



Asymptotic properties of Kneser solutions to nonlinear second order ODEs with regularly varying coefficients

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Abstract

In this work, we investigate properties of a class of solutions to the second order ODE,

$$(p(t)u'(t))' + q(t)f(u(t)) = 0$$

on the interval $[a, \infty)$, $a \geq 0$, where p and q are functions regularly varying at infinity, and f satisfies $f(L_0) = f(0) = f(L) = 0$, with $L_0 < 0 < L$. Our aim is to describe the asymptotic behaviour of the non-oscillatory solutions satisfying one of the following conditions:

$$u(a) = u_0 \in (0, L), \quad 0 \leq u(t) \leq L, \quad t \in [a, \infty),$$

$$u(a) = u_0 \in (L_0, 0), \quad L_0 \leq u(t) \leq 0, \quad t \in [a, \infty).$$

The existence of Kneser solutions on $[a, \infty)$ is investigated and asymptotic properties of such solutions and their first derivatives are derived. The analytical findings are illustrated by numerical simulations using the collocation method.

Keywords: second order ordinary differential equations, regular variation, asymptotic properties, non-oscillatory solutions, Kneser solutions

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1. Introduction

In this paper, within the framework of regular variation, we study the existence and asymptotic behaviour of Kneser solutions to the nonlinear second order ODE,

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \tag{1}$$

where $t \in [a, \infty)$, $a \geq 0$. This equation is closely related to the extensively studied Emden-Fowler equation,

$$(p(t)\Phi_\alpha(u'))' + q(t)\Phi_\gamma(u) = 0,$$

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where $\Phi_\alpha(u) =: |u|^\alpha \operatorname{sgn} u$, $\alpha \geq 1$. The Emden-Fowler equation is called sub-half-linear, half-linear or super-half-linear if $\alpha > \gamma$, $\alpha = \gamma$ or $\alpha < \gamma$, respectively. The results for sub-half-linear case can be found in [17, 20] those for the half-linear case in [3, 12, 18]. The super-half-linear case was studied in [4, 22], where a different sign condition was posed on the nonlinear term when compared to the present paper (cf. (22) and (29)). According to the above terminology, equation (1) can be investigated in a neighbourhood of the origin as a super-linear equation, since in our case $\alpha = 1$ and $\gamma = r > 1 = \alpha$, see (42).

In the following analysis, we assume that the data functions p and q are regularly varying at infinity and $a \geq 0$. Moreover, we discuss solutions of (1) satisfying

$$u(a) = u_0 \in (0, L), \quad 0 \leq u(t) \leq L \text{ for } t \in [a, \infty), \quad (2)$$

or

$$u(a) = u_0 \in (L_0, 0), \quad L_0 \leq u(t) \leq 0 \text{ for } t \in [a, \infty), \quad (3)$$

where the interval $[L_0, L]$ is specified in the following way:

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0.$$

Note that for $a > 0$ equation (1) is regular, while for $a = 0$, we assume that $p(0) = 0$, and therefore there is a time singularity at $t = 0$.

Definition 1.1. A function u is called a *solution of equation (1) on $[a, \infty)$* if $u \in C^1[a, \infty)$, $pu' \in C^1[a, \infty)$, and u satisfies equation (1) for all $t \in [a, \infty)$. The solution u of equation (1) on $[a, \infty)$ is called a *solution of problem (1), (2) or problem (1), (3)* if u additionally satisfies condition (2) or (3), respectively.

Definition 1.2. A solution u of equation (1) on $[a, \infty)$ is called a *Kneser solution* if there exists $t_0 > a$ such that

$$u(t)u'(t) < 0 \text{ for } t \in [t_0, \infty). \quad (4)$$

The aim of the paper is twofold. First of all, we investigate the existence of the Kneser solutions to problems (1), (2) and (1), (3). Moreover, we describe the asymptotic properties of the Kneser solutions in the framework of regularly varying functions. The provided asymptotic formulas are generalizations of those discussed in [31], where the case $p \equiv q$ was investigated. The existence of various types of solutions to (1) with $p \equiv q$ has been also studied in [26, 27, 28].

Other asymptotic results for related equations or systems which are characterized by regularly varying functions can be found in [6, 10, 11, 16, 17, 18, 19, 23, 24, 29, 30]. We also refer to [5, 13], where Kneser solutions for two-dimensional systems of ODEs were studied.

2. Regularly varying functions

In this section, functions regularly varying at infinity are introduced and some of their basic properties necessary for the analysis are shown. For a complete treatment of the theory of regularly varying functions, see for instance [1], and [21] for the applications to asymptotics of ordinary differential equation.

Definition 2.1. A function g , which is positive and measurable on $[\tau_0, \infty)$, $\tau_0 > 0$, is called *regularly varying of index $\alpha \in \mathbb{R}$* if for each $\lambda > 0$

$$\lim_{t \rightarrow \infty} \frac{g(\lambda t)}{g(t)} = \lambda^\alpha.$$

The set of all regularly varying functions of index α is denoted by $RV(\alpha)$.

Remark 2.2. A regularly varying function of index $\alpha = 0$ is called a *slowly varying* function and the set of those functions is denoted by SV . A slowly varying function can be bounded or unbounded, but as $t \rightarrow \infty$ it can neither grow too fast to infinity, nor decay too fast to zero. This means that

$$\lim_{t \rightarrow \infty} t^\varepsilon L(t) = \infty, \quad \lim_{t \rightarrow \infty} t^{-\varepsilon} L(t) = 0$$

for any $\varepsilon > 0$ holds.

Remark 2.3. Definition 2.1 implies that a regularly varying function g of index α can be represented as

$$g(t) = t^\alpha L(t), \quad t \in [\tau_0, \infty), \quad (5)$$

where L is a slowly varying function.

In what follows, the symbol \sim is used to denote the asymptotic equivalence,

$$f(t) \sim g(t) \text{ as } t \rightarrow \infty \Leftrightarrow \lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 1.$$

Theorem 2.4. (Karamata Integration Theorem)

Let $L(t) \in SV$, $c > 0$.

(i) If $\alpha > -1$, then

$$\int_c^t s^\alpha L(s) ds \sim \frac{1}{\alpha + 1} t^{\alpha+1} L(t) \text{ as } t \rightarrow \infty.$$

(ii) If $\alpha < -1$, then

$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha + 1} t^{\alpha+1} L(t) \text{ as } t \rightarrow \infty.$$

(iii) If $\alpha = -1$, then

$$l(t) = \int_c^t \frac{L(s)}{s} ds \in SV \text{ and } \lim_{t \rightarrow \infty} \frac{L(t)}{l(t)} = 0.$$

In order to investigate the asymptotic behaviour of the non-oscillatory solutions of problem (1), (2) and problem (1), (3), we first need to provide auxiliary lemmas for regularly varying functions.

Lemma 2.5. Let $\alpha > 0$ and $g \in RV(\alpha)$. Then,

$$\lim_{t \rightarrow \infty} g(t) \int_t^\infty \frac{ds}{g(s)} = \infty. \quad (6)$$

Proof: According to Remark 2.3, the function g can be represented as $g(t) = t^\alpha L(t)$, $t \in [\tau_0, \infty)$, where $L \in SV$. For $\alpha > 1$, property (6) is a simple consequence of Theorem 2.4 (ii), since $-\alpha < -1$. Indeed, noticing that $L^{-1} \in SV$, for $t \rightarrow \infty$, we have

$$\int_t^\infty \frac{ds}{g(s)} = \int_t^\infty s^{-\alpha} L^{-1}(s) ds \sim \frac{1}{\alpha - 1} t^{-\alpha+1} L^{-1}(t)$$

and therefore, the function

$$g(t) \int_t^\infty \frac{ds}{g(s)} = t^\alpha L(t) \int_t^\infty s^{-\alpha} L^{-1}(s) ds$$

is asymptotically equivalent to

$$\frac{1}{\alpha - 1} t L(t) L^{-1}(t) = \frac{t}{\alpha - 1}.$$

Thus, for $\alpha > 1$, property (6) follows.

