COMPACT FAMILIES OF PIECEWISE CONSTANT FUNCTIONS IN $L^p(0,T;B)$

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Abstract. A strong compactness result in the spirit of the Lions-Aubin-Simon lemma is proven for piecewise constant functions in time (u_τ) with values in a Banach space. The main feature of our result is that it is sufficient to verify one uniform estimate for the time shifts $u_\tau - u_\tau(\cdot - \tau)$ instead of all time shifts $u_\tau - u_\tau(\cdot - h)$ for $h > 0$, as required in Simon's compactness theorem. This simplifies significantly the application of the Rothe method in the existence analysis of parabolic problems.

1. Introduction

A useful technique to prove the existence of weak solutions to nonlinear evolution equations and their systems is to semi-discretize the equations in time by the implicit Euler method (also called Rothe method [5]):

$$\frac{1}{\tau}\big(u_\tau(t) - u_\tau(t-\tau)\big) + A(u_\tau(t)) = f_\tau(t), \quad \tau \le t < T, \quad u_\tau(0) \text{ given}, \tag{1}$$

where $\tau > 0$ is the time step, A is an abstract (nonlinear) operator defined on a certain Banach space, and f_τ is some (piecewise constant) function with values in a Banach space. In this way, nonlinear elliptic problems are obtained which are sometimes easier to solve. In order to pass to the limit of vanishing time steps, $\tau \to 0$, (relative) compactness for the sequence of piecewise constant approximate solutions (u_τ) is needed. Since the problem is nonlinear, we need strong convergence of (a subsequence of) (u_τ) to identify the limit. If the governing operator is monotone, the limit can be identified using Minty's trick (see, e.g., [6, Lemma 2.13]). Having suitable *a priori* estimates at hand, strong compactness can be concluded from the Aubin (or Lions-Aubin-Simon) lemma [7] which is a consequence of a compactness criterium due to Kolmogorov. However, the results of [7] are not directly applicable. Indeed, typically one can derive the uniform estimate

$$\|u_\tau - u_\tau(\cdot - \tau)\|_{L^1(\tau,T;Y)} = \tau\, \|{-A}(u_\tau) + f_\tau\|_{L^1(\tau,T;Y)} \le C\tau, \tag{2}$$

Date: November 8, 2011.

2000 *Mathematics Subject Classification.* 46B50, 35A35.

Key words and phrases. Compactness, Aubin lemma, Rothe method.

The first author is supported by DFG (446 CHV 113/170/0-2) and the Center of Evolution Equations of the University of Konstanz. The second author acknowledges partial support from the Austrian Science Fund (FWF), grants P20214, P22108, and I395; the Austrian-Croatian Project HR 01/2010; the Austrian-French Project FR 07/2010; and the Austrian-Spanish Project ES 08/2010 of the Austrian Exchange Service (ÖAD). Both authors thank Etienne Emmrich for useful discussions and the hint on the paper [3] and the anonymous referees for their comments which helped to improve this paper.

where $C > 0$ does not depend on τ, and Y is some Banach space. On the other hand, in order to apply the Aubin lemma, one needs [7, Theorem 3]

$$\|u_\tau - u_\tau(\cdot - h)\|_{L^1(h,T;Y)} \to 0 \quad \text{as } \ h \to 0, \ \text{ uniformly in } \tau > 0. \tag{3}$$

A possible way to avoid this problem is to construct linear interpolants of u_τ, say $\widetilde{u}_\tau$, for which a continuous time-derivative version of the Aubin lemma can be applied, giving $\widetilde{u} \to u$ in $L^1(0,T;B)$ as $\tau \to 0$ for some Banach space B [7, Corollary 4]. Since we need strong convergence of (u_τ), one has to show that $u_\tau - \widetilde{u}_\tau \to 0$ in $L^1(0,T;B)$, which might be difficult to prove (see Section 4 for a situation in which such a proof is possible).

In this note, we show that estimate (2) suffices to infer strong compactness of (u_τ). The main feature of our result is that it is sufficient to study the time shifts $u_\tau - u_\tau(\cdot - \tau)$ instead of all time shifts $u_\tau - u_\tau(\cdot - h)$ for all $h > 0$. This simplifies the proof of the limit $\tau \to 0$ in (1) significantly.

For our main results, let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, and set $t_k = k\tau$, $k = 0, \ldots, N$. Furthermore, let $(S_h u)(x,t) = u(x, t-h)$, $t \ge h > 0$, be the shift operator. We notice that quasi-uniform time steps may be considered too [3], but they are of minor interest in the existence analysis.

