LYAPUNOV FUNCTIONALS, WEAK SEQUENTIAL STABILITY, AND UNIQUENESS ANALYSIS FOR ENERGY-TRANSPORT SYSTEMS

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Abstract. A class of strongly coupled parabolic systems, modeling the energy transport of electrons in semiconductors, is analyzed. The variables are the electron density and the thermal energy. First, some Lyapunov functionals are derived, which yields the weak sequential stability for smooth solutions in the sense of Feireisl, using weak compactness results. Second, by the H^{-1} method, the uniqueness of bounded weak solutions is proved.

1. Introduction

The semiconductor Boltzmann equation describes the evolution of the electron distribution function in a semiconductor crystal subject to elastic electron-phonon, inelastic electron-phonon, and electron-electron scattering. Ben Abdallah and Degond [3] have shown that a Chapman-Enskog expansion of the distribution function around the thermal equilibrium leads to a system of diffusive equations for the electron density n and the thermal energy $\frac{3}{2}n\theta$ (with the electron temperature θ). The so-called *energy-transport model* consists of strongly coupled parabolic equations which, without relaxation, read as

$$n_t = \operatorname{div} J_n, \quad \frac{3}{2}(n\theta)_t = \operatorname{div} J_e - J_n \cdot \nabla V, \quad t > 0, \tag{1}$$

$$J_n = \nabla(n\theta^{1/2-\beta}) - n\theta^{-1/2-\beta}\nabla V, \tag{2}$$

$$J_e = (2-\beta)\big(\nabla(n\theta^{3/2-\beta}) - n\theta^{1/2-\beta}\nabla V\big) \quad \text{in } \Omega,\ t > 0, \tag{3}$$

$$n(\cdot,0) = n_0, \quad (n\theta)(\cdot,0) = n_0\theta_0 \quad \text{in } \Omega, \tag{4}$$

where J_n and J_e are the particle and energy current densities, respectively, V is the electric potential, $\Omega \subset \mathbb{R}^d$ $(d \le 3)$ is the bounded semiconductor domain, and $\beta < 2$ is a parameter which appears in the elastic scattering rate. In fact, the scattering rate is assumed to be proportional to E^β, with E being the semiconductor band energy.

The main assumptions in the derivation are that the elastic collisions are dominant, that the semiconductor energy band is of parabolic type, and that the semiconductor material is non-degenerate such that Maxwell-Boltzmann statistics can be employed. We refer to

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[3, 11, 17] for details of the derivation. For a rigorous derivation of energy-transport models in a linear framework, we refer to [4].

Certain values of β have been used in the physics literature. For instance, $\beta = \frac{1}{2}$ gives the so-called *Chen model* [5], $\beta = 0$ leads to the *Lyumkis model* [19], and $\beta = -\frac{1}{2}$ appears in the diffusion approximation of the hydrodynamic semiconductor model [17].

System (1)-(3) possesses some interesting mathematical features. First, it can be written in a "symmetric" form by introducing the so-called dual entropy variables $w_1 = (\mu - V)/\theta$ and $w_2 = -1/\theta$, where the chemical potential μ is defined by $n = \theta^{3/2}\exp(\mu/\theta)$ [9, 17]. This formulation eliminates the terms involving the electric field $-\nabla V$. Second, when appropriate boundary conditions are imposed, the above system possesses a Lyapunov functional, namely the entropy

$$H = \int_\Omega n \log \frac{n}{\theta^{3/2}} dx, \tag{5}$$

i.e., it holds that $dH/dt \leq 0$ along positive smooth solutions, for all $t > 0$. We notice that these two properties – the existence of an entropy functional and the symmetrization property – are strongly related, see [17] for details.

There are only a few analytical results for (1)-(4) with appropriate boundary conditions, mainly due to the strong nonlinear coupling in the equations and the lack of a maximum principle which might provide the positivity or boundedness of the physical variables. In earlier works, drift-diffusion equations with temperature-dependent mobilities but without temperature gradients [22] (also see [15]) or with nonisothermal systems containing simplified thermodynamic forces [2] have been studied. Later, existence results for the complete energy-transport equations have been achieved, see [12, 14] for stationary solutions near thermal equilibrium, [6, 7, 8] for transient solutions close to equilibrium, and [9, 10] for systems with uniformly positive definite diffusion matrices. Furthermore, the nonlinear stability of classical bounded solutions to the one-dimensional equations was investigated [1]. All these results give only partial answers to the well-posedness problem, and a complete global existence result for any data and with physical transport coefficients is still missing.

In view of these difficulties, we analyze in this paper only a simplification of (1)-(4), namely without electric field and subject to periodic boundary conditions:

$$n_t = \Delta\big(n\theta^{1/2-\beta}\big), \quad t > 0, \quad n(\cdot, 0) = n_0 \quad \text{in } \mathbb{T}^d, \tag{6}$$

$$\frac{3}{2}(n\theta)_t = (2-\beta)\Delta\big(n\theta^{3/2-\beta}\big), \quad t > 0, \quad (n\theta)(\cdot, 0) = n_0\theta_0 \quad \text{in } \mathbb{T}^d, \tag{7}$$

where $\mathbb{T}^d$ is the d-dimensional torus. We will prove three results.

