The Relaxation-Time Limit in the Quantum Hydrodynamic Equations for Semiconductors

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Abstract

The relaxation-time limit from the quantum hydrodynamic model to the quantum drift-diffusion equations in $\mathbb{R}^3$ is shown for solutions which are small perturbations of the steady state. The quantum hydrodynamic equations consist of the isentropic Euler equations for the particle density and current density including the quantum Bohm potential and a momentum relaxation term. The momentum equation is highly nonlinear and contains a dispersive term with third-order derivatives. The equations are self-consistently coupled to the Poisson equation for the electrostatic potential. The relaxation-time limit is performed both in the stationary and the transient model. The main assumptions are that the steady-state velocity is irrotational, that the variations of the doping profile and the velocity at infinity are sufficiently small and, in the transient case, that the initial data are sufficiently close to the steady state. As a by-product, the existence of global-in-time solutions to the quantum drift-diffusion model in $\mathbb{R}^3$ close to the steady-state is obtained.

Keywords: quantum hydrodynamic equations, third-order derivatives, global relaxation-time limit, quantum drift-diffusion equations.

1 Introduction

For the numerical simulation of modern ultra-small semiconductor devices, model equations based on quantum mechanical phenomena, like the Schrödinger or Wigner equation, have to be employed. However, the numerical solution of these models is usually extremely time consuming. Recently, macroscopic quantum semiconductor models have been derived with the intention to find a compromise between the contradictory requirements of computational efficiency and physical accuracy.

For instance, so-called *quantum hydrodynamic models* have been derived from Wigner-Boltzmann equations by using a moment method and appropriate closure conditions [7, 8, 10, 12]. The zero-temperature quantum hydrodynamic model is formally equivalent to the single-state Schrödinger equation leading to the so-called Madelung's equations [32]. Temperature terms are then obtained from a system of mixed-state Schrödinger equations and appropriate closure conditions [11].

Macroscopic quantum models have the advantages that they are solved in the (3+1)-dimensional position-time space instead of, for instance, the (3+3+1)-dimensional phase space of the Wigner equation and that the macroscopic particle and current densities are a direct solution of the equations and do not need to be computed from the microscopic variables. In particular, this helps in formulating appropriate boundary conditions.

Quantum hydrodynamic models contain highly nonlinear and dispersive terms with third-order derivatives and therefore, its analytical and numerical treatment is quite involved. However, in certain physical regimes, these models can be reduced formally to simpler models. More precisely, when performing a diffusive scaling, the convective term can be formally neglected and the model reduces to the so-called *quantum drift-diffusion model* whose analysis and numerical solution is much simpler than for the original model since it is parabolic and of fourth order. Up to now, the model reduction is only formal. In this paper we prove the reduction limit, which is referred to as the *relaxation-time limit*, rigorously. This is the first result on the rigorous relaxation-time limit in the quantum hydrodynamic model.

More specifically, we study the following (scaled) isentropic quantum hydrodynamic model:

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho E + \frac{\varepsilon^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \frac{\rho\mathbf{u}}{\tau}, \tag{1.2}$$

$$\lambda^2\nabla\cdot E = \rho - \mathcal{C}, \quad \nabla\times E = 0, \quad \mathbf{x}\in\mathbb{R}^3,\ t>0, \tag{1.3}$$

with initial conditions

$$\rho(\mathbf{x},0) = \rho_1(\mathbf{x}), \quad \mathbf{u}(\mathbf{x},0) = \mathbf{u}_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3.$$

The variables are the electron density ρ, the mean velocity $\mathbf{u}$, and the electric field E. Furthermore, $P = P(\rho)$ is the pressure function and $\mathcal{C} = \mathcal{C}(\mathbf{x})$ the doping concentration. The parameters are the (scaled) Planck constant ε, the momentum relaxation time τ, and the Debye length λ. The quantum hydrodynamic equations (1.1)–(1.3) can be interpreted as Euler equations for a charged isentropic gas, containing the quantum Bohm potential $\Delta\sqrt{\rho}/\sqrt{\rho}$ and the relaxation term $\rho\mathbf{u}/\tau$. We refer to [22] and the references therein for details on the derivation and scaling of the above equations.

In this paper we are interested in the small relaxation-time analysis. For this, we rescale the equations like in [33]:

$$\mathbf{x} \to \mathbf{x}, \quad t \to \frac{t}{\tau}, \quad (\rho_\tau, \mathbf{u}_\tau, E_\tau)(\mathbf{x}, t) = \Big(\rho, \frac{1}{\tau}\mathbf{u}, E\Big)\Big(\mathbf{x}, \frac{t}{\tau}\Big). \tag{1.4}$$

Then (1.1)–(1.3) can be rewritten as

$$\partial_t \rho_\tau + \nabla\cdot(\rho_\tau \mathbf{u}_\tau) = 0, \tag{1.5}$$

$$\tau^2 \partial_t(\rho_\tau \mathbf{u}_\tau) + \tau^2 \nabla\cdot(\rho_\tau \mathbf{u}_\tau \otimes \mathbf{u}_\tau) + \nabla P(\rho_\tau) = \rho E_\tau + \frac{\varepsilon^2}{2}\rho_\tau \nabla\left(\frac{\Delta\sqrt{\rho_\tau}}{\sqrt{\rho_\tau}}\right) - \rho_\tau \mathbf{u}_\tau, \tag{1.6}$$

$$\lambda^2 \nabla\cdot E_\tau = \rho_\tau - \mathcal{C}, \quad \nabla\times E = 0, \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0. \tag{1.7}$$

In the formal limit $\tau \to 0$ we obtain the *quantum drift-diffusion equations*

$$\partial_t \rho + \nabla\cdot\left[\rho\left(E - \nabla h(\rho) + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\right)\right] = 0, \tag{1.8}$$

$$\lambda^2 \nabla\cdot E = \rho - \mathcal{C}, \quad \nabla\times E = 0, \quad \mathbf{x} \in \mathbb{R}^3,\ t > 0, \tag{1.9}$$

where the enthalpy $h(\rho)$ is defined by $\rho h'(\rho) = P'(\rho)$, $h(1) = 0$.

The quantum drift-diffusion equations can be also derived via a quantum entropy minimization principle and through a diffusion scaling limit from a BGK-type Wigner model [6]. The stationary multi-dimensional equations are analyzed in bounded domains with mixed Dirichlet-Neumann boundary conditions in [2]. For the transient equations, general existence results have been only obtained in the one-dimensional case [14, 28, 29] (however, see [13] for the multi-dimensional zero-temperature zero-field approximation).

In this paper we make the limit $\tau \to 0$ rigorous both in the stationary and the time-dependent equations. The dispersive third-order quantum term in the moment equation (1.6) is responsible for formidable mathematical difficulties. For instance, no maximum principle is available in order to show the non-negativity of the particle density ρ which is necessary to define the quantum term. Up to now, there is no satisfactory theory to deal with this difficulty (see, however, [11]). In particular, the existence of solutions of (1.5)-(1.7) has been shown only under additional assumptions. The well-posedness of steady state "subsonic" solutions has been proved in [9, 15, 21, 23, 35]. Transient solutions are shown to exist either locally in time [16, 17, 25] or globally in time for data close to a steady state [18, 19, 24, 31], using different boundary conditions. It is not surprising that only partial results have been obtained up to now since also for the classical Euler equations, there is no complete existence theory in several space dimensions.

Relaxation-time limits in the classical hydrodynamic equations have been performed first in [33], when uniform L^∞ bounds are available. Without this assumption, the limit has been proved in [4] for smooth solutions which are small perturbations of a steady state and then for weak solutions in [26, 27] (for the isentropic equations) and in [20] (for the isothermal model). The multi-dimensional equations are considered in [30]. In [1, 5] the relaxation-time limit in the hydrodynamic model including an energy equation has been shown. The idea of [26, 27] was to derive estimates uniform in the relaxation time by employing so-called higher-order entropies which allow to obtain L^p bounds for any $p < \infty$. Unfortunately, this idea cannot be used here since we are not able to control the dispersive quantum term. On the other hand, the usual energy/entropy estimates are not enough to conclude the limit. Therefore, our approach is to use smooth solutions and to impose (smallness) assumptions on the data ensuring the positivity of the particle density. The small perturbation condition allows to derive uniform estimates in Sobolev spaces for higher-order derivatives.

Our first result is an existence result for the stationary version of (1.5)–(1.7), essentially under the conditions that the steady-state velocity $\bar{\mathbf{u}}_\tau$ is irrotational and that $\nabla\mathcal{C}$ and $\nabla\bar{\mathbf{u}}_\tau$ at infinity are sufficiently small (see Theorem 2.3). The first assumption allows to reformulate the equations for the steady-state density $\bar{\rho}_\tau$ and the velocity $\bar{\mathbf{u}}_\tau$ as elliptic second-order equations for $\sqrt{\bar{\rho}_\tau}$ and the velocity potential, thus avoiding the nonlinear third-order term. From the second assumption, estimates for $\sqrt{\bar{\rho}_\tau}$, $\bar{\mathbf{u}}_\tau$, and the steady-state electric field $\bar{E}_\tau$ in some Sobolev norms uniformly in τ can be derived. The bounds are independent of τ since the only term involving τ, written as $\tau^2(\bar{\mathbf{u}}_\tau\cdot\nabla)\bar{\mathbf{u}}_\tau$, is of lower order and can be thus controlled by elliptic estimates. We notice that the stationary density $\bar{\rho}_\tau$ does not need to be close to a constant. In fact, no restriction on the difference $|\sup_{\mathbf{x}\in\mathbb{R}^3}\mathcal{C}(\mathbf{x}) - \inf_{\mathbf{x}\in\mathbb{R}^3}\mathcal{C}(\mathbf{x})|$ for the doping profile $\mathcal{C}(\mathbf{x})$ is necessary (see Remark 2.2).

The relaxation-time limit in the transient equations is more involved. The main idea is to reformulate the momentum equation (1.6) as a nonlinear fourth-order wave equation for the square root of the particle density $\sqrt{\rho_\tau}$ (as in [24]) and to analyze the evolution equation for the vorticity $\nabla\times\mathbf{u}_\tau$. In both cases we get rid of the third-order term: It becomes a Bi-Laplacian in the wave equation and it disappears in the vorticity equation. Then, the equations for the differences $w = \sqrt{\rho_\tau} - \sqrt{\bar{\rho}_\tau}$ and $\mathbf{z} = \mathbf{u}_\tau - \bar{\mathbf{u}}_\tau$, where $(\bar{\rho}_\tau, \bar{\mathbf{u}}_\tau)$ is a steady state solution, are of the form

$$\tau^2\partial_t^2 w + \partial_t w + \frac{\varepsilon^2}{4}\Delta^2 w + \bar{\rho}_\tau w + 2\tau^2\mathbf{u}_\tau\partial_t\nabla w - \nabla\cdot\big((P'(\bar{\rho}_\tau) - \tau^2|\mathbf{u}_\tau|)\nabla w\big) = f, \quad (1.10)$$

$$\tau^2\partial_t(\nabla\times\mathbf{z}) + \nabla\times\mathbf{z} = g, \quad (1.11)$$

where f and g depend on w, $\mathbf{u}$ and their derivatives. A priori estimates independent of τ are obtained by multiplying (1.10) by $w+2\partial_t w$ and (1.11) by $\nabla\times\mathbf{z}$, taking the sum of both equations, and by integrating over $\mathbb{R}^3$. The fifth term on the left-hand side of (1.10) can be controled by the first four terms on the left-hand side for small perturbations. A uniform bound for the last term on the left-hand side of (1.10) is obtained from a subsonic-type condition which yields positivity of the difference $P'(\bar{\rho}_\tau) - \tau^2|\mathbf{u}_\tau|$. Assuming solutions which are small perturbations of the order of $O(\delta)$, we are able to arrive to the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left(w^2+|\Delta w|^2+\tau^2(\partial_t w)^2+\tau^2|\nabla\times\mathbf{z}|^2\right)d\mathbf{x} + (c_1-c_2\delta)\int_{\mathbb{R}^3}\left(w^2+|\Delta w|^2+(\partial_t w)^2+|\nabla\times\mathbf{z}|^2\right)d\mathbf{x}\le 0$$

for some constants c_1, $c_2>0$ independent of τ and δ. Choosing δ sufficiently small, we obtain uniform estimates for w and $\nabla\times\mathbf{z}$ in some Sobolev or Lebesgue spaces with L^2 regularity in time, respectively. A uniform bound for $\nabla\cdot\mathbf{z}$ follows from (1.5) and the above estimates for w. Thus we conclude a uniform bound for the derivative $D\mathbf{z}$ in some Lebesgue norm. With these estimates it is possible to obtain uniform bounds also for higher-order derivatives allowing to pass to the limit $\tau\to 0$ in (1.5)-(1.7) (see Theorem 2.5). We remark that here, we do not need to assume that the transient velocity $\mathbf{u}_\tau$ is irrotational.