Theorem 1. *Let X, B, and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Furthermore, let either $1 \le p < \infty$, $r = 1$ or $p = \infty$, $r > 1$, and let (u_τ) be a sequence of functions, which are constant on each subinterval (t_{k-1}, t_k), satisfying*

$$\tau^{-1} \|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} + \|u_\tau\|_{L^p(0,T;X)} \le C_0 \quad \text{for all } \tau > 0, \tag{4}$$

where $C_0 > 0$ is a constant which is independent of τ. If $p < \infty$, then (u_τ) is relatively compact in $L^p(0,T;B)$. If $p = \infty$, there exists a subsequence of (u_τ) which converges in each space $L^q(0,T;B)$, $1 \le q < \infty$, to a limit which belongs to $C^0([0,T];B)$.

A related result in finite-dimensional spaces was recently proven by Gallouët and Latché [4, Theorem 3.4]. The same setting for degenerate elliptic-parabolic equations in L^1 was considered by Andreianov [2]. In view of (3), one may conjecture that the condition $\|u_\tau - S_\tau u_\tau\|_{L^r(\tau,T;Y)} = \mathcal{O}(\tau^\alpha)$ as $\tau \to 0$ with $0 < \alpha < 1$ instead of $\mathcal{O}(\tau)$ is sufficient to obtain relative compactness. The following result shows that this is not the case (also see Theorem 5 below).

Proposition 2. *The factor τ^{-1} in inequality (4) cannot be replaced by $\tau^{-\alpha}$ for $0 < \alpha < 1$.*

This note is organized as follows. In Section 2, Theorem 1 is shown; the proof of Proposition 2 is presented in Section 3. Finally, we comment these results in Section 4.

2. Proof of Theorem 1

The proof of Theorem 1 is based on a characterisation of the norm of fractional Sobolev spaces. Let $1 \le q < \infty$, $0 < \sigma < 1$, and let Y be a Banach space. The fractional Sobolev space $W^{\sigma,q}(0,T;Y)$ is the space of (equivalence classes of) functions $u \in L^q(0,T;Y)$ with finite Slobodeckii norm

$$\|u\|_{W^{\sigma,q}(0,T;Y)} = \Big(\|u\|^q_{L^q(0,T;Y)} + |u|^q_{\dot{W}^{\sigma,q}(0,T;Y)}\Big)^{1/q},$$

where

$$|u|_{\dot{W}^{\sigma,q}(0,T;Y)} = \bigg(\int_0^T\int_0^T \frac{\|u(t)-u(s)\|^q_Y}{|t-s|^{1+\sigma q}}\,\mathrm{d}s\,\mathrm{d}t\bigg)^{1/q}$$

is the Slobodeckii semi-norm. Fractional Sobolev spaces in time have also been proven to be a useful tool in [3].

Lemma 3. *Let $1 \le q < \infty$, $0 < \sigma < 1$ with $\sigma q < 1$ and let $u \in L^q(0,T;Y)$ be a piecewise constant function with (a finite number of) jumps of height $[u]_k \in Y$ at points t_k, $k = 1,\dots,N-1$. Then $u \in W^{\sigma,q}(0,T;Y)$ and*

$$\|u\|_{W^{\sigma,q}(0,T;Y)} \le \|u\|_{L^q(0,T;Y)} + C^{1/q}_{\sigma q,T}\sum_{k=1}^{N-1}\|[u]_k\|_Y\,,$$

where $C_{\sigma q,T} = 2(2^{\sigma q}-1)T^{1-\sigma q}/(\sigma q(1-\sigma q))$ does not depend on N.

Proof. We may assume that $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and that $u(t) = u_k$ for $t_{k-1} < t \le t_k$ where $k = 1,\dots,N$. Then $[u]_k = u_{k+1} - u_k$, $k = 1,\dots,N-1$, and

$$u(t) = u_k = u_1 + \sum_{j=1}^{k-1}(u_{j+1}-u_j) = u_1 + \sum_{j=1}^{N-1}[u]_j H_{t_j}(t)$$

for $t_{k-1} < t \le t_k$, where H_{t_j} is the shifted Heaviside function

$$H_{t_j}(t) = \begin{cases} 0 & \text{for } \ 0 < t \le t_j, \\ 1 & \text{for } \ t_j < t < T. \end{cases}$$

