ANALYSIS OF A PARABOLIC CROSS-DIFFUSION POPULATION MODEL WITHOUT SELF-DIFFUSION

LI CHEN

Institut für Mathematik, Johannes Gutenberg-Universität Mainz,
Staudingerweg 9, 55099 Mainz, Germany.
Permanent address: Department of Mathematical Sciences, Tsinghua University,
Beijing, 100084, People's Republic of China.
lchen@math.tsinghua.edu.cn

ANSGAR JÜNGEL

Institut für Mathematik, Johannes Gutenberg-Universität Mainz,
Staudingerweg 9, 55099 Mainz, Germany.
juengel@mathematik.uni-mainz.de

The global existence of non-negative weak solutions to a strongly coupled parabolic system arising in population dynamics is shown. The cross-diffusion terms are allowed to be arbitrarily large, whereas the self-diffusion terms are assumed to vanish. The last assumption complicates the analysis since these terms usually provide H^1 estimates of the solutions. The existence proof is based on a positivity-preserving backward Euler-Galerkin approximation, discrete entropy estimates, and L^1 weak compactness arguments. Furthermore, employing the entropy-entropy production method, we show for special stationary solutions that the transient solution converges exponentially fast to its steady state. As a by-product, we prove that only constant steady states exist if the inter-specific competition parameters vanish no matter how strong the cross-diffusion constants are.

Keywords: Population equations, strong cross-diffusion, weak competition, relative entropy, global-in-time existence of weak solutions, long-time behavior of solutions.

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1. Introduction

In their pioneering work, Shigesada, Kawasaki, and Teramoto[23] proposed a generalization of the Lotka-Volterra differential equations in order to describe spatial segregation of interacting population species. Denoting by $u_1 = u_1(x,t)$ and $u_2 = u_2(x,t)$ the densities of the two competing species, the equations read as follows:

$$\partial_t u_i - \mathrm{div} J_i = (a_i - b_i u_1 - c_i u_2) u_i, \quad J_i = \nabla\big((d_i + \rho_{i1} u_1 + \rho_{i2} u_2) u_i\big), \qquad (1.1)$$

with homogeneous Neumann boundary and initial conditions

$$J_i \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad u(\cdot,0) = u_i^0 \geq 0 \quad \text{in } \Omega, \qquad i = 1,2. \qquad (1.2)$$

Problem (1.1)–(1.2) has to be solved in $Q_T = \Omega \times (0,T)$, where $T > 0$ and $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain. In (1.1), d_1, $d_2 \geq 0$ are the diffusion rates, ρ_{11}, $\rho_{22} \geq 0$ the self-diffusion coefficients, and ρ_{12}, $\rho_{21} \geq 0$ are the cross-diffusion constants making the parabolic problem strongly coupled. Furthermore, the nonnegative coefficients a_1 and a_2 denote the intrinsic growth rates, b_1 and c_2 the intra-specific competition constants, and b_2 and c_1 the rates of inter-specific competition. Equations (1.1) have the interesting feature that they allow for pattern formation depending on the relative sizes of the interaction coefficients.[19] For vanishing coefficients d_i and ρ_{ij}, we obtain the classical Lotka-Volterra differential equations.

The above system possesses the diffusion matrix

$$\begin{pmatrix} d_1 + 2\rho_{11}u_1 + \rho_{12}u_2 & \rho_{12}u_1 \\ \rho_{21}u_2 & d_2 + 2\rho_{22}u_2 + \rho_{21}u_1 \end{pmatrix}.$$

Nonlinear problems with a full diffusion matrix are difficult to deal with since, for instance, maximum principles, employed for the derivation of a priori estimates, generally cannot be applied. Moreover, the above matrix is not symmetric and generally not positive definite. In Ref.[5,9] it has been shown that the problem (1.1)–(1.2) can be transformed to a system with a symmetric, positive definite diffusion matrix via the change of variables $w_1 = \ln(u_1)/\rho_{12}$ and $w_2 = \ln(u_2)/\rho_{21}$. This symmetrization property is strongly connected to the existence of the entropy

$$E(t) = \int_\Omega \left(\frac{1}{\rho_{12}}\Phi(u_1) + \frac{1}{\rho_{21}}\Phi(u_2) \right) dx,$$

where $\Phi(x) = x(\ln x - 1) + 1$, $x \geq 0$ (see Ref.[7,12]). Differentiating this function formally, the a priori estimate

$$\begin{aligned} E(t) &+ 2\int_{Q_t} \left(\frac{d_1}{\rho_{12}}|\nabla\sqrt{u_1}|^2 + \frac{d_2}{\rho_{21}}|\nabla\sqrt{u_2}|^2 + |\nabla\sqrt{u_1u_2}|^2 \right) dxd\tau \\ &+ \int_{Q_t} \left(\frac{b_1}{\rho_{12}}(u_1^2\ln u_1^2 + 1) + \frac{c_2}{\rho_{21}}(u_2^2\ln u_2^2 + 1) + \Big(\frac{c_1}{\rho_{12}} + \frac{b_2}{\rho_{21}}\Big)u_1u_2 \right) dxd\tau \quad (1.3) \\ &+ 2\int_{Q_t} \left(\frac{\rho_{11}}{\rho_{12}}|\nabla u_1|^2 + \frac{\rho_{22}}{\rho_{21}}|\nabla u_2|^2 \right) dxd\tau \leq C(E(0)+1), \quad 0 < t \leq T, \end{aligned}$$

for some $C > 0$ depending on T is obtained. In particular, if $\rho_{11} > 0$ and $\rho_{22} > 0$, we obtain $L^2(0,T;H^1(\Omega))$ bounds for u_1 and u_2.

The above inequality can be derived by employing the test functions $\ln(u_1)/\rho_{12}$ and $\ln(u_2)/\rho_{21}$ in the weak formulation of (1.1) for $i = 1$ and $i = 2$, respectively. Clearly, this derivation is only rigorous if the densities u_1 and u_2 are positive. However, since we are lacking a maximum (or minimum) principle, it is not clear how to prove this property. This problem can be fixed by working in the variables w_1, w_2 since then $u_1 = \exp(\rho_{12}w_1)$ and $u_2 = \exp(\rho_{21}w_2)$ are automatically positive. In order to make the estimate (1.3) rigorous, the idea of Ref.[5] was to semi-discretize the system (1.1) in time and to approximate the cross-diffusion terms by finite differences in such a way that a discrete entropy inequality analogous to (1.3) holds.

In this paper we extend the results and improve the method of Ref.[5]. First, we allow for vanishing self-diffusion coefficients $\rho_{11} = \rho_{22} = 0$. This complicates the analysis since we do not conclude $L^2(0,T;H^1(\Omega))$ bounds for u_1 and u_2 from (1.3) but only $L^2(0,T;H^1(\Omega))$ bounds for the nonlinear functions $\sqrt{u_1}$ and $\sqrt{u_2}$. Furthermore, we are able to give a much simpler proof than presented in Ref.[5] by using a Galerkin approximation. Our approximate problem provides a positivity-preserving fully discrete scheme, which is interesting from a numerical point of view. Finally, we also study the long-time behavior of the transient solutions to (1.1)-(1.2) for special (constant) steady states. We prove that the solutions converge exponentially fast to their steady state in the entropy and in the L^1 norm.