As a by-product, we conclude the existence of a strong global-in-time solution of the quantum drift-diffusion model in $\mathbb{R}^3$. In the literature, up to now, only the one-dimensional transient quantum drift-diffusion equations or its zero-temperature zero-field approximation is analyzed [3, 13, 14, 28]. Therefore, this is the first result on the multi-dimensional model with solutions which are small perturbations of the steady state.

This paper is organized as follows. In the next section we reformulate the stationary equations and present our main theorems for the steady-state and the transient model. Section 3 is concerned with the existence result and the proof of uniform estimates for the steady-state problem. Finally, the relaxation-time limit in the time-dependent equations is shown in Section 4.

Notation. Throughout this paper, c and c_i denote generic positive constants. The spaces $L^p(\mathbb{R}^3)$, $H^k(\mathbb{R}^3)$, and $W^{k,p}(\mathbb{R}^3)$ ($k\ge 1$, $1\le p\le\infty$) denote the usual Lebesgue and Sobolev spaces, respectively. The norm of $H^k(\mathbb{R}^3)$ is denoted by $\|\cdot\|_{H^k}$ or $\|\cdot\|_k$, and the norm of $L^p(\mathbb{R}^3)$ is $\|\cdot\|_{L^p}$. If $p=2$ we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$. Furthermore, $\mathcal{H}^k(\mathbb{R}^3)$ is defined as the subspace of all functions $f\in L^6(\mathbb{R}^3)$ such that $Df\in H^{k-1}(\mathbb{R}^3)$ ($k\ge 1$).

2 Preliminaries and main results

2.1 The stationary model

First we consider the stationary version of the scaled equations (1.5)–(1.7). For simplicity, we set $\lambda=1$. It is convenient to make use of the transformation $\rho_\tau=\psi_\tau^2$. Then

$$\nabla\cdot(\psi_\tau^2\mathbf{u}_\tau)=0,\quad J_\tau=\psi_\tau^2\mathbf{u}_\tau, \tag{2.1}$$

$$\tau^2(\mathbf{u}_\tau\cdot\nabla)\mathbf{u}_\tau+\nabla h(\psi_\tau^2)+\mathbf{u}_\tau=E_\tau+\frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi_\tau}{\psi_\tau}\right), \tag{2.2}$$

$$\nabla\cdot E_\tau=\psi_\tau^2-\mathcal{C},\quad \nabla\times E_\tau=0,\quad \mathbf{x}\in\mathbb{R}^3, \tag{2.3}$$

where the enthalpy $h(\rho)$ is defined by $\rho h'(\rho) = P'(\rho)$, $h(1) = 0$. Our approach to solve this problem is to integrate (2.2) in order to derive a system of second-order equations. For this, we need to assume that the quantum fluid is irrotational, i.e., the velocity is assumed to be the gradient of the so-called Fermi potential S_τ, $\mathbf{u}_\tau = \nabla S_\tau$. The second equation in (2.3) implies that also the electric field is a gradient of the electrostatic potential V_τ, $E_\tau = -\nabla V_\tau$. Then, introducing the function

$$F(\rho, J) = h(\rho) + \frac{|J|^2}{2\rho^2},$$

we can write (2.2) equivalently as

$$\nabla F(\psi_\tau^2, \tau J_\tau) = -\nabla(V_\tau + S_\tau) + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi_\tau}{\psi_\tau}\right).$$

Therefore, integrating this equation, we obtain the elliptic equation

$$\frac{\varepsilon^2}{2}\Delta\psi_\tau = \psi_\tau(F(\psi_\tau^2, \tau J_\tau) - \phi_\tau), \tag{2.4}$$

where $\phi_\tau = -(V_\tau + S_\tau)$. The integration constant can be set to zero by defining the reference point for the electrostatic potential. By (2.1) and (2.3), the function ϕ_τ satisfies the elliptic equation

$$\Delta\phi_\tau = \psi_\tau^2 - \mathcal{C} + \frac{2}{\psi_\tau^3} J_\tau \cdot \nabla\psi_\tau. \tag{2.5}$$

The system of equations (2.1), (2.4), and (2.5) for the variables $(\psi_\tau, J_\tau, \phi_\tau)$ is formally equivalent to (2.1)–(2.3) for the variables $(\psi_\tau, \mathbf{u}_\tau, E_\tau)$ or $(\psi_\tau, \mathbf{u}_\tau, \phi_\tau)$.

In order to specify the conditions at infinity for the functions $(\psi_\tau, \mathbf{u}_\tau, \phi_\tau)$, we impose the following assumptions. The doping profile is assumed to satisfy the bounds

$$0 < \rho_- \le \inf_{\mathbf{x}\in\mathbb{R}^3} \mathcal{C}(\mathbf{x}) \le \sup_{\mathbf{x}\in\mathbb{R}^3} \mathcal{C}(\mathbf{x}) \le \rho_+. \tag{2.6}$$

We suppose further that there exist a function $S_0 \in C_b^1(\mathbb{R}^3, \mathbb{R})$ and two positive constants u_+, $\tilde{u}_0$ such that

$$\sup_{\mathbf{x}\in\mathbb{R}^3} |\mathbf{u}_0(\mathbf{x})| \le u_+ < \tilde{u}_0. \tag{2.7}$$

Notice that in $\mathbb{R}^3$ the stationary equation of mass conservation does not necessarily gives a constant current density J_τ. Here we set

$$J_0 = \mathcal{C}\mathbf{u}_0. \tag{2.8}$$

Then, defining the stationary profile

$$(\psi_0, \mathbf{u}_0, \phi_0^\tau)(\mathbf{x}) = \Big(\sqrt{\mathcal{C}}, \mathbf{u}_0, F(\mathcal{C}, \tau J_0)\Big)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{2.9}$$

we impose the following condition at infinity

$$|(\psi_\tau - \psi_0, \mathbf{u}_\tau - \mathbf{u}_0, \phi_\tau - \phi_0^\tau)(\mathbf{x})| \to 0 \quad \text{as } |\mathbf{x}| \to \infty \tag{2.10}$$

or, equivalently,

$$|(\bar\psi_\tau - \psi_0, \bar{\mathbf{u}}_\tau - \mathbf{u}_0, \bar E_\tau - E_0^\tau)(\mathbf{x})| \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.11}$$

where the electric field $\bar E_\tau$ is expressed in terms of $\bar\phi_\tau$ and $\bar{\mathbf{u}}_\tau$ by

$$\bar E_\tau = \nabla\phi_\tau + \bar{\mathbf{u}}_\tau, \quad E_0^\tau = \nabla\phi_0^\tau + \mathbf{u}_0. \tag{2.12}$$

Our first result is an existence theorem for the problem (2.1), (2.4), (2.5), and (2.10).

Theorem 2.1 *Let* (2.6)–(2.9) *hold and let* $P \in C^3(0,\infty)$ *such that*

$$\inf_{\mathbf{x}\in\mathbb{R}^3} \{\varepsilon\sqrt{\mathcal{C}} + P'(\mathcal{C})\} > 0. \tag{2.13}$$

Assume that $\nabla\mathcal{C}, \nabla\mathbf{u}_0 \in H^3 \cap L^{6/5}(\mathbb{R}^3)$ *with*

$$\delta_0 := \|\nabla\mathcal{C}\|_{H^3\cap L^{6/5}(\mathbb{R}^3)} + \|\nabla\mathbf{u}_0\|_{H^3\cap L^{6/5}(\mathbb{R}^3)} < \infty. \tag{2.14}$$

Then there exist positive constants τ_*, u_*, *and* δ_* *such that if* $0 < \tau \le \tau_*$, $u_+ \le u_*$, *and* $\delta_0 \le \delta_*$, *there exists a unique strong solution* $(\bar\psi_\tau, \bar{\mathbf{u}}_\tau, \bar\phi_\tau)$ *of* (2.1), (2.4), (2.5), *and* (2.10) *satisfying* $\nabla\times\bar{\mathbf{u}}_\tau = 0$ *and*

$$\|\bar\psi_\tau - \psi_0\|_4 + \|\bar{\mathbf{u}}_\tau - \mathbf{u}_0\|_3 + \|\bar\phi_\tau - \phi_0^\tau\|_4 \le c_0\delta_0, \tag{2.15}$$

where $c_0 > 0$ *is a constant independent of* τ *and* δ_0.

Furthermore, there exists a unique strong solution $(\bar\psi_\tau, \bar{\mathbf{u}}_\tau, \bar E_\tau)$ *of* (2.1)-(2.3) *and* (2.11) *satisfying* $\nabla\times\bar{\mathbf{u}}_\tau = 0$ *and*

$$\|\bar\psi_\tau - \psi_0\|_4 + \|\bar{\mathbf{u}}_\tau - \mathbf{u}_0\|_3 + \|\bar E_\tau - E_0^\tau\|_3 \le c_4\delta_0, \tag{2.16}$$

where the constant $c_4 > 0$ *is independent of* τ *and* δ_0.

Remark 2.2 Notice that the condition on the doping profile does not necessarily require that the doping function is close to a constant or that the difference $|\rho_+ - \rho_-|$ (see (2.6)) is small. Indeed, define the one-dimensional function $\mathcal{C}_0$ by

$$\mathcal{C}_0(x) = \rho_- + \int_0^{(\rho_+-\rho_-)^{-1}\alpha x} (\rho_+ - \rho_-)m^{3/2}(\xi)d\xi, \quad x \in \mathbb{R},$$

where $m \in C_0^\infty(\mathbb{R}_+)$, $m > 0$, and $\int_0^\infty m(\xi)d\xi = 1$. Then the function $\mathcal{C}(\mathbf{x}) = \mathcal{C}_0(|\mathbf{x}|^2)$, $\mathbf{x} \in \mathbb{R}^3$, satisfies

$$\mathcal{C}(0) = \rho_-, \quad \mathcal{C}(\mathbf{x}) \to \rho_+ \text{ as } |\mathbf{x}| \to \infty, \quad \|\nabla\mathcal{C}\| \le c\alpha^{1/4}.$$

The uniform bounds (2.15) or (2.16) allow to perform the relaxation-time limit $\tau \to 0$. Indeed, there exists $(\hat{\psi}_0, \hat{E}_0)$ such that, maybe for a subsequence which is not relabeled,

$$\begin{aligned} \bar{\psi}_\tau \to \hat{\psi}_0 \quad & \text{in } \ H^{4-s_0}_{\rm loc} \cap C^2_b(\mathbb{R}^3), \quad s_0 \in (0,1/2), \\ \bar{E}_\tau \to \hat{E}_0 \quad & \text{in } \ H^{3-s_0}_{\rm loc} \cap C^1_b(\mathbb{R}^3), \quad s_0 \in (0,1/2), \\ \tau^2 |\bar{\mathbf{u}}_\tau|^2 \to 0 \quad & \text{in } \ W^{2,3}_{\rm loc} \cap C^1_b(\mathbb{R}^3) \quad \text{as } \tau \to 0. \end{aligned} \tag{2.17}$$

Since ψ_0 is bounded from below by a positive constant, the estimate (2.16) implies that, for sufficiently small δ_0, $\bar{\psi}_\tau \geq c_* > 0$ for some $c_* > 0$ not depending on τ. These convergence results allow to pass to the limit $\tau \to 0$ in (2.1)–(2.3), showing that the limit functions $\hat{\psi}_0$ and $\hat{E}_0$ are solutions of

$$\nabla\cdot(\hat{\psi}^2 \hat{\mathbf{u}}) = 0, \quad \hat{\mathbf{u}} = -\nabla h(\hat{\psi}^2) + \hat{E} + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\hat{\psi}}{\hat{\psi}}\right),$$

$$\nabla\cdot \hat{E} = \hat{\psi}^2 - \mathcal{C}, \quad \nabla\times\bar{E}_\tau = 0.$$

These equations are equivalent to the stationary quantum drift-diffusion model,

$$\nabla\cdot\left(\hat{\psi}^2\left[-\nabla h(\hat{\psi}^2) + \hat{E} + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\hat{\psi}}{\hat{\psi}}\right)\right]\right) = 0, \tag{2.18}$$

$$\nabla\cdot \hat{E} = \hat{\psi}^2 - \mathcal{C}, \quad \nabla\times\hat{E} = 0, \quad \mathbf{x}\in\mathbb{R}^3. \tag{2.19}$$

We have shown the following result.