The first result is the derivation of certain Lyapunov functionals which are of the form (5) or $H = \int_{\mathbb{T}^d} n^a\theta^b dx$ for certain (a, b); see Section 2. The corresponding dissipation terms $-dH/dt$ yield gradient estimates for the solutions and allow us, as a second result, to prove in Section 3 the weak sequential stability in the sense of Feireisl [13]. Weak sequential stability means that, given a sequence of (smooth) solutions to a system of equations, there exists a subsequence which converges to a (weak) solution to this problem.

Typically, the sequence of solutions solves an approximate system of equations, obtained from the original one by a Galerkin scheme or a semidiscretization in time, for instance, and the index of the sequence is related to the approximation parameter. Then weak stability implies that the limiting solution is a solution to the original system. Although the construction of the sequence of approximating solutions is an open problem, the weak sequential stability constitutes an important step in the global existence analysis of the energy-transport equations. The weak sequential stability for the case $\beta = -\frac{1}{2}$ has been treated in [18]. The case $\beta = \frac{1}{2}$ (Chen model) is trivial here since this choice decouples the system (6)-(7). In this paper, we will consider the range $-\frac{1}{2} < \beta < \frac{1}{2}$.

The third result (Section 4) is the proof of the uniqueness of bounded weak solutions, which is based on the observation that (6)-(7) can be written as

$$\rho(u)_t = \Delta a(u) \quad \text{in } \mathbb{T}^d,\ t > 0,$$

where the primal entropy variables are given by $u = (u_1, u_2) = (\log(n\theta^{-3/2}), -\theta^{-1})$ and

$$\rho(u) = (-u_2)^{-3/2} e^{u_1} \begin{pmatrix} 1 \\ -\frac{3}{2}u_2^{-1} \end{pmatrix}, \quad a(u) = (-u_2)^{-(2-\beta)} e^{u_1} \begin{pmatrix} 1 \\ -(2-\beta)u_2^{-1} \end{pmatrix}.$$

In fact, the above formulation allows us to apply the H^{-1} method to prove the uniqueness result. We remark that the H^{-1} method has been also employed to show the uniqueness of solutions to (1)-(4) subject to truncated diffusion coefficients in [16]. The above formulation, however, is new.

2. Lyapunov functionals

We begin by proving that the entropy (5) is a Lyapunov functional, and we estimate its entropy dissipation.

Lemma 1. *Let $(n_\varepsilon, \theta_\varepsilon)$ be a positive smooth solution to (6)-(7) and let $\beta < 2$. Then there exists a constant $\kappa > 0$ depending on β but not on $\varepsilon > 0$ such that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} n_\varepsilon \log(n_\varepsilon\theta_\varepsilon^{-3/2})dx + \kappa\int_{\mathbb{T}^d}\big(n_\varepsilon^{-1}\theta_\varepsilon^{1/2-\beta}|\nabla n_\varepsilon|^2 + n_\varepsilon\theta_\varepsilon^{-3/2-\beta}|\nabla\theta_\varepsilon|^2\big)dx \le 0.$$

Proof. We differentiate as follows:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d} n_\varepsilon \log(n_\varepsilon\theta_\varepsilon^{-3/2})dx &= \frac{d}{dt}\int_{\mathbb{T}^d}\Big(\frac{5}{2}n_\varepsilon\log n_\varepsilon - \frac{3}{2}n_\varepsilon\log(n_\varepsilon\theta_\varepsilon)\Big)dx \\
&= \int_{\mathbb{T}^d}\Big(\frac{5}{2}\log\Big(\frac{n_\varepsilon}{\theta_\varepsilon^{3/2}}\Big)\partial_t n_\varepsilon - \frac{3}{2}\frac{1}{\theta_\varepsilon}\partial_t(n_\varepsilon\theta_\varepsilon)\Big)dx \\
&= -\int_{\mathbb{T}^d} n_\varepsilon\theta_\varepsilon^{1/2-\beta}\left(A\Big|\frac{\nabla n_\varepsilon}{n_\varepsilon}\Big|^2 + 2B\frac{\nabla n_\varepsilon}{n_\varepsilon}\cdot\frac{\nabla\theta_\varepsilon}{\theta_\varepsilon} + C\Big|\frac{\nabla\theta_\varepsilon}{\theta_\varepsilon}\Big|^2\right)dx,
\end{aligned}$$

where

$$A = 1, \quad B = \frac{1}{2}(1-2\beta), \quad C = \beta^2 - 2\beta + \frac{9}{4}.$$

Now, we use the following result: If $A > 0$ and $AC - B^2 > 0$, then there exists $\kappa > 0$ such that for all x, $y \in \mathbb{R}^d$, $A|x|^2 + 2Bx\cdot y + C|y|^2 \ge \kappa(|x|^2 + |y|^2)$. For the above choice of A,