Let us now consider $\alpha \in (0, 1]$. Then,

$$\int_t^\infty \frac{ds}{g(s)} = \int_t^\infty s^{-\alpha} L^{-1}(s) ds = \int_t^\infty s^{-\alpha-1} s L^{-1}(s) ds \geq t \int_t^\infty s^{-\alpha-1} L^{-1}(s) ds, \quad t \in [\tau_0, \infty).$$

According to Theorem 2.4 (ii) for $-\alpha - 1 < -1$, this is asymptotically equivalent to

$$\frac{t^{1-\alpha} L^{-1}(t)}{\alpha}.$$

Therefore, for $t \rightarrow \infty$

$$g(t) \int_t^\infty \frac{ds}{g(s)} \geq t^{\alpha+1} L(t) \int_t^\infty s^{-\alpha-1} L^{-1}(s) ds \sim \frac{t}{\alpha}$$

and (6) follows for any $\alpha \in (0, 1]$. \square

Lemma 2.6. *Let us assume that the functions g and h satisfy $g \in RV(\alpha)$ and $h \in RV(\beta)$, where $\alpha > 0$, $\beta > 0$, $\beta - \alpha > -1$, and $c > \tau_0$. Then,*

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \int_c^t h(s) ds = \infty.$$

Proof: According to Remark 2.3, the functions g and h can be represented as

$$g(t) = t^\alpha L_g(t), \quad h(t) = t^\beta L_h(t), \quad t \in [\tau_0, \infty),$$

where $L_g, L_h \in SV$. Therefore,

$$\frac{1}{g(t)} \int_c^t h(s) ds = t^{-\alpha} L_g^{-1}(t) \int_c^t s^\beta L_h(s) ds. \quad (7)$$

Due to Theorem 2.4 (i), the function given by (7) is asymptotically equivalent to the function

$$\frac{1}{\beta+1} t^{-\alpha} L_g^{-1}(t) t^{\beta+1} L_h(t) = \frac{1}{\beta+1} t^{\beta-\alpha+1} L(t),$$

where $L(t) = L_g^{-1}(t) L_h(t) \in SV$. Now, Remark 2.2 and the assumption $\beta - \alpha > -1$ imply

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \int_c^t h(s) ds = \infty. \quad \square$$

3. Existence of Kneser solutions to regular equation (1)

In this section, the existence of the Kneser solutions to problems (1), (2) and (1), (3) with $a > 0$ is discussed. We assume

$$L_0 < 0 < L, \quad f \in C[L_0, L], \quad f(L_0) = f(0) = f(L) = 0, \quad (8)$$

$$p \in C[a, \infty), \quad p > 0 \text{ on } [a, \infty), \quad (9)$$

$$q \in C[a, \infty), \quad q > 0 \text{ on } (a, \infty). \quad (10)$$

The existence result is proved using the Diagonalization Lemma.

Lemma 3.1. (Diagonalization Lemma)

Let $u_n \in C^1[a, n]$, $n \in \mathbb{N}$, $n > a$ be such that for each $b > a$ there exists $\rho_b > 0$ satisfying

$$|u_n^{(j)}(t)| \leq \rho_b \text{ for } t \in [a, b], n \geq b, j = 0, 1,$$

and

$$\{u'_n\}_{n \geq b} \text{ is equicontinuous on } [a, b].$$

Then, there exists a subsequence $\{u_{k_n}\} \subset \{u_n\}$ and $u \in C^1[a, \infty)$ such that

$$\lim_{n \rightarrow \infty} u_{k_n}^{(j)}(t) = u^{(j)}(t) \text{ locally uniformly on } [a, \infty), j = 0, 1.$$

Proof: Let $\{b_n\} \in \mathbb{N}$ be increasing, $b_n > a$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} b_n = \infty$. Then, for $b_1 \in \mathbb{N}$, we have $|u_n^{(j)}(t)| < \rho_{b_1}$ for $t \in [a, b_1], n \geq b_1, j = 0, 1$. Moreover, $\{u_n\}_{n \geq b_1}$ is equicontinuous on $[a, b_1]$ and hence, by the Arzelà-Ascoli Theorem, there is a subsequence $\{u_{k_1,n}\} \subset \{u_n\}_{n \geq b_1}$, for which $\{u_{k_1,n}^{(j)}(t)\}$ is uniformly convergent on $[a, b_1]$ for $j = 0, 1$.

Next, note that there exists a subsequence $\{u_{k_2,n}\} \subset \{u_{k_1,n}\}$ such that $\{u_{k_2,n}^{(j)}\}$ is uniformly convergent on $[a, b_2]$ for $j = 0, 1$ and that we can proceed inductively to obtain a subsequence $\{u_{k_i,n}\} \subset \{u_{k_{i-1},n}\}$ such that $\{u_{k_i,n}^{(j)}\}$ is uniformly convergent on $[a, b_i]$ for $j = 0, 1$.

Let $k_n := k_{n,n}$ for $n \in \mathbb{N}$ and consider the diagonal sequence $\{u_{k_n}\}$. Choose $\beta > a$. Then, $[a, \beta] \subset [a, b_m]$ for some $m \in \mathbb{N}$. Since $\{u_{k_n}\}_{n \geq m}$ is extracted from $\{u_{k_{m,n}}\}$ and $\{u_{k_{m,n}}^{(j)}\}$ is uniformly convergent on $[a, b_m]$ for $j = 0, 1$, it follows that $\{u_{k_n}^{(j)}\}$ is uniformly convergent on $[a, \beta]$ for $j = 0, 1$. Consequently, $\{u_{k_n}^{(j)}\}_{n \geq m}$ is locally uniformly convergent on $[a, \infty)$. Let $\lim_{n \rightarrow \infty} u_{k_n}(t) = u(t)$ and $\lim_{n \rightarrow \infty} u'_{k_n}(t) = v(t)$ for $t \in [a, \infty)$. Then, $u, v \in C[a, \infty)$ and letting $n \rightarrow \infty$ in

$$u_{k_n}(t) = u_{k_n}(a) + \int_a^t u'_{k_n}(s) ds, \quad t \in [a, n], n \in \mathbb{N},$$

yields

$$u(t) = u(a) + \int_a^t v(s) ds, \quad t \in [a, \infty).$$

Hence, $u \in C^1[a, \infty), v = u'$ on $[a, \infty)$ and the result follows. □

The fact that $a > 0$ is crucial for the existence result stated in Theorem 3.2 since in this case equation (1) is regular on $[a, \infty)$. Before proceeding, we define the function f^* by

$$f^*(x) = \begin{cases} \frac{L-x}{x-L+1}, & x > L, \\ f(x), & x \in [0, L], \\ \frac{x}{x-1}, & x < 0, \end{cases} \tag{11}$$

and we consider the following auxiliary equation:

$$(p(t)u'(t))' + q(t)f^*(u(t)) = 0. \tag{12}$$

Theorem 3.2. *Let assumptions (8)–(10) be satisfied and $a > 0$. Then, problem (1), (2) (and problem (1), (3)) has at least one solution.*

Proof:

Step 1. *Showing solvability of an auxiliary Dirichlet problem:*

Let $n > a, u_0 \in (0, L)$, and consider the Dirichlet boundary conditions

$$u(a) = u_0, u(n) = 0 \tag{13}$$

to hold. We first prove the existence of a solution to problem (12), (13). The linear homogeneous problem

$$(p(t)u'(t))' = 0, u(a) = 0, u(n) = 0, \tag{14}$$

has only the trivial solution $u \equiv 0$. Indeed, the assumption of the existence of a nontrivial solution results in the following contradiction. Let u be a nontrivial solution of (14). Then, there exists $\theta \in (a, n)$ such that $u(\theta) \neq 0, u'(\theta) = 0$. Integrating (14) from θ to $t \in [a, n]$, yields

$$p(t)u'(t) = p(\theta)u'(\theta) = 0, \quad t \in [a, n].$$

Since p is positive on $[a, \infty)$, $u' = 0$ on $[a, n]$ follows and hence, u has to be a constant function on $[a, n]$. Therefore, $u \equiv 0$ is the only solution to (14). Consequently, there exists the unique Green's function $G(t, s)$ for problem (14),

$$G(t, s) = \begin{cases} \left(1 - \frac{P(s)}{P(n)}\right)P(t) & \text{for } a \leq t \leq s \leq n, \\ \left(1 - \frac{P(t)}{P(n)}\right)P(s) & \text{for } a \leq s \leq t \leq n, \end{cases} \quad (15)$$

where $P(t) = \int_a^t \frac{d\tau}{p(\tau)}$, $t \in [a, n]$. The Green's function (15) is bounded,

$$|G(t, s)| \leq P(n) \text{ for } t, s \in [a, n].$$

Moreover, the partial derivative of $G(t, s)$ has the form

$$\frac{\partial G(t, s)}{\partial t} = \begin{cases} \left(1 - \frac{P(s)}{P(n)}\right) \frac{1}{p(t)} & \text{for } a \leq t < s \leq n, \\ -\frac{P(s)}{P(n)} \frac{1}{p(t)} & \text{for } a \leq s < t \leq n, \end{cases}$$

and is also bounded,

$$\left| \frac{\partial G(t, s)}{\partial t} \right| \leq \frac{1}{p_{\min}}, \quad t \in [a, s] \cup (s, n], s \in [a, n],$$

where $p_{\min} = \min\{p(t) : t \in [a, n]\} > 0$. Consequently, for every $y \in C[a, n]$, the inhomogeneous linear problem

$$(p(t)u'(t))' = y(t), \quad u(a) = u_0, u(n) = 0,$$

has the unique solution

$$u(t) = \frac{u_0}{P(n)} \int_t^n \frac{d\tau}{p(\tau)} - \int_a^n G(t, s)y(s) ds, \quad t \in [a, n].$$

Let us define the operator $T : C[a, n] \rightarrow C[a, n]$,

$$(Tu)(t) = \frac{u_0}{P(n)} \int_t^n \frac{d\tau}{p(\tau)} + \int_a^n G(t, s)q(s)f^*(u(s)) ds, \quad t \in [a, n]$$

and let u be a fixed point of T . Then,

$$\begin{aligned} u(t) &= \frac{u_0}{P(n)} \int_t^n \frac{d\tau}{p(\tau)} + \int_a^t \left(1 - \frac{P(t)}{P(n)}\right)P(s)q(s)f^*(u(s)) ds \\ &\quad + \int_t^n \left(1 - \frac{P(s)}{P(n)}\right)P(t)q(s)f^*(u(s)) ds, \\ u'(t) &= -\frac{u_0}{P(n)p(t)} - \frac{1}{p(t)} \int_a^t \frac{1}{P(n)}P(s)q(s)f^*(u(s)) ds \\ &\quad + \frac{1}{p(t)} \int_t^n \left(1 - \frac{P(s)}{P(n)}\right)q(s)f^*(u(s)) ds, \\ (p(t)u'(t))' &= -\frac{1}{P(n)}P(t)q(t)f^*(u(t)) - \left(1 - \frac{P(t)}{P(n)}\right)q(t)f^*(u(t)) = -q(t)f^*(u(t)), \quad t \in [a, n]. \end{aligned}$$

Therefore, $u \in C^1[a, n]$, $pu' \in C^1[a, n]$ and u is a solution of equation (12) in $[a, n]$. Moreover, since $P(a) = 0$, we have

$$\begin{aligned} u(a) &= \frac{u_0}{P(n)}P(n) + \int_a^n \left(1 - \frac{P(s)}{P(n)}\right)P(a)q(s)f^*(u(s)) ds = u_0, \\ u(n) &= \int_a^n \left(1 - \frac{P(n)}{P(n)}\right)P(s)q(s)f^*(u(s)) ds = 0, \end{aligned}$$

and conditions (13) hold.

