Zurich Summer School 2014

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Adaptive Boundary Element Methods

Part 1: h-h/2 Error Estimators

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Adaptive Boundary Element Methods: Error Estimation, Convergence, and Optimality

- part 1: introduction & h-h/2 error estimators
- part 2: estimator reduction & convergence of ABEM
- part 3: linear convergence of ABEM
- part 4: optimal convergence rate of ABEM
- part 5: local inverse estimates for nonlocal operators

- http://www.asc.tuwien.ac.at/∼dirk/zss/

2D Model Problem

2D Model Problem

- weakly-singular integral equation

$$Vu(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,u(y)\,dy = f(x) \quad \text{for } x \in \Gamma$$

- $\Gamma \subseteq \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ bounded and Lipschitz

Variational Formulation

Find solution $u \in \mathcal{H}$ of

$$\langle\!\langle u\,,v\rangle\!\rangle := \langle Vu\,,v\rangle_{L^2(\Gamma)} = \langle f\,,v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \mathcal{H}$$

- $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ is scalar product
- induced norm $|\!|\!|v|\!|\!| := \langle\!\langle v\,,v\rangle\!\rangle^{1/2} \simeq \|v\|_{\widetilde{H}^{-1/2}(\Gamma)}$
- Theorem of Riesz $\implies$ unique solution $u \in \mathcal{H}$
- u in general *not* computable numerically

A Priori Error Estimate

- suppose regularity $u \in \mathcal{H} \cap H^t(\Gamma)$
- $U_\ell \in \mathcal{X}_\ell = \mathcal{P}^p(\mathcal{T}_\ell)$ Galerkin solution

$$\implies \|u - U_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|u\|_{H^t(\Gamma)} \max_{T\in\mathcal{T}_\ell} \mathrm{diam}(T)^{\min\{p+1,t\}+1/2}$$

Optimal Convergence Behavior

- suppose smooth solution u
- suppose $h \simeq \mathrm{diam}(T)$ for all $T \in \mathcal{T}_\ell$
- let $N := \#\mathcal{T}_\ell \simeq h^{-1}$ (because of 2D)

$\implies |\!|\!|u - U_\ell|\!|\!| \simeq \|u - U_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h^{p+3/2} \simeq N^{-(p+3/2)}$

Galerkin BEM

- $\mathcal{T}_\ell$ partition of Γ into affine line segments
- $h_\ell \in L^\infty(\Gamma)$ local mesh-size $h_\ell|_T := \mathrm{diam}(T)$
- $\mathcal{P}^p(\mathcal{T}_\ell)$ space of $\mathcal{T}_\ell$-piecewise polynomials of degree $\le p$
- $\mathcal{X}_\ell := \mathcal{P}^p(\mathcal{T}_\ell)$ discrete subspace of $\mathcal{H} = \widetilde{H}^{-1/2}(\Gamma)$

Galerkin Formulation

Find solution $U_\ell \in \mathcal{X}_\ell$ of

$$\langle\!\langle U_\ell\,,V_\ell\rangle\!\rangle = \langle f\,,V_\ell\rangle_{L^2(\Gamma)} \quad \text{for all } V_\ell \in \mathcal{X}_\ell$$

- Theorem of Riesz $\implies$ unique solution $u \in \mathcal{H}$
- U_ℓ computable by solving a linear SPD system
- Céa lemma $|\!|\!|u - U_\ell|\!|\!| = \min_{V_\ell\in\mathcal{X}_\ell} |\!|\!|u - V_\ell|\!|\!|$

Empirical Convergence Rates

- suppose: $\Phi(N) = cN^{-\alpha}$ with $c, \alpha > 0$ unknown
  - e.g., $\Phi(N) = |\!|\!|u - U_\ell|\!|\!|$ with $N = \#\mathcal{T}_\ell$
- suppose: can compute $\Phi(N_j)$ for certain N_j
- goal: determine empirical value of $\alpha > 0$
- $\Phi(N_j) = cN_j^{-\alpha}$