In the following we explain our results in more detail. We set $\rho_{11} = \rho_{22} = 0$ and we rescale the equations such that $\rho_{12} = \rho_{21} = 1$. Then the equations to be studied are as follows:

$$\partial_t u_i - \Delta(d_i u_i + u_1 u_2) = (a_i - b_i u_1 - c_i u_2)u_i, \quad (x,t) \in Q_T, \ i = 1,2, \qquad (1.4)$$

with boundary and initial conditions

$$\nabla u_i \cdot \nu = 0 \quad \text{on } \partial\Omega \times (0,\infty), \qquad u_i(\cdot,0) = u_i^0 \quad \text{in } \Omega, \qquad i = 1,2. \qquad (1.5)$$

Our first main result is contained in the following existence theorem.

Theorem 1.1. *Let $s = 1 + d^2/(2d+2)$ and $\partial\Omega \in C^{\ell,1}$ with $\ell \in \mathbb{N}$, $\ell \geq s$. Furthermore, let $a_i, b_i, c_i \geq 0$, $d_i > 0$, and $u_i^0 \in L_\Psi(\Omega)$ be such that $u_i^0 \geq 0$ in Ω, $i = 1,2$. Then there exists a weak solution (u_1, u_2) of problem (1.4)–(1.5) satisfying*

$$u_i \geq 0 \ \text{in } \Omega \times (0,\infty), \quad \partial_t u_i \in L^1_{\text{loc}}(0,\infty;(H^s(\Omega))'),$$
$$u_i \in L^{4/3}_{\text{loc}}(0,\infty;W^{1,4/3}(\Omega)) \cap L^\infty_{\text{loc}}(0,\infty;L_\Psi(\Omega)).$$

The equations (1.4) are satisfied in the sense of distributions and the initial data (1.5) are satisfied in the sense of the dual space $(H^s(\Omega))'$.

The space $L_\Psi(\Omega)$ is the Orlicz space with function $\Psi(x) = \Phi(x+1) = (1+x)\ln(1+x) - x$, $x \geq 0$. We refer to Ref.[1,14] for its definition and properties.

In order to prove Theorem 1.1 we use a semi-discretization in time (backward Euler method) so that problem (1.4)–(1.5) becomes a recursive sequence of elliptic equations. Then we perform the change of unknowns $u_i = e^{w_i}$ $(i = 1,2)$. The advantage of this transformation is that the property $w_i \in L^\infty(\Omega)$ implies the positivity of u_i. In Ref.[9] the problem (1.1)–(1.2) has been considered in one space dimension only, since then the solution satisfies $w_1, w_2 \in H^1(\Omega) \hookrightarrow L^\infty(\Omega)$. Clearly, this argument cannot be used in several space dimensions. Our new idea is to employ a Galerkin approximation. More precisely, we solve the semi-discrete elliptic problem in a sequence of finite-dimensional spaces whose union is dense in $H^s(\Omega)$ with $s > d/2$. Then $w_i \in H^s(\Omega) \hookrightarrow L^\infty(\Omega)$ and the transformation $u_i = e^{w_i}$ is well defined and yields positive discrete solutions.

The discrete entropy inequality and Aubin's lemma allow us to conclude the strong convergence in $L^1(Q_T)$ of a subsequence of the discrete solutions $u_i^{(\tau)}$, where

τ denotes the discretization parameters. However, from the entropy estimates, we obtain a uniform estimate for the discrete time derivative of $u_i^{(\tau)}$ only in the space $L^1(0,T;(H^s(\Omega))')$. Since L^1 is not reflexive, generally, we cannot extract a converging subsequence. In order to prove the weak compactness in L^1 we use a variant of a result of Yosida[25] (see Lemma 3.2).

We turn to the study of the long-time behavior of the solutions to (1.4)–(1.5). The case of the Lotka-Volterra equations with diffusion (i.e. $\rho_{ij} = 0$ for $i,j = 1,2$) has been studied in Ref.[3,8]. It turns out that the asymptotic behavior depends on the relative sizes of the quantities $A = a_1/a_2$, $B = b_1/b_2$, and $C = c_1/c_2$:

$$\begin{aligned}
&\text{(i)} && A > \max\{B,C\},\\
&\text{(ii)} && A < \max\{B,C\},\\
&\text{(iii)} && B > A > C \text{ (weak competition)},\\
&\text{(iv)} && B < A < C \text{ (strong competition)}.
\end{aligned}$$

The solution $(u_1(\cdot,t), u_2(\cdot,t))$ converges, as $t \to \infty$, uniformly to $(a_1/b_1, 0)$ in case (i), to $(0, a_2/c_2)$ in case (ii), and to $u^* = ((a_1c_2 - a_2c_1)/(b_1c_2 - b_2c_1), (b_1a_2 - b_2a_1)/(b_1c_2 - b_2c_1))$ in case (iii). Thus, in cases (i) or (ii), one of the species is wiped out whereas in case (iii), both species coexist. Case (iv) is more involved. For instance, the constant steady states $(a_1/b_1, 0)$ and $(0, a_2/c_2)$ are locally stable and u^* is unstable,[19] and the stability of *positive* steady states depends on the shape of the domain Ω.[13,20,21]

In the triangular cross-diffusion case (i.e. $\rho_{ij} \geq 0$ but $\rho_{21} = 0$), Le et al. proved the existence of a global attractor of the system.[16,17] However, only a few results are available on the asymptotic behavior of the solutions to the cross-diffusion model with *full* diffusion matrix, since in this situation, the influences from both the Lotka-Volterra and the self- and cross-diffusion terms need to be taken into account. The interesting topic here is the question if the system admits non-constant steady states, expressing spatial segregation of the species. For some results in this direction, we refer to Ref.[11,20,22]. Lou and Ni investigate this question extensively in Ref.[19]. Roughly speaking, their results can be summarized as follows.

- If the diffusion or self-diffusion rates are sufficiently large, there exist only constant steady states (no segregation).
- In the weak competition case and if the self-diffusion and/or cross-diffusion rates are weaker than the diffusion coefficients, there still exist only constant stationary solutions.
- In the weak or strong competition case, fixing one of the cross-diffusion parameters ρ_{12} or ρ_{21}, there exists a non-constant steady state if the other cross-diffusion constant is sufficiently large (and if the diffusion and Lotka-Volterra parameters are appropriately chosen, see Ref.[19]).

These results indicate that diffusion and self-diffusion seem to prevent pattern formation, whereas cross-diffusion seems to support the segregation process. In Ref.[19]

the following question remained unsolved: *Do non-constant steady states still exist if both cross-diffusion coefficients are strong but qualitatively similar?* In this paper, we give a partial answer to this question. More precisely, we show that *in the case of vanishing inter-specific competition $b_2 = c_1 = 0$ (special case of weak competition), only constant solutions exist no matter how strong the cross-diffusion coefficients are.* Furthermore, we prove that the solution, constructed in Theorem 1.1, converges exponentially fast to its (constant) steady state if $a_i = b_i = c_i = 0$ $(i = 1, 2)$.