By definition of the $W^{\sigma,q}(0,T;Y)$ norm and the semi-norm property of $|\cdot|_{\dot{W}^{\sigma,q}(0,T;Y)}$, we find that

$$\begin{aligned}
\|u\|_{W^{\sigma,q}(0,T;Y)} &= \Big(\|u\|^q_{L^q(0,T;Y)} + |u|^q_{\dot{W}^{\sigma,q}(0,T;Y)}\Big)^{1/q} \\
&\le \|u\|_{L^q(0,T;Y)} + |u|_{\dot{W}^{\sigma,q}(0,T;Y)} \\
&\le \|u\|_{L^q(0,T;Y)} + |u_1|_{\dot{W}^{\sigma,q}(0,T;Y)} + \sum_{j=1}^{N-1}\|[u]_j\|_Y\ |H_{t_j}|_{\dot{W}^{\sigma,q}(0,T)} \\
&= \|u\|_{L^q(0,T;Y)} + \sum_{j=1}^{N-1}\|[u]_j\|_Y\ |H_{t_j}|_{\dot{W}^{\sigma,q}(0,T)}. \qquad (5)
\end{aligned}$$

It remains to compute the seminorm of H_{t_j}:

$$|H_{t_j}|^q_{\dot W^{\sigma,q}} = \int_0^T\int_0^T \frac{|H_{t_j}(t)-H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}}\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^{t_j}\int_{t_j}^T \frac{|H_{t_j}(t)-H_{t_j}(s)|^q}{|t-s|^{1+\sigma q}}\,\mathrm{d}s\,\mathrm{d}t$$
$$= 2\int_0^{t_j}\int_{t_j}^T |t-s|^{-1-\sigma q}\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{\sigma q(1-\sigma q)}\big((T-t_j)^{1-\sigma q}+t_j^{1-\sigma q}-T^{1-\sigma q}\big).$$

The right-hand side can be interpreted as a function of $\vartheta = t_j\in[0,T]$ whose maximum is achieved at $\vartheta = T/2$. Therefore,

$$|H_{t_j}|^q_{\dot W^{\sigma,q}} \le \frac{2}{\sigma q(1-\sigma q)}\left(2\Big(\frac{T}{2}\Big)^{1-\sigma q}-T^{1-\sigma q}\right) = \frac{2}{\sigma q(1-\sigma q)}(2^{\sigma q}-1)T^{1-\sigma q} = C_{\sigma q,T}.$$

Inserting this estimate in (5), the result follows. □

For later use, we remark that the calculations in (5) and below show that

$$|u|_{\dot W^{\sigma,1}(0,T;Y)} \le C^{1/q}_{\sigma q,T}\sum_{k=1}^{N-1}\|[u]_k\|_Y\,. \tag{6}$$

Proof of Theorem 1. The idea of the proof is to apply Corollary 5 in [7]: If (u_τ) is bounded in $L^p(0,T;X)\cap W^{\sigma,\varrho}(0,T;Y)$, where $\sigma>\max\{0,1/\varrho-1/p\}$, then (u_τ) is relatively compact in $L^p(0,T;B)$ if $p<\infty$, $\varrho=1$ and in $C^0([0,T];B)$ if $p=\infty$, $\varrho>1$.

First we consider the case $p<\infty$ and $\varrho=1$. Let $\sigma\in(0,1)$ satisfy $1-1/p<\sigma<1$ and let $u_\tau(t)=u_k$ for $t_{k-1}<t<t_k$, $k=1,\ldots,N$. Then

$$\begin{aligned}\sum_{k=1}^{N-1}\|[u_\tau]_k\|_Y &= \sum_{k=1}^{N-1}\|u_{k+1}-u_k\|_Y = \tau^{-1}\sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}}\|u_{k+1}-u_k\|_Y\,\mathrm{d}t\\ &= \tau^{-1}\|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T;Y)} \le C_0.\end{aligned}\tag{7}$$

Since $L^p(0,T;X)\hookrightarrow L^1(0,T;Y)$, Lemma 3 shows that (u_τ) is bounded in $W^{\sigma,\varrho}(0,T;Y)$, and the corollary applies.

