=1}^N \frac{\|\rho_i^0\|_{L^1(\Omega)}}{M_i} \leq M_0\gamma\|c^0\|_{L^1(\Omega)}.$$

which finishes the proof. □

Now, we turn to the proof of Theorem 3 which is divided into several steps.

Step 1: Relative entropy dissipation inequality. Let $w^k \in \widetilde{H}^2(\Omega;\mathbb{R}^N)$ be a solution to (31) which exists according to Lemma 13. We introduce the following notation:

$$\rho^k = (\rho_1^k,\ldots,\rho_{N+1}^k) = (\rho_1(w^k),\ldots,\rho_{N+1}(w^k)), \quad w^k = (w_1^k,\ldots,w_N^k),$$
$$\bar\rho^k = (\bar\rho_1^k,\ldots,\bar\rho_{N+1}^k), \quad \bar x^k = (\bar x_1^k,\ldots,\bar x_{N+1}^k), \quad \bar w^k = (\bar w_1^k,\ldots,\bar w_N^k),$$

where $\bar\rho_i^k = \text{meas}(\Omega)^{-1}\|\rho_i^k\|_{L^1(\Omega)}$, $\bar c^k = \text{meas}(\Omega)^{-1}\|c^k\|_{L^1(\Omega)}$, $\bar x_i^k = \bar\rho_i^k/(\bar c^k M_i)$ for $i = 1,\ldots,N+1$, and $\bar w_i^k = \ln(\bar x_i^k)/M_i - \ln(\bar x_{N+1}^k)/M_{N+1}$ for $i=1,\ldots,N$. It holds that

$$\bar c^k = \sum_{i=1}^{N+1}\frac{\bar\rho_i^k}{M_i}, \quad \sum_{i=1}^{N+1}\bar\rho_i^k = \sum_{i=1}^{N+1}\bar x_i^k = 1.$$

With the test function $w^k - \bar w^k$ in (31) we obtain

$$\frac{1}{\tau}\int_\Omega(\rho'(w^k)-\rho'(w^{k-1}))\cdot(w^k-\bar w^k)dz + \int_\Omega \nabla w^k : B(w^k)\nabla w^k dz$$
$$+ \int_\Omega((u\cdot\nabla)\rho'(w^k))\cdot(w^k-\bar w^k)dz + \varepsilon\int_\Omega(|\Delta w^k|^2 + w^k\cdot(w^k-\bar w^k))dz = 0. \tag{45}$$

If $k=1$, we write $(\rho_1,\ldots,\rho_N)$ instead of $\rho'(w^{k-1})$ in the first integral. The second integral can be estimated according to Lemma 12 and the third integral vanishes in view of Lemma 11. Furthermore, using $w^k\cdot(w^k-\bar w^k) \geq \frac12(|w^k|^2-|\bar w^k|)^2$, the fourth integral can be written as

$$\int_\Omega(|\Delta w^k|^2 + w^k\cdot(w^k-\bar w^k))dz \geq \frac12\int_\Omega(|w^k|^2-|\bar w^k|^2)dz \geq -\frac12\int_\Omega|\bar w^k|^2dz.$$

It remains to treat the first integral in (45). For this, we employ the formulation (19) of w^k and $\rho_{N+1}^k = 1-\sum_{i=1}^N\rho_i^k$:

$$(\rho'(w^k)-\rho'(w^{k-1}))\cdot w^k = \sum_{i=1}^N(\rho_i^k-\rho_i^{k-1})\left(\frac{\ln x_i^k}{M_i} - \frac{\ln x_{N+1}^k}{M_{N+1}}\right) = \sum_{i=1}^{N+1}(\rho_i^k-\rho_i^{k-1})\frac{\ln x_i^k}{M_i}$$
$$= \sum_{i=1}^{N+1}(c^kx_i^k - c^{k-1}x_i^{k-1})\ln x_i^k = (c^kx^k - c^{k-1}x^{k-1})\cdot\ln x^k.$$