Theorem 2.3 *Let the assumptions of Theorem 2.1 hold and let $(\bar{\psi}_\tau, \bar{\mathbf{u}}_\tau, \bar{E}_\tau)$ be the strong solution of* (2.1)-(2.3) *and* (2.11) *satisfying* (2.16). *Then there exist functions $(\hat{\psi}_0, \hat{E}_0)$ and a subsequence (not relabeled) $(\bar{\psi}_\tau, \bar{E}_\tau)$ such that, as $\tau \to 0$,*

$$(\bar{\psi}_\tau, \bar{E}_\tau) \to (\hat{\psi}_0, \hat{E}_0) \quad \textit{in } (H^{4-s_0}_{\rm loc} \cap C^2_b) \times (H^{3-s_0}_{\rm loc} \cap C^1_b)(\mathbb{R}^3), \quad s_0 \in (0,1/2), \tag{2.20}$$

and $(\hat{\psi}_0, \hat{E}_0)$ is a strong solution of (2.18)–(2.19) *with the conditions at infinity*

$$|(\hat{\psi}_0 - \sqrt{\mathcal{C}}, \hat{E}_0 - \nabla h(\mathcal{C}))(\mathbf{x})| \to 0 \quad \textit{as } |\mathbf{x}| \to \infty. \tag{2.21}$$

Moreover,

$$\|\hat{\psi}_0 - \sqrt{\mathcal{C}}\,\|_4 + \|\hat{E}_0 - \nabla h(\mathcal{C})\|_3 \leq c\delta_0, \tag{2.22}$$

where $c > 0$ is a constant independent of δ_0 (δ_0 is defined in (2.14)*).*

2.2 The transient model

In dealing with the transient model, it is again convenient to make use of the variable transformation $\rho_\tau = \psi_\tau^2$ in the scaled equations (1.5)–(1.7), yielding

$$2\psi_\tau\,\partial_t\psi_\tau + \nabla\cdot(\psi_\tau^2 \mathbf{u}_\tau) = 0, \tag{2.23}$$

$$\tau^2\partial_t\mathbf{u}_\tau + \tau^2(\mathbf{u}_\tau\cdot\nabla)\mathbf{u}_\tau + \nabla h(\psi_\tau^2) + \mathbf{u}_\tau = E_\tau + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi_\tau}{\psi_\tau}\right), \tag{2.24}$$

$$\nabla\cdot E_\tau = \psi_\tau^2 - \mathcal{C}, \quad \nabla\times E_\tau = 0, \tag{2.25}$$

$$\psi_\tau(\mathbf{x},0) = \psi_1(\mathbf{x}), \quad \mathbf{u}_\tau(\mathbf{x},0) = \hat{\mathbf{u}}_1^\tau(\mathbf{x}) = \frac{1}{\tau}\mathbf{u}_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3. \tag{2.26}$$

Then, in the formal limit $\tau \to 0$, we obtain the reformulated quantum drift-diffusion equations (1.8)–(1.9),

$$2\hat{\psi}\,\partial_t\hat{\psi} + \nabla\cdot\left(\hat{\psi}^2\left[-\nabla h(\hat{\psi}^2) + \hat{E} + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\hat{\psi}}{\hat{\psi}}\right)\right]\right) = 0, \tag{2.27}$$

$$\nabla\cdot\hat{E} = \hat{\psi}^2 - \mathcal{C}, \quad \nabla\times\hat{E} = 0, \tag{2.28}$$

$$\hat{\psi}(\mathbf{x},0) = \psi_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3. \tag{2.29}$$

Our first main result is concerned with the existence of "small" global-in-time solutions together with an estimate uniform in the asymptotic parameter τ.

Theorem 2.4 *Suppose that $P(\rho)\in C^5(0,\infty)$. Let* (2.6)–(2.9) *and* (2.13)–(2.14) *hold, and let $(\bar{\psi}_\tau, \bar{\mathbf{u}}_\tau, \bar{E}_\tau)$ be the unique steady-state solution of* (2.1)–(2.3) *and* (2.11) *given by Theorem 2.3 for sufficiently small δ_0. Assume that $(\psi_1 - \bar{\psi}_\tau, \hat{\boldsymbol{u}}_1^\tau - \bar{\mathbf{u}}_\tau) \in H^6(\mathbb{R}^3)\times\mathcal{H}^5(\mathbb{R}^3)$ with $\inf_{x\in\mathbb{R}^3}\psi_1(\mathbf{x}) > 0$ and define*

$$\delta_1 := \|\psi_1 - \bar{\psi}_\tau\|_6 + \|\tau(\hat{\boldsymbol{u}}_1^\tau - \bar{\mathbf{u}}_\tau)\|_{\mathcal{H}^5} < \infty. \tag{2.30}$$

Then there exist positive constants $m_1, \tau_1, c_2, \Lambda_0$ independent of τ such that if $0<\delta_1\le m_1$ and $\tau\le\tau_1\le\min\{1,\tau_*\}$, a solution $(\psi_\tau, \mathbf{u}_\tau, E_\tau)$ of (2.23)–(2.26) *exists globally in time and satisfies the uniform estimate*

$$\begin{aligned}
&\|(\psi_\tau - \bar{\psi}_\tau)(t)\|_6 + \|\tau\partial_t\psi_\tau(t)\|_2 + \|\tau^2\partial_t^2\psi_\tau(t)\| \\
&\quad + \|\tau(\mathbf{u}_\tau - \bar{\mathbf{u}}_\tau)(t)\|_{\mathcal{H}^5} + \|\tau^2\partial_t\mathbf{u}_\tau(t)\|_{\mathcal{H}^3} + \|(E_\tau - \bar{E}_\tau)(t)\|_{\mathcal{H}^3} \\
&\le c_2(\|\psi_1 - \bar{\psi}_\tau\|_6 + \|\tau(\hat{\boldsymbol{u}}_1^\tau - \bar{\mathbf{u}}_\tau)\|_{\mathcal{H}^5})e^{-\Lambda_0 t} \quad \textit{for all } t>0.
\end{aligned} \tag{2.31}$$

We recall that $\mathcal{H}^5(\mathbb{R}^3) = \{f\in L^6(\mathbb{R}^3),\ Df\in H^4(\mathbb{R}^3)\}$. Further uniform bounds are given in the proof of Theorem 2.4 (see (4.47)–(4.50)). The higher regularity assumption is needed in order to prove the positivity of the particle density. The above result extends the local existence theorem of [18].

Our second main result is concerned with the relaxation-time limit $\tau\to 0$.

Theorem 2.5 *Let the assumptions* (2.6)–(2.9) *and* (2.13)–(2.14) *hold. Let $(\psi_\tau, \mathbf{u}_\tau, E_\tau)$ be a global solution of* (2.23)–(2.26) *given by Theorem 2.4. Then there exists a pair of functions $(\hat{\psi}, \hat{E})$ such that, as $\tau\to 0$,*

$$\psi_\tau \to \hat{\psi} \quad \textit{in } C(0,T; C_b^2\cap H_{\rm loc}^{4-s_0}(\mathbb{R}^3)), \quad E_\tau \to \hat{E} \quad \textit{in } C(0,T; C_b^2\cap \mathcal{H}_{\rm loc}^{3-s_0}(\mathbb{R}^3)), \tag{2.32}$$

where $s_0 \in (0, 1/2)$. The limit functions $\hat\psi$, $\hat E$ are a strong solution of (2.27)–(2.29) *and satisfy for two positive constants c_3, Λ_3,*

$$\|(\hat\psi - \hat\psi_0)(t)\|_4 + \|(\hat E - \hat E_0)(t)\|_{\mathcal{H}^3(\mathbb{R}^3)} \le c_3\delta_3 e^{-\Lambda_3 t} \quad \text{for all } t > 0, \tag{2.33}$$

where c_3, $\Lambda_3 > 0$, $\delta_3 = \|\psi_1 - \hat\psi_0\|_{H^4(\mathbb{R}^3)}$, and $(\hat\psi_0, \hat E_0)$ is the stationary solution of (2.18)–(2.19) *and* (2.21).

2.3 Auxiliary results

We need the following standard results.

Lemma 2.6 *Let $f \in H^s(\mathbb{R}^3)$, $s \ge 3/2$. There exists a unique solution u of the divergence equation*

$$\nabla\cdot \mathbf{u} = f, \quad \nabla\times\mathbf{u} = 0, \quad \mathbf{u}(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.34}$$

satisfying

$$\|\mathbf{u}\|_{L^6(\mathbb{R}^3)} \le c\|f\|_{L^2(\mathbb{R}^3)}, \quad \|D\mathbf{u}\|_{H^s(\mathbb{R}^3)} \le c\|f\|_{H^s(\mathbb{R}^3)}. \tag{2.35}$$

Lemma 2.7 (1) Let $f, g \in L^\infty \cap H^s(\mathbb{R}^3)$, $s \ge 3/2$. Then, for some constant $c > 0$,

$$\begin{aligned} \|D^\alpha(fg)\| &\le c\|g\|_{L^\infty}\|D^\alpha f\| + c\|f\|_{L^\infty}\|D^\alpha g\|, && (2.36)\\ \|D^\alpha(fg) - fD^\alpha g\| &\le c\|g\|_{L^\infty}\|D^\alpha f\| + c\|f\|_{L^\infty}\|D^{\alpha-1} g\|, && (2.37) \end{aligned}$$

for all $1 \le |\alpha| \le s$.

(2). Let $\mathbf{u} \in \mathcal{H}^1(\mathbb{R}^3) = \{\mathbf{u} \in L^6(\mathbb{R}^3),\ D\mathbf{u} \in L^2(\mathbb{R}^3)\}$. Then, for some constant $c > 0$,

$$\|\mathbf{u}\|_{L^6} \le c\|D\mathbf{u}\|. \tag{2.38}$$

3 Proof of Theorem 2.1

The proof is based on Banach's fixed-point theorem.