It remains to discuss the case $p=\infty$ and $\varrho>1$. We define the piecewise linear interpolants

$$\widetilde u_\tau(t) = \begin{cases} u_1 & \text{for } 0\le t\le t_1,\\ u_k - \frac{t_k-t}{\tau}(u_k-u_{k-1}) & \text{for } t_{k-1}\le t\le t_k,\quad 2\le k\le N.\end{cases}$$

Let $(S_\tau u_\tau)(t)=u_1$ for $0\le t<t_1$. We observe that

$$\begin{cases} \widetilde u_\tau'(t) = \dfrac{1}{\tau}\big(u_\tau(t)-(S_\tau u_\tau)(t)\big), \quad 0\le t\le T,\quad t\ne t_k,\\ \|\widetilde u_\tau(t)\|_X \le \|u_\tau(t)\|_X + \|(S_\tau u_\tau)(t)\|_X\,,\quad 0\le t\le T,\end{cases}\tag{8}$$

which implies that $\|\widetilde u_\tau\|_{L^p(0,T;X)}\le 2\|u_\tau\|_{L^p(0,T;X)}$. Now we apply Theorem 1 to (u_τ) with $p=1$ instead of $p=\infty$, and we apply Corollary 5 in [7] to $(\widetilde u_\tau)$ with $\sigma=1$. We end up with a subsequence of (u_τ) (not relabeled) such that $u_\tau\to u^*$ in $L^1(0,T;B)$, and we may assume that the associated subsequence $(\widetilde{u}_\tau)$ of piecewise linear interpolants converges to a limit $\widehat{u}$ in the topology of $C^0([0,T];B)$. Next we know, for $k=1,\dots,N$ and $t_{k-1}<t<t_k$, that

$$\|u_\tau(t)-\widetilde{u}_\tau(t)\|_Y = \frac{t_k-t}{\tau}\,\|u_\tau(t)-(S_\tau u_\tau)(t)\|_Y \le \|u_\tau(t)-(S_\tau u_\tau)(t)\|_Y\,, \tag{9}$$

from which we infer that $\|u_\tau-\widetilde{u}_\tau\|_{L^1(0,T;Y)}\le C_0\tau$. Notice that the embeddings $L^1(0,T;B) \hookrightarrow L^1(0,T;Y)$ and $C^0([0,T];B)\hookrightarrow L^1(0,T;Y)$ are both continuous, hence $u^*=\widehat{u}$.

Since $(\widetilde{u}_\tau)$ converges in $C^0([0,T];B)$ to $\widehat{u}$, there exists a constant $\widehat{C}>0$ such that $\|\widetilde{u}_\tau\|_{L^\infty(0,T;B)}\le\widehat{C}$ for all τ, and then also $\|u_\tau\|_{L^\infty(0,T;B)}\le\widehat{C}$ for all τ. The desired convergence of (u_τ) to u^* in any space $L^q(0,T;B)$ for $1\le q<\infty$ follows from interpolation between $\|u_\tau-u^*\|_{L^1(0,T;B)}\to 0$ and $\|u_\tau-u^*\|_{L^\infty(0,T;B)}\le 2\widehat{C}$, which completes the proof. □

Remark 4. Estimates (6) and (7) imply that, for all piecewise constant functions $u\in L^1(0,T;Y)$ with jumps at $t_k=k\tau$,

$$|u|_{\dot W^{\sigma,1}(0,T;Y)}\le C_{\sigma q,T}^{1/q}\sum_{k=1}^{N-1}\|[u]_k\|_Y\le \tau^{-1}C_{\sigma q,T}^{1/q}\,\|u-S_\tau u\|_{L^1(\tau,T;Y)}\,.$$

By Lemma 5 of [7], there exists an inverse inequality for all $u\in W^{\sigma,1}(0,T;Y)$ and all $\sigma\in(0,1)$:

$$\|u-S_\tau u\|_{L^1(\tau,T;Y)}\le C_3\tau^\sigma|u|_{\dot W^{\sigma,1}(0,T;Y)},$$

where $C_3>0$ depends on σ and T. In this sense, the chain of inequalities

$$\tau|u|_{\dot W^{\sigma,1}(0,T;Y)}\le\tau^\sigma C_{\sigma q,T}^{1/q}C_3|u|_{\dot W^{\sigma,1}(0,T;Y)}$$

is almost sharp since we can choose σ as close to one as we wish. □

3. Proof of Proposition 2

We construct a sequence (u_τ) satisfying the assumptions of Theorem 1 with $\tau^{-\alpha}$ $(0<\alpha<1)$ in (4) instead of τ^{-1}, but not possessing a convergent subsequence in $L^p(0,T;B)$, where $p<\infty$.