In order to prove the long-time behavior we employ the so-called entropy-entropy production method (see, e.g., Ref.[2,4]). The relative entropy of the population system with stationary solution (U_1, U_2) equals

$$E(t; U_1, U_2) = \int_\Omega \left(\frac{U_1}{\rho_{12}} \Phi\Big(\frac{u_1}{U_1}\Big) + \frac{U_2}{\rho_{21}} \Phi\Big(\frac{u_2}{U_2}\Big) \right) dx,$$

where we recall that $\Phi(x) = x(\ln x - 1) + 1$, $x \geq 0$. If $a_i = b_i = c_i = 0$ $(i = 1, 2)$ the steady state is given by

$$(\bar{U}_1, \bar{U}_2) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega (u_1^0, u_2^0) dx, \tag{1.6}$$

and we are able to show that

$$E(t; \bar{U}_1, \bar{U}_2) - E(s; \bar{U}_1, \bar{U}_2) \leq -C \int_s^t \sum_{i=1}^2 \|\nabla \sqrt{u_i}\|^2_{L^2(\Omega)} d\tau, \quad 0 \leq s < t < \infty, \tag{1.7}$$

The logarithmic Sobolev inequality allows to relate the L^2 norm of $\nabla\sqrt{u_i}$ with the relative entropy and then, the Gronwall inequality yields the exponential decay in the entropy.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold and let $a_i = b_i = c_i = 0$, $i = 1, 2$. Furthermore, let (u_1, u_2) be the weak solution constructed in Theorem 1.1. Then there exists a constant $C > 0$ such that $(u_1(\cdot, t), u_2(\cdot, t))$ converges exponentially fast to its steady state (1.6) as $t \to \infty$. More precisely, we have the entropy decay*

$$E(t; \bar{U}_1, \bar{U}_2) \leq E(0; \bar{U}_1, \bar{U}_2) e^{-Ct}, \quad t > 0,$$

and the L^1 decay

$$\sum_{i=1}^2 \frac{1}{2 meas(\Omega) \bar{U}_i} \|u_i(\cdot, t) - \bar{U}_i\|_{L^1(\Omega)} \leq \sqrt{E(0; \bar{U}_1, \bar{U}_2)} e^{-Ct/2}, \quad t > 0.$$

Our final result for the case $b_2 = c_1 = 0$ is obtained by considering the steady state $(a_1/b_1, a_2/c_2)$ and proving the entropy inequality (1.7) for this situation.

Proposition 1.1. *If $a_1, a_2, b_1, c_2 > 0$ and $b_2 = c_1 = 0$ then there exist only constant stationary solutions to (1.4)–(1.5).*

The paper is organized as follows. In section 2 we formulate the fully discretized equations and prove the existence of an approximate positive solution. The limit of vanishing approximation parameters and hence the existence of a weak solution to (1.4)–(1.5) is proven in section 3. Finally, the long-time behavior of the solution is analyzed in section 4.

2. An approximate problem

In this section we prove the existence of solutions to an approximate problem which can be seen as a positivity-preserving fully discretized numerical scheme.

Let (v_j) be a dense subset of $H^s(\Omega)$ with $s = 1 + d^2/(2d+2)$ being orthogonal in the L^2 scalar product. For instance, one may choose v_j as the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions. We may assume that $v_1 = 1$ in Ω. Then, by the regularity of $\partial\Omega$, $v_j \in H^s(\Omega)$, and, since the Laplace operator is self-adjoint and compact, (v_j) is dense in $L^2(\Omega)$ and therefore also in $H^s(\Omega)$. Notice that $v_j \in W^{1,r'}(\Omega) \hookrightarrow L^\infty(\Omega)$, $r' = 2d+2$.

Let $V_n = \operatorname{span}\{v_1, \ldots, v_n\}$, $n \in \mathbb{N}$, be a finite-dimensional subspace of $H^s(\Omega)$, and let $w_i^{(0,n)} \in V_n$ be such that $\exp(w_i^{(0,n)}) \to u_i^0$ strongly in $L_\Psi(\Omega)$, as $n \to \infty$, $i = 1, 2$.

We decompose $(0,T] = \cup_{k=1}^K ((k-1)\tau, k\tau]$ for $\tau = T/K$, $K \in \mathbb{N}$. Let $w_i^{(k-1,n)} \in V_n$ be given and set $u_i^{(k-1,n)} = \exp(w_i^{(k-1,n)})$, $i = 1, 2$. This definition makes sense since $w_i^{(k-1,n)} \in V_n \subset L^\infty(\Omega)$. In the following, we solve the approximate problem

$$
\begin{aligned}
&\int_\Omega \left(\varepsilon \nabla w_i^{(k,n)} + d_i u_i^{(k,n)} \nabla w_i^{(k,n)} + u_1^{(k,n)} u_2^{(k,n)} \nabla (w_1^{(k,n)} + w_2^{(k,n)})\right) \cdot \nabla \chi dx \\
&\quad + \varepsilon \int_\Omega w_i^{(k,n)} \chi dx \qquad\qquad (2.1)\\
&= -\frac{1}{\tau} \int_\Omega \left(u_i^{(k,n)} - u_i^{(k-1,n)}\right) \chi dx + \int_\Omega u_i^{(k,n)} \left(a_i - b_i u_i^{(k,n)} - c_i u_i^{(k,n)}\right) \chi dx
\end{aligned}
$$

for all $\chi \in V_n$, where $\varepsilon > 0$, $u_i^{(k,n)} = \exp(w_i^{(k,n)})$, $i = 1, 2$, and we show that the discrete entropy

$$
E^{(k,n)} = \sum_{i=1}^2 \int_\Omega \left(u_i^{(k,n)} (\ln u_i^{(k,n)} - 1) + 1\right) dx
$$

is uniformly bounded.

Lemma 2.1. *For sufficiently small fixed $\tau > 0$ and for all $k = 1, \ldots, K$, there exists a solution $(w_1^{(k,n)}, w_2^{(k,n)}) \in V_n^2$ to (2.1), satisfying the discrete entropy estimate*

$$
\begin{aligned}
&E^{(k,n)} + \varepsilon\tau \sum_{j=1}^k \sum_{i=1}^2 \int_\Omega \left(|\nabla w_i^{(j,n)}|^2 + (w_i^{(j,n)})^2\right) dx \\
&\quad + \tau \sum_{j=1}^k \int_\Omega \Big(\sum_{i=1}^2 d_i u_i^{(j,n)} |\nabla w_i^{(j,n)}|^2 + u_1^{(j,n)} u_2^{(j,n)} |\nabla (w_1^{(j,n)} + w_2^{(j,n)})|^2 \Big) dx
\end{aligned}
$$

$$
\begin{aligned}
&+\tau\sum_{j=1}^{k}\int_\Omega\Big(\frac{b_1}{2}\big((u_1^{(j,n)})^2\ln(u_1^{(j,n)})^2+1\big)+\frac{c_2}{2}\big((u_2^{(j,n)})^2\ln(u_2^{(j,n)})^2+1\big)\\
&+(b_2+c_1)u_1^{(j,n)}u_2^{(j,n)}\Big)dx\\
&\le C(E^{(0,n)}+1), \qquad (2.2)
\end{aligned}
$$

where the constant $C>0$ is independent of τ, n, and ε (but depending on T).