B, and C, the assumptions $A > 0$ and $AC - B^2 = 2 - \beta > 0$ are satisfied. Thus, choosing $x = n_\varepsilon^{-1/2}\theta_\varepsilon^{1/4-\beta/2}\nabla n_\varepsilon$ and $y = n_\varepsilon^{1/2}\theta_\varepsilon^{-3/4-\beta/2}\nabla\theta_\varepsilon$, we infer that

$$\frac{d}{dt}\int_{\mathbb{T}^d} n_\varepsilon \log(n_\varepsilon\theta_\varepsilon^{-3/2})dx \le -\kappa\int_{\mathbb{T}^d}\big(n_\varepsilon^{-1}\theta_\varepsilon^{1/2-\beta}|\nabla n_\varepsilon|^2 + n_\varepsilon\theta_\varepsilon^{-3/2-\beta}|\nabla\theta_\varepsilon|^2\big)dx,$$

which shows the lemma. □

The following lemma shows that there exist Lyapunov functionals of the form $\int_{\mathbb{T}^d} n^a\theta^b dx$ for certain choices of (a,b). To this end, define the set M_β of all $(a,b)\in\mathbb{R}^2$ such that

$$\begin{aligned} &C_1a^4 + C_2a^3b + C_3a^2b^2 + C_4ab^3 + C_5b^4 + C_6a^3 \\ &\qquad + C_7a^2b + C_8ab^2 + C_9b^3 + C_{10}a^2 + C_{11}ab + C_{12}b^2 > 0 \quad \text{and} \end{aligned} \tag{8}$$

$$a^2 - a + \frac{1}{3}(1-2\beta)ab + \frac{1}{3}(-1+2\beta)b > 0, \tag{9}$$

where

$$\begin{aligned} C_1 &= -\frac{1}{16}(2\beta-1)^2, & C_2 &= \frac{1}{24}(2\beta-3)(2\beta-1)^2, \\ C_3 &= -\frac{1}{144}(4\beta^2-12\beta-3)(2\beta-1)^2, & C_4 &= -\frac{1}{36}(2\beta-3)(2\beta-1)^2, \\ C_5 &= -\frac{1}{36}(2\beta-1)^2, & C_6 &= \frac{1}{8}(2\beta-1)^2, \\ C_7 &= -\frac{2}{3}\beta^3 + \frac{4}{3}\beta^2 - \frac{1}{6}\beta - \frac{7}{6}, & C_8 &= \frac{1}{72}(2\beta-3)(8\beta^3-12\beta^2-6\beta-31), \\ C_9 &= \frac{1}{36}(2\beta-1)^3, & C_{10} &= -\frac{1}{16}(2\beta-1)^2, \\ C_{11} &= \frac{1}{3}\beta^3 - \frac{1}{2}\beta^2 - \frac{5}{12}\beta + \frac{31}{24}, & C_{12} &= -\frac{1}{9}\beta^4 + \frac{2}{9}\beta^3 - \frac{1}{6}\beta^2 + \frac{13}{18}\beta - \frac{193}{144}. \end{aligned}$$

The set M_β is illustrated in Figure 1 for various values of β.

Lemma 2. *Let $(n_\varepsilon,\theta_\varepsilon)$ be a positive smooth solution to (6)-(7) and let $-1/2 < \beta < 1/2$, $(a,b)\in M_\beta$. Then there exists a constant $\kappa > 0$ depending on β but not on $\varepsilon > 0$ such that*

$$\frac{d}{dt}\int_{\mathbb{T}^d} n_\varepsilon^a\theta_\varepsilon^b dx + \kappa\int_{\mathbb{T}^d}\big(n_\varepsilon^{a-2}\theta_\varepsilon^{1/2+b-\beta}|\nabla n_\varepsilon|^2 + n_\varepsilon^a\theta_\varepsilon^{-3/2+b-\beta}|\nabla\theta_\varepsilon|^2\big)dx \le 0.$$

It holds that

$$(2,\beta-\tfrac{1}{2}),\ (2,3-2\beta),\ (2,5),\ (\tfrac{6}{5},\beta-\tfrac{1}{2}) \in M_\beta \quad \textit{for all } -\frac{1}{2} < \beta < \frac{1}{2}.$$

We notice that $(a,0)$, $(1,1)\notin M_\beta$ for all $a > 0$ and $\beta < 2$. Indeed, by mass and energy conservation, the integrals $\int_{\mathbb{T}^d} n_\varepsilon dx$ and $\int_{\mathbb{T}^d} n_\varepsilon\theta_\varepsilon dx$ are constant in time, and the dissipation vanishes. The choice $(a,b) = (2,\beta-1/2)$ yields an estimate for ∇n_ε in L^2. Unfortunately, the choice $(a,b) = (2, 3/2+\beta)$ for $-1/2 < \beta < 1/2$, which would give a bound for $\nabla(n_\varepsilon\theta_\varepsilon)$ in L^2, is not admissible.