Similarly, $(\rho'(w^k)-\rho'(w^{k-1}))\cdot\bar w^k = (c^kx^k-c^{k-1}x^{k-1})\cdot\ln\bar x^k$. Therefore, the first integral becomes

$$\int_\Omega(\rho'(w^k)-\rho'(w^{k-1}))\cdot(w^k-\bar w^k)dz = \int_\Omega(c^kx^k-c^{k-1}x^{k-1})\cdot\ln\frac{x^k}{\bar x^k}dz$$

$$= \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot \ln\frac{x^k}{\bar{x}^0}dz + \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot \ln\frac{\bar{x}^0}{\bar{x}^k}dz = I_1 + I_2.$$

First, we estimate I_1 . To this end, we use the convexity of $h(\rho')$:

$$h(\rho'(w^k)) - h(\rho'(w^{k-1})) \le w^k\cdot(\rho'(w^k)-\rho'(w^{k-1})) = (c^k x^k - c^{k-1}x^{k-1})\cdot \ln x^k. \tag{46}$$

Then definitions (12) of the relative entropy H^* and (7) of the entropy density $h(\rho')$ give

$$\begin{aligned} H^*(\rho^k) - H^*(\rho^{k-1}) &= \sum_{i=1}^{N+1}\int_\Omega (c^k x_i^k \ln x_i^k - c^{k-1}x_i^{k-1}\ln x_i^{k-1})dz \\ &\quad - \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot\ln\bar{x}^0 dz \\ &= \int_\Omega \big(h(\rho'(w^k)) - h(\rho'(w^{k-1}))\big)dz + \sum_{i=1}^{N+1}\int_\Omega (c^k x_i^k - c^{k-1}x_i^{k-1})dz \\ &\quad - \int_\Omega (c^k - c^{k-1})dz - \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot \ln\bar{x}^0 dz. \end{aligned}$$

Since $\sum_{i=1}^{N+1}\int_\Omega c^k x_i^k dz = \int_\Omega c^k dz$, the second and third integrals on the right-hand side cancel. We employ (46) to find that

$$H^*(\rho^k) - H^*(\rho^{k-1}) \le \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot\ln x^k dz - \int_\Omega (c^k x^k - c^{k-1}x^{k-1})\cdot \ln\bar{x}_i^0 dz = I_1.$$

Next, we estimate I_2. Let $0<\gamma<\min\{\frac12,\gamma_0,(2M_0)^{-1}\}$, where M_0 is defined in Lemma 17. We infer from Lemmas 16 and 17 and from the definition (44) of γ_0 the following bounds:

$$\frac{1-M_0\gamma}{1+\gamma} \le \frac{\bar{x}_i^0}{\bar{x}_i^k} = \frac{\|\rho_i^0\|_{L^1(\Omega)}\|c^k\|_{L^1(\Omega)}}{\|\rho_i^k\|_{L^1(\Omega)}\|c^0\|_{L^1(\Omega)}} \le \frac{1+M_0\gamma}{1-\gamma}, \quad i=1,\dots,N, \tag{47}$$

$$\frac{1-M_0\gamma}{1+\gamma/(2\gamma_0)} \le \frac{\bar{x}_{N+1}^0}{\bar{x}_{N+1}^k} = \frac{\|\rho_{N+1}^0\|_{L^1(\Omega)}\|c^k\|_{L^1(\Omega)}}{\|\rho_{N+1}^k\|_{L^1(\Omega)}\|c^0\|_{L^1(\Omega)}} \le \frac{1+M_0\gamma}{1-\gamma/(2\gamma_0)}. \tag{48}$$

Thus, taking into account $\sum_{i=1}^{N+1} x_i^k = 1$, we obtain

$$\begin{aligned} I_2 &\ge \sum_{i=1}^N \int_\Omega c^k x_i^k dz \ln\frac{1-M_0\gamma}{1+\gamma} + \int_\Omega c^k x_{N+1}^k dz \ln\frac{1-M_0\gamma}{1+\gamma/(2\gamma_0)} \\ &\quad - \sum_{i=1}^N \int_\Omega c^{k-1}x_i^{k-1}dz \;\ln\frac{1+M_0\gamma}{1-\gamma} - \int_\Omega c^{k-1}x_{N+1}^{k-1}dz \ln\frac{1+M_0\gamma}{1-\gamma/(2\gamma_0)} \\ &\ge \int_\Omega c^k dz \ln\frac{1-M_0\gamma}{(1+\gamma)(1+\gamma/(2\gamma_0))} - \int_\Omega c^{k-1}dz\ln\frac{1+M_0\gamma}{(1-\gamma)(1-\gamma/(2\gamma_0))}. \end{aligned}$$