Proof. In order to simplify the presentation, we omit the indices k and n. The idea is to employ the Leray-Schauder fixed-point theorem. For this, we construct a mapping $S : V_n^2\times[0,1]\to V_n^2$ by solving, for given $(\bar w_1,\bar w_2)\in V_n^2$, $\bar u_1=e^{\bar w_1}$, $\bar u_2=e^{\bar w_2}$, and $\sigma\in[0,1]$, the problem

$$
\begin{aligned}
&\varepsilon\int_\Omega(\nabla w_i\cdot\nabla\chi+w_i\chi)dx+\sigma\int_\Omega\big(d_i\bar u_i\nabla\bar w_i+\bar u_1\bar u_2\nabla(\bar w_1+\bar w_2)\big)\cdot\nabla\chi dx\\
&=-\frac{\sigma}{\tau}\int_\Omega(\bar u_i-u_i^{(k-1,n)})\chi dx+\sigma\int_\Omega\bar u_i(a_i-b_i\bar u_1-c_i\bar u_2)\chi dx \qquad (2.3)
\end{aligned}
$$

for all $\chi\in V_n$, where $i=1,2$. Since $\bar u_i\in L^\infty(\Omega)$, we can apply the lemma of Lax-Milgram to obtain a unique solution $(w_1,w_2)\in V_n^2$ to (2.3). Thus, setting $S(\bar w_1,\bar w_2,\sigma)=(w_1,w_2)$ defines the fixed-point operator S.

Notice that $S(\bar w_1,\bar w_2,0)=(0,0)$. Furthermore, by standard arguments, S is continuous. Since V_n is finite dimensional, $S(\cdot,\sigma)$ is a compact operator for all $\sigma\in[0,1]$. It remains to establish uniform estimates for every fixed point of $S(\cdot,\sigma)$. Let (w_1,w_2) be a fixed point, i.e., (w_1,w_2) solves (2.3) with $\bar w_i=w_i$ and $\bar u_i=u_i=e^{w_i}$, $i=1,2$. We use $\chi=w_1$ as a test function in (2.3) for $i=1$ and $\chi=w_2$ in (2.3) for $i=2$ and add both equations. This gives

$$
\begin{aligned}
&\frac{\sigma}{\tau}\sum_{i=1}^{2}\int_\Omega(u_i-u_i^{(k-1,n)})w_idx+\varepsilon\sum_{i=1}^{2}\int_\Omega(|\nabla w_i|^2+w_i^2)dx\\
&+\sigma\int_\Omega\Big(\sum_{i=1}^{2}d_iu_i|\nabla w_i|^2+u_1u_2|\nabla(w_1+w_2)|^2\Big)dx\\
&=\sigma\sum_{i=1}^{2}\int_\Omega(a_i-b_iu_1-c_iu_2)u_i\ln u_idx. \qquad (2.4)
\end{aligned}
$$

The first integral on the left-hand side can be estimated by means of the elementary inequality $x(\ln x-\ln y)\ge x-y$ for all $x,y>0$ as

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega(u_i-u_i^{(k-1,n)})w_idx\\
&=\sum_{i=1}^{2}\int_\Omega\big(u_i\ln u_i-u_i^{(k-1,n)}\ln u_i^{(k-1,n)}+u_i^{(k-1,n)}(\ln u_i^{(k-1,n)}-\ln u_i)\big)dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{2}\int_\Omega \big(u_i\ln u_i - u_i^{(k-1,n)}\ln u_i^{(k-1,n)} + u_i^{(k-1,n)} - u_i\big)dx \\
&= E^{(k,n)} - E^{(k-1,n)}.
\end{aligned}
$$

For the estimate of the right-hand side of (2.4) we employ the elementary inequality $x\ln x \geq x-1$ for $x\geq 0$:

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega (a_i - b_iu_1 - c_iu_2)u_i\ln u_i dx \\
&= \int_\Omega\Big(\sum_{i=1}^{2} a_i\big(u_i(\ln u_i - 1)+1\big) + a_1(u_1-1) + a_2(u_2-1) + \frac{b_1}{2} + \frac{c_2}{2} \\
&\quad - \frac{b_1}{2}(u_1^2\ln u_1^2 + 1) - \frac{c_2}{2}(u_2^2\ln u_2^2+1) - b_2u_1u_2\ln u_2 - c_1u_2u_1\ln u_1\Big)dx \\
&\leq \int_\Omega\Big(\max\{a_1,a_2\}\sum_{i=1}^{2}\big(u_i(\ln u_i - 1)+1\big) - \frac{b_1}{2}(u_1^2\ln u_1^2+1) - \frac{c_2}{2}(u_2^2\ln u_2^2+1) \\
&\quad + \frac{b_1}{2} + \frac{c_2}{2} - a_1 - a_2 + (a_1+b_2)u_1 + (a_2+c_1)u_2 - (b_2+c_1)u_1u_2\Big)dx.
\end{aligned}
$$

The linear terms in u_1 and u_2 can be estimated in terms of the entropy such that we obtain, for some constant $C>0$ which is independent of ε, τ, and n,

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_\Omega (a_i - b_iu_1 - c_iu_2)u_i\ln u_i dx \\
&\leq \int_\Omega\Big(C\sum_{i=1}^{2}\big(u_i(\ln u_i - 1)+1\big) - \frac{b_1}{2}(u_1^2\ln u_1^2+1) - \frac{c_2}{2}(u_2^2\ln u_2^2+1) \\
&\quad - (b_2+c_1)u_1u_2 + C\Big)dx.
\end{aligned}
$$

Thus, (2.4) gives

$$
\begin{aligned}
&\frac{\sigma}{\tau}E^{(k,n)} + \varepsilon\sum_{i=1}^{2}\int_\Omega(|\nabla w_i|^2 + w_i^2)dx \\
&\quad + \sigma\int_\Omega\Big(\sum_{i=1}^{2} d_iu_i|\nabla w_i|^2 + u_1u_2|\nabla(w_1+w_2)|^2\Big)dx \\
&\quad + \sigma\int_\Omega\Big(\frac{b_1}{2}(u_1^2\ln u_1^2+1) + \frac{c_2}{2}(u_2^2\ln u_2^2+1) + (b_2+c_1)u_1u_2\Big)dx \\
&\leq \frac{\sigma}{\tau}E^{(k-1,n)} + \sigma C(E^{(k,n)}+1), \qquad (2.5)
\end{aligned}
$$

and the discrete Gronwall inequality for sufficiently small $\tau>0$ (and $\sigma=1$) implies (2.2), using $k\tau\leq T$. The estimate (2.2) provides a uniform H^1 estimate for w_1 and w_2 and shows the lemma. □