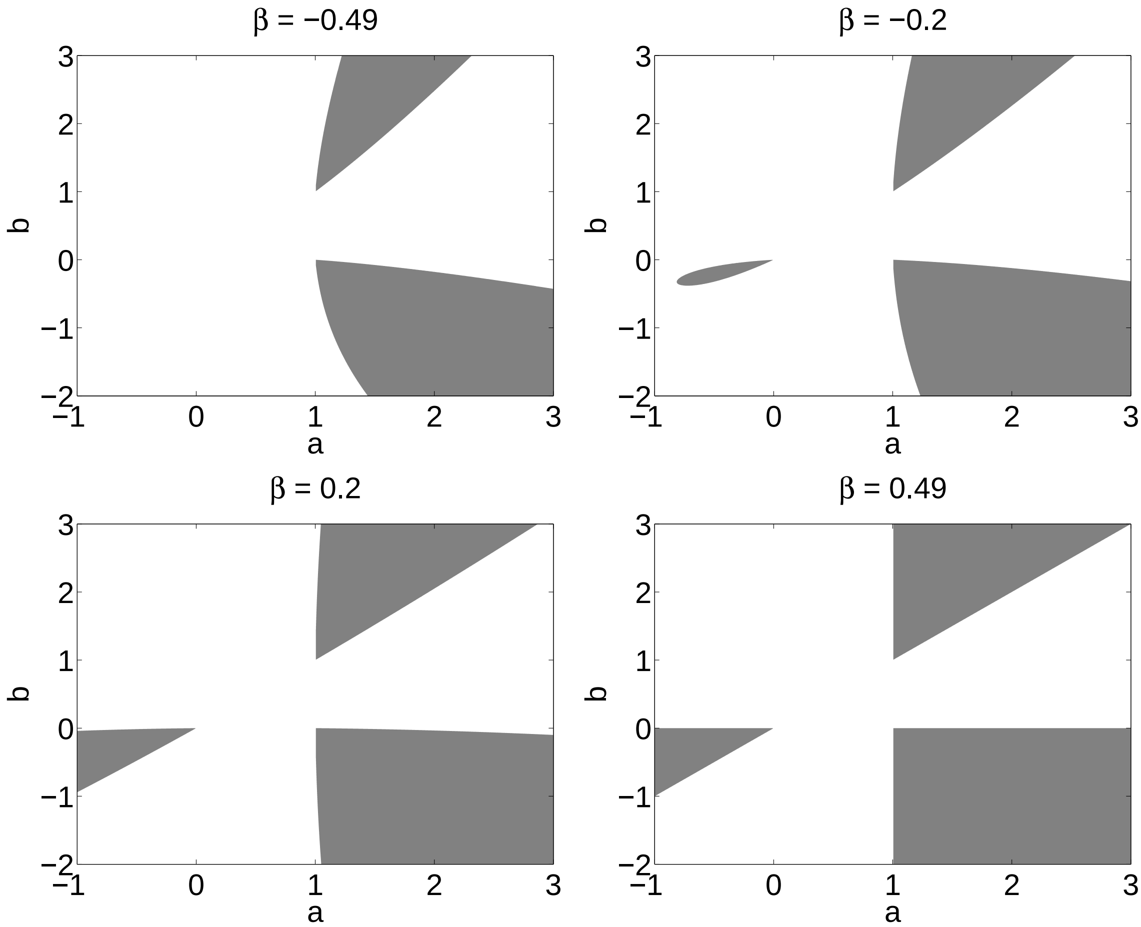

FIGURE 1. The gray regions represent all $(a, b) \in M_\beta$ for different choices of β.

Proof. We differentiate:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d} n_\varepsilon^a \theta_\varepsilon^b dx &= \int_{\mathbb{T}^d} \big((a-b)n_\varepsilon^{a-1}\theta_\varepsilon^b \partial_t n_\varepsilon + b n_\varepsilon^{a-1}\theta_\varepsilon^{b-1}\partial_t(n_\varepsilon\theta_\varepsilon)\big)dx \\ &= -\int_{\mathbb{T}^d}\left(A\Big|\frac{\nabla n_\varepsilon}{n_\varepsilon}\Big|^2 + 2B\frac{\nabla n_\varepsilon}{n_\varepsilon}\cdot\frac{\nabla\theta_\varepsilon}{\theta_\varepsilon} + C\Big|\frac{\nabla\theta_\varepsilon}{\theta_\varepsilon}\Big|^2\right)dx,\end{aligned}$$

where

$$\begin{aligned}A &= (a-1)\Big(a + \frac{b}{3}(1-2\beta)\Big), \\ B &= \frac{1}{2}\Big(\frac{1}{2}-\beta\Big)a^2 - \frac{1}{2}\Big(\frac{1}{2}-\beta\Big)a + \left(\frac{1}{2}(2-\beta)\Big(1-\frac{2}{3}\beta\Big) + \frac{1}{4} + \frac{1}{2}\beta\right)ab \\ &\quad - \left(\frac{5}{12} + \frac{1}{6}\beta + \frac{1}{2}(2-\beta)\Big(1-\frac{2}{3}\beta\Big)\right)b + \frac{1}{3}\Big(\frac{1}{2}-\beta\Big)b^2,\end{aligned}$$

$$C = b\left(\left(\frac{1}{2}-\beta\right)(a-b) + (2-\beta)\left(1-\frac{2}{3}\beta\right)(b-1)\right).$$

The result follows for all choices of (a,b) and β for which $AC-B^2>0$ and $A>0$ (see the proof of Lemma 1). A tedious computation shows that $AC-B^2>0$ is equivalent to (8) and $A>0$ is equivalent to (9), which shows the first statement.