Because of $c^k \le (\min_{1\le i\le N+1} M_i)^{-1} = M_*^{-1}$ (see (17)), we conclude that

$$I_2 \ge -C_2(\gamma) := -\text{meas}(\Omega) M_*^{-1} \ln \frac{(1+M_0\gamma)(1+\gamma)(1+\gamma/(2\gamma_0))}{(1-M_0\gamma)(1-\gamma)(1-\gamma/(2\gamma_0))}. \tag{49}$$

Therefore, the first integral in (45) is bounded as follows:

$$\int_\Omega (\rho'(w^k) - \rho'(w^{k-1}))\cdot(w^k - \bar w^k)dz \ge H^*(\rho^k) - H^*(\rho^{k-1}) - C_2(\gamma).$$

Summarizing, (45) can be estimated as

$$H^*(\rho^k) - H^*(\rho^{k-1}) + C_B\tau \int_\Omega \|\nabla\sqrt{x^k}\|^2 dz \le \frac{\varepsilon\tau}{2}\int_\Omega |\bar w^k|^2 dz + C_2(\gamma). \tag{50}$$

Step 2: Estimate of the relative entropy. We split the relative entropy into two integrals:

$$H^*(\rho^k) = \sum_{i=1}^{N+1}\int_\Omega c^k x_i^k \ln\frac{x_i^k}{\bar x_i^k}dz + \sum_{i=1}^{N+1}\int_\Omega c^k x_i^k \ln\frac{\bar x_i^k}{\bar x_i^0}dz = J_1 + J_2.$$

It follows from (47) and (48) that

$$\begin{aligned} J_2 &\le \sum_{i=1}^{N}\int_\Omega c^k x_i^k dz \ln\frac{1+\gamma}{1-M_0\gamma} + \int_\Omega c^k x_{N+1}^k dz \ln\frac{1+\gamma/(2\gamma_0)}{1-M_0\gamma} \\ &\le C_3(\gamma) := \text{meas}(\Omega) M_*^{-1} \ln\frac{(1+\gamma)(1+\gamma/(2\gamma_0)}{1-M_0\gamma}. \end{aligned} \tag{51}$$

The integral J_1 is also split into two parts:

$$J_1 = \sum_{i=1}^{N+1}\int_\Omega c^k x_i^k \ln\frac{c^k x_i^k \text{meas}(\Omega)}{\|c^k x_i^k\|_{L^1(\Omega)}}dz + \sum_{i=1}^{N+1}\int_\Omega c^k x_i^k \ln\frac{\|c^k x_i^k\|_{L^1(\Omega)}}{c^k \bar x_i^k \text{meas}(\Omega)}dz = J_{11} + J_{12}.$$

Inserting the definitions $x_i^k = \rho_i^k/(c^k M_i)$ and $\bar x_i^k = \bar\rho_i^k/(\bar c^k M_i)$ and using Jensen's inequality for the convex function $s \mapsto s\ln s$ ($s > 0$), we obtain

$$J_{12} = \sum_{i=1}^{N+1}\int_\Omega c^k x_i^k \ln\frac{\bar c^k}{c^k}dz = \int_\Omega c^k \ln\frac{\bar c^k}{c^k}dz = \|c^k\|_{L^1(\Omega)}\ln \bar c^k - \|c^k \ln c^k\|_{L^1(\Omega)} \le 0.$$

The estimate of J_{11} is more involved. We employ the logarithmic Sobolev inequality

$$\int_\Omega v^2 \ln\frac{v^2}{\bar v^2}dz \le C_L \int_\Omega |\nabla v|^2 dz, \quad \bar v^2 = \frac{1}{\text{meas}(\Omega)}\int_\Omega v^2 dz,$$

where $v \in H^1(\Omega)$ and $C_L > 0$ depends only on Ω [15]. Then

$$J_{11} \le C_L \sum_{i=1}^{N+1}\int_\Omega |\nabla\sqrt{c^k x_i^k}|^2 dz.$$

Since

$$\sum_{i=1}^{N+1}|\nabla\sqrt{c^k x_i^k}|^2 \le 2\sum_{i=1}^{N+1} x_i^k|\nabla\sqrt{c^k}|^2 + 2\sum_{i=1}^{N+1} c^k|\nabla\sqrt{x_i^k}|^2 = 2|\nabla\sqrt{c^k}|^2 + 2c^k\|\nabla\sqrt{x^k}\|^2,$$

we obtain

$$J_{11} \le 2C_L\int_\Omega |\nabla\sqrt{c^k}|^2dz + 2C_LM_*^{-1}\int_\Omega \|\nabla\sqrt{x^k}\|^2dz.$$