3. Existence of weak solutions

The solution of the fully discrete system (2.1) also depends on ε and will be denoted by $(w_1^{(k,n,\varepsilon)}, w_2^{(k,n,\varepsilon)})$. We also introduce the piecewise constant function $w_i^{(\tau)}(x,t) = w_i^{(k,n,\varepsilon)}(x)$ if $x \in \Omega$, $t \in ((k-1)\tau, k\tau]$, $i = 1,2$. Setting $Q_t = \Omega \times (0,t)$, $u_i^{(\tau)} = \exp(w_i^{(\tau)})$ for $i = 1,2$ and

$$E^{(\tau)}(t) = \sum_{i=1}^{2} \int_\Omega \Big(u_i^{(\tau)}(x,t)\big(\ln u_i^{(\tau)}(x,t) - 1\big) + 1\Big)\, dx,$$

we can rewrite the estimate (2.2) as

$$\begin{aligned} & E^{(\tau)}(t) + \int_{Q_t} \Big(\sum_{i=1}^{2} d_i \big|\nabla \sqrt{u_i^{(\tau)}}\big|^2 + \big|\nabla \sqrt{u_1^{(\tau)} u_2^{(\tau)}}\big|^2 \Big) dx d\sigma \\ & \quad + \varepsilon \sum_{i=1}^{2} \int_{Q_t} (|\nabla w_i^{(\tau)}|^2 + w_i^2) dx + (b_2 + c_1) \int_{Q_t} u_1^{(\tau)} u_2^{(\tau)} dx d\sigma \\ & \quad + \int_{Q_t} \Big(\frac{b_1}{2} \big((u_1^{(\tau)})^2 \ln(u_1^{(\tau)})^2 + 1\big) + \frac{c_2}{2} \big((u_2^{(\tau)})^2 \ln(u_2^{(\tau)})^2 + 1\big) \Big) dx d\sigma \\ & \le C(E^{(\tau)}(0) + 1). \end{aligned} \tag{3.1}$$

The constant $C > 0$ is independent of τ, ε, and n.

For the limit $(\varepsilon, \tau) \to 0$, $n \to \infty$, we employ the following convergence results.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ be a bounded domain and let $u_n \in L^p(\Omega)$, $1 \le p \le \infty$, such that (u_n) is bounded in $L^p(\Omega)$ and $u_n \to u$ pointwise a.e. in Ω as $n \to \infty$. Then $u_n \to u$ strongly in $L^q(\Omega)$ as $n \to \infty$ for all $q < p$.*

A proof of this lemma can be found in Ref.[18] (Ch. 1.3 and p. 144).

Lemma 3.2. *Let X be a reflexive Banach space, $T > 0$, and $(u_n) \subset L^1(0,T;X)$ be a sequence such that (u_n) is bounded in $L^1(0,T;X)$ and $\int \chi\langle \phi, u_n\rangle_{X',X} dt$ converges for every $\phi \in X'$ and $\chi \in L^\infty(0,T)$ as $n \to \infty$, where $\langle \cdot,\cdot\rangle_{X',X}$ denotes the duality product of X and its dual space X'. Then $u_n \rightharpoonup u$ weakly in $L^1(0,T;X)$ for some $u \in L^1(0,T;X)$ as $n \to \infty$.*

Proof. The lemma is a consequence of Theorem 4 (Ch. 5.1) in Ref.[25] Indeed, let $\phi \in X'$ and $f_n[\phi](t) = \langle \phi, u_n(t)\rangle_{X',X}$ for $t \in (0,T)$. Then $f_n[\phi] \in L^1(0,T)$, $(f_n[\phi])$ is bounded in $L^1(0,T)$, and $\lim_{n\to\infty} \int f_n[\phi]\chi dt$ exists for all $\chi \in L^\infty(0,T)$. Thus, by Theorem 4 (Ch. 5.1) of Ref.[25], there exists $f[\phi] \in L^1(0,T)$ such that $f_n[\phi] \rightharpoonup f[\phi]$ weakly in $L^1(0,T)$ as $n \to \infty$. The function u, defined by $u(\phi) = f[\phi]$ for $\phi \in X'$, satisfies $u \in L^1(0,T;X'')$. Since X is reflexive, u can be interpreted as a function in $L^1(0,T;X)$ and $f[\phi] = \langle \phi, u\rangle_{X',X}$. Hence, as $n \to \infty$,

$$\int_0^T \chi\langle \phi, u_n\rangle_{X',X} dt = \int_0^T f_n[\phi]\chi dt \to \int_0^T \chi\langle \phi, u\rangle_{X',X} dt$$

for all $\phi \in X'$ and $\chi \in L^\infty(0,T)$. This implies the conclusion. $\square$

In the following lemma we show that the sequences $(w_i^{(\tau)})$ and $(u_i^{(\tau)})$ have convergent subsequences. For this we define

$$\partial_t^\tau u_i(\cdot,t) = \frac{1}{\tau}\left(u_i^{(k,n)} - u_i^{(k-1,n)}\right) \quad \text{if } t \in ((k-1)\tau, k\tau].$$

Lemma 3.3. *As $(\varepsilon,\tau)\to 0$, $n\to\infty$, it holds for $i=1,2$, up to subsequences which are not relabeled,*

$$\begin{aligned}
u_i^{(\tau)} &\to u_i && \text{strongly in } L^1(0,T;L^\alpha(\Omega)), && (3.2)\\
\nabla u_i^{(\tau)} = u_i^{(\tau)}\nabla w_i^{(\tau)} &\rightharpoonup \nabla u_i && \text{weakly in } L^{4/3}(\Omega), && (3.3)\\
u_1^{(\tau)}u_2^{(\tau)} &\rightharpoonup u_1u_2 && \text{weakly in } L^{1+1/d}(Q_T), && (3.4)\\
u_1^{(\tau)}u_2^{(\tau)}\nabla(w_1^{(\tau)}+w_2^{(\tau)}) &\rightharpoonup \nabla(u_1u_2) && \text{weakly in } L^r(Q_T), && (3.5)\\
\big(u_i^{(\tau)}\big)^2 &\rightharpoonup u_i^2 && \text{weakly in } L^1(Q_T), && (3.6)\\
\varepsilon w_i^{(\tau)}, \varepsilon\nabla w_i^{(\tau)} &\rightharpoonup 0 && \text{weakly in } L^2(Q_T), && (3.7)\\
\partial_t^\tau u_i^{(\tau)} &\rightharpoonup \partial_t u_i && \text{weakly in } L^1(0,T;(H^s(\Omega))'), && (3.8)
\end{aligned}$$

for some functions u_1, u_2, where $1\le\alpha<4/3$ and $r=(2d+2)/(2d+1)$.