Take $X=Y=B=\mathbb{C}$ and $(0,T)=(0,1)$. For $\beta\ge1$, define the function

$$f_\beta(t):=(\beta p+1)^{1/p}t^\beta,\quad 0\le t\le 1.$$

Then we have $\|f_\beta\|_{L^p(0,T)}=1$. For later use, we remark that

$$\lim_{\beta\to\infty}f_\beta(t)=0, \tag{10}$$

for each fixed $t\in[0,1)$, uniformly on compact sub-intervals $[0,t_*]\subset[0,1)$.

Since $\alpha<1$, we may choose a real number $0<\gamma\le\min\{1,p(1-\alpha)\}$. We set $\beta(\tau)=\tau^{-\gamma}$ and

$$u_\tau(t):=\begin{cases} f_{\beta(\tau)}(k\tau) & \text{for}\quad k\tau\le t<(k+1)\tau,\ k\in\{0,1,\dots,N-1\},\\ f_{\beta(\tau)}((N-1)\tau) & \text{for}\quad t=1.\end{cases}$$

The function u_τ has jumps of height $[u_\tau]_k$ at the values $t = k\tau$ for $1 \leq k \leq N-1$, and all jumps have the same sign. In particular,

$$\sum_{k=1}^{N-1} \|[u_\tau]_k\|_Y = \sum_{k=1}^{N-1} [u_\tau]_k = f_{\beta(\tau)}(1-\tau) = (\tau^{-\gamma}p+1)^{1/p}(1-\tau)^{\tau^{-\gamma}},$$
$$1 \geq (1-\tau)^{\tau^{-\gamma}} \geq (1-\tau)^{1/\tau} \geq \frac{1}{2e}.$$

Therefore, it follows that

$$\tau^{-\alpha}\|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T,Y)} = \tau^{1-\alpha}\sum_{k=1}^{N-1}\|[u_\tau]_k\|_Y = \tau^{1-\alpha}(\tau^{-\gamma}p+1)^{1/p}(1-\tau)^{\tau^{-\gamma}}$$
$$\leq \tau^{1-\alpha}(\tau^{-\gamma}p+1)^{1/p} \leq \left(\frac{1}{2}\right)^{1-\alpha}\left(\left(\frac{1}{2}\right)^{-\gamma}+1\right)^{1/p},$$

for all $\tau \in (0,1/2)$, since $1-\alpha-\gamma/p \geq 0$. Hence, (4) holds. But the sequence $(u_\tau) \subset L^p(0,T;B)$ does not possess a converging subsequence, which can be seen as follows. Fix $t \in [0,1)$. Then $0 \leq u_\tau(t) \leq f_{\beta(\tau)}(t)$, and (10) implies the pointwise convergence $\lim_{\tau\to 0} u_\tau(t) = 0$, uniform on compact sub-intervals $[0,t_*] \subset [0,1)$. Thus, the pointwise limit of the subsequence must be the zero function. However, this is impossible, because of the following uniform lower bound:

$$\int_0^1 |u_\tau(t)|^p\,\mathrm{d}t \geq \int_0^{1-\tau} |f_{\beta(\tau)}(t)|^p\,\mathrm{d}t = (1-\tau)^{\tau^{-\gamma}p+1} \geq \frac{1}{2}\left((1-\tau)^{\tau^{-\gamma}}\right)^p \geq \frac{1}{2}(2e)^{-p},$$

showing the claim.

4. Comments

Let X, B, and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is dense and compact, the embedding $B \hookrightarrow Y$ is continuous, and there exist $0 < \theta < 1$, $C_\theta > 0$ such that for all $u \in X$, the interpolation inequality

$$\|u\|_B \leq C_\theta \|u\|_X^\theta \|u\|_Y^{1-\theta} \tag{11}$$

holds. The setting which we have in mind relates to (1), with given $u(0) \in B$. In this situation, a slightly weaker version of Theorem 1 can be derived directly from the Aubin lemma.[1] Indeed, since X is dense in B, we may approximate $u(0) \in B$ by $u_0 \in X$, and we define the piecewise linear interpolant by

$$\widetilde{u}_\tau(t) = u_k - \frac{t_k - t}{\tau}(u_k - u_{k-1}), \quad t_{k-1} \leq t \leq t_k, \quad 1 \leq k \leq N.$$

We suppose that u_0 and u_1 satisfy

$$\tau\|u_0\|_X^p \leq C_1, \quad \|u_0 - u_1\|_Y \leq C_1 \tag{12}$$

[1]The authors are grateful to one of the referees for this observation.

for some constant $C_1 > 0$ independent of τ. The first bound can always be satisfied; the second bound is a mild condition related to the construction of the sequence (u_k). If this sequence is defined according to (1), the second bound can be replaced by the regularity assumption $\tau\|A(u_1)\|_Y \leq C$ for some constant $C > 0$ independent of τ since $\|u_1 - u_0\|_Y \leq \tau\|A(u_1)\|_Y + \tau\|f_\tau(\tau)\|_Y$.