We claim that the first integral can be estimated by a multiple of the second one. Indeed, by the Cauchy-Schwarz inequality, the definition of c^k according to Lemma 7, and the bound (17), it follows that

$$\begin{aligned}|\nabla\sqrt{c^k}|^2 &= \frac{1}{4c^k}\left|\frac{-\sum_{i=1}^{N+1}M_i\nabla x_i^k}{(\sum_{i=1}^{N+1}M_ix_i^k)^2}\right|^2 = (c^k)^3\left|\sum_{i=1}^{N+1}M_i\sqrt{x_i^k}\nabla\sqrt{x_i^k}\right|^2 \\ &\le (c^k)^3\sum_{i=1}^{N+1}M_i^2x_i^k\sum_{i=1}^{N+1}|\nabla\sqrt{x_i^k}|^2 \le M_*^{-3}M^{*2}\|\nabla\sqrt{x^k}\|^2,\end{aligned}$$

recalling that $M_* = \min_{1\le i\le N+1}M_i$ and setting $M^* = \max_{1\le i\le N+1}M_i$. Thus, we can estimate J_{11} as follows:

$$J_{11} \le 2C_LM_*^{-1}(M_*^{-2}M^{*2}+1)\int_\Omega \|\nabla\sqrt{x^k}\|^2dz.$$

Combining the above estimates, we conclude that

$$H^*(\rho^k) \le C_3(\gamma) + 2C_LM_*^{-1}(M_*^{-2}M^{*2}+1)\int_\Omega \|\nabla\sqrt{x^k}\|^2dz.$$

Step 3: End of the proof. Replacing the entropy dissipation term involving $\sqrt{x^k}$ in (50) by the above estimate for $H^*(\rho^k)$, we find that

$$(1+C_4\tau)H^*(\rho^k) \le H^*(\rho^{k-1}) + \frac{\varepsilon\tau}{2}\int_\Omega |\bar w^k|^2dz + C_\gamma, \tag{52}$$

where $C_4 = \frac12 C_BC_L^{-1}M_*(M_*^{-2}M^{*2}+1)^{-1}$ and $C_\gamma = C_2(\gamma) + \frac12 C_3(\gamma)C_L^{-1}M_*(M_*^{-2}M^{*2}+1)^{-1}$. Note that according to definitions (49) and (51), we have $C_\gamma \to 0$ as $\gamma\to 0$.

We need to estimate the integral involving w^k. For this, we observe that (47)-(48) and the upper bound for γ imply that $\frac13 \le \bar x_i^0/\bar x_i^k \le 3$ for $i = 1,\ldots,N+1$. This provides some uniform bounds for $\bar x_i^k$,

$$0 < \frac{M_*\min_{1\le i\le N+1}\|\rho_i^0\|_{L^1(\Omega)}}{3M^*\sum_{i=1}^{N+1}\|\rho_i^0\|_{L^1(\Omega)}} \le \frac{\bar x_i^0}{3} \le \bar x_i^k \le 3\bar x_i^0 \le 3, \quad i = 1,\ldots,N+1,$$

which allow us to estimate w^k:

$$\int_\Omega |\bar w^k|^2dz \le \sum_{i=1}^{N}\int_\Omega\left(\left|\frac{\ln\bar x_i^k}{M_i}\right| + \left|\frac{\ln\bar x_{N+1}^k}{M_{N+1}}\right|\right)^2dz \le C_5,$$

where $C_5 > 0$ depends on Ω, ρ^0, M_*, and M^*. Hence, (52) becomes

$$H^*(\rho^k) \le (1 + C_4\tau)^{-1} H^*(\rho^{k-1}) + \Big(\frac{\varepsilon\tau}{2} C_5 + C_\gamma\Big)(1 + C_4\tau)^{-1}.$$