Step 1: Reformulation of the problem. The stationary equations for the difference

$$(n, \mathbf{v}, q) = (\bar\psi_\tau, \bar{\mathbf{u}}_\tau, \bar\phi_\tau) - (\psi_0, \mathbf{u}_0, \phi_0^\tau)$$

read as follows:

$$\nabla\cdot(\mathcal{C}\mathbf{v}) + \nabla\cdot(2n\mathcal{C}^{1/2}\mathbf{u}_0) = \nabla\cdot r_0(n, \mathbf{v}, q), \quad \nabla\times\mathbf{v} = 0, \tag{3.1}$$

$$\frac{\varepsilon^2}{2}\Delta n - 2\big(P'(\mathcal{C}) - |\tau\mathbf{u}_0|^2\big)n - \tau^2\mathcal{C}^{1/2}\mathbf{u}_0\cdot\mathbf{v} - 2|\tau\mathbf{u}_0|^2 n + \mathcal{C}^{1/2}\, q = r_1(n, \mathbf{v}, q), \tag{3.2}$$

$$\Delta q - 2\mathcal{C}^{1/2} n - \frac{2}{\mathcal{C}^{3/2}} J_0\cdot\nabla n = r_2(n, \mathbf{v}, q), \tag{3.3}$$

$$(n, \mathbf{v}, q)(\mathbf{x}) \to 0, \quad |\mathbf{x}| \to \infty. \tag{3.4}$$

where r_0, r_1 and r_2 are defined by

$$
\begin{aligned}
r_0(n,\mathbf{v},q) &= -((2\mathcal{C}^{1/2}+n)n\mathbf{v}+n^2\mathbf{u}_0+J_0),\\
r_1(n,\mathbf{v},q) &= 2\big(P'(\mathcal{C})-|\tau\mathbf{u}_0|^2\big)n^2-\frac{\varepsilon^2}{2}\Delta\sqrt{\mathcal{C}}-nq\\
&\quad+\frac{\tau^2}{2}(\sqrt{\mathcal{C}}+n)^{-3}(|J_\tau|^2-|J_0|^2)-\tau^2\sqrt{\mathcal{C}}\mathbf{u}_0\cdot\mathbf{v}-2|\tau\mathbf{u}_0|^2n\\
&\quad+(\mathcal{C}^{1/2}+n)[F((\mathcal{C}^{1/2}+n)^2,\tau J_0)-F(\mathcal{C},\tau J_0)-2\sqrt{\mathcal{C}}F_{\mathcal{C}}(\mathcal{C},\tau J_0)n],\\
r_2(n,\mathbf{v},q) &= -\Delta\phi_0^\tau+\mathcal{C}^{-1/2}(\mathcal{C}^{1/2}+n)^{-3}J_\tau\cdot\nabla\mathcal{C}\\
&\quad+2((\mathcal{C}^{1/2}+n)^{-3}-\mathcal{C}^{-3/2})J_\tau\cdot\nabla n+n^2+\frac{2}{\mathcal{C}^{3/2}}(J_\tau-J_0)\cdot\nabla n,
\end{aligned}
$$

and

$$J_\tau=J_0+[\mathcal{C}^{1/2}+n]^2\mathbf{v}+2n\mathcal{C}^{1/2}\mathbf{u}_0+n^2\mathbf{u}_0=\mathcal{C}\mathbf{v}+2n\mathcal{C}^{1/2}\mathbf{u}_0-r_0,\quad \nabla\cdot J_\tau=0. \tag{3.5}$$

Equations (3.1)–(3.4) can be written in a more compact form as

$$\mathcal{T}(U)=\vec{r}(U),\quad \nabla\times\mathbf{v}=0, \tag{3.6}$$

where $U=(n,\mathbf{v},q)$, $\mathcal{T}(U)$ denotes the terms on the left-hand side of (3.1)–(3.4), and $\vec{r}(U)=(r_0(n,\mathbf{v},q),r_1(n,\mathbf{v},q),r_2(n,\mathbf{v},q))$. For given $\widetilde{U}=(\tilde{n},\tilde{\mathbf{v}},\tilde{q})\in\mathcal{M}:=\{U;\ \|n\|_2^2+\|q\|_2^2+\|\mathbf{v}\|_1^2\le\eta_0^2\}$, where $\eta_0>0$ will be determined later (at the end of step 2 below), we define a map

$$\mathcal{S}:\widetilde{U}=(\tilde{n},\tilde{\mathbf{v}},\tilde{q})\in\mathcal{M}\longmapsto U=(n,\mathbf{v},q)$$

by solving the linear problem

$$\mathcal{T}(U)=\vec{r}(\widetilde{U}),\quad \nabla\times\mathbf{v}=0. \tag{3.7}$$

This linear problem can be solved by standard methods; the details are left to the reader. Thus, the fixed-point map $\mathcal{S}$ is well defined.

Step 2: A priori estimates. Let $\widetilde{U}=(\tilde{n},\tilde{\mathbf{v}},\tilde{q})\in\mathcal{M}$. Then

$$\|\tilde{n}\|_2^2+\|\tilde{q}\|_2^2+\|\tilde{\mathbf{v}}\|_1^2\le\eta_0^2<1, \tag{3.8}$$

where the constant $\eta_0>0$ is determined later. The function $\mathcal{S}(\tilde{U})=U=(n,\mathbf{v},q)$ solves

$$\nabla\cdot(\mathcal{C}\mathbf{v})+\nabla\cdot(2n\mathcal{C}^{1/2}\mathbf{u}_0)=\nabla\cdot\tilde{r}_0,\quad \nabla\times\mathbf{v}=0, \tag{3.9}$$

$$\frac{\varepsilon^2}{2}\Delta n-2\big(P'(\mathcal{C})-|\tau\mathbf{u}_0|^2\big)n-\tau^2\mathcal{C}^{1/2}\mathbf{u}_0\cdot\mathbf{v}-2|\tau\mathbf{u}_0|^2n+\mathcal{C}^{1/2}q=\tilde{r}_1, \tag{3.10}$$

$$\Delta q-2\mathcal{C}^{1/2}n-\frac{2}{\mathcal{C}^{3/2}}J_0\cdot\nabla n=\tilde{r}_2, \tag{3.11}$$

$$(n,\mathbf{v},q)(\mathbf{x})\to 0,\quad |\mathbf{x}|\to\infty. \tag{3.12}$$

where $\tilde{r}_i = r_i(\tilde{n}, \tilde{\mathbf{v}}, \tilde{q})$, $i = 0, 1, 2$.

Writing $\mathbf{v} = \nabla \widetilde{S}$, we multiply (3.9) by $\widetilde{S}$, $\nabla\cdot \mathbf{v}$ respectively, integrate over $\mathbb{R}^3$, and use

$$\|\tilde{r}_0 + J_0\| \le c\eta_0 \|(\tilde{n}, \tilde{\mathbf{v}})\| \le c\eta_0^2$$

and

$$\|\nabla\cdot \tilde{r}_0\| \le c\|(\nabla\mathcal{C}, \nabla\mathbf{u}_0)\| + c\eta_0(\|\tilde{n}\|_2 + \|\tilde{\mathbf{v}}\|_1) \le c\delta_0 + c\eta_0^2$$

to obtain finally the estimates

$$\int \mathcal{C}|\mathbf{v}|^2 d\mathbf{x} \le 4u_+^2\|n\|^2 + c(\delta_0 + \eta_0^2)\|\mathbf{v}\|_{L^2} \quad \Rightarrow \quad \int \mathcal{C}|\mathbf{v}|^2 d\mathbf{x} \le 8u_+^2\|n\|^2 + c(\delta_0^2 + \eta_0^3), \tag{3.13}$$

$$\int \mathcal{C}|\nabla\cdot \mathbf{v}|^2 d\mathbf{x} \le 4u_+^2\|\nabla n\|^2 + c\delta_0(\|\mathbf{v}\|^2 + \|\nabla\cdot \mathbf{v}\|^2 + \|n\|^2) + c\|\nabla\cdot \mathbf{v}\|(\delta_0 + \eta_0^2). \tag{3.14}$$

Here and in the following, we integrate over $\mathbb{R}^3$ if no integration domain is indicated. Thus, by the above inequalities (3.13)–(3.14) and Lemma 2.6,

$$\|\mathbf{v}\|^2 + \|D\mathbf{v}\|^2 \le c(\delta_0^2 + \eta_0^3) + cu_+^2(\|n\|^2 + \|\nabla n\|^2) + c\delta_0\|n\|^2, \tag{3.15}$$

where we recall that $\rho_+ \ge \sup_{\mathbf{x}\in\mathbb{R}^3}\mathcal{C}(x)$, $\rho_- \le \inf_{\mathbf{x}\in\mathbb{R}^3}\mathcal{C}(x)$, and $u_+ \ge \sup_{\mathbf{x}\in\mathbb{R}^3}|\mathbf{u}_0(\mathbf{x})|$.

For the estimate of n and q we remark first that

$$\begin{aligned} \left\|\tilde{r}_1 + \frac{\varepsilon^2}{2}\Delta\sqrt{\mathcal{C}}\right\| + \|\tilde{r}_2 + \Delta\phi_0^\tau\| &\le c\eta_0(\|\tilde{n}\|_2 + \|\tilde{q}\|_2 + \|\tilde{\mathbf{v}}\|_1) \le c\eta_0^2, \\ \|\tilde{r}_1\| + \|\tilde{r}_2\| &\le c(\delta_0 + \eta_0(\|\tilde{n}\|_2 + \|\tilde{q}\|_2 + \|\tilde{\mathbf{v}}\|_1)) \le c(\delta_0 + \eta_0^2). \end{aligned} \tag{3.16}$$

Multiply (3.10) by n and (3.11) by $q/2$, integrate over $\mathbb{R}^3$ and sum the resulting equations. This leads to

$$\begin{aligned} &\frac{\varepsilon^2}{2}\|\nabla n\|^2 + \frac{1}{2}\|\nabla q\|^2 + 2\int \big(P'(\mathcal{C}) - |\tau\mathbf{u}_0|^2\big) n^2\, d\mathbf{x} \\ &\qquad + 2\int |\tau\mathbf{u}_0|^2 n^2 d\mathbf{x} + \tau^2 \int \mathcal{C}^{1/2} n\mathbf{u}_0\cdot \mathbf{v}\, d\mathbf{x} \\ &= \int \frac{n}{\mathcal{C}^{3/2}} J_0\cdot\nabla q\, d\mathbf{x} + \int qn\nabla\cdot (\mathcal{C}^{-3/2}J_0)\, d\mathbf{x} - \int \Big(n\tilde{r}_1 + \frac{1}{2}q\tilde{r}_2\Big)\, d\mathbf{x} \qquad (3.17) \\ &\le \frac{1}{2}u_+\rho_-^{-1/2}(\|n\|^2 + \|\nabla q\|^2) + \frac{\varepsilon^2}{4}\|\nabla n\|^2 + \frac{1}{8}\|\nabla q\|^2 \\ &\qquad + c(\eta_0 + \delta_0)(\|n\|^2 + \|q\|^2) + c_*(\rho_\pm, \varepsilon)\delta_0^2 + c\eta^3, \qquad (3.18) \end{aligned}$$

where $c_*(\rho_\pm, \varepsilon) > 0$ denotes a constant which may depend on $\rho_\pm$ and ε. Furthermore, multiplying (3.11) by $n/2$ and integrating over $\mathbb{R}^3$ gives

$$\begin{aligned} \int \mathcal{C}^{1/2} n^2\, d\mathbf{x} &= -\int n^2\nabla\cdot (\mathcal{C}^{-3/2}J_0)\, d\mathbf{x} - \frac{1}{2}\int (\nabla n\cdot\nabla q + n\tilde{r}_2)\, d\mathbf{x} \qquad (3.19) \\ &\le c(\eta_0 + \delta_0)\|n\|^2 + c_*(\rho_\pm, \varepsilon)\delta_0^2 + \int n\Delta\phi_0^\tau\, d\mathbf{x} - \frac{1}{2}\int \nabla n\cdot\nabla q\, d\mathbf{x}. \end{aligned}$$