Proof. We first show (3.2). Since $\|u_i^{(\tau)}\|_{L^\infty(0,T;L_\Psi(\Omega))}$ is uniformly bounded, we obtain from (3.1)

$$\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^\infty(0,T;L^1(\Omega))} \le C, \quad \Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^2(0,T;H^1(\Omega))} \le C,$$

where here and in the following, $C>0$ denotes a generic constant which is independent of ε, τ, and n. By Gagliardo-Nirenberg's inequality with $p=2+2/d$ and $\theta=2d(p-1)/(d+2)p$ (and thus $\theta p=2$), we infer

$$\begin{aligned}
\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^p(Q_T)} &\le C\Big(\int_0^T \Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^1(\Omega)}^{(1-\theta)p}\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{H^1(\Omega)}^{\theta p}dt\Big)^{1/p}\\
&\le C\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^\infty(0,T;L^1(\Omega))}^{1-\theta}\Big(\int_0^T \Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{H^1(\Omega)}^{\theta p}dt\Big)^{1/p}\\
&\le C. \qquad (3.9)
\end{aligned}$$

Therefore, with $r=(2d+2)/(2d+1)$,

$$\begin{aligned}
\|u_1^{(\tau)}u_2^{(\tau)}\nabla(w_1^{(\tau)}+w_2^{(\tau)})\|_{L^r(Q_T)} &= 2\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\nabla\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^{(2d+2)/(2d+1)}(Q_T)}\\
&\le 2\Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^{2+2/d}(Q_T)}\Big\|\nabla\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|_{L^2(Q_T)}\\
&\le C, \qquad (3.10)
\end{aligned}$$

$$\|\nabla u_i^{(\tau)}\|_{L^{4/3}(Q_T)} = \|u_i^{(\tau)}\nabla w_i^{(\tau)}\|_{L^{4/3}(Q_T)} = 2\Big\|\sqrt{u_i^{(\tau)}}\nabla\sqrt{u_i^{(\tau)}}\Big\|_{L^{4/3}(Q_T)}$$

$$\leq 2\Big\|\sqrt{u_i^{(\tau)}}\Big\|_{L^4(Q_T)}\Big\|\nabla\sqrt{u_i^{(\tau)}}\Big\|_{L^2(Q_T)} \leq C, \tag{3.11}$$

$$\|u_1^{(\tau)}u_2^{(\tau)}\|_{L^{1+1/d}(Q_T)} = \Big\|\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\Big\|^2_{L^{2+2/d}(Q_T)} \leq C. \tag{3.12}$$

Let $P_n : H^s(\Omega) \to V_n$ be the projection on V_n. Then, for all $\psi = \phi\eta$ with $\phi \in H^s(\Omega)$ and $\eta \in L^\infty(0,T)$,

$$\begin{aligned}
&\left|\int_{Q_T} \partial_t^\tau u_i^{(\tau)}\psi dxdt\right| = \left|\int_{Q_T} \partial_t^\tau u_i^{(\tau)}(P_n\phi)\eta dxdt\right| \\
&= \left|\int_{Q_T} \big(\nabla(d_iu_i^{(\tau)} + u_1^{(\tau)}u_2^{(\tau)})\cdot\nabla(P_n\phi) + u_i^{(\tau)}(a_i - b_iu_1^{(\tau)} - c_iu_2^{(\tau)})P_n\phi\big)\eta dxdt\right| \\
&\quad \leq \int_0^T \Big(\|\nabla(d_iu_i^{(\tau)} + u_1^{(\tau)}u_2^{(\tau)})\|_{L^r(Q_T)}\|\nabla\phi\|_{L^{r'}(\Omega)}\|\eta\|_{L^{r'}(0,T)} \\
&\qquad + \|u_i^{(\tau)}(a_i - b_iu_1^{(\tau)} - c_iu_2^{(\tau)})\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}\|\eta\|_{L^\infty(0,T)}\Big)\, dxdt \\
&\quad \leq C\|\phi\|_{W^{1,r'}(\Omega)}\|\eta\|_{L^\infty(\Omega)} \leq C\|\psi\|_{L^\infty(0,T;H^s(\Omega))},
\end{aligned}$$

where $r' = 2d+2$. By density, this inequality also holds for all $\psi \in L^\infty(0,T;H^s(\Omega))$. This shows that

$$\|\partial_t^\tau u_i^{(\tau)}\|_{L^1(0,T;(H^s(\Omega))')} \leq C, \quad i = 1,2.$$

Summarizing, this bound and (3.11) give

$$\|u_i^{(\tau)}\|_{L^{4/3}(0,T;W^{1,4/3}(\Omega))} + \|\partial_t^\tau u_i^{(\tau)}\|_{L^1(0,T;(H^s(\Omega))')} \leq C.$$

Since $W^{1,4/3}(\Omega)$ injects compactly into $L^{4/3}(\Omega)$ and the latter space injects continuously into $(H^s(\Omega))'$, we can apply the version of Aubin's lemma in Ref.[24] (Thm. 5) to conclude, maybe passing to a subsequence which is not relabeled, that (3.2) holds.

In particular, (a subsequence of) the sequence $(u_i^{(\tau)})$ converges pointwise a.e. in Q_T to u_i as $(\varepsilon,\tau) \to 0$, $n \to \infty$. Since $(\sqrt{u_i^{(\tau)}})$ is bounded in $L^4(Q_T)$ (again a consequence of (3.1)), Lemma 3.1 implies that

$$\sqrt{u_i^{(\tau)}} \to \sqrt{u_i} \quad \text{strongly in } L^q(Q_T) \text{ for all } q < 4.$$

With this strong convergence result and the boundedness of $(\nabla\sqrt{u_i^{(\tau)}})$ in $L^2(Q_T)$ we conclude that

$$\nabla\sqrt{u_i^{(\tau)}} \rightharpoonup \nabla\sqrt{u_i} \quad \text{weakly in } L^2(Q_T).$$

Thus,

$$\nabla u_i^{(\tau)} = 2\sqrt{u_i^{(\tau)}}\nabla\sqrt{u_i^{(\tau)}} \rightharpoonup 2\sqrt{u_i}\nabla\sqrt{u_i} = \nabla u_i \quad \text{weakly in } L^q(Q_T) \tag{3.13}$$

for all $1 < q < 4/3$. In fact, since

$$\Big\|\sqrt{u_i^{(\tau)}}\nabla\sqrt{u_i^{(\tau)}}\Big\|^{4/3}_{L^{4/3}(Q_T)} = \Big\|\nabla\sqrt{u_i^{(\tau)}}\Big\|^{4/3}_{L^2(Q_T)}\|u_i^{(\tau)}\|^{2/3}_{L^2(Q_T)} \leq C,$$

by (3.1), the sequence $(\nabla u_i^{(\tau)})$ is bounded in $L^{4/3}(Q_T)$, and the weak convergence (3.13) also holds true for $q = 4/3$. This shows (3.3).

The bound (3.12) and the pointwise convergence of $(u_i^{(\tau)})$ imply that

$$\sqrt{u_1^{(\tau)}u_2^{(\tau)}} \rightharpoonup \sqrt{u_1u_2} \quad \text{weakly in } L^{2+2/d}(Q_T),$$

which proves (3.4).