Solving this recursion, we infer that

$$H^*(\rho^k) \le (1 + C_4\tau)^{-1} H^*(\rho^0) + \Big(\frac{\varepsilon\tau}{2} C_5 + C_\gamma\Big)\sum_{i=1}^{k}(1 + C_4\tau)^{-i}.$$

Using $\sum_{i=1}^k (1 + C_4\tau)^{-i} \le 1/(C_4\tau)$, it follows that

$$H^*(\rho^{(\tau)}(\cdot, t)) \le (1 + C_4\tau)^{-t/\tau} H^*(\rho^0) + \frac{\varepsilon C_5}{2C_4} + \frac{C_\gamma}{C_4\tau}, \quad 0 < t < T.$$

Now, we take $\tau = \tau(\gamma) = \sqrt{C_\gamma}$ and $\varepsilon = \varepsilon(\gamma)$ according to (43). In the limit $\gamma \to 0$, it follows that $C_\gamma/\tau(\gamma) \to 0$, $\varepsilon(\gamma) \to 0$, and $\tau(\gamma) \to 0$ so that $\rho_i^{(\tau)} \to \rho_i$ strongly in $L^2(0, T; L^2(\Omega))$ for $i = 1, \ldots, N + 1$. This gives in the limit $\gamma \to 0$

$$H^*(\rho(\cdot, t)) \le e^{-C_4 t} H^*(\rho^0), \quad t \ge 0, \tag{53}$$

and, taking into account Lemmas 16 and 17, we conclude the L^1 conservation for ρ_i and c:

$$\int_\Omega \rho_i dz = \int_\Omega \rho_i^0 dz, \quad \int_\Omega c dz = \int_\Omega c^0 dz,$$

where $c^0 = \sum_{j=1}^{N+1} \rho_j^0/M_j$ and $i = 1, \ldots, N + 1$.

It remains to estimate $x_i - \bar{x}_i^0$ in the L^1 norm. Defining

$$f_i = \frac{cx_i}{\int_\Omega c^0 x_i^0 dz}, \quad g_i = \frac{c}{\int_\Omega c^0 dz},$$

the entropy $H^*(\rho) = \sum_{i=1}^{N+1} \int_\Omega cx_i \ln(x_i/\bar{x}_i^0) dz$ can be written as

$$H^*(\rho) = \sum_{i=1}^{N+1} \int_\Omega c^0 x_i^0 dz \int_\Omega f_i \ln\frac{f_i}{g_i} dz,$$

where we employed the identity

$$\frac{f_i}{g_i} = \frac{x_i \int_\Omega c^0 dz}{\int_\Omega c^0 x_i^0 dz} = \frac{M_i x_i \int_\Omega c^0 dz}{\int_\Omega \rho_i^0 dz} = \frac{M_i x_i \bar{c}^0}{\bar{\rho}_i^0} = \frac{x_i}{\bar{x}_i^0}.$$

Finally, using

$$c\bar{x}_i^0 = \frac{c\bar{\rho}_i^0}{\bar{c}^0 M_i} = \frac{c\int_\Omega \rho_i^0 dz}{\int_\Omega c^0 dz M_i} = \frac{c\int_\Omega c^0 x_i^0 dz}{\int_\Omega c^0 dz} = \int_\Omega c^0 x_i^0 dz\, g_i$$

and the Csiszár-Kullback inequality with constant $C_K > 0$ (see, e.g., [15, 22]), we find that

$$\begin{aligned}\|cx_i - c\bar{x}_i^0\|_{L^1(\Omega)}^2 &= \Big(\int_\Omega c^0 x_i^0 dz\Big)^2 \|f_i - g_i\|_{L^1(\Omega)}^2 \le \int_\Omega c^0 x_i^0 dz \Big(\int_\Omega \frac{\rho_i^0}{M_i} dz\Big) C_K \int_\Omega f_i \ln\frac{f_i}{g_i} dz \\ &\le M_i^{-1} C_K \|\rho_i^0\|_{L^1(\Omega)} H^*(\rho).\end{aligned}$$

Together with (53), the conclusion of the theorem follows.

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