To show the existence of a fixed point of the operator T , we now use the Schauder Fixed Point Theorem. Let $\Omega \subset C[a, b]$,

$$\Omega = \{x \in C[a, n] : \|x\|_{C[a, n]} \leq R\}, \quad (16)$$

where

$$R = |u_0| + P(n)MQ, \quad M = \sup\{|f^*(x)| : x \in \mathbb{R}\}, \quad Q = \int_a^n q(s) ds.$$

Then,

$$\begin{aligned} \|Tu\|_{C[a, n]} &= \max_{t \in [a, n]} \left\{ \frac{u_0}{P(n)} \int_t^n \frac{d\tau}{p(\tau)} + \int_a^n G(t, s)q(s)f^*(u(s)) ds \right\} \\ &\leq \frac{|u_0|}{P(n)}P(n) + P(n)M \int_a^n q(s) ds = R. \end{aligned}$$

Consequently, $T(\Omega)$ is bounded in $C[a, n]$. Due to (16), $T(\Omega) \subset \Omega$. Since f^* is a continuous function, it follows from

$$\|Tu_m - Tu\|_{C[a, n]} \leq \max_{t \in [a, b]} \left\{ \int_a^n |G(t, s)q(s)| |f^*(u_m(s)) - f^*(u(s))| ds \right\}, \quad \{u_m\} \subset \Omega, u \in \Omega,$$

that T is continuous on Ω . Moreover, for $u \in \Omega$, $t_1, t_2 \in [a, n]$, there exists $\xi \in (t_1, t_2)$ such that

$$\begin{aligned} |(Tu)(t_1) - (Tu)(t_2)| &\leq \left| \frac{u_0}{P(n)} \int_{t_1}^{t_2} \frac{d\tau}{p(\tau)} \right| + \int_a^n |G(t_1, s) - G(t_2, s)|q(s)|f^*(u(s))| ds \\ &\leq \frac{|u_0|}{P(n)} \frac{|t_1 - t_2|}{p_{\min}} + \int_a^n \frac{|\partial G(\xi, s)|}{\partial t} |t_1 - t_2|q(s)|f^*(u(s))| ds \\ &\leq |t_1 - t_2| \left(\frac{|u_0|}{P(n)} \frac{1}{p_{\min}} + M \int_a^n \frac{q(s)}{p_{\min}} ds \right) \\ &\leq |t_1 - t_2| \left(\frac{|u_0|}{P(n)} \frac{1}{p_{\min}} + \frac{MQ}{p_{\min}} \right). \end{aligned}$$

This implies the compactness of T on Ω , due to the Arzelà-Ascoli Theorem. Since the operator T is continuous and compact on Ω and $T(\Omega) \subset \Omega$, there exists a fixed point $u = Tu$ due to the Schauder Fixed Point Theorem.

Step 2. *Showing solvability of problem (1), (13):*

Let u be a solution of problem (12), (13). We now show that

$$0 \leq u(t) \leq L \text{ for } t \in [a, n]. \quad (17)$$

Let us assume that

$$u(t_0) = \max\{u(t) : t \in [a, n]\} > L. \quad (18)$$

Since $u(a) = u_0 \in (0, L)$ and $u(n) = 0$, it follows that $t_0 \in (a, n)$ and $u'(t_0) = 0$. Therefore, we can find $\delta > 0$ such that $u(t) > L$ on $(t_0, t_0 + \delta) \subset (a, n)$ and, by (11),

$$(p(t)u'(t))' = -q(t) \frac{L - u(t)}{u(t) - L + 1} > 0, \quad t \in (t_0, t_0 + \delta) \quad (19)$$

follows. After integrating (19) over (t_0, t) , $t \in (t_0, t_0 + \delta)$, we obtain

$$0 < - \int_{t_0}^t q(s) \frac{L - u(s)}{u(s) - L + 1} ds = p(t)u'(t).$$

Thus, $u' > 0$ on $(t_0, t_0 + \delta)$, which contradicts (18).

Analogously, the contradiction follows when we assume

$$\min\{u(t) : t \in [a, n]\} < 0$$

to hold. Finally, it follows from (11) and (17) that u is a solution of equation (1) on $[a, n]$.

Step 3. *Showing solvability of problem (1), (2):*

It follows from Step 2 that for each $n \in \mathbb{N}$, $n \geq a$, we have a solution u_n of equation (1) on $[a, n]$. This solution satisfies

$$u_n(a) = u_0, \quad 0 \leq u_n(t) \leq L \text{ for } t \in [a, n]. \quad (20)$$

We now show that there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ which locally uniformly converges on $[a, \infty)$ to a solution u of problem (1), (2). To this aim, we consider an arbitrary compact interval $[a, b] \subset [a, \infty)$. Then, the following holds

$$0 \leq u_n(t) \leq L, \quad t \in [a, b], \quad n > b.$$

Consequently, there exists $\tau_n \in [a, b]$ such that $|u'_n(\tau_n)| \leq \frac{L}{b-a}$.

Let us estimate the first derivative of the solution u_n on $[a, b]$. After integrating equation (1) from $t \in [a, b]$ to τ_n we obtain

$$\begin{aligned} p(t)u'_n(t) &= p(\tau_n)u'_n(\tau_n) + \int_t^{\tau_n} q(s)f(u_n(s)) \, ds, \\ u'_n(t) &= \frac{p(\tau_n)}{p(t)}u'_n(\tau_n) + \frac{1}{p(t)} \int_t^{\tau_n} q(s)f(u_n(s)) \, ds, \\ |u'_n(t)| &\leq \frac{p_{\max}}{p_{\min}} \frac{L}{b-a} + \frac{1}{p_{\min}} q_{\max} f_{\max}(b-a) =: R_b, \quad t \in [a, b], \end{aligned} \quad (21)$$

where

$$\begin{aligned} p_{\max} &= \max\{p(t) : t \in [a, b]\}, \\ p_{\min} &= \min\{p(t) : t \in [a, b]\}, \\ q_{\max} &= \max\{q(t) : t \in [a, b]\}, \\ f_{\max} &= \max\{|f(x)| : 0 \leq x \leq L\}. \end{aligned}$$

According to (21), the sequence $\{u_n\}$ is equicontinuous on $[a, b]$. Equation (1) yields

$$|(p(t)u'_n(t))'| \leq |q(t)f(u_n(t))| \leq q_{\max}f_{\max}, \quad t \in [a, b].$$

Since the sequence $\{(pu'_n)'\}$ is bounded on $[a, b]$, the sequence $\{pu'_n\}$ is equicontinuous on $[a, b]$. Since $p_{\min} > 0$, we have by (21) for $t_1, t_2 \in [a, b]$

$$|u'_n(t_1) - u'_n(t_2)| \leq \frac{1}{p_{\min}} (|p(t_1)u'_n(t_1) - p(t_2)u'_n(t_2)| + R_b|p(t_2) - p(t_1)|).$$

This implies that the sequence $\{u'_n\}$ is also equicontinuous on $[a, b]$. From the Arzelà-Ascoli Theorem and the Diagonalization Lemma 3.1 it follows, that there exists a subsequence $u_{n_m} \rightrightarrows^{loc} u$, $u'_{n_m} \rightrightarrows^{loc} u'$ on $[a, \infty)$ and u is a solution of equation (1) on $[a, \infty)$. By (20), u satisfies (2).

For problem (1), (3) we consider $u_0 \in (L_0, 0)$ and use the dual argument. □

Imposing some additional assumptions on f , p , and q , permits to derive two different limits of solutions to problems (1), (2) and (1), (3).