Moreover, the discrete entropy estimate (3.1) gives

$$\nabla\sqrt{u_1^{(\tau)}u_2^{(\tau)}} \rightharpoonup \nabla\sqrt{u_1u_2} \quad \text{weakly in } L^2(Q_T).$$

Thus

$$\sqrt{u_1^{(\tau)}u_2^{(\tau)}}\nabla\sqrt{u_1^{(\tau)}u_2^{(\tau)}} \rightharpoonup \sqrt{u_1u_2}\nabla\sqrt{u_1u_2} \quad \text{weakly in } L^q(Q_T) \text{ for all } q < r.$$

In fact, this convergence also holds true for $q = r$ in view of the uniform bound provided by (3.10). Hence, (3.5) is shown.

Furthermore, (3.1) shows that $(u_i^{(\tau)})$ is bounded in $L^2(Q_T)$ and therefore, the pointwise convergence of $(u_i^{(\tau)})$ gives (3.6). The convergence (3.7) is a consequence of the uniform bound for $(\sqrt{\varepsilon}w_i^{(\tau)})$ in $L^2(0,T;H^1(\Omega))$ which follows from (3.1).

It remains to show that (3.8) holds. For this, let $\phi \in H^s(\Omega)$, $\eta \in L^\infty(0,T)$. Let $\delta > 0$ be arbitrary and let $n \in \mathbb{N}$ be so large that there exists $\chi \in V_n$ such that $\|\phi - \chi\|_{H^s(\Omega)} \le \delta$. Then

$$\begin{aligned}\int_{Q_T} \partial_t^\tau u_i^{(\tau)}\phi\eta dxdt &= \int_{Q_T} \partial_t^\tau u_i^{(\tau)}(\phi - \chi)\eta dxdt \\ &\quad + \int_{Q_T} \big(\nabla(d_iu_i^{(\tau)} + u_1^{(\tau)}u_2^{(\tau)})\cdot\nabla\chi + u_i^{(\tau)}(a_i - b_iu_1^{(\tau)} - c_iu_2^{(\tau)})\chi\big)\eta dxdt.\end{aligned} \tag{3.14}$$

The first term on the right-hand side can be estimated by

$$\left|\int_{Q_T} \partial_t^\tau u_i^{(\tau)}(\phi-\chi)\eta dxdt\right| \le \delta\|\partial_t^\tau u_i^{(\tau)}\|_{L^1(0,T;(H^s(\Omega))')}\|\eta\|_{L^\infty(\Omega)} \le \delta C.$$

In view of the above convergence results, the second term on the right-hand side of (3.14) is also converging. Therefore,

$$\lim_{(\varepsilon,\tau)\to 0, n\to\infty} \int_{Q_T} \partial_t^\tau u_i^{(\tau)}\psi dxdt$$

exists for all $\psi = \phi\eta$ and hence, by density, also for all $\psi \in L^\infty(0,T;H^s(\Omega))$. Thus, Lemma 3.2 implies (3.8). □

Proof. (Theorem 1.1.) The approximate problem (2.1) can be written as

$$\begin{aligned}\frac{1}{\tau}\int_{Q_T} \partial_t^\tau u_i^{(\tau)}\chi\eta dxdt &+ \varepsilon\int_{Q_T}\left(\nabla w_i^{(\tau)}\cdot\nabla\chi + w_i^{(\tau)}\chi\right)\eta dxdt \\ &+ \int_{Q_T}\left(d_iu_i^{(\tau)}\nabla w_i^{(\tau)} + u_1^{(\tau)}u_2^{(\tau)}\nabla(w_1^{(\tau)} + w_2^{(\tau)})\right)\nabla\chi\eta dxdt \\ &= \int_{Q_T} u_i^{(\tau)}\left(a_i - b_iu_1^{(\tau)} - c_iu_2^{(\tau)}\right)\chi\eta dxdt, \quad i = 1,2,\end{aligned}$$

where $\chi \in V_n$ and $\eta \in L^\infty(0,T)$. Lemma 3.3 allows to pass to the limit $(\varepsilon, \tau) \to 0$, $n \to \infty$ in the above equation which yields

$$\int_0^T \langle \partial_t u_i, \psi \rangle dt + \int_{Q_T} \nabla(d_i u_i + u_1 u_2) \cdot \nabla \psi dx dt = \int_{Q_T} u_i(a_i - b_i u_1 - c_i u_2)\psi dx dt,$$

for $i = 1, 2$ and for all $\psi \in L^\infty(0, T; H^s(\Omega))$, where $\langle \cdot, \cdot \rangle$ denotes the duality product between $(H^s(\Omega))'$ and $H^s(\Omega)$.

The initial data are satisfied in the sense of $(H^s(\Omega))'$ since

$$u_i \in W^{1,1}(0, T; (H^s(\Omega))') \subset C^0([0, T]; (H^s(\Omega))').$$

This proves Theorem 1.1. □

4. Long-time behavior of the solutions

The exponential decay of the transient solutions $(u_1, u_2)(\cdot, t)$ to its steady state (U_1, U_2) as $t \to \infty$ will be proven by means of the entropy–entropy production method. For this, we introduce the relative entropy

$$E(t; U_1, U_2) = \sum_{i=1}^2 \int_\Omega U_i \Phi\Big(\frac{u_i(t)}{U_i}\Big) dx,$$

where we recall that $\Phi(x) = x(\ln x - 1) + 1$. We only consider the special steady states

$$(\bar{U}_1, \bar{U}_2) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega (u_1^0, u_2^0) dx \quad \text{if } a_i = b_i = c_i = 0 \ (i = 1, 2) \text{ and}$$
$$(U_1^*, U_2^*) = \left(\frac{a_1}{b_1}, \frac{a_2}{c_2}\right) \quad \text{if } b_2 = c_1 = 0.$$

Lemma 4.1. *Let $0 < s < t$ and (u_1, u_2) be the weak solution to (1.4)–(1.5) obtained by Theorem 1.1. Then*

$$E(t; \bar{U}_1, \bar{U}_2) - E(s; \bar{U}_1, \bar{U}_2) \le -C \sum_{i=1}^2 \int_s^t \|\nabla \sqrt{u_i}\|_{L^2(\Omega)}^2 d\tau \tag{4.1}$$

if $a_i = b_i = c_i = 0$ $(i = 1, 2)$ and

$$E(t; U_1^*, U_2^*) - E(s; U_1^*, U_2^*) \le -C \sum_{i=1}^2 \int_s^t \|\nabla \sqrt{u_i}\|_{L^2(\Omega)}^2 d\tau \tag{4.2}$$
$$- \int_s^t \int_\Omega \big(b_1 u_1 (u_1 - U_1^*)(\ln u_1 - \ln U_1^*) + c_2 u_2 (u_2 - U_2^*)(\ln u_2 - \ln U_2^*)\big) dx dt$$

if $b_2 = c_1 = 0$.