Now, taking the sum of (3.17), multiplied by 1/2, and (3.19), multiplied by ε, and using Cauchy's inequality and (3.13), we find

$$
\begin{aligned}
&\int \big(\varepsilon \mathcal{C}^{1/2} + P'(\mathcal{C}) - |\tau \mathbf{u}_0|^2\big) n^2 \, d\mathbf{x} \\
&\quad \le \ \varepsilon \int n^2 \nabla\cdot (\mathcal{C}^{-3/2} J_0)\, d\mathbf{x} - \frac{\varepsilon}{2}\int n\tilde{r}_2 \, d\mathbf{x} - \int |\tau \mathbf{u}_0|^2 n^2 d\mathbf{x} - \frac{\tau^2}{2}\int \mathcal{C}^{1/2} n \mathbf{u}_0 \cdot \mathbf{v}\, d\mathbf{x} \\
&\qquad + \int \frac{n}{2\mathcal{C}^{3/2}} J_0 \cdot \nabla q\, d\mathbf{x} + \frac{1}{2}\int q n \nabla\cdot (\mathcal{C}^{-3/2} J_0)\, d\mathbf{x} - \frac{1}{2}\int \Big(n\tilde{r}_1 + \frac{1}{2} q \tilde{r}_2\Big)\, d\mathbf{x} \\
&\quad \le \ c_*(\rho_\pm, \varepsilon)\delta_0 + \frac{1}{4} u_+ \rho_-^{-1/2}(\|n\|^2 + \|\nabla q\|^2) + \frac{\varepsilon}{8} u_+ \rho_-^{-1/2}\|n\|^2 + \frac{\tau^2}{16}\int \mathcal{C}|\mathbf{v}|^2\, d\mathbf{x} \\
&\qquad + a_1 \|n\|^2 - \frac{1}{4}\int \nabla q \cdot \nabla \phi_0^\tau \, d\mathbf{x} + c(\eta_0 + \delta_0)(\|n\|^2 + \|q\|^2) + c\eta_0^3 \\
&\quad \le \ c_*(\rho_\pm, \varepsilon)\delta_0 + \frac{1}{4} u_+ \rho_-^{-1/2}(\|n\|^2 + \|\nabla q\|^2) + \frac{\varepsilon}{8} u_+ \rho_-^{-1/2}\|n\|^2 + \frac{1}{2}|\tau u_+|^2 \|n\|^2 \\
&\qquad + a_1 \|n\|^2 - \frac{1}{4}\int \nabla q \cdot \nabla \phi_0^\tau \, d\mathbf{x} + c(\eta_0 + \delta_0)(\|n\|^2 + \|q\|^2) + c\eta_0^3, \qquad (3.20)
\end{aligned}
$$

where $a_1 > 0$ is a constant which will be specified below. Notice that in view of assumption (2.13), there exist constants $\tau_* \in (0, 1/2]$ and $u_* > 0$ such that

$$
A_0 := \inf_{\mathbf{x}\in\mathbb{R}^3} \Big(\varepsilon\sqrt{\mathcal{C}(\mathbf{x})} + P'(\mathcal{C}(\mathbf{x})) - \frac{3}{2}|\tau u_+|^2\Big) > 0, \quad \tau \in (0, \tau_*], \quad u_+ \in (0, u_*],
$$

and $u_+(1 + \varepsilon^2/2)\rho_-^{-1/2} \le A_0$. Then we choose $a_1 = A_0/4$.

Taking the sum of (3.20), multiplied by $A_1 := (\rho_- + 8u_+^2)/(\rho_- A_0)$, and (3.13), multiplied by ρ_-^{-1}, gives, after some manipulations,

$$
\begin{aligned}
\|\mathbf{v}\|^2 + \|n\|^2 \ &\le \ c_*(\rho_\pm, \varepsilon)\delta_0^2 + c\eta_0^3 + \frac{A_1}{2} u_+ \rho_-^{-1/2}\|\nabla q\|^2 \\
&\quad + c(\eta_0 + \delta_0)(\|n\|^2 + \|q\|^2) - \frac{A_1}{2}\int \nabla q \cdot \nabla \phi_0^\tau \, d\mathbf{x} \qquad (3.21)
\end{aligned}
$$

for $\tau \in (0, \tau_*]$ and $u_+ \in (0, u_*]$. Furthermore, we obtain from (3.18),

$$
\begin{aligned}
\frac{\varepsilon^2}{4}\|\nabla n\| + \frac{1}{4}\|\nabla q\| \ &\le \ A_2 \|n\|^2 + \frac{\tau^2}{2}\int \mathcal{C}|\mathbf{v}|^2\, d\mathbf{x} + c(\eta_0 + \delta_0)(\|q\|^2 + \|n\|^2) \\
&\quad + \frac{1}{2} u_+ \rho_-^{-1/2}\|\nabla q\|^2 + c_*(\rho_\pm, \varepsilon)\delta_0^2 + c\eta_0^3, \qquad (3.22)
\end{aligned}
$$

where $A_2 := 2\sup_{\mathbf{x}\in\mathbb{R}^3} |P'(\mathcal{C}(\mathbf{x})) + |\tau u_+|^2 + u_+ \rho_-^{-1/2}$. Thus, taking the sum of (3.22) and $2A_2$ times (3.21), we can show that there exist constants $u_* > 0$ and $\tau_* \in (0, 1/2]$ such that for $u_+ \in (0, u_*]$ and $\tau \in (0, \tau_*]$,

$$
\frac{\varepsilon^2}{4}\|\nabla n\|^2 + \frac{1}{8}\|\nabla q\|^2 + A_2\|n\|^2 + A_2 \|\mathbf{v}\|^2 \le c_*(\rho_\pm, \varepsilon)\delta_0^2 + c\eta_0^3 + c(\eta_0 + \delta_0)(\|n\|^2 + \|q\|^2). \quad (3.23)
$$

We multiply (3.10) by q and integrate over $\mathbb{R}^3$, using (3.16) and (3.23), to infer

$$\begin{aligned}\int \mathcal{C}^{1/2}q^2\,d\mathbf{x} &= \frac{1}{2}\varepsilon^2\int \nabla n\cdot\nabla q\,d\mathbf{x} + 2\int P'(\mathcal{C})nq\,d\mathbf{x} + \tau^2\int \mathcal{C}^{1/2}q\mathbf{u}_0\cdot\mathbf{v}\,d\mathbf{x} \\ &\quad + \int q\tilde{r}_1\,d\mathbf{x}. \qquad (3.24) \\ &\le \frac{\varepsilon^2}{4}(\|\nabla n\|^2+\|\nabla q\|^2) + \frac{1}{4}(2+\tau^2)\int \mathcal{C}^{1/2}q^2\,d\mathbf{x} \\ &\quad + 4\sup_{\mathbf{x}\in\mathbb{R}^3}|P'(\mathcal{C})|^2\mathcal{C}^{-1/2}\|n\|^2 + \tau^2\sup_{\mathbf{x}\in\mathbb{R}^3}\mathcal{C}^{1/2}|\mathbf{u}_0|^2\|\mathbf{v}\| + c\|\tilde{r}_1\|^2 \\ &\le c_*(\rho_\pm,\varepsilon)\delta_0^2 + c\eta_0^3 + c(\eta_0+\delta_0)(\|n\|^2+\|q\|^2) + \frac{1}{4}(2+\tau^2)\int \mathcal{C}^{1/2}q^2\,d\mathbf{x},\end{aligned}$$

from which we conclude that

$$\int \mathcal{C}^{1/2}q^2\,d\mathbf{x} \le c_*(\rho_\pm,\varepsilon)\delta_0^2 + c\eta_0^3 + c(\eta_0+\delta_0)(\|n\|^2+\|q\|^2) \qquad (3.25)$$

for $\tau\in(0,\tau_*]$.

From the sum of (3.23) and (3.25) we obtain, for sufficiently small η_0 and δ_0,

$$\|\nabla n\|^2 + \|\nabla q\|^2 + \|n\|^2 + \|q\|^2 + \|\mathbf{v}\|^2 \le c_*(\rho_\pm,\varepsilon)\delta_0^2 + c\eta_0^3, \qquad (3.26)$$

and then, by (3.15),

$$\|D\mathbf{v}\|^2 \le c_*(\rho_\pm,\varepsilon)\delta_0^2 + c\eta_0^3.$$

Equations (3.10) and (3.11) allow to derive estimates for higher-order derivatives of n and q, for sufficiently small η_0 and δ_0. After some computations, we arrive to

$$\|n\|_2^2 + \|q\|_2^2 + \|\mathbf{v}\|_1^2 \le c_*(\rho_\pm,\varepsilon)\delta_0^2 + c\eta_0^3 =: \eta_1^2, \qquad (3.27)$$

where we remark that $c_*(\rho_\pm,\varepsilon)$ is independent of η_0, δ_0, and τ.

Now we can choose η_0 and δ_0. We take

$$\eta_0 = \delta_0\sqrt{2c_*(\rho_\pm,\varepsilon)},$$

where δ_0 is so small that $2c\delta_0\sqrt{2c_*(\rho_\pm,\varepsilon)} \le 1/2$ in order to guarantee that

$$\eta_1 \le \eta_0 = \delta_0\sqrt{2c_*(\rho_\pm,\varepsilon)}.$$

This shows that $(n,\mathbf{v},q)\in\mathcal{M}$.

Step 3: End of the proof. Let $\widetilde{U}_1 = (\tilde{n}_1,\tilde{\mathbf{v}}_1,\tilde{q}_1)$, $\widetilde{U}_2 = (\tilde{n}_2,\tilde{\mathbf{v}}_2,\tilde{q}_2)\in\mathcal{M}$ and set $U_1 = \mathcal{S}(\widetilde{U}_1)$, $U_2 = \mathcal{S}(\widetilde{U}_2)$. Then the difference $\bar{U} = U_1 - U_2 = (n_1-n_2,\mathbf{v}_1-\mathbf{v}_2,q_1-q_2)$ solves the problem

$$\mathcal{T}(\bar{U}) = \vec{r}(\widetilde{U}_1) - \vec{r}(\widetilde{U}_2).$$

A computation as in step 2 shows that $\bar{U}$ satisfies the estimate

$$\|U_1-U_2\|_{H^2\times H^1\times H^2} \le \kappa\|\widetilde{U}_1-\widetilde{U}_2\|_{H^2\times H^1\times H^2} \qquad (3.28)$$

with $\kappa = c\delta_0$ for some $c>0$. In particular, $\kappa\in(0,1)$ for sufficiently small δ_0. This shows that $\mathcal{S}$ is a contraction and, by Banach's fixed-point theorem, we conclude the existence and uniqueness of a solution of (3.1)–(3.4) or, equivalently, of (2.1), (2.4), (2.5), and (2.10).

4 Proof of Theorems 2.4 and 2.5

First we recall a local existence result proved in [18]. In this reference, the doping concentration is assumed to be sufficiently close to a constant. However, it can be seen that the result is still true for doping profiles satisfying the assumptions of Theorem 2.4.

Lemma 4.1 *Assume that $P(\rho) \in C^5(0,\infty)$ and $(D\psi_1, \hat{\mathbf{u}}_1^\tau) \in H^5(\mathbb{R}^3) \times \mathcal{H}^5(\mathbb{R}^3)$ such that $\inf_{x\in\mathbb{R}^3} \psi_1(\mathbf{x}) > 0$. Then, for fixed relaxation time $\tau > 0$, there exists $T_{**} > 0$ and a unique solution $(\psi_\tau, \mathbf{u}_\tau, E_\tau)$ of (2.23)–(2.26) with $\psi_\tau > 0$ in the time interval $[0, T_{**}]$ satisfying*

$$D\psi_\tau \in C^i([0,T_{**}]; H^{5-2i}(\mathbb{R}^3)) \quad i = 0,1, \quad \psi_\tau \in C^3([0,T_{**}]; L^2(\mathbb{R}^3)),$$
$$\mathbf{u}_\tau \in C^i([0,T_{**}]; \mathcal{H}^{5-2i}(\mathbb{R}^3)), \ i = 0,1,2, \quad E_\tau \in C^1([0,T_{**}]; \mathcal{H}^3(\mathbb{R}^3)).$$

We recall that $\mathcal{H}^5(\mathbb{R}^3) = \{f \in L^6(\mathbb{R}^3),\ Df \in H^4(\mathbb{R}^3)\}$.