Now we make the agreement that $(S_\tau u_\tau)(t) = u_0$ for $0 \leq t < t_1 = \tau$. Then (8) still holds. It follows from (4) that

$$\|\widetilde{u}'_\tau\|_{L^1(0,T;Y)} = \|u_1 - u_0\|_Y + \tau^{-1}\|u_\tau - S_\tau u_\tau\|_{L^1(\tau,T;Y)} \leq C_1 + C_0.$$

Furthermore, using (8) and (4) again,

$$\|\widetilde{u}_\tau\|_{L^p(0,T;X)} \leq \tau^{1/p}\|u_0\|_X + 2\|u_\tau\|_{L^p(0,T;X)} \leq C_1^{1/p} + 2C_0.$$

Hence, by the Aubin lemma [7, Corollary 4], up to a subsequence, $\widetilde{u}_\tau \to u$ in $L^p(0,T;B)$ as $\tau \to 0$. By the interpolation inequality (11) and by (9),

$$\begin{aligned}\|u_\tau - \widetilde{u}_\tau\|_{L^1(0,T;B)} &\leq C_\theta \|u_\tau - \widetilde{u}_\tau\|^\theta_{L^1(0,T;X)} \|u_\tau - \widetilde{u}_\tau\|^{1-\theta}_{L^1(0,T;Y)} \\ &\leq C_\theta\big(\|u_\tau\|_{L^1(0,T;X)} + \|\widetilde{u}_\tau\|_{L^1(0,T;X)}\big)^\theta \|u_\tau - S_\tau u_\tau\|^{1-\theta}_{L^1(0,T;Y)}.\end{aligned}$$

We remark that $\|u_\tau - S_\tau u_\tau\|_{L^1(0,T;Y)} \leq \tau(\|u_1 - u_0\|_Y + C_0)$, which implies that $u_\tau - \widetilde{u}_\tau \to 0$ in $L^1(0,T;B)$. Since $\widetilde{u}_\tau \to u$ in $L^p(0,T;B)$, we find that $u_\tau \to u$ in $L^q(0,T;B)$ for all $q < p$. Notice, however, that Theorem 1 allows us to conclude this result up to $q = p$ without assuming (11) and (12).

Proposition 2 shows that the exponent of the factor τ in (4) cannot be raised. However, when allowing for arbitrary time shifts S_h, the factor can be replaced by $h^{-\alpha}$, where $0 < \alpha < 1$, under some conditions. An example, adapted to our situation, can be found in [1, Theorem 1.1]:

Theorem 5 (Amann)**.** *Let* (11) *hold. Furthermore, let* $0 < s < 1$, $1 \leq p < \infty$, *and* $F \subset L^p(0,T;Y)$. *Assume that there exists* $C_2 > 0$ *such that each* $u \in F$ *satisfies the following infinite collection of inequalities:*

$$h^{-s}\|u - S_h u\|_{L^1(\tau,T;Y)} + \|u\|_{L^p(0,T;X)} \leq C_2 \quad \text{for all } h > 0.$$

Then F *is relatively compact in* $L^q(0,T;B)$ *for all* $q < p/((1-\theta)(1-s)p + \theta)$.

Notice that $q = p$ is admissible if $(1-\theta)(1-s)p+\theta < 1$ which is equivalent to $s > 1-1/p$. Thus, if we wish to allow for arbitrary large $p \geq 1$, we have to require the condition $s = 1$, which corresponds to the result of Theorem 1. On the other hand, in applications, often $p = 2$, and compactness follows even for $s < 1$, namely for any $s > 1/2$.

In the special situation when we have the triple $X \hookrightarrow B \hookrightarrow X'$, where $Y = X'$ is the dual space of X and B is a Hilbert space, the assumptions of Amann's theorem hold with $\theta = 1/2$. Then $q < 2p/((1-s)p + 1)$, and we see that $2p$ is an upper bound for q. This corresponds to the result of Walkington [8, Theorem 3.1 (1)].

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