$\implies \Phi(N_1)/\Phi(N_2) = (N_1/N_2)^{-\alpha}$

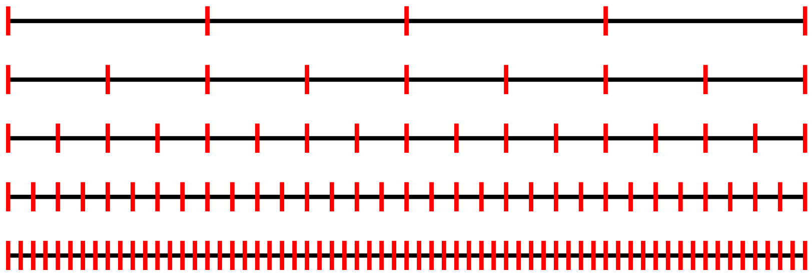
$\implies \alpha = -\log\frac{\Phi(N_1)}{\Phi(N_2)}/\log\frac{N_1}{N_2} = -\frac{\log\Phi(N_1)-\log\Phi(N_2)}{\log N_1 - \log N_2}$

- consequence: plot $\log\Phi(N_j)$ over $\log N_j$

$\implies \alpha$ is negative slope of corresponding curve

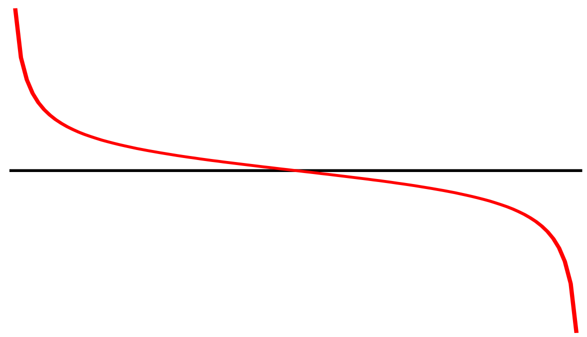
Uniform Mesh-Refinement

- in each step, all elements $T \in \mathcal{T}_\ell$ are bisected



Uniform vs. Adaptive Mesh-Refinement 1/2

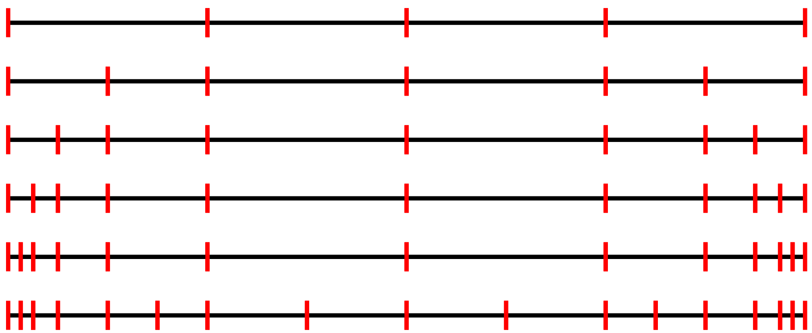
- 1D boundary piece $\Gamma = (-1,1) \times \{0\}$
- $u(x,y) = -2x/\sqrt{1-x^2}$ solution to $f(x,y) = -x$



- singularities at $\pm(1,0)$

Adaptive Mesh-Refinement

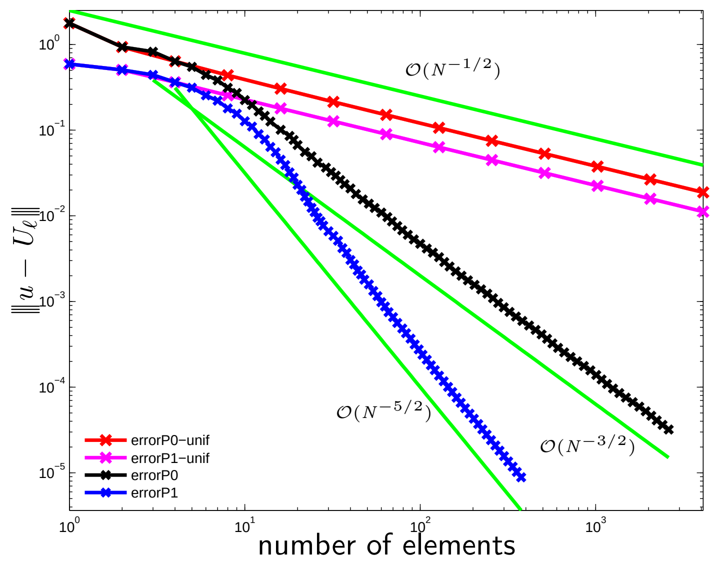
- in each step, only certain elements are bisected



- usually, take care of K-mesh property

$$\frac{\mathrm{diam}(T)}{\mathrm{diam}(T')} \le K \quad \text{for all } T, T' \in \mathcal{T}_\ell \text{ neighbours}$$

Uniform vs. Adaptive Mesh-Refinement 2/2



Observation

- although RHS is smooth, solution exhibits singularities
- uniform refinement can lead to poor convergence rates
- higher p does not help on uniform meshes
- appropriate adaptive refinement recovers optimal rates

- question: how to grade the mesh automatically?
- answer: a posteriori error estimation + adaptive algorithm!