Proof. We only prove the second inequality (4.2) since the proof of the first one is similar (and, in fact, simpler).

We choose $\chi = w_i^{(k,n)} - \ln U_i^* \in V_n$ in (2.1) and add the equations for $i=1$ and $i=2$. Since $b_2 = c_1 = 0$, we arrive, after a similar computation as in the proof of Lemma 2.1, to

$$\begin{aligned}
&\frac{1}{\tau}(E_*^{(k,n)} - E_*^{(k-1,n)}) + \varepsilon\sum_{i=1}^2\int_\Omega(|\nabla w_i^{(k,n)}|^2 + |w_i^{(k,n)}|^2)dx \\
&\quad + \sum_{i=1}^2\int_\Omega d_i u_i^{(k,n)}|\nabla w_i^{(k,n)}|^2dx + \int_\Omega u_1^{(k,n)}u_2^{(k,n)}|\nabla(w_1^{(k,n)} + w_2^{(k,n)})|^2dx \qquad (4.3)\\
&\quad + \int_\Omega\Big[b_1u_1^{(k,n)}(u_1^{(k,n)} - U_1^*)\ln\Big(\frac{u_1^{(k,n)}}{U_1^*}\Big) + c_2u_2^{(k,n)}(u_2^{(k,n)} - U_2^*)\ln\Big(\frac{u_2^{(k,n)}}{U_2^*}\Big)\Big]dx \\
&\le \varepsilon\sum_{i=1}^2\int_\Omega w_i^{(k,n)}\ln U_i^*dx,
\end{aligned}$$

where

$$E_*^{(k,n)} = \sum_{i=1}^2\int_\Omega U_i^*\Phi\Big(\frac{u_i^{(k,n)}}{U_i^*}\Big)dx.$$

As in the proof of Lemma 2.1 we can rewrite the above estimate in terms of the variables $u_i^{(\tau)}$, $i=1,2$, which are piecewise constant in time.

We claim now that for all $0<s<t$, up to subsequences which are not relabeled, the following limits hold as $(\varepsilon,\tau)\to 0$ and $n\to\infty$:

(i) $E(t;U_1^*,U_2^*) = \lim E_*^{(k,n)}$ if $t\in(t_{k-1},t_k]$,

(ii) $\displaystyle\int_s^t\int_\Omega|\nabla\sqrt{u_i}|^2dxd\tau \le \liminf\int_s^t\int_\Omega u_i^{(\tau)}|\nabla w_i^{(\tau)}|^2dxd\tau$,

(iii) $\displaystyle\int_s^t\int_\Omega u_i(u_i - U_i^*)\ln\big(\tfrac{u_i}{U_i^*}\big)dxd\tau \le \liminf\textstyle\int_s^t\int_\Omega u_i^{(\tau)}(u_i^{(\tau)} - U_i^*)\ln\big(\frac{u_i^{(\tau)}}{U_i^*}\big)dxd\tau$,

(iv) $\displaystyle\varepsilon\int_s^t\int_\Omega w_i^{(\tau)}\ln U_i^*dxd\tau \to 0$.

Indeed, the convergence result (i) follows from the strong L^p ($p<2$) convergence of $u_i^{(\tau)}(\cdot,t)$ to $u_i(\cdot,t)$ for a.e. $t>0$ and Lebesgue's dominated convergence theorem. The result (ii) is a consequence of the weakly lower semi-continuity of the L^2 norm. Furthermore, Fatou's lemma and the pointwise convergence of $(u_i^{(k,n)})$ imply (iii). Finally, the estimate (3.1) shows that

$$\Big|\varepsilon\int_\Omega w_i^{(k,n)}\ln U_i^*dx\Big| \le \sqrt{\varepsilon}C\|\sqrt{\varepsilon}w_i^{(k,n)}\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}$$

from which we conclude (iv). Thus, the limit $(\varepsilon,\tau)\to 0$, $n\to\infty$ in (4.3) finishes the proof. $\square$

Proof. (Theorem 1.2.) Inequality (4.1) implies

$$E(t;\bar U_1,\bar U_2) \le E(0;\bar U_1,\bar U_2) - C\sum_{i=1}^{2}\int_0^t \|\nabla\sqrt{u_i}\|^2_{L^2(\Omega)}d\tau.$$

Thus, employing the logarithmic Sobolev inequality,[4,10]

$$\int_\Omega g^2\ln\Big(\frac{g^2}{\bar g}\Big)dx \le C\int_\Omega |\nabla g|^2dx$$

for all $g\in H^1(\Omega)$ such that $\bar g = \text{meas}(\Omega)^{-1}\|g\|^2_{L^2(\Omega)}$, and the conservation of mass,

$$\int_\Omega u_i(\cdot,t)dx = \int_\Omega u_i^0 dx = \text{meas}(\Omega)\bar U_i, \quad t>0,$$

we arrive to

$$E(t;\bar U_1,\bar U_2) \le E(0;\bar U_1,\bar U_2) - C\int_0^t E(\tau;\bar U_1,\bar U_2)d\tau, \quad t>0.$$

Hence, by Gronwall's inequality,

$$E(t;\bar U_1,\bar U_2) \le E(0;\bar U_1,\bar U_2)e^{-Ct}, \quad t>0.$$

The L^1 decay is derived by applying the Csiszár-Kullback inequality[4,6,15]

$$\|g-G\|^2_{L^1(\Omega)} \le 4M\int_\Omega G\Phi\Big(\frac{g}{G}\Big)dx,$$

for all non-negative g, $G\in L^1(\Omega)$ such that $\int G\Phi(g/G)dx$ exists and satisfying $\int g dx = \int G dx = M$. Indeed, we obtain

$$\begin{aligned}\sum_{i=1}^{2}\frac{1}{2\text{meas}(\Omega)\bar U_i}\|u_i(\cdot,t)-\bar U_i\|_{L^1(\Omega)} &\le \sqrt{E(t;\bar U_1,\bar U_2)}\\ &\le \sqrt{E(0;\bar U_1,\bar U_2)}e^{-Ct/2}.\end{aligned}$$

This proves the theorem. □

Proof. (Proposition 1.1.) Since the function $x\mapsto \ln x$ is non-decreasing, inequality (4.2) implies

$$\frac{d}{dt}E(t;U_1^*,U_2^*) + \sum_{i=1}^{2}\|\nabla\sqrt{u_i}\|^2_{L^2(\Omega)} \le 0.$$

Moreover, if $\frac{d}{dt}E(t_0;U_1^*,U_2^*) = 0$ for some $t_0>0$ then (4.2) proves that $u_i(\cdot,t_0) = U_i^*$, $i=1,2$. Consequently, since $t\mapsto E(t;U_1^*,U_2^*)$ is non-negative and non-increasing, $E(t;U_1^*,U_2^*) = E(t_0;U_1^*,U_2^*) = 0$ for $t\ge t_0$. This shows that $u_i(\cdot,t) = U_i^*$, $i=1,2$, for $t\ge t_0$, which means that there only exist constant stationary solutions. □

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