It remains to consider the special cases for (a,b):

- $a=2$ and $b=\beta-1/2$:
$$\begin{aligned} AC-B^2 &= \frac{77}{48} - \frac{59}{12}\beta + \frac{149}{36}\beta^2 - \frac{4}{9}\beta^3 - 3\beta^4 + \frac{20}{9}\beta^5 - \frac{4}{9}\beta^6 > 0 \\ &\quad \text{for } \frac{1}{2}-\sqrt{3} < \beta < \frac{1}{2}, \\ A &= \frac{2}{3}\left(3 - \left(\beta-\frac{1}{2}\right)^2\right) > 0 \quad \text{for } \frac{1}{2}-\sqrt{3} < \beta < \frac{1}{2}+\sqrt{3}. \end{aligned}$$
- $a=2$ and $b=3-2\beta$:
$$\begin{aligned} AC-B^2 &= \frac{63}{16} - \frac{51}{4}\beta + \frac{413}{36}\beta^2 - \frac{22}{9}\beta^3 - 3\beta^4 + \frac{20}{9}\beta^5 - \frac{4}{9}\beta^6 > 0 \\ &\quad \text{for } -1.728 < \beta < \frac{1}{2}, \\ A &= \frac{4}{3}\left((\beta-1)^2 + \frac{5}{4}\right) > 0 \quad \text{for all } \beta\in\mathbb{R}. \end{aligned}$$
- $a=2$ and $b=5$:
$$\begin{aligned} AC-B^2 &= \frac{2519}{144} - \frac{47}{18}\beta - \frac{1}{6}\beta^2 - \frac{70}{9}\beta^3 - \frac{25}{9}\beta^4 > 0 \quad \text{for } -3.092 < \beta < \frac{11}{10}, \\ A &= \frac{10}{3}\left(\frac{11}{10} - \beta\right) > 0 \quad \text{for } \beta < \frac{11}{10}. \end{aligned}$$
- $a=6/5$ and $b=\beta-1/2$:
$$\begin{aligned} AC-B^2 &= \frac{6557}{30000} - \frac{1489}{2500}\beta + \frac{109}{2500}\beta^2 + \frac{412}{375}\beta^3 - \frac{183}{125}\beta^4 + \frac{4}{5}\beta^5 - \frac{4}{25}\beta^6 > 0 \\ &\quad \text{for } -0.635 < \beta < 1/2, \\ A &= \frac{2}{15}\left(\frac{9}{5} - \left(\beta-\frac{1}{2}\right)^2\right) > 0 \quad \text{for } \frac{1}{2} - \frac{3}{5}\sqrt{5} < \beta < \frac{1}{2} + \frac{3}{5}\sqrt{5}. \end{aligned}$$

Under these conditions, we have $A|x|^2 + 2Bx\cdot y + C|y|^2 \geq \kappa(|x|^2+|y|^2)$ for all x, $y\in\mathbb{R}^d$ and for some $\kappa>0$, and the proof continues analogously to the proof of Lemma 1. □

3. Weak sequential stability

We assume that there exists a sequence $(n_\varepsilon,\theta_\varepsilon)$, $\varepsilon>0$, of positive smooth solutions to (6)-(7). The lower bound for n_ε and θ_ε may depend on the approximation parameter, $n_\varepsilon \geq c(\varepsilon) > 0$ and $\theta_\varepsilon \geq c(\varepsilon) > 0$ in $\mathbb{T}^d$. Our aim is to show that $(n_\varepsilon,\theta_\varepsilon)$ converges to a weak solution to (6)-(7). The main result is as follows.

Theorem 3. *Let $d \le 3$, $T > 0$, and let $(n_\varepsilon, \theta_\varepsilon)$ be a sequence of positive smooth solutions to* (6)-(7)*. Then there exists a subsequence (which is not relabeled) such that*

$$\begin{aligned} n_\varepsilon \to n,\ n_\varepsilon\theta_\varepsilon \to n\theta \quad & \text{strongly in } L^2(0,T;L^6(\mathbb{T}^d)), \\ n_\varepsilon\theta_\varepsilon \to n\theta \quad & \text{strongly in } L^4(0,T;L^p(\mathbb{T}^d)), \\ n_\varepsilon\theta^{1/2-\beta} \rightharpoonup n\theta^{1/2-\beta} \quad & \text{weakly in } L^{4/(1+2\beta)}(0,T;L^{4/(3-2\beta)}(\mathbb{T}^d)), \\ n_\varepsilon\theta^{3/2-\beta} \rightharpoonup^* z \quad & \text{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^d)), \end{aligned}$$

and $z = n\theta^{3/2-\beta}$ in $\{n > 0\}$.

More precisely, we will prove that $n^{1/2-\beta}(z - n\theta^{3/2-\beta}) = 0$ in $\mathbb{T}^d$, $t > 0$. It is an open problem if $z = n\theta^{3/2-\beta} = 0$ in $\{n = 0\}$. We know (see the proof below) that $n\theta = 0$ in $\{n = 0\}$ but the difficulty is that we do not have any control on higher exponents of θ. For the proof of this theorem, we show first some a priori estimates, based on the results of the previous section.

Lemma 4. *The following uniform estimates hold:*

$$\|n_\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^d))} + \|\partial_t n_\varepsilon\|_{L^2(0,T;H^{-s}(\mathbb{T}^d))} \le C,$$

where $C > 0$ is here and in the following a generic constant independent of ε and $s \ge 2 + d/2$.

Proof. Lemma 2 with $a = 2$ and $b = \beta - 1/2$ gives that ∇n_ε is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^d))$. Moreover, by mass conservation, $\int_{\mathbb{T}^d} n_\varepsilon dx$ is constant for all $t \ge 0$, thus n_ε is uniformly bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$. Therefore, by the Poincaré inequality, n_ε is uniformly bounded in $L^2(0,T;H^1(\mathbb{T}^d))$.