3D Model Problem

- weakly-singular integral equation

$$Vu(x) := \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\,u(y)\,dy = f(x) \quad \text{for } x \in \Gamma$$

- $\Gamma \subseteq \partial\Omega$ with $\Omega \subset \mathbb{R}^3$ bounded and Lipschitz

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Find solution $u \in \mathcal{H}$ of

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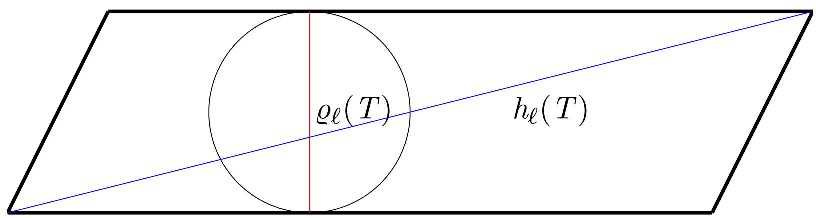
- $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ is scalar product
- induced norm $|\!|\!|v|\!|\!| := \langle\!\langle v\,,v\rangle\!\rangle^{1/2} \simeq \|v\|_{\widetilde{H}^{-1/2}(\Gamma)}$
- Theorem of Riesz $\Longrightarrow$ unique solution $u \in \mathcal{H}$
- u in general *not* computable numerically

3D Model Problem

Galerkin Discretization

- as before $U_\ell \in \mathcal{X}_\ell := \mathcal{P}^p(\mathcal{T}_\ell)$
- mesh $\mathcal{T}_\ell$ of Γ with local mesh-widths $h_\ell, \varrho_\ell \in L^\infty(\Gamma)$



A Priori Error Estimate

- suppose regularity $u \in \mathcal{H} \cap H^t(\Gamma)$
- $U_\ell \in \mathcal{X}_\ell = \mathcal{P}^p(\mathcal{T}_\ell)$ Galerkin solution

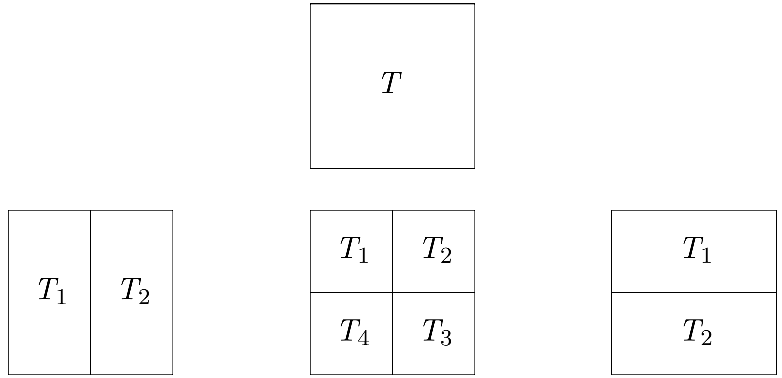
$$\Longrightarrow \|u - U_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|u\|_{H^t(\Gamma)} \max_{T \in \mathcal{T}_\ell} \mathrm{diam}(T)^{\min\{p+1,t\}+1/2}$$

Optimal Convergence Behavior

- suppose smooth solution u
- suppose $h \simeq \mathrm{diam}(T) \simeq |T|^{1/2}$ for all $T \in \mathcal{T}_\ell$
- let $N := \#\mathcal{T}_\ell \simeq h^{-2}$ (because of 3D)

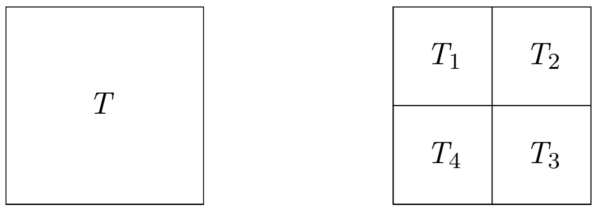
$$\Longrightarrow |||u - U_\ell||| \simeq \|u - U_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h^{p+3/2} \simeq N^{-(p+3/2)/2}$$