Theorem 3.3. Let us assume that $a > 0$, conditions (8)–(10) hold, and

$$f \in \text{Lip}_{loc}(0, L), \quad f(x) > 0 \text{ for } x \in (0, L). \quad (22)$$

Then, problem (1), (2) has a solution u , such that

$$0 < u(t) < L \quad \text{for } t \in [a, \infty). \quad (23)$$

If in addition

$$\lim_{t \rightarrow \infty} \frac{1}{p(t)} \int_a^t q(s) \, ds = \infty, \quad (24)$$

with

$$\liminf_{t \rightarrow \infty} p(t) > 0, \quad (25)$$

then, either

$$u'(t) > 0 \text{ for } t \geq a \text{ and } \lim_{t \rightarrow \infty} u(t) = L, \quad (26)$$

or

$$u \text{ is a Kneser solution.} \quad (27)$$

Proof: According to Theorem 3.2, problem (1), (2) has a solution u . Let us assume that $u(b) = L$ for some $b > a$. Due to (2), $u'(b) = 0$. By virtue of the first condition in (22), $u \equiv L$ is the only solution satisfying $u(b) = L, u'(b) = 0$. Therefore, $u(t) < L$ for $t \in [a, \infty)$. Assume that $u(c) = 0$ for some $c > a$. Due to (2), $u'(c) = 0$. Integrating (1) over $(c, t), t \in [a, \infty)$, and using the second condition in (22) yields $u'(t) \leq 0$ for $t > c$ and $u'(t) \geq 0$ for $t < c$. Therefore, $u \equiv 0$ which contradicts $u(a) = u_0 > 0$ and consequently, (23) holds. By (1), (2), (10), and (22)

$$(pu')'(t) = -q(t)f(u(t)) < 0, \quad t \geq a, \quad (28)$$

and thus, pu' is decreasing on $[a, \infty)$.

Assume that $u' > 0$ on $[a, \infty)$. Then, there exists $\lim_{t \rightarrow \infty} u(t) =: l_0 \in (u_0, L]$. Let $l_0 \in (u_0, L)$ and let us denote $m_0 := \min\{f(x) : x \in [u_0, l_0]\} > 0$. Integration of (1) over $[a, t]$ yields

$$\begin{aligned} p(t)u'(t) - p(a)u'(a) &\leq -m_0 \int_a^t q(s) \, ds, \quad t \in [a, \infty), \\ 0 < u'(t) &\leq \frac{1}{p(t)} (p(a)u'(a) - m_0 \int_a^t q(s) \, ds), \quad t \in [a, \infty). \end{aligned}$$

Letting $t \rightarrow \infty$ and using (24), (25), we arrive at $0 \leq \liminf_{t \rightarrow \infty} u'(t) \leq -\infty$, which is a contradiction. Therefore $l_0 = L$ and (26) holds.

Assume now that there exists $b \geq a$ such that $u'(b) \leq 0$. Then, $(pu')(b) \leq 0$ and since pu' is decreasing, we can find $t_0 > b$ such that $pu' < 0$ on $[t_0, \infty)$. By (9), $u' < 0$ on $[t_0, \infty)$ and (27) follows. \square

The dual theorem formulated below can be proved using similar arguments.

Theorem 3.4. Let (8)–(10) hold and $a > 0$. Moreover, let us assume

$$f \in \text{Lip}_{loc}[L_0, 0), \quad f(x) < 0 \text{ for } x \in (L_0, 0). \quad (29)$$

Then, problem (1), (3) has a solution u , such that

$$L_0 < u(t) < 0 \quad \text{for } t \in [a, \infty).$$

If in addition (24) and (25) hold, then either

$$u'(t) < 0 \text{ for } t \geq a \text{ and } \lim_{t \rightarrow \infty} u(t) = L_0,$$

or

$$u \text{ is a Kneser solution.}$$

4. Existence of Kneser solutions to singular equation (1)

In this section, the existence of Kneser solutions to problems (1), (2) and (1), (3) for the singular case $a = 0$ is studied, under the assumptions (8) and

$$p \in C[0, \infty), p > 0 \text{ on } (0, \infty), p(0) = 0, \quad (30)$$

$$q \in C[0, \infty), q > 0 \text{ on } (0, \infty). \quad (31)$$

It is important to notice that for $a = 0$, equation (1) becomes singular, since $p(0) = 0$, and the results obtained in the previous section cannot be immediately extended to cover this case. To see this, note that for $p(t) = t^\alpha$, $\alpha \in (0, 1]$, $\int_1^\infty \frac{ds}{p(s)} = \infty$ follows, and thus, problems (1), (2) and (1), (3) have no Kneser solutions, see Remark 5.1. Therefore, α has to be greater than 1. However, in such a case $\int_0^1 \frac{ds}{p(s)} = \infty$ and the functions P and G in (15) are not defined at $t = a = 0$. Consequently, the approach used for $a > 0$ has to be modified for the case when $a = 0$.

The existence results for the Kneser solution of problems (1), (2) and (1), (3) with $a = 0$ and $p \equiv q$, were studied in [31], see Theorems 3.4 and 3.5. The following results are corollaries to Theorems 3.4 and 3.5 [31].

Theorem 4.1. *Let us assume that (8), (22), (30) and the following assumptions:*

$$p \equiv q, \quad (32)$$

$$p \in C^1(0, \infty), p' > 0 \text{ on } (0, \infty), \lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} = 0, \quad (33)$$

$$\frac{p'(t) \int_0^t p(s) ds}{p^2(t)} \geq c, \quad t \in (0, \infty), \quad (34)$$

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \quad x \in (0, A_0], \quad (35)$$

hold for some $c > \frac{1}{2}$ and $A_0 \in (0, L)$, where $F(x) = \int_0^x f(z) dz$. Then, for each $u_0 \in (0, A_0]$ there exists a unique Kneser solution u to problem (1), (2) with $a = 0$. This solution has the following properties:

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) < 0, \quad t \in (0, \infty).$$

A dual statement for an initial condition u_0 from a negative neighbourhood of zero is given in the following theorem.

Theorem 4.2. *Assume (8), (29), (30), (32), and (33) to hold. Let condition (34) hold with a constant $c > \frac{1}{2}$ and assume that there exists $B_0 \in (L_0, 0)$ such that the inequality*

$$\frac{xf(x)}{F(x)} \geq \frac{2}{2c-1}, \quad x \in [B_0, 0) \quad (36)$$

is satisfied. Then, for each $u_0 \in [B_0, 0)$, there exists a unique Kneser solution u to problem (1), (3) with $a = 0$. This solution has the following properties:

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0, \quad u'(0) = 0, \quad u'(t) > 0, \quad t \in (0, \infty).$$

To our knowledge, the existence of Kneser solutions for the case $a = 0$ and $p \neq q$ under assumptions (8), (30), and (31) remains an open problem.

Remark 4.3. Let us note, that the condition $u'(0) = 0$ is necessary for the smoothness of the solution in the case where $p \equiv q$ is an increasing function. To see this, let us consider a solution u of (1), (2) or (1), (3). Since $u \in C^1[0, \infty)$, the assumption $p(0) = 0$ yields $p(0)u'(0) = 0$. Since f is continuous on $[L_0, L]$ and $u(0) \in (L_0, L)$, there exist $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. We now integrate (1) and use the monotonicity of p to obtain

$$|u'(t)| = \left| \frac{1}{p(t)} \int_0^t p(s)f(u(s)) ds \right| \leq \frac{M}{p(t)} \int_0^t p(s) ds \leq Mt, \quad t \in (0, \delta).$$

Consequently, $u'(0) = 0$ holds.

Remark 4.4. In Theorems 3.4 [31] additional restrictions on f were made. Those assumptions are related to the interval $\mathbb{R} \setminus (0, L)$. Since the Kneser solution obtained in Theorem 4.1 satisfies $u(t) \in (0, L)$ for $t \in [0, \infty)$, we can appropriately change f on $\mathbb{R} \setminus (0, L)$ and hence, we can omit those additional assumptions and still refer to Theorem 3.4 [31]. Similar remark can be made on context of Theorem 4.2.

5. Asymptotic properties of Kneser solutions

This section focuses on properties of Kneser solutions of problems (1), (2) and (1), (3) in the neighbourhood of infinity. Asymptotic formulas for the solutions and for their first derivatives are provided. These asymptotic properties apply to the Kneser solutions of regular differential equations (1) with $a > 0$, as well as to Kneser solutions of singular equation (1) with $a = 0$, provided that $u(a) \in [L_0, L]$.

We assume that function f satisfies condition (8) and the second conditions in (22) and (29),

$$L_0 < 0 < L, f \in C[L_0, L], f(L_0) = f(0) = f(L) = 0, xf(x) > 0 \text{ for } x \in (L_0, 0) \cup (0, L). \quad (37)$$

Remark 5.1. Problems (1), (2) and (1), (3) have no Kneser solutions in case that

$$\int_1^\infty \frac{ds}{p(s)} = \infty. \quad (38)$$

This follows from (30), (31), (37) and the following arguments: Let u be a solution of (1), (2). Then, pu' is decreasing for $t \geq a$. Assume that $pu' \leq 0$ for $t \geq t_1 \geq a$. By integrating inequality $p(t)u'(t) < p(t_1)u'(t_1) = K < 0$, we obtain

$$u(t) \leq u(t_1) + K \int_{t_1}^t \frac{ds}{p(s)}.$$

Therefore, as t tends to infinity, $\lim_{t \rightarrow \infty} u(t) \leq -\infty$ contradicting (2). This means that $u' > 0$ on $[t_0, \infty)$. Hence, any solution of (1), (2) is increasing and there exists no Kneser solution to (1), (2). Similar arguments can be given for problem (1), (3).

According to Theorem 2.4, condition (38) is satisfied when $p \in RV(\alpha)$ with $\alpha < 1$. For $\alpha = 1$, the integral may be convergent (or may not) and hence Kneser solutions of the problem could exist. Therefore, in the following asymptotic analysis, we restrict our attention to the case $\alpha \geq 1$.

We first formulate the asymptotic properties of Kneser solutions to problem (1), (2), or (1), (3), see lemma below.