By conservation of the thermal energy, $(n_\varepsilon\theta_\varepsilon)$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$. In particular, $(n_\varepsilon\theta_\varepsilon)^{1/2-\beta}$ is uniformly bounded in $L^\infty(0,T;L^{2/(1-2\beta)}(\mathbb{T}^d))$ for $\beta < 1/2$. Since (n_ε) is bounded in $L^2(0,T;L^2(\mathbb{R}^d))$ and, as a consequence, $n_\varepsilon^{1/2+\beta}$ is uniformly bounded $L^{4/(1+2\beta)}(0,T;\ L^{4/(1+2\beta)}(\mathbb{T}^d))$, we infer that

$$n_\varepsilon\theta_\varepsilon^{1/2-\beta} = n_\varepsilon^{1/2+\beta}(n_\varepsilon\theta_\varepsilon)^{1/2-\beta}$$

is uniformly bounded in $L^{4/(1+2\beta)}(0,T;L^{4/(3-2\beta)}(\mathbb{T}^d))$ and, in particular, in $L^2(0,T;L^1(\mathbb{T}^d))$ when $-1/2 < \beta < 1/2$. Hence,

$$\|\partial_t n_\varepsilon\|_{L^2(0,T;W^{-2,1}(\mathbb{T}^d))} = \|\Delta(n_\varepsilon\theta_\varepsilon^{1/2-\beta})\|_{L^2(0,T;W^{-2,1}(\mathbb{T}^d))} \le \|n_\varepsilon\theta_\varepsilon^{1/2-\beta}\|_{L^2(0,T;L^1(\mathbb{T}^d))} \le C.$$

The continuous embedding $W^{2,-1}(\mathbb{T}^d) \hookrightarrow H^{-s}(\mathbb{T}^d)$ for $s \ge 2 + d/2$ finishes the proof. □

Lemma 5. *The following uniform estimates hold:*

$$\|n_\varepsilon\theta_\varepsilon\|_{L^2(0,T;W^{1,10/7}(\mathbb{T}^d))} + \|\partial_t(n_\varepsilon\theta)\|_{L^\infty(0,T;H^{-2}(\mathbb{T}^d))} \le C.$$

Proof. We apply Lemma 2 with $a = 2$ and $b = 3 - 2\beta$: $(n_\varepsilon^2\theta_\varepsilon^{3-2\beta})$ is bounded in $L^\infty(0,T;$ $L^1(\mathbb{T}^d))$ and, in particular, $n_\varepsilon\theta_\varepsilon^{3/2-\beta}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$. Hence,

$$\|\partial_t(n_\varepsilon\theta_\varepsilon)\|_{L^\infty(0,T;H^{-2}(\mathbb{T}^d))} \le (2-\beta)\|n_\varepsilon\theta_\varepsilon^{3/2-\beta}\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \le C.$$

In order to estimate $\nabla(n_\varepsilon\theta_\varepsilon)$, we first observe that, by Lemma 2, with $a = 6/5$ and $b = \beta - 1/2$, $n_\varepsilon^{-4/5}|\nabla n_\varepsilon|^2$ and $n_\varepsilon^{6/5}\theta_\varepsilon^{-2}|\nabla\theta_\varepsilon|^2$ are uniformly bounded in $L^1(0,T;L^1(\mathbb{T}^d))$. Furthermore, by Lemma 2, with $a = 2$ and $b = 5$, $n_\varepsilon^2\theta_\varepsilon^5$ is uniformly bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$ and $n_\varepsilon^{2/5}\theta_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^5(\mathbb{T}^d))$. Hence,

$$\nabla(n_\varepsilon\theta_\varepsilon) = n_\varepsilon^{2/5}\theta_\varepsilon\big(n_\varepsilon^{-2/5}\nabla n_\varepsilon + n_\varepsilon^{3/5}\theta_\varepsilon^{-1}\nabla\theta_\varepsilon\big)$$

is the product of an $L^\infty(0,T;L^5(\mathbb{T}^d))$ function with an $L^2(0,T;L^2(\mathbb{T}^d))$ function, which implies that $\nabla(n_\varepsilon\theta_\varepsilon) \in L^2(0,T;L^{10/7}(\mathbb{T}^d))$. Since $(n_\varepsilon\theta_\varepsilon)$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}^d))$, by the Poincaré inequality, we conclude that $(n_\varepsilon\theta_\varepsilon)$ is bounded in $L^2(0,T;W^{1,10/7}(\mathbb{T}^d))$. $\square$

The uniform estimates in Lemmas 4 and 5 allow us to apply the Aubin lemma [21, Corollary 4], yielding for $\varepsilon \to 0$, up to subsequences, the strong convergences

$$\begin{aligned} n_\varepsilon &\to n \quad \text{strongly in } L^2(0,T;L^p(\mathbb{T}^d)), \\ n_\varepsilon\theta_\varepsilon &\to w \quad \text{strongly in } L^2(0,T;L^q(\mathbb{T}^d)) \end{aligned}$$

for all $1 \le p < 6$, $1 \le q < 33/10$ (and $d \le 3$). Moreover, up to subsequences,

$$\begin{aligned} \partial_t n_\varepsilon &\rightharpoonup n_t \quad \text{weakly in } L^2(0,T;H^{-s}(\mathbb{T}^d)), \\ \partial_t(n_\varepsilon\theta_\varepsilon) &\rightharpoonup^* w_t \quad \text{weakly* in } L^\infty(0,T;H^{-2}(\mathbb{T}^d)). \end{aligned}$$