Anisotropic Mesh-Refinement



- marked element T can be split anisotropically

Isotropic Mesh-Refinement

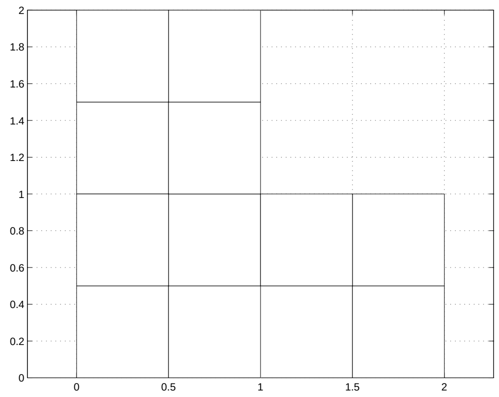


- marked element T is split into $T_1, \dots, T_4$
- either all elements are refined (uniform refinement)
- or only some elements (adaptive isotropic refinement)

K-mesh property

- $\leq K$ elements meet at each node
- $\leq K$ hanging nodes per edge
- $\dfrac{h_\ell|_T}{h_\ell|_{T'}} \leq K$ for all neighboring elements $T \cap T' \neq \emptyset$
- $\dfrac{\varrho_\ell|_T}{\varrho_\ell|_{T'}} \leq K$ for all neighboring elements $T \cap T' \neq \emptyset$
- satisfied for each mesh with $K = K(\mathcal{T}_\ell)$
- important: $K(\mathcal{T}_\ell) \leq C < \infty$ uniformly bdd under refinement
  - usually required for numerical analysis

Numerical Experiment



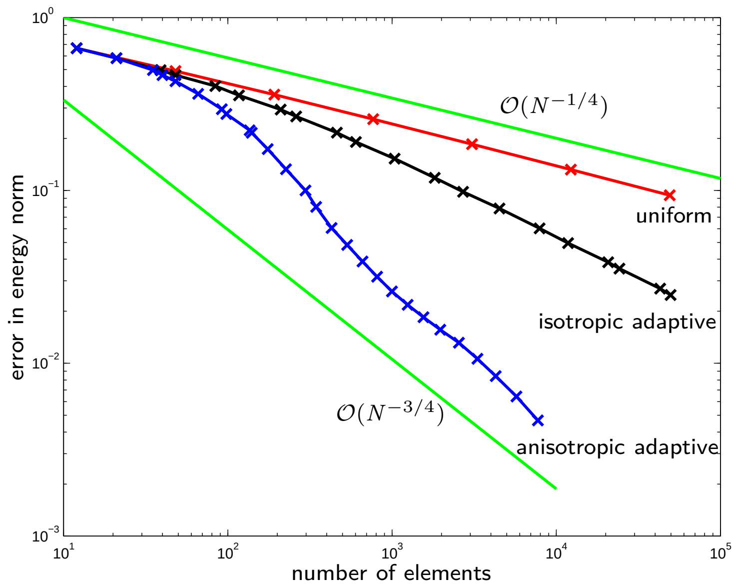
- consider $\color{red}{Vu = 1}$ on L-shaped screen $L \times \{0\} \subset \mathbb{R}^3$
- initial mesh $\mathcal{T}_0$ with $N = 12$ elements
- lowest-order BEM $p = 0$

Observation

- although RHS is smooth, solution u exhibits singularities
  - singularities along edges and at corners
- for 3D BEM, convergence for uniform refinement can be poor
- isotropic adaptive refinement improves convergence rate
- but: optimal convergence requires anisotropic elements

- question: how to grade the mesh automatically?
- answer: a posteriori error estimation + adaptive algorithm!

Uniform vs. Adaptive Mesh-Refinement



Abstract h-h/2 Error Estimation

Abstract Setting

- Hilbert space $\mathcal{H}$ with scalar product $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$
- induced norm $|\!|\!| \cdot |\!|\!|$
- $F(\cdot)$ linear, continuous

Variational Formulation

Find solution $u \in \mathcal{H}$ of

$$\langle\!\langle u\,,v \rangle\!\rangle = F(v) \quad \text{for all } v \in \mathcal{H}$$

- Theorem of Riesz $\Longrightarrow$ unique solution $u \in \mathcal{H}$

Characteristics of Galerkin Methods

- U_ℓ characterized by Galerkin orthogonality

$$\langle\!\langle u - U_\ell\,, V_\ell \rangle\!\rangle = 0 \quad \text{for all} \quad V_\ell \in \mathcal{X}_\ell$$

- Galerkin projection $\mathbb{G}_\ell : \mathcal{H} \to \mathcal{X}_\ell$, $u \mapsto U_\ell$
  - linear projection $\mathbb{G}_\ell^2 = \mathbb{G}_\ell$
  - orthogonal with respect to $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$