Theorem 5.2. Assume that (37) holds and $a \geq 0$. Moreover, assume that $p \in RV(\alpha) \cap C[a, \infty)$, $q \in RV(\beta) \cap C[a, \infty)$, $\alpha \geq 1$, $\beta > 0$, $\beta - \alpha > -1$. Let u be a Kneser solution of problem (1), (2) or (1), (3). Then,

$$\lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0. \quad (39)$$

Proof: Let u be a Kneser solution of problem (1), (2). By (4), there exists $t_0 > a$ such that

$$0 < u(t) < L, \quad u'(t) < 0, \quad t \in [t_0, \infty). \quad (40)$$

Hence, there exists

$$\lim_{t \rightarrow \infty} u(t) =: l_1 \in [0, u(t_0)).$$

Assume that $l_1 \in (0, u(t_0))$ and let us introduce $m_1 := \min\{f(x) : x \in [l_1, u(t_0)]\} > 0$. The integration of (1) over $[t_0, t]$ gives

$$\begin{aligned} p(t)u'(t) - p(t_0)u'(t_0) &\leq -m_1 \int_{t_0}^t q(s) ds, \quad t \in [t_0, \infty), \\ u'(t) &\leq \frac{p(t_0)u'(t_0)}{p(t)} - m_1 \frac{1}{p(t)} \int_{t_0}^t q(s) ds, \quad t \in [t_0, \infty). \end{aligned}$$

Since p satisfies (5) with $\alpha \geq 1$, we see that $\liminf_{t \rightarrow \infty} p(t) > 0$. Consequently, it follows from Lemma 2.6 $\lim_{t \rightarrow \infty} u'(t) = -\infty$, which implies $\lim_{t \rightarrow \infty} u(t) = -\infty$, a contradiction. Therefore $l_1 = 0$, i.e.,

$$\lim_{t \rightarrow \infty} u(t) = 0. \quad (41)$$

The second condition in (40) implies $\liminf_{t \rightarrow \infty} u'(t) \leq 0$. Assume that $\liminf_{t \rightarrow \infty} u'(t) < 0$. Then, there exist a sequence $\{t_n\} \subset (a, \infty)$ and $\varepsilon > 0$ such that

$$\lim_{n \rightarrow \infty} t_n = \infty, \quad \lim_{n \rightarrow \infty} u'(t_n) = -\varepsilon.$$

By (1), (2), (10), and (37) the function $(pu')'(t) < 0$ for $t \geq a$. Therefore, the inequalities

$$u'(t) \leq -\frac{\varepsilon}{2}, \quad p(t)u'(t) < p(t_n)u'(t_n), \quad t > t_n,$$

hold for sufficiently large $n \geq n_0$. Then,

$$u'(t) < -\frac{\varepsilon}{2}p(t_n)\frac{1}{p(t)}, \quad t > t_n, \quad n \geq n_0.$$

Integrating the above inequality results in

$$u(t) - u(t_n) \leq -\frac{\varepsilon}{2}p(t_n) \int_{t_n}^t \frac{1}{p(s)} ds, \quad t > t_n, \quad n \geq n_0.$$

Let $t \rightarrow \infty$. Then, according to (41),

$$-u(t_n) \leq -\frac{\varepsilon}{2}p(t_n) \int_{t_n}^{\infty} \frac{1}{p(s)} ds, \quad n \geq n_0.$$

Now, letting $n \rightarrow \infty$ and using Lemma 2.5 yields

$$0 = -\lim_{n \rightarrow \infty} u(t_n) \leq -\frac{\varepsilon}{2} \lim_{n \rightarrow \infty} p(t_n) \int_{t_n}^{\infty} \frac{ds}{p(s)} = -\frac{\varepsilon}{2} \lim_{t \rightarrow \infty} p(t) \int_t^{\infty} \frac{ds}{p(s)} = -\infty.$$

This is again a contradiction. Therefore, $\liminf_{t \rightarrow \infty} u'(t) = 0$. It follows from (40), $\limsup_{t \rightarrow \infty} u'(t) \leq 0$, and consequently,

$$\lim_{t \rightarrow \infty} u'(t) = 0.$$

The proof for a Kneser solution of problem (1), (3) can be given in an analogous way. □

Finally, we specify the asymptotic behaviour of Kneser solutions in a more precise way.

Theorem 5.3. Assume that (37) holds and $a \geq 0$. Moreover, let us assume that $p \in RV(\alpha) \cap C[a, \infty)$, $\alpha \geq 1$, $q \in RV(\beta) \cap C[a, \infty)$, $\beta > 0$, $\beta - \alpha > -1$, and

$$\exists r > 1 : \liminf_{x \rightarrow 0} \frac{|f(x)|}{|x|^r} > 0, \quad \limsup_{x \rightarrow 0} \frac{|f(x)|}{|x|^r} < \infty. \quad (42)$$

Let u be a Kneser solution of problem (1), (2) or (1), (3). Then, for any $\varepsilon > 0$

$$\lim_{t \rightarrow \infty} t^{\frac{\beta - \alpha + 2}{r - 1}} - \varepsilon |u(t)| = 0. \quad (43)$$

Proof: Let u be a Kneser solution of problem (1), (2) or (1), (3). Consider $t_0 > a$ from (4). According to (42) and (39), there exist $k, K, \delta > 0$ and $t_1 \geq t_0$ such that

$$k < \frac{|f(x)|}{|x|^r} < K, \quad x \in (0, \delta) \quad \text{and} \quad 0 < |u(t)| < \delta, \quad t \geq t_1.$$

Hence, we have

$$k|u(t)|^r < |f(u(t))| < K|u(t)|^r, \quad t \geq t_1. \quad (44)$$

We now integrate equation (1) from t_1 to $t \geq t_1$ and obtain

$$p(t)u'(t) - p(t_1)u'(t_1) + \int_{t_1}^t q(s)f(u(s)) \, ds = 0.$$

Since $u(t)u'(t) < 0$ and $u(t)$ is monotone for $t > t_1$,

$$p(t)|u'(t)| > \int_{t_1}^t q(s)|f(u(s))| \, ds > k|u(t)|^r \int_{t_1}^t q(s) \, ds$$

follows. Therefore,

$$\frac{|u'(t)|}{k|u(t)|^r} > \frac{1}{p(t)} \int_{t_1}^t q(s) \, ds, \quad t > t_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^\alpha L_p(t)$ and $q(t) = t^\beta L_q(t)$, respectively. Functions L_p, L_q always exist due to Remark 2.3. According to Theorem 2.4 (i), there exists a sufficiently large $b \geq t_1$ such that

$$\frac{1}{p(t)} \int_{t_1}^t q(s) \, ds = \frac{\int_{t_1}^t s^\beta L_q(s) \, ds}{t^\alpha L_p(t)} \geq \frac{1}{2(\beta+1)} t^{\beta-\alpha+1} \frac{L_q(t)}{L_p(t)}, \quad t > b.$$

Therefore,

$$\frac{|u'(t)|}{k|u(t)|^r} > c_1 t^{\beta-\alpha+1} L(t), \quad t > b,$$

where $c_1 = \frac{1}{2(\beta+1)}$ and $L(t) = \frac{L_q(t)}{L_p(t)}$. Again by Theorem 2.4 (i), there exists a sufficiently large $T \geq b$ such that

$$\begin{aligned} \frac{1}{k(r-1)} \left(\frac{1}{|u(t)|^{r-1}} - \frac{1}{|u(T)|^{r-1}} \right) &> c_1 \int_T^t s^{\beta-\alpha+1} L(s) \, ds \\ &\geq \frac{c_1}{2(\beta-\alpha+2)} t^{\beta-\alpha+2} L(t) \end{aligned}$$

holds for $t > T$. Consequently,

$$0 < |u(t)| < \left(k(r-1)c_2 t^{\beta-\alpha+2} L(t) \right)^{-\frac{1}{r-1}}, \quad t > T,$$

where $c_2 = \frac{c_1}{2(\beta-\alpha+2)}$. Let us define $L_2(t) := (L(t))^{-\frac{1}{r-1}}$ and $c_3 := k(r-1)c_2$, then

$$0 < t^{\frac{\beta-\alpha+2}{r-1}} |u(t)| < c_3 L_2(t), \quad t > T. \quad (45)$$

Finally, we choose an $\varepsilon > 0$ and multiply inequality (45) by $t^{-\varepsilon}$. Then, Remark 2.2 yields

$$\lim_{t \rightarrow \infty} t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon} |u(t)| = 0$$

which completes the proof. □

We finally focus our attention on the first derivatives of Kneser solutions.

Theorem 5.4. Let all assumptions of Theorem 5.3 be satisfied. Then, for any $\varepsilon > 0$ the following statements hold.