In the proof of Lemma 4 we have shown that $n_\varepsilon\theta_\varepsilon^{1/2-\beta}$ is uniformly bounded in $L^{4/(1+2\beta)}(0,T;L^{4/(3-2\beta)}(\mathbb{T}^d))$. Then, up to a subsequence,

$$n_\varepsilon\theta_\varepsilon^{1/2-\beta} \rightharpoonup y \quad \text{weakly in } L^{4/(1+2\beta)}(0,T;L^{4/(3-2\beta)}(\mathbb{T}^d)).$$

In the proof of Lemma 5 we have shown that $n_\varepsilon\theta^{3/2-\beta}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$. Thus, up to a subsequence,

$$n_\varepsilon\theta_\varepsilon^{3/2-\beta} \rightharpoonup^* z \quad \text{weakly* in } L^\infty(0,T;L^2(\mathbb{T}^d)).$$

In the following, we wish to identify w with $n\theta$, y with $n\theta^{1/2-\beta}$, and z with $n\theta^{3/2-\beta}$. We employ ideas from [20]. By the Fatou lemma and the a.e. convergence of (subsequences of) (n_ε) and $(n_\varepsilon\theta_\varepsilon)$, we infer that (for $-1/2 < \beta < 1/2$)

$$\int_{\mathbb{T}^d} \liminf_{\varepsilon\to 0} \frac{(n_\varepsilon\theta_\varepsilon)^{3/2-\beta}}{n_\varepsilon^{1/2-\beta}} dx \le \liminf_{\varepsilon\to 0} \int_{\mathbb{T}^d} n_\varepsilon\theta_\varepsilon^{3/2-\beta} dx < \infty,$$

in view of the uniform $L^\infty(0,T;L^2(\mathbb{T}^d))$ bound for $n_\varepsilon\theta_\varepsilon^{3/2-\beta}$. This shows that $w = 0$ in $\{n = 0\}$. We define $\theta := w/n$ for $n > 0$ and $\theta := 0$ if $n = 0$. Then $w = n\theta$.

The decomposition $n_\varepsilon\theta_\varepsilon^{1/2-\beta} = n_\varepsilon^{1/2+\beta}(n_\varepsilon\theta_\varepsilon)^{1/2-\beta}$ shows, after passing to the limit $\varepsilon \to 0$, that

$$y = n^{1/2+\beta}w^{1/2-\beta} = n^{1/2+\beta}(n\theta)^{1/2-\beta} = n\theta^{1/2-\beta}.$$

The strong convergence of (n_ε) in $L^2(0,T;L^p(\mathbb{T}^d))$ $(p < 6)$ and the weak* convergence of $(n_\varepsilon\theta_\varepsilon^{3/2-\beta})$ in $L^\infty(0,T;L^2(\mathbb{T}^d))$ imply that

$$(n_\varepsilon\theta_\varepsilon)^{3/2-\beta} = n_\varepsilon^{1/2-\beta}(n_\varepsilon\theta_\varepsilon^{3/2-\beta}) \rightharpoonup n^{1/2-\beta}z \quad \text{weakly in } L^2(0,T;L^1(\mathbb{T}^d)).$$

On the other hand, $(n_\varepsilon\theta_\varepsilon)^{3/2-\beta}$ converges a.e. to $w^{3/2-\beta} = (n\theta)^{3/2-\beta}$. This shows that $n^{1/2-\beta}(z - n\theta^{3/2-\beta}) = 0$ a.e. Thus, $z = n\theta^{3/2-\beta}$ if $n > 0$.

We can pass to the limit in the energy-transport equations to obtain Theorem 3.

4. Uniqueness of positive weak solutions

We prove the following result:

Theorem 6. *There exists at most one bounded weak solution* (n, θ) *to* (6)-(7) *in the class of functions* $(n, \theta) \in (H^1(0, T; H^{-1}(\mathbb{T}^d)) \cap L^2(0, T; H^1(\mathbb{T}^d)))^2$ *for which there exist constants* $0 < m \le M$ *such that* $0 < m \le n, \theta \le M$ *in* $\mathbb{T}^d \times (0, T)$.

Proof. The proof is based on the H^{-1} method. Let (n, θ) and $(\bar n, \bar\theta)$ be two weak solutions to (6)-(7) with the same initial data (n_0, θ_0) and with the regularity stated in the theorem. We define the entropy variables $u = (u_1, u_2)$ and $\bar u = (\bar u_1, \bar u_2)$ by

$$u_1 = \log(n\theta^{-3/2}), \quad u_2 = -\theta^{-1},$$
$$\bar u_1 = \log(\bar n\bar\theta^{-3/2}), \quad \bar u_2 = -\bar\theta^{-1},$$

and the vector-valued functions

$$\rho(u) = \begin{pmatrix} \rho_1(u) \\ \rho_2(u) \end{pmatrix} = (-u_2)^{-3/2} e^{u_1} \begin{pmatrix} 1 \\ -\frac{3}{2}u_2^{-1} \end{pmatrix},$$
$$a(u) = \begin{pmatrix} a_1(u) \\ a_2(u) \end{pmatrix} = (-u_2)^{-(2-\beta)} e^{u_1} \begin{pmatrix} 1 \\ -(2-\beta)u_2^{-1} \end{pmatrix}.$$