4.1 Reformulation of the equations

In the following we omit the index τ to simplify the presentation and use the index t for the time derivative ∂_t. We reformulate the quantum hydrodynamic equations as a fourth-order wave equation which avoids the dispersive third-order term. For this, we differentiate (2.23) with respect to time, multiply the resulting equation by τ^2 and replace the term $\tau^2 \mathbf{u}_t$ by using (2.24). We end up with the following wave equation for ψ:

$$\begin{aligned}\tau^2\psi_{tt} + \psi_t + \frac{\varepsilon^2}{4}\Delta^2\psi + \frac{1}{2\psi}\nabla\cdot(\psi^2 E) - \frac{\tau^2}{2\psi}\nabla^2\cdot(\psi^2\mathbf{u}\otimes\mathbf{u}) \\ -\frac{1}{2\psi}\Delta P(\psi^2) + \tau^2\frac{\psi_t^2}{\psi} - \frac{\varepsilon^2}{4}\frac{|\Delta\psi|^2}{\psi} = 0\end{aligned} \tag{4.1}$$

with initial data

$$\psi(\mathbf{x},0) = \psi_1(\mathbf{x}), \quad \psi_t(\mathbf{x},0) = -\hat{\mathbf{u}}_1^\tau\cdot\nabla\psi_1 - \frac{1}{2}\psi_1\nabla\cdot\hat{\mathbf{u}}_1^\tau. \tag{4.2}$$

Employing the identity $(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla(|\mathbf{u}|^2) - \mathbf{u}\times(\nabla\times\mathbf{u})$, we can write (2.24) as

$$\tau^2\partial_t\mathbf{u} + \mathbf{u} + \tau^2\frac{1}{2}\nabla(|\mathbf{u}|^2) - \tau^2\mathbf{u}\times\varphi + \nabla h(\psi^2) = E + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\psi}{\psi}\right), \tag{4.3}$$

where $\varphi = \nabla\times\mathbf{u}$ denotes the vorticity vector of the velocity $\mathbf{u}$. Taking the curl of (4.3), we obtain an equation for φ:

$$\tau^2\partial_t\varphi + \varphi + \tau^2(\mathbf{u}\cdot\nabla)\varphi + \tau^2\varphi\nabla\cdot\mathbf{u} - \tau^2(\varphi\cdot\nabla)\mathbf{u} = 0. \tag{4.4}$$

We wish to establish uniform a priori estimates for $(\psi, \mathbf{u}, E)$ around the steady state $(\bar{\psi}_\tau, \bar{\mathbf{u}}_\tau, \bar{E}_\tau)$. For this, we introduce

$$w = \psi - \bar{\psi}_\tau, \quad \mathbf{z} = \mathbf{u} - \bar{\mathbf{u}}_\tau, \quad \theta = E - \bar{E}_\tau. \tag{4.5}$$

Then (2.25), (4.1), and (4.3) can be rewritten as a system of equations for the new variables $(w, \mathbf{u}, \theta)$:

$$\tau^2 \mathbf{z}_t + \tau^2([\bar{\mathbf{u}}_\tau + \mathbf{z}] \cdot \nabla)\mathbf{z} + \mathbf{z} = f_1, \tag{4.6}$$

$$\tau^2 w_{tt} + w_t + \frac{\varepsilon^2}{4}\Delta^2 w + \bar{\psi}_\tau^2 w + 2\tau^2[\bar{\mathbf{u}}_\tau + \mathbf{z}] \cdot \nabla w_t - \nabla\cdot\left((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla w\right) = f_2, \tag{4.7}$$

$$\nabla\cdot\theta = (2\bar{\psi}_\tau + w)w, \quad \nabla\times\theta = 0, \tag{4.8}$$

where

$$\begin{aligned} f_1(\mathbf{x},t) = & -\tau^2(\mathbf{z}\cdot\nabla)[\bar{\mathbf{u}}_\tau + \mathbf{z}] + \theta - \nabla(h((\bar{\psi}_\tau + w)^2) - h(\bar{\psi}_\tau^2)) \\ & + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta(w+\bar{\psi}_\tau)}{w+\bar{\psi}_\tau} - \frac{\Delta\bar{\psi}_\tau}{\bar{\psi}_\tau}\right), \\ f_2(\mathbf{x},t) = & -\frac{\tau^2 w_t^2}{w+\bar{\psi}_\tau} - \frac{1}{2(\bar{\psi}_\tau + w)}\nabla\cdot\big((\bar{\psi}_\tau + w)^2(\bar{E}_\tau + \theta)\big) + \frac{1}{2\bar{\psi}_\tau}\nabla\cdot\big(\bar{\psi}_\tau^2\bar{E}_\tau\big) \\ & + \frac{\tau^2}{2(\bar{\psi}_\tau + w)}\nabla^2\cdot\big([\bar{\psi}_\tau + w]^2[\bar{\mathbf{u}}_\tau + \mathbf{z}]\otimes[\bar{\mathbf{u}}_\tau + \mathbf{z}]\big) - \frac{\tau^2}{2\bar{\psi}_\tau}\nabla^2\cdot\big(\bar{\psi}_\tau^2\bar{\mathbf{u}}_\tau\otimes\bar{\mathbf{u}}_\tau\big) \\ & + 2\tau^2[\bar{\mathbf{u}}_\tau + \mathbf{z}]\cdot\nabla w_t + \bar{\psi}_\tau^2 w + \frac{\varepsilon^2}{4}\frac{|\Delta(\bar{\psi}_\tau + w)|^2}{(\bar{\psi}_\tau + w)} - \frac{\varepsilon^2}{4}\frac{|\Delta\bar{\psi}_\tau|^2}{\bar{\psi}_\tau} \\ & + \frac{1}{2(\bar{\psi}_\tau + w)}\Delta P((\bar{\psi}_\tau + w)^2) - \frac{1}{2\bar{\psi}_\tau}\Delta P(\bar{\psi}_\tau^2) \\ & - \nabla\cdot\big((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla w\big). \end{aligned}$$

The initial values are given by

$$w(\mathbf{x},0) = w_1(\mathbf{x}), \quad w_t(\mathbf{x},0) = w_2(\mathbf{x}), \quad \mathbf{z}(\mathbf{x},0) = \mathbf{z}_1(\mathbf{x}) := \hat{\mathbf{u}}_1^\tau - \bar{\mathbf{u}}_\tau,$$

where

$$w_1(\mathbf{x}) := \psi_1 - \bar{\psi}_\tau, \quad w_2(\mathbf{x}) := -\hat{\mathbf{u}}_1^\tau\cdot\nabla\psi_1 - \frac{1}{2}\psi_1\nabla\cdot\hat{\mathbf{u}}_1^\tau + \bar{\mathbf{u}}_\tau\cdot\nabla\bar{\psi}_\tau + \frac{1}{2}\bar{\psi}_\tau\nabla\cdot\bar{\mathbf{u}}_\tau. \tag{4.9}$$

For future reference we notice the following equation:

$$2w_t + 2[\bar{\mathbf{u}}_\tau + \mathbf{z}]\cdot\nabla w + 2\mathbf{z}\cdot\nabla\bar{\psi}_\tau + w\nabla\cdot\bar{\mathbf{u}}_\tau + (\bar{\psi}_\tau + w)\nabla\cdot\mathbf{z} = 0. \tag{4.10}$$

Since $\bar{\mathbf{u}}_\tau$ is irrotational by assumption, $\nabla\times\mathbf{z} = \nabla\times\mathbf{u} = \varphi$ solves

$$\tau^2\partial_t\varphi + \varphi + \tau^2([\bar{\mathbf{u}}_\tau + \mathbf{z}]\cdot\nabla)\varphi + \tau^2\varphi\nabla\cdot[\bar{\mathbf{u}}_\tau + \mathbf{z}] - \tau^2(\varphi\cdot\nabla)[\bar{\mathbf{u}}_\tau + \mathbf{z}] = 0, \tag{4.11}$$

$$\varphi(\mathbf{x},0) = \nabla\times\hat{\mathbf{u}}_1^\tau(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3. \tag{4.12}$$

4.2 A priori estimates

Let $T > 0$ and define the space

$$X(T) = \{(w, \mathbf{z}, \theta) \in L^\infty(0, T; H^6 \times \mathcal{H}^5 \times \mathcal{H}^3(\mathbb{R}^3))\}.$$

We assume that for a local-in-time solution of (4.6)–(4.8), which exists thanks to Lemma 4.1, the quantity

$$\delta_T =: \max_{0\le t\le T} \sum_{j=0}^{2} \|\tau^j \partial_t^j w(t)\|_{H^{6-2j}(\mathbb{R}^3)} + \max_{0\le t\le T} \sum_{j=0}^{1} \|\tau^{j+1} \partial_t^j \mathbf{z}(t)\|_{\mathcal{H}^{5-2i}(\mathbb{R}^3)} \tag{4.13}$$

is "small". It is not difficult to check that for sufficiently small δ_T, it holds

$$\frac{1}{2}\sqrt{\rho_-} \le w + \bar{\psi}_\tau \le \frac{3}{2}\sqrt{\rho_+} \tag{4.14}$$

and

$$\sum_{|\alpha|=0}^{4-2j} \sum_{j=0}^{2} \|\tau^j D^\alpha \partial_t^j w\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}_+)} + \sum_{|\alpha|=0}^{3-2j} \sum_{j=0}^{1} \|\tau^j D^\alpha \partial_t^{j+1} \mathbf{z}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}_+)} \tag{4.15}$$

is "small".

From Lemma 2.6 and equation (4.8) we obtain the following estimates for θ.

Lemma 4.2 *It holds*

$$|\theta| + \|\theta\|_{L^6} + \|D\theta\|_3 \le c\|w\|_3, \quad |\,\theta_t| + \|\theta_t\|_{L^6} + |D\theta_t|_2 \le c\|w_t\|_2, \tag{4.16}$$

provided that $\delta_T + \delta_0$ is sufficiently small.

The estimates (4.16) together with (4.13) gives

$$|\theta| + |\theta_t| + \|(\theta, \theta_t)\|_{L^6} + \|D(\theta, \theta_t)\|_2 \le c\delta_T. \tag{4.17}$$

Then we have the following main estimates.

Lemma 4.3 *It holds for $(w, \mathbf{u}, E) \in X(T)$,*

$$\begin{aligned} \|\theta(t)\|_{\mathcal{H}^3} + \sum_{j=0}^{2} \|\tau^j \partial_t^j w(t)\|_{6-2j} + \sum_{j=0}^{1} \|\tau^{j+1} \partial_t^j \mathbf{z}(t)\|_{\mathcal{H}^{5-2i}} &\le c\delta_1 e^{-a_0 t}, & (4.18)\\ \int_0^t \big(\|(w, w_t, \tau w_{tt})(s)\|_{H^6\times H^4\times H^2} + \|\mathbf{z}(s)\|_{\mathcal{H}^5} + \|(\theta, \tau\mathbf{z}_t)(s)\|_{\mathcal{H}^3}\big) ds &\le c\delta_1, & (4.19) \end{aligned}$$

provided that $\delta_T + \delta_0 + \tau_1$ is sufficiently small. Here, $a_0 > 0$ is a constant independent of $\tau > 0$ and δ_1 is given by (2.30)*.*

Proof: **Step 1: Basic estimates.** We multiply (4.7) by $w + 2w_t$ and integrate over $\mathbb{R}^3$. Then, using (2.12), (2.16), (4.10), (4.13), (4.14), Lemma 2.7, and Lemma 4.2, we conclude after a tedious but straightforward computation that

$$
\begin{aligned}
&\frac{d}{dt}\int\Big(\frac{1}{2}w^2+\tau^2 w w_t+\tau^2 w_t^2+\bar{\psi}_\tau^2 w^2+\frac{\varepsilon^2}{4}|\Delta w|^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla w|^2\Big)d\mathbf{x}\\
&\qquad+\frac{\varepsilon^2}{4}\|\Delta w\|^2+(2-\tau^2)\|w_t\|^2+\int\bar{\psi}_\tau^2 w^2 d\mathbf{x}+\int(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla w|^2 d\mathbf{x}\\
&\le\ c\delta_T\|(w,\tau w_t,\nabla w)\|^2+\int f_2(w+2w_t)d\mathbf{x} \qquad (4.20)\\
&\le\ c(\delta_T+\delta_0)\|(w,\tau w_t,\nabla w,\Delta w)\|^2+c(\delta_T+\delta_0)\|\nabla\times\mathbf{z}\|^2, \qquad (4.21)
\end{aligned}
$$

where we have expressed $\nabla\cdot\mathbf{z}$ by w_t and ∇w in terms of (4.10) and used the estimate

$$\|\nabla w\|\le c(\|w\|+\|\Delta w\|). \tag{4.22}$$

Moreover, by (2.16), one can verify that under the condition (2.13) there exist two constants A_0, $a_0>0$ independent of τ such that

$$
\begin{aligned}
&\int\Big(\bar{\psi}_\tau^2 w^2+\frac{\varepsilon^2}{4}|\Delta w|^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla w|^2\Big)d\mathbf{x}\\
&\ge(A_0-c\delta_0)(\|w\|^2+\|\Delta w\|^2)\ge a_0(\|w\|^2+\|\Delta w\|^2).
\end{aligned}
$$

provided that δ_0 is small enough.