(i) If $\beta > r\alpha - r - 1$, then

$$\lim_{t \rightarrow \infty} t^{\alpha-\varepsilon} |u'(t)| = 0. \quad (46)$$

(ii) If $\beta \leq r\alpha - r - 1$, then

$$\lim_{t \rightarrow \infty} t^{\frac{\beta-\alpha+r+1}{r-1} - \varepsilon} |u'(t)| = 0. \quad (47)$$

Proof: Let u be a Kneser solution of problem (1), (2) or (1), (3) and let $a \leq t_0 \leq t_1$ be the points from the proof of Theorem 5.3. Then, $uu' < 0$ on $[t_1, \infty)$ and (44) holds. Let us choose $\varepsilon_1 > 0$. Due to (43), for each $c > 0$ there exists $T_1 \geq t_1$ such that

$$0 < t^{\frac{\beta-\alpha+2}{r-1} - \varepsilon_1} |u(t)| < c, \quad t > T_1. \quad (48)$$

We first integrate equation (1) over (T_1, t) and set $A_1 := p(T_1)|u'(T_1)|$. Then, by (44),

$$0 < p(t)|u'(t)| = A_1 + \int_{T_1}^t q(s)|f(u(s))| ds < A_1 + K \int_{T_1}^t q(s)|u(s)|^r ds, \quad t > T_1.$$

Let L_p and L_q be slowly varying functions such that $p(t) = t^\alpha L_p(t)$ and $q(t) = t^\beta L_q(t)$, respectively. This implies

$$0 < t^\alpha L_p(t)|u'(t)| < A_1 + K \int_{T_1}^t s^\beta L_q(s)|u(s)|^r ds, \quad t > T_1.$$

Due to (48),

$$\begin{aligned} 0 < t^\alpha L_p(t)|u'(t)| &< A_1 + K \int_{T_1}^t s^{\beta-r\mu+r\varepsilon_1} L_q(s)|s^{\mu-\varepsilon_1} u(s)|^r ds \\ &< A_1 + Kc^r \int_{T_1}^t s^{\beta-r\mu+\varepsilon_2} L_q(s) ds, \end{aligned}$$

where $\mu = \frac{\beta-\alpha+2}{r-1}$, $\varepsilon_2 = r\varepsilon_1 > 0$.

(i) Let $\beta > r\alpha - r - 1$. Then,

$$\begin{aligned} \beta - r\mu &= \frac{(r\alpha - r - 1) + 1 - \beta - r}{r - 1} \\ &< \frac{\beta - \beta + 1 - r}{r - 1} = -1. \end{aligned}$$

Now, we choose ε_1 and accordingly ε_2 which are sufficiently small for $\beta - r\mu + \varepsilon_2 < -1$ to hold. By the Karamata Integration Theorem 2.4 (ii), there exists $A_2 < \infty$ such that

$$0 < t^\alpha L_p(t)|u'(t)| < A_1 + Kc^r \int_{T_1}^\infty s^{\beta-r\mu+\varepsilon_2} L_q(s) ds = A_2$$

holds for $t > T_1$. We choose an arbitrary $\varepsilon > 0$ and multiply the above inequality by $t^{-\varepsilon} L_p^{-1}$. Thus,

$$0 < t^{\alpha-\varepsilon} |u'(t)| < A_2 t^{-\varepsilon} L_p^{-1}(t),$$

for $t > T_1$. Due to Remark 2.2, asymptotic formula (46) follows.

(ii) Let $\beta \leq r\alpha - r - 1$. Then, for arbitrary $\varepsilon_2 > 0$,

$$\begin{aligned}\beta - r\mu + \varepsilon_2 &= \frac{r\beta - \beta}{r-1} - \frac{r}{r-1}(\beta - \alpha + 2) + \varepsilon_2 \\ &= \frac{(r\alpha - r - 1) + 1 - \beta - r}{r-1} + \varepsilon_2 \\ &\geq \frac{\beta - \beta + 1 - r}{r-1} + \varepsilon_2 \geq -1 + \varepsilon_2 > -1.\end{aligned}$$

By the Karamata Integration Theorem 2.4 (i), there exists a sufficiently large $T > T_1$, such that

$$0 < t^\alpha L_p(t)|u'(t)| < A_1 + \frac{2Kc^r}{\beta - r\mu + \varepsilon_2 + 1} t^{\beta - r\mu + \varepsilon_2 + 1} L_q(t) = A_1 + A_2 t^\omega L_q(t)$$

holds for $t > T$, where $A_2 = \frac{2Kc^r}{\beta - r\mu + \varepsilon_2 + 1}$ and $\omega = \beta - r\mu + \varepsilon_2 + 1 > 0$. Therefore,

$$t^{\alpha - \omega}|u'(t)| < A_1 t^{-\omega} L_p^{-1}(t) + A_2 L(t),$$

where $t > T$, $L(t) = \frac{L_q(t)}{L_p(t)}$. Finally, we choose an arbitrary $\varepsilon_3 > 0$ and multiply the above inequality by $t^{-\varepsilon_3}$. Consequently, we obtain

$$0 < t^{\frac{\beta - \alpha + r + 1}{r-1} - \varepsilon} |u'(t)| < A_1 t^{-\omega - \varepsilon_3} L_p^{-1}(t) + A_2 t^{-\varepsilon_3} L(t),$$

where $\varepsilon = \varepsilon_2 + \varepsilon_3$. By Remark 2.2, the asymptotic formula (47) holds and the result follows. \square

6. Numerical simulations

We now use the open domain MATLAB Code `bvpsuite` to numerically simulate three model problems in order to illustrate theoretical statements made above. The aim is to give numerical evidence for the existence of Kneser solutions. We focus on the singular problems (1), (2) and (1), (3) with $a = 0$ and simulate Kneser solutions on the interval $[0, \infty)$ which contains the singular point $t = 0$. Moreover, asymptotic properties of such solutions are investigated and compared with the analytically derived asymptotic formulas (43) and (47).

The MATLAB software package `bvpsuite` [15] is designed to solve BVPs in ODEs and differential algebraic equations. The solver routine is based on a class of collocation methods whose orders may vary from 2 to 8. Collocation has been investigated in the context of singular differential equations of first and second order in [8, 32], respectively. This method could be shown to be robust with respect to *singularities in time* and retains its high convergence order if the analytical solution is appropriately smooth. The code also provides an asymptotically correct estimate for the global error of the numerical approximation. To enhance the efficiency of the method, a mesh adaptation strategy is implemented, which attempts to choose grids related to the solution behaviour, making sure that the tolerance is satisfied with the least possible effort. Error estimate procedure and mesh adaptation work dependably provided that the solution of the problem and its global error are appropriately smooth¹. Both the code and the manual can be downloaded from <http://www.math.tuwien.ac.at/~ewa>. For further information see [15]. This software has proved useful for the approximation of numerous singular BVPs important for applications, see for example [2, 7, 14, 25].

Since we intend to solve a scalar second order differential equation, we have to specify two boundary/initial conditions which are correctly posed to guarantee the uniqueness of the solution, at least locally. More precisely, we try to solve problems (1), (2) and (1), (3) with $a = 0$, but we do not know the values of $u(0)$. Therefore, we solve the differential equation (1),

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in (0, \infty),$$

¹The required smoothness of higher derivatives is related to the order of the used collocation method.

subject to the following boundary conditions:

$$u'(0) = 0, \quad u(\infty) := \lim_{t \rightarrow \infty} u(t) = 0. \quad (49)$$

The first condition in (49) is motivated by the results obtained for $p \equiv q$, where *this condition is necessary for any solution to be continuous*, cf. Remark 4.3. The second condition in (49) has to be satisfied by any Kneser solution under the assumptions of Theorem 5.2. It turns out that from the numerical point of view, the problem is very involved and the numerical treatment is by no means straightforward.

For the first tests, we choose the simplest regularly varying functions p and q ,

$$p(t) = t^\alpha, \quad \alpha > 1, \quad q(t) = t^\beta, \quad \beta \geq \alpha, \quad t \in [0, \infty). \quad (50)$$

In order to recover the solution asymptotics specified in (43), the parameter β has to satisfy $\beta > \alpha - 1$. Here, we restrict our attention to the case $\beta \geq \alpha$.²

For p, q from (50), we rewrite equation (1) and obtain

$$u''(t) + \frac{\alpha}{t}u'(t) + t^{\beta-\alpha}f(u(t)) = 0, \quad t \in (0, \infty), \quad u'(0) = 0, \quad u(\infty) = 0. \quad (51)$$

To solve this boundary value problem, the differential equation is reduced to the finite interval $[0, 1]$. To this aim, we rewrite the problem as follows:

$$v_1''(t) + \frac{\alpha}{t}v_1'(t) + t^{\beta-\alpha}f(v_1(t)) = 0, \quad t \in (0, 1], \quad v_2''(t) + \frac{\alpha}{t}v_2'(t) + t^{\beta-\alpha}f(v_2(t)) = 0, \quad t \in [1, \infty), \quad (52)$$

and use the transformation $\tau = \frac{1}{t}$ in the equation for v_2 . Then, the problem is solved on $(0, 1]$ subject to boundary conditions

$$v_1'(0) = 0, \quad v_2(0) = 0, \quad v_1(1) = v_2(1), \quad v_1'(1) = -v_2'(1).$$

Example 1

The following model is used to illustrate the existence of positive and negative Kneser solutions of equation (51). The problem data reads:

$$p(t) = t^5, \quad q(t) = t^7, \quad t \in [0, \infty),$$

$$f(x) = \begin{cases} -12 - 2x & \text{for } x < -2, \\ x^3 & \text{for } x \in [-2, 1], \\ 2 - x & \text{for } x > 1, \end{cases} \quad (53)$$

and $L_0 = -6$, $L = 2$, $r = 3$. As shown in Figure 1, we have found two different Kneser solutions u_1, u_2 , lying in the regions indicated in (2) and (3), respectively. The solutions satisfy $\lim_{t \rightarrow \infty} u_i'(t) = 0$, $i = 1, 2$ in correspondence to the theory. According to Theorem 5.3, the asymptotic behaviour of any Kneser solution u of (1), (2) or (1), (3) is specified by (43),

$$\lim_{t \rightarrow \infty} t^{\frac{\beta-\alpha+2}{r-1}} - \varepsilon |u(t)| = 0.$$