These functions are well defined since u_j and $\bar u_j$ are bounded from below and above by assumption. Moreover, it holds $\rho_j(u)$, $\rho_j(\bar u)$, $a_j(u)$, $a_j(\bar u) \in L^2(0, T; H^1(\mathbb{T}^d))$ for $j = 1, 2$. Noticing that $\rho(u) = (n, \frac{3}{2}n\theta)^\top$ and $a(u) = (n\theta^{1/2-\beta}, n\theta^{3/2-\beta})^\top$, (6)-(7) for (n, θ) and $(\bar n, \bar\theta)$ respectively, can be equivalently formulated as

$$\rho(u)_t = \Delta a(u), \quad t > 0, \quad \rho(u(\cdot, 0)) = \rho(u_0) \quad \text{in } \mathbb{T}^d, \tag{10}$$
$$\rho(\bar u)_t = \Delta a(\bar u), \quad t > 0, \quad \rho(\bar u(\cdot, 0)) = \rho(u_0) \quad \text{in } \mathbb{T}^d, \tag{11}$$

where $u_0 = \log(n_0\theta_0^{-3/2})$. Furthermore, there exist unique weak solutions w_1, $w_2 \in L^2(0, T; H^1(\mathbb{T}^d))$ to

$$-\Delta w_j = \rho_j(u) - \rho_j(\bar u) \quad \text{in } \mathbb{T}^d, \quad \int_{\mathbb{T}^d} w_j dx = 0, \quad j = 1, 2.$$

We set $w = (w_1, w_2)^\top$. Notice that $w(\cdot, 0) = 0$ since $\rho_j(u(\cdot, 0)) - \rho_j(\bar u(\cdot, 0)) = 0$. Using w as a test function in the weak formulation of (10)-(11) and taking the difference of both equations leads to

$$\int_0^t \langle(\rho(u) - \rho(\bar u))_t, w\rangle ds + \int_0^t \int_{\mathbb{T}^d} \nabla(a(u) - a(\bar u)) \cdot \nabla w dx ds = 0, \tag{12}$$

where $\langle\cdot,\cdot\rangle$ is the dual product between $(H^{-1}(\mathbb{T}^d))^2$ and $(H^1(\mathbb{T}^d))^2$. The first integral equals

$$-\int_0^t \langle \Delta w_t, w\rangle ds = \int_0^t \int_{\mathbb{T}^d} \nabla w_t \cdot \nabla w dx ds = \frac{1}{2}\int_{\mathbb{T}^d} |\nabla w(\cdot,t)|^2 dx.$$

Taking $a(u)-a(\bar{u})$ as a test function in the weak formulation of the equation for w, we find that the second integral in (12) equals

$$\int_0^t \int_{\mathbb{T}^d} (\rho(u)-\rho(\bar{u}))\cdot(a(u)-a(\bar{u}))dxds.$$

By the mean-value theorem, we infer that

$$\rho(u)-\rho(\bar{u}) = \rho'(\xi)(u-\bar{u}), \quad a(u)-a(\bar{u}) = a'(\eta)(u-\bar{u}),$$

where ξ, $\eta \in \mathbb{R}^2$ depend on (x,t) through u and $\bar{u}$ and the prime denotes the derivative with respect to u. Collecting the above equations, (12) becomes

$$\frac{1}{2}\int_{\mathbb{T}^d} |\nabla w(\cdot,t)|^2 dx + \int_0^t \int_{\mathbb{T}^d} (u-\bar{u})^\top (\rho'(\xi)^\top a'(\eta))(u-\bar{u})dxds = 0.$$

The components of the matrix product $\rho'(\xi)^\top a'(\eta)$ read as

$$\sum_{j=1}^2 \frac{\partial \rho_j}{\partial u_k}(\xi)\frac{\partial a_j}{\partial u_\ell}(\eta).$$

Now, the derivatives ρ' and a' are

$$\rho'(\xi) = (-\xi_2)^{-3/2} e^{\xi_1} \begin{pmatrix} 1 & \frac{3}{2}(-\xi_2)^{-1} \\ \frac{3}{2}(-\xi_2)^{-1} & \frac{15}{4}(-\xi_2)^{-2} \end{pmatrix},$$
$$a'(\eta) = (-\eta_2)^{-(2-\beta)} e^{\eta_1} \begin{pmatrix} 1 & (2-\beta)(-\eta_2)^{-1} \\ (2-\beta)(-\eta_2)^{-1} & (3-\beta)(2-\beta)(-\eta_2)^{-2} \end{pmatrix}.$$

Since u and $\bar{u}$ are bounded from below and above, these matrices are positive definite (for $\beta < 2$), and we conclude that there exists a constant $c > 0$ depending on the lower and upper bounds m and M such that

$$\frac{1}{2}\int_{\mathbb{T}^d} |\nabla w(\cdot,t)|^2 dx + c\int_0^t \int_{\mathbb{T}^d} |u-\bar{u}|^2 dxds \le 0.$$

This implies that $w(\cdot,t) = 0$ for all $t > 0$ and hence, $\rho(u) = \rho(\bar{u})$ and $n = \bar{n}$, $n\theta = \bar{n}\bar{\theta}$. $\square$

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