In order to estimate the L^2 norm of $\nabla\times\mathbf{z}$, we multiply (4.11) by $\varphi=\nabla\times\mathbf{z}$ and integrate over $\mathbb{R}^3$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\tau\nabla\times\mathbf{z}\|^2+(1-c\delta_T)\|\nabla\times\mathbf{z}\|^2\le c\delta_T\|(\tau w_t,w,\nabla w,\Delta w)\|^2. \tag{4.23}$$

Now, taking the sum of (4.21) and (4.23) and using $\tau<1$, we arrive to the differential inequality

$$\frac{d}{dt}A(t)+(1-c(\delta_T+\delta_0))B(t)\le 0, \tag{4.24}$$

where

$$
\begin{aligned}
A(t)=&\|\tau\nabla\times\mathbf{z}(t)\|^2+\frac{1}{2}\|w(t)\|^2+\tau^2\|w_t(t)\|^2+\frac{\varepsilon^2}{4}\|\Delta w(t)\|^2\\
&+\int\big(\tau^2 w w_t+\bar{\psi}_\tau^2 w^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla w|^2\big)(x,t)d\mathbf{x},\\
B(t)=&a_0\|\Delta w(t)\|^2+a_0\|w(t)\|^2+\|w_t(t)\|^2+\|\nabla\times\mathbf{z}(t)\|^2.
\end{aligned}
$$

Direct integration of the above differential inequality and (4.22) lead to

$$\int_0^t\|(w,w_t,\nabla w,\Delta w,\nabla\times\mathbf{z})(s)\|^2ds\le cI_1, \tag{4.25}$$

provided that δ_T and δ_0 are so small that $1 - c(\delta_T + \delta_0) > 0$, and I_1 is given by

$$I_1 = \|w_1\|_2^2 + \|\tau w_2\|^2 + \|\tau D\mathbf{z}_1\|^2 \leq c(\|\psi_1 - \bar{\psi}_\tau\|_2^2 + \|\tau D(\hat{\mathbf{u}}_1^\tau - \bar{\mathbf{u}}_\tau)\|^2) \leq c\delta_1, \tag{4.26}$$

where δ_1 is defined by (2.30). On the other hand, it holds

$$\|(w, \tau w_t, \nabla w, \Delta w, \tau\nabla\times\mathbf{z})(t)\|^2 \leq cA(t) \leq cB(t).$$

Replacing $B(t)$ in (4.24) by $cA(t)$ and applying Gronwall's lemma to the resulting inequality, we obtain the following inequality

$$\|(w, \tau w_t, \nabla w, \Delta w, \tau\nabla\times\mathbf{z})(t)\|^2 \leq cI_1 e^{-a_1 t} \tag{4.27}$$

for sufficiently small δ_T and δ_0, and $a_1 > 0$ is some constant.

Now we employ the inequalities (4.27), (4.25), (4.26), and the equation (4.10) to compute

$$\|w(t)\|_2^2 + \|\tau w_t(t)\|^2 + \|\tau D\mathbf{z}(t)\|^2 \leq c(\|w_1\|_2^2 + \|\tau D\mathbf{z}_1\|^2)e^{-a_1 t} \tag{4.28}$$

$$\int_0^t (\|w(s)\|_2^2 + \|w_t(s)\|^2 + \|D\mathbf{z}(s)\|^2) ds \leq c(\|w_1\|_2^2 + \|D\mathbf{z}_1\|^2) \tag{4.29}$$

where $a_1 > 0$ is some constant, provided that δ_T and $\delta_0 + \tau_1$ are sufficiently small.

Step 2: Higher-order estimates. To obtain higher order estimates for w and $\mathbf{z}$ we differentiate (4.7) with respect to x and t. The functions $\tilde{w} = D^\alpha w$ $(1 \leq |\alpha| \leq 2)$ and $\bar{w} = D^\gamma w_t$ $(0 \leq |\gamma| \leq 2)$ satisfy the equations

$$\tau^2 \tilde{w}_{tt} + \tilde{w}_t + \frac{1}{4}\varepsilon^2 \Delta^2 \tilde{w} + \bar{\psi}_\tau^2 \tilde{w} + 2\tau^2 [\bar{\mathbf{u}}_\tau + \mathbf{z}] \cdot \nabla \tilde{w}_t - \nabla\cdot ((P'(\bar{\psi}_\tau^2) - \tau^2 |\bar{\mathbf{u}}_\tau|^2)\nabla \tilde{w}) = f_3 + D^\alpha f_2, \tag{4.30}$$

where

$$\begin{aligned} f_3(\mathbf{x}, t) := &- (D^\alpha(\bar{\psi}_\tau^2 w) - \bar{\psi}_\tau^2 D^\alpha w) - 2\tau^2 (D^\alpha(\mathbf{u}\cdot\nabla w_t) - \mathbf{u}\cdot\nabla\tilde{w}_t) \\ &+ D^\alpha(\nabla\cdot ((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla w)) - \nabla\cdot ((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla\tilde{w}), \end{aligned}$$

and

$$\begin{aligned} \tau^2 \bar{w}_{tt} + \bar{w}_t + \frac{1}{4}\varepsilon^2\Delta^2\bar{w} + \bar{\psi}_\tau^2\bar{w} + 2\tau^2\mathbf{u}\cdot\nabla\bar{w}_t - \nabla\cdot ((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla\bar{w}) \\ = f_4 := -\tau^2 2\mathbf{u}_t\cdot\nabla\tilde{w}_t + D^\alpha(f_2)_t + (f_3)_t, \end{aligned} \tag{4.31}$$

respectively.

Set $\hat{\varphi} = D^\beta(\nabla\times\mathbf{z})$ $(1 \leq |\beta| \leq 4)$ and $\check{\varphi} = D^\gamma(\nabla\times\mathbf{z}_t)$ $(0 \leq |\gamma| \leq 2)$. Applying the differential operators D^β and $D^\beta\partial_t$ to (4.11), respectively, we find

$$\tau^2\partial_t\hat{\varphi} + \hat{\varphi} + \tau^2([\bar{\mathbf{u}}_\tau + \mathbf{z}]\cdot\nabla)\hat{\varphi} + \tau^2\hat{\varphi}\nabla\cdot [\bar{\mathbf{u}}_\tau + \mathbf{z}] - \tau^2(\hat{\varphi}\cdot\nabla)[\bar{\mathbf{u}}_\tau + \mathbf{z}] = f_5, \tag{4.32}$$

$$\tau^2\partial_t\check{\varphi} + \check{\varphi} + \tau^2([\bar{\mathbf{u}}_\tau + \mathbf{z}]\cdot\nabla)\check{\varphi} + \tau^2\check{\varphi}\nabla\cdot [\bar{\mathbf{u}}_\tau + \mathbf{z}] - \tau^2(\check{\varphi}\cdot\nabla)[\bar{\mathbf{u}}_\tau + \mathbf{z}] = f_6, \tag{4.33}$$

where

$$\begin{aligned} f_5(\mathbf{x},t) :=& -\tau^2 D^\beta(([\bar{\mathbf{u}}_\tau+\mathbf{z}]\cdot\nabla)\varphi)+\tau^2([\bar{\mathbf{u}}_\tau+\mathbf{z}]\cdot\nabla)\hat{\varphi}-\tau^2 D^\beta(\varphi\nabla\cdot[\bar{\mathbf{u}}_\tau+\mathbf{z}]) \\ &+\tau^2\hat{\varphi}\nabla\cdot[\bar{\mathbf{u}}_\tau+\mathbf{z}]+\tau^2 D^\beta((\varphi\cdot\nabla)[\bar{\mathbf{u}}_\tau+\mathbf{z}])-\tau^2(\hat{\varphi}\cdot\nabla)[\bar{\mathbf{u}}_\tau+\mathbf{z}], \\ f_6(\mathbf{x},t) :=& (f_5)_t-\tau^2(\mathbf{z}_t\cdot\nabla)D^\gamma\varphi-\tau^2 D^\gamma\varphi\nabla\cdot\mathbf{z}_t+\tau^2(D^\gamma\varphi\cdot\nabla)\mathbf{z}_t. \end{aligned}$$

Then, multiplying (4.30) by $\tilde{w}+2\tilde{w}_t$, (4.31) by $\tau^2(\bar{w}+2\bar{w}_t)$, (4.32) by $\hat{\varphi}$, and (4.33) by $\tau^2\check{\varphi}$ and integrating over $\mathbb{R}^3$, we infer the following four estimates:

$$\begin{aligned} &\frac{d}{dt}\int\left(\frac{1}{2}\tilde{w}^2+\tau^2\tilde{w}\tilde{w}_t+\tau^2\tilde{w}_t^2+\bar{\psi}_\tau^2\tilde{w}^2+\frac{\varepsilon^2}{4}|\Delta\tilde{w}|^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla\tilde{w}|^2\right)d\mathbf{x} \\ &\qquad+\frac{\varepsilon^2}{4}\|\Delta\tilde{w}\|^2+(2-\tau^2)\|\tilde{w}_t\|^2+\int\bar{\psi}_\tau^2\tilde{w}^2 d\mathbf{x}+\int(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla\tilde{w}|^2 d\mathbf{x} \\ &\leq\ c\delta_T\|(\tilde{w},\tau\tilde{w}_t,\nabla\tilde{w})\|^2+\int(f_3+D^\alpha f_2)(\tilde{w}+2\tilde{w}_t)d\mathbf{x}, \end{aligned} \tag{4.34}$$

where $1\leq|\alpha|\leq 2$,

$$\begin{aligned} &\tau^2\frac{d}{dt}\int\left(\frac{1}{2}\bar{w}^2+\tau^2\bar{w}\bar{w}_t+\tau^2\bar{w}_t^2+\bar{\psi}_\tau^2\bar{w}^2+\frac{\varepsilon^2}{4}|\Delta\bar{w}|^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla\bar{w}|^2\right)d\mathbf{x} \\ &\qquad+\tau^2\int\left(\frac{\varepsilon^2}{4}|\Delta\bar{w}|^2+(2-\tau^2)\bar{w}_t^2+\bar{\psi}_\tau^2\bar{w}^2+(P'(\bar{\psi}_\tau^2)-\tau^2|\bar{\mathbf{u}}_\tau|^2)|\nabla\bar{w}|^2\right)d\mathbf{x} \\ &\leq\ c\delta_T\|\tau(\bar{w},\tau\bar{w}_t,\nabla\bar{w})\|^2+\tau^2\int(\bar{w}+2\bar{w}_t)f_4 d\mathbf{x}, \end{aligned} \tag{4.35}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\tau\hat{\varphi}\|^2+(1-c\delta_T)\|\tau\hat{\varphi}\|^2\ \leq\ c\delta_T\|(\tau w_t,w,\nabla w,\Delta w)\|^2+c\int\hat{\varphi}f_5 d\mathbf{x}, \tag{4.36}$$

$$\frac{1}{2}\frac{d}{dt}\|\tau^2\check{\varphi}\|^2+(1-c\delta_T)\|\tau\check{\varphi}\|^2\ \leq\ c\delta_T\|(\tau w_t,w,\nabla w,\Delta w)\|^2+c\tau^2\int\check{\varphi}f_6 d\mathbf{x}. \tag{4.37}$$