Thus, for $\alpha = 5$, $\beta = 7$, and $r = 3$ this formula becomes

$$\lim_{t \rightarrow \infty} t^{2-\varepsilon} |u_i(t)| = 0, \quad i = 1, 2.$$

The first derivative of the Kneser solution behaves asymptotically as specified in (47). Therefore,

$$\lim_{t \rightarrow \infty} t^{\frac{\beta-\alpha+r+1}{r-1}} - \varepsilon |u_i'(t)| = \lim_{t \rightarrow \infty} t^{3-\varepsilon} |u_i'(t)| = 0, \quad i = 1, 2.$$

We illustrate the asymptotic behaviour of the Kneser solutions using graphs with double logarithmic scales, where the power k in the relation $y = ax^k$ corresponds to the slope of the line. Figure 1 clearly indicates that not only solutions u_1, u_2 , but also the expressions $t^2 u_i(t)$, $i = 1, 2$ tend to zero for $t \rightarrow \infty$. Similar observations can be made for the first derivatives of both solutions u_1', u_2' , where $t^3 u_i'(t)$, $i = 1, 2$ tends to zero for $t \rightarrow \infty$.

²For $\beta \in (\alpha - 1, \alpha)$ no Kneser solutions were found.

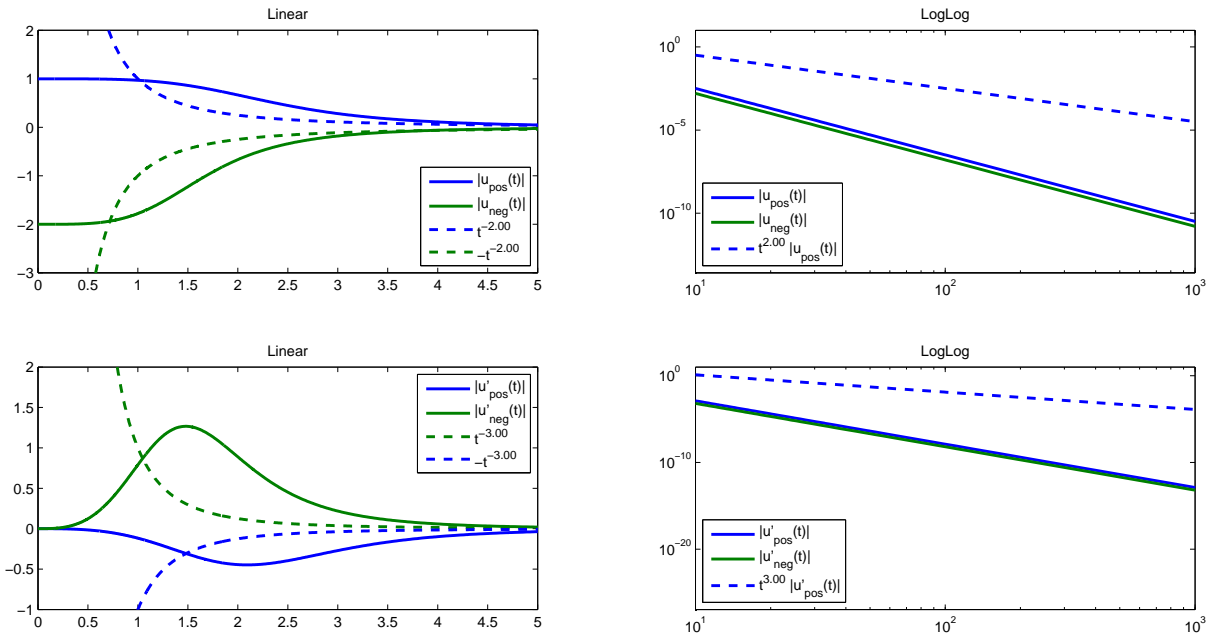


Figure 1. Example 1: Solutions u_i , $i = 1, 2$, of (51), (49) corresponding to $\alpha = 5$ and $\beta = 7$ plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions u'_i , $i = 1, 2$ are shown in lower graphs.

Numerical simulations for problem (51), (49) with parameters $(\alpha, \beta) \in \{(4, 4), (4, 5), (5, 6)\}$ and function f satisfying condition (42) with $r = 2, 3, 4$ show similar behaviour and are not discussed here. For more details see [9].

Example 2

Using this example, we illustrate how the difference $\beta - \alpha$ affects the asymptotic behaviour of the Kneser solutions. According to (43), if $\beta - \alpha$ grows we expect that the solution decay towards zero becomes faster. To see this, we consider problem (51), (49) with f specified in (53) and

$$(\alpha, \beta) \in \{(3, 3), (4, 5), (5, 7)\}.$$

We can observe in Figure 2 that larger difference $\beta - \alpha$ indeed results in a steeper decline of the solution towards zero.

Example 3

Here, Kneser solutions of problem (1), (49) with a function f given in (53) and

$$p(t) = t^\alpha \in RV(\alpha), \quad q(t) = t^\beta(1 + \exp(-t)) \in RV(\beta), \quad t \in [0, \infty)$$

are discussed. Then, equation (1) takes the form

$$u''(t) + \frac{\alpha}{t}u'(t) + t^{\beta-\alpha}(1 + \exp(-t))f(u(t)) = 0.$$

Two Kneser solutions and their first derivatives can be found in Figure 3 for the parameters $(\alpha, \beta) = (4, 5)$, $(\alpha, \beta) = (5, 7)$, respectively.

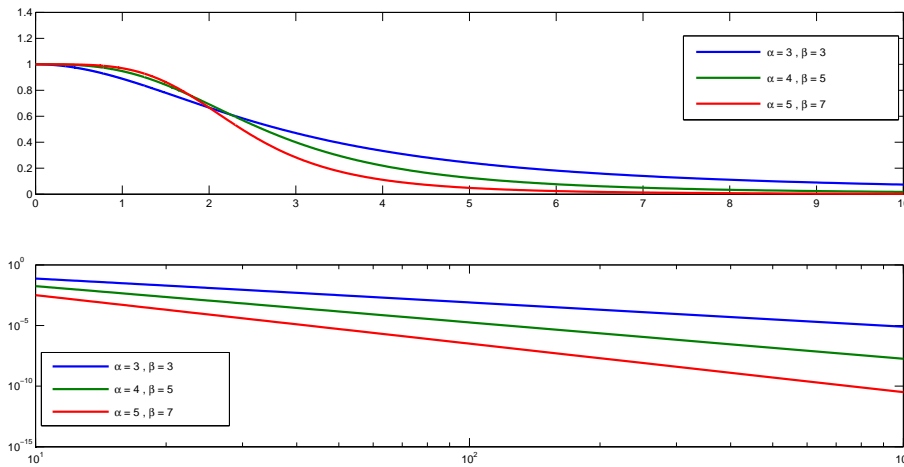


Figure 2. Example 2: Comparison of Kneser solutions depending on the difference $\beta - \alpha$. Graph of the Kneser solutions using linear scales (above), double logarithmic scales (below).

Example 4

We designed this example to illustrate the influence of parameters α and β . We choose

$$p(t) = t^\alpha \log(1 + t) \in RV(\alpha), \quad q(t) = t^\beta \in RV(\beta), \quad f(x) = \operatorname{sgn}(x)x^4(2 + x)(1 - x), \quad x \in \mathbb{R}.$$

This yields

$$u''(t) + \left(\frac{\alpha}{t} + \frac{1}{(1 + t) \log(1 + t)} \right) u'(t) + \frac{t^{\beta-\alpha}}{\log(1 + t)} \operatorname{sgn}(u(t))u^4(t)(2 + u(t))(1 - u(t)) = 0. \tag{54}$$

First, we fix $\alpha = 4$ and vary $\beta \in \{4, 5, 6, 7\}$. Numerical results are shown in Figure 4. All solutions seem to have a similar asymptotic behaviour. Moreover, we observe that the function $t^{5/3}u(t)$ for u corresponding to $\alpha = 4, \beta = 7$ tends to zero for large values of t . The same holds for all other solutions.

We now fix $\beta = 10$ and vary $\alpha \in \{5, 6, 7, 8\}$. Figure 5 shows the related Kneser solutions and their first derivatives. A closer look at the solution u with the slowest decay towards zero shows that the limit of $t^{7/3}u$ is zero for $t \rightarrow \infty$. Other solutions show faster convergence towards zero.

The above observations mean that the asymptotic formula (43) can be applied to all numerical Kneser solutions of problem (54), (49), but it does not optimally recover the speed of their decay.

Asymptotic behaviour of the numerically computed solutions indicates that the second term in equation (51),

$$\frac{\alpha}{t} u'(t), \quad t \in [0, \infty),$$

becomes dominant as $t \rightarrow \infty$, and therefore, properties of the solutions seem to be mainly controlled by the parameter α .

7. Conclusions

In this paper, we investigated the existence and asymptotic properties of the Kneser solutions to the second order ODE (1),

$$(p(t)u'(t))' + q(t)f(u(t)) = 0,$$

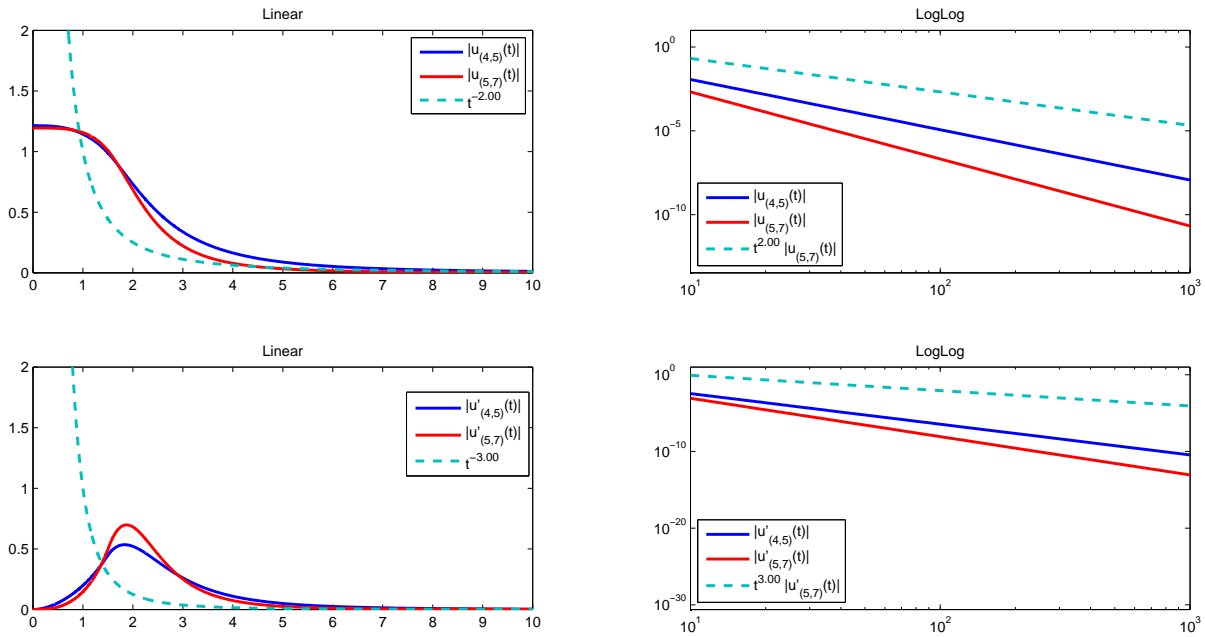


Figure 3. Example 3: Solutions corresponding to $\alpha = 4, \beta = 5$ and $\alpha = 5, \beta = 7$, respectively, plotted using linear scales, upper graph (left), and double logarithmic scales, upper graph (right). First derivatives of the solutions are shown in lower graphs.

which was considered to be super-linear in the neighbourhood of the origin. More precisely, two classes of initial value problems were discussed,

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in [a, \infty), \quad u(a) = u_0 \in (0, L), \quad 0 \leq u(t) \leq L \text{ for } t \in [a, \infty), \quad (55)$$

and

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in [a, \infty), \quad u(a) = u_0 \in (L_0, 0), \quad L_0 \leq u(t) \leq 0 \text{ for } t \in [a, \infty). \quad (56)$$

In the regular case ($a > 0$), it turned out that there exists a non-oscillatory solution of problem (55) (and problem (56)), which is either a Kneser solution or a monotonically increasing solution whose limit is L for t tending to infinity (and a monotonically decreasing solution whose limit is L_0 for t tending to infinity).

In the singular case where $a = 0$, the existence of the Kneser solutions was shown for $p \equiv q$ [31], while for $p \neq q$ this is still an open question.

In both the regular and the singular case, we have provided asymptotic formulas for Kneser solutions and their first derivatives to the ODE (1) with regularly varying coefficients.

The aim of further investigations is to show the existence of Kneser solutions to equation (1) with a time singularity at $a = 0$ and $p \neq q$, and to more precisely describe the speed of decay towards zero for Kneser solutions to equation (1).

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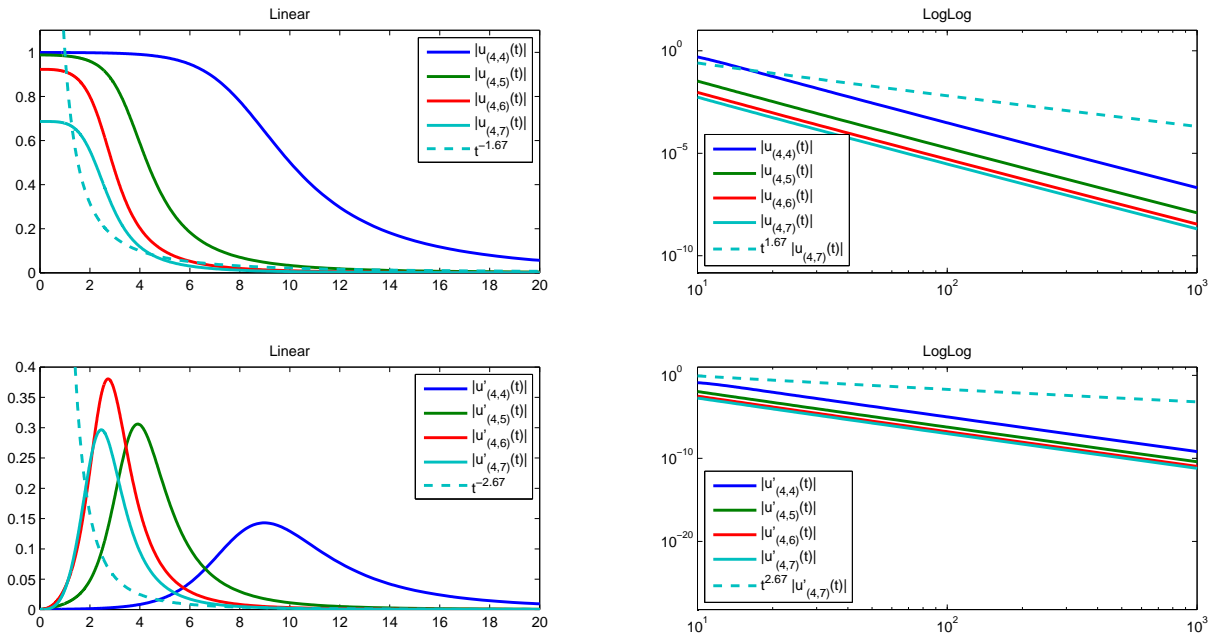


Figure 4. Example 4: Comparison of Kneser solution with fixed α plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of solutions are shown in the lower graphs.

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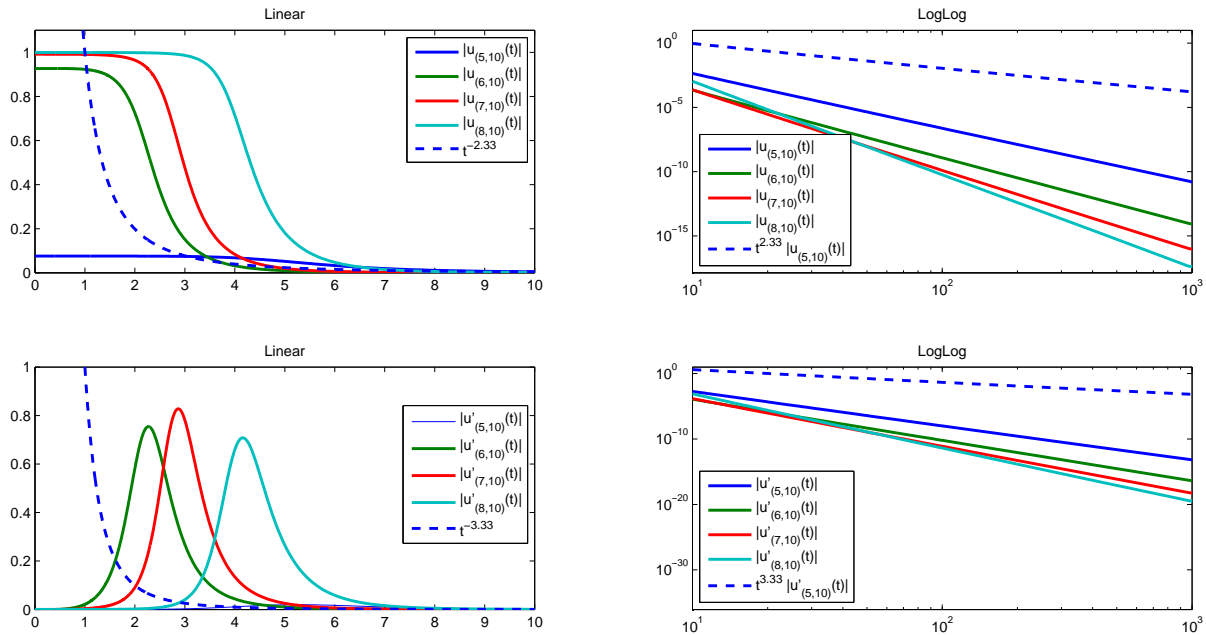


Figure 5. Example 4: Comparison of Kneser solution with fixed β plotted in a graph with linear scales (left above) and double logarithmic scales (right above). First derivatives of the solutions are shown in the lower graphs.

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