The last terms on the right-hand sides of (4.34)–(4.37) can be estimated, respectively, by using (2.12), (2.16), (4.13)–(4.16), Lemma 2.7, Lemma 4.2, and Cauchy's inequality. This gives

$$\begin{aligned} \int(f_3+D^\alpha f_2)(\tilde{w}+2\tilde{w}_t)d\mathbf{x}\ \leq&\ c(\delta_T+\delta_0)\|(\tau w_t,w,\nabla w,\Delta w)\|_2^2 \\ &+c(\delta_T+\delta_0)\|\tau\nabla\times\mathbf{z}\|_2^2+c(\delta_T+\delta_0)\|(\tilde{w},\tau\tilde{w}_t)\|^2, \end{aligned} \tag{4.38}$$

$$\begin{aligned} \tau^2\int(\bar{w}+2\bar{w}_t)f_4 d\mathbf{x}\ =&\ \int\tau^2(\bar{w}+2\bar{w}_t)(\tau^2 2\mathbf{u}_t\cdot\nabla\tilde{w}_t-D^\alpha(f_2)_t-(f_3)_t)d\mathbf{x} \\ \leq&\ c(\delta_T+\delta_0)\|\tau(w,\nabla w,\Delta w,\tau w_t,\nabla w_t,\tau\Delta w_t,\tau w_{tt})\|_2^2 \\ &+c(\delta_T+\delta_0)\|\tau(\bar{w},\bar{w}_t)\|^2 \\ &+c(\delta_T+\delta_0)\|\tau(\tau\nabla\times\mathbf{z},\tau\nabla\times\mathbf{z}_t)\|_2^2, \end{aligned} \tag{4.39}$$

$$\begin{aligned} \int\hat{\varphi}f_5 d\mathbf{x}\ \leq&\ c(\delta_T+\delta_0)(\|\tau\hat{\varphi}\|^2+\|\tau\nabla\times\mathbf{z}\|_{|\beta|-1}^2 \\ &+\|(\tau w_t,\nabla w)\|_{|\beta|-1}^2), \end{aligned} \tag{4.40}$$

where $1 \leq |\beta| \leq 4$, and

$$\int \tau^2 \check{\varphi} f_6 d\mathbf{x} \leq c(\delta_T + \delta_0)(\|\tau^2 \hat{\varphi}\|^2 + \|\tau^2 \nabla\times\mathbf{z}_t\|_1^2 + \|(\tau w_t, \nabla w, \tau^2 w_{tt}, \tau\nabla w_t)\|_1^2). \quad (4.41)$$

Here, we have estimated $D^\alpha \mathbf{z}$ and $D^\alpha \mathbf{z}_t$ by

$$\|\tau D\mathbf{z}\|_m^2 \leq c\|\tau(\nabla\times\mathbf{z}, \nabla\cdot\mathbf{z})\|_m^2, \quad \|\tau^2 D\mathbf{z}_t\|_m^2 \leq c\|\tau^2(\nabla\times\mathbf{z}_t, \nabla\cdot\mathbf{z}_t)\|_m^2, \quad m \geq 1, \quad (4.42)$$

and replaced $\nabla\cdot\mathbf{z}_t$ and $\nabla\cdot\mathbf{z}$ through (4.6) and (4.10). Taking the sum of the differential inequalities (4.34), (4.37) (with $0 \leq |\alpha| \leq 2$ and $1 \leq |\beta| \leq 2$) and (4.36)–(4.37) (with $0 \leq |\gamma| \leq 2$), and using (4.28)–(4.29), (4.38)–(4.40), we are able to obtain, by arguing similar as in the proof of (4.28)–(4.29),

$$\begin{aligned}
\|w(t)\|_4^2 + \|\tau w_t(t)\|_2^2 + \|\tau\nabla\times\mathbf{z}(t)\|_2^2 &\leq cI_2 e^{-a_2 t}, && (4.43)\\
\int_0^t (\|w(s)\|_4^2 + \|w_t(s)\|_2^2 + \|\nabla\times\mathbf{z}(s)\|_2^2) ds &\leq cI_2, && (4.44)\\
\|\tau w_t(t)\|_4^2 + \|\tau^2 w_{tt}(t)\|_2^2 + \|\tau^2\nabla\times\mathbf{z}_t(t)\|_2^2 &\leq cI_3 e^{-a_2 t}, && (4.45)\\
\int_0^t (\|\tau w_t(s)\|_4^2 + \|\tau w_{tt}(s)\|_2^2 + \|\tau\nabla\times\mathbf{z}_t(s)\|_2^2) ds &\leq cI_3, && (4.46)
\end{aligned}$$

for $(w, \mathbf{z}, \theta) \in X(T)$, provided that δ_T and $\delta_0 + \tau_1$ are small enough. Here, $a_2 > 0$ is some constant, I_2 and I_3 are defined by

$$\begin{aligned}
I_2 =& \|w_1\|_4^2 + \|\tau w_2\|_2^2 + \|\tau D\mathbf{z}_1\|_2^2 \leq c(\|\psi_1 - \bar{\psi}_\tau\|_4^2 + \|\tau D(\hat{\mathbf{u}}_1^\tau - \bar{\mathbf{u}}_\tau)\|_3^2) \leq c\delta_1,\\
I_3 =& \|\tau w_2\|_4^2 + \|\tau^2 w_3\|_2^2 + \|\tau^2 D\mathbf{z}_2\|_2^2 \leq c(\|\psi_1 - \bar{\psi}_\tau\|_6^2 + \|\tau D(\hat{\mathbf{u}}_1^\tau - \bar{\mathbf{u}}_\tau)\|_4^2) \leq c\delta_1,
\end{aligned}$$

and we recall that δ_1 is defined by (2.30). Furthermore, $w_3(\mathbf{x}) = w_{tt}(\mathbf{x}, 0)$ and $\mathbf{z}_2(\mathbf{x}) = \mathbf{z}_t(\mathbf{x}, 0)$ are obtained from (4.7) and (4.6) with $t = 0$, i.e.,

$$\begin{aligned}
\tau^2\mathbf{z}_2 =& -\tau^2(\mathbf{u}_1\cdot\nabla)\mathbf{z}_1 - \mathbf{z}_1 + f_1(\mathbf{x}, 0),\\
\tau^2 w_3 =& -w_2 - \frac{1}{4}\varepsilon^2\Delta^2 w_1 - \bar{\psi}_\tau^2 w_1 - 2\tau^2\mathbf{u}_1\cdot\nabla w_1\\
&+ \nabla\cdot((P'(\bar{\psi}_\tau^2) - \tau^2|\bar{\mathbf{u}}_\tau|^2)\nabla w_1) + f_2(\mathbf{x}, 0).
\end{aligned}$$

Finally, since we can estimate $D^5 w$ and $D^6 w$ from (4.7) and $D^\alpha\mathbf{z}$ with $3 \leq |\alpha| \leq 4$ from (4.32), we obtain the assertion (4.18) and (4.19) in view of Lemma 4.2, (4.28), (4.29), (4.43), (4.44). The proof of Lemma 4.3 is complete. □

4.3 End of the proof

Proof of Theorem 2.4. Lemma 4.3 shows that any local-in-time solution satisfying (4.13) is bounded and thus can be extended to a global-in-time solution. Moreover, the uniform bounds (4.18)–(4.19) of Lemma 4.3 imply the estimate (2.31).

Proof of Theorem 2.5. Let $(\psi_\tau, \mathbf{u}_\tau, E_\tau)(\mathbf{x}, t)$ be a solution of (2.23)–(2.26). Then, by (4.5), (4.43)–(4.46), and (2.35), the following uniform estimates holds:

$$\begin{aligned}
\|(\psi_\tau - \bar{\psi}_\tau)(t)\|_4^2 + \|\tau\partial_t\psi_\tau(t)\|_2^2 + \|\tau D\partial_t E_\tau(t)\|_2^2 & \\
+\|D(E_\tau - \bar{E}_\tau)(t)\|_4^2 + \|\tau D(\mathbf{u}_\tau - \bar{\mathbf{u}}_\tau)(t)\|_2^2 &\leq cI_2 e^{-a_2 t}, \qquad (4.47)\\
\int_0^t (\|(\psi_\tau - \bar{\psi}_\tau)(s)\|_4^2 + \|\partial_t\psi_\tau(s)\|_2^2 + \|D\partial_t E_\tau(s)\|_2^2 & \\
+\|D(E_\tau - \bar{E}_\tau)(s)\|_2^2 + \|D(\mathbf{u}_\tau - \bar{\mathbf{u}}_\tau)(s)\|_2^2)ds &\leq cI_2, \qquad (4.48)\\
\|\tau\partial_t\psi_\tau(t)\|_4^2 + \|\tau^2\partial_t^2\psi_\tau(t)\|_2^2 + \|\tau D\partial_t E_\tau(t)\|_2^2 & \\
+\|\tau^2\partial_t^2 E_\tau(t)\|_2^2 + \|\tau^2 D\partial_t\mathbf{u}_\tau(t)\|_2^2 &\leq cI_3 e^{-a_2 t}, \qquad (4.49)\\
\int_0^t (\|\tau\partial_t\psi_\tau(s)\|_4^2 + \|\tau\partial_t^2\psi_\tau(s)\|_2^2 + \|\tau D\partial_t E_\tau(s)\|_2^2 & \\
+\|\tau D\partial_t^2 E_\tau(s)\|_2^2 + \|\tau D\partial_t\mathbf{u}_\tau(s)\|_2^2)ds &\leq cI_3, \qquad (4.50)
\end{aligned}$$

for $t \geq 0$, where I_2 and I_3 can be bounded independently of τ. These uniform estimates and Aubin's lemma [34] imply the existence of subsequences (not relabeled) such that

$$\begin{aligned}
\psi_\tau \to \hat{\psi} \quad &\text{in} \quad L^2(0,T; C_b^2 \cap H_{\mathrm{loc}}^{4-s_0}(\mathbb{R}^3)),\\
E_\tau \to \hat{E} \quad &\text{in} \quad L^2(0,T; C_b^2 \cap \mathcal{H}_{\mathrm{loc}}^{3-s_0}(\mathbb{R}^3)),\\
\mathbf{u}_\tau \rightharpoonup \hat{\mathbf{u}} \quad &\text{in} \quad L^2(0,T; \mathcal{H}^3(\mathbb{R}^3)) \quad \text{as } \tau \to 0,
\end{aligned}$$

for any $T > 0$ and $s_0 \in (0, 1/2)$. From (4.47) we know that there is a positive constant c independent of $\tau > 0$ such that $\psi_\tau \geq c > 0$ in $(0,T) \times \mathbb{R}^3$ which implies that $\hat{\psi} \geq c > 0$. From (4.47)–(4.48) follows that

$$\tau^2|\mathbf{u}_\tau|^2 \to 0 \quad \text{in } L^1(0,T; W_{\mathrm{loc}}^{2,3}(\mathbb{R}^3)). \qquad (4.51)$$

Hence, the above convergence results allow to the pass to the limit $\tau \to 0$ in the quantum hydrodynamic equations and we obtain

$$\hat{\psi}\hat{\psi}_t + \nabla\cdot(\hat{\psi}^2\hat{\mathbf{u}}) = 0, \quad \hat{\mathbf{u}} = -\nabla h(\hat{\psi}^2) + \hat{E} + \frac{\varepsilon^2}{2}\nabla\left(\frac{\Delta\hat{\psi}}{\hat{\psi}}\right),$$
$$\nabla\cdot\hat{E} = \hat{\psi}^2 - \mathcal{C}, \quad \nabla\times E_\tau = 0,$$

which implies that $(\hat{\psi}, \hat{E})$ is a global weak solution of (2.27)–(2.29) in the sense of distributions. The proof of Theorem 2.5 is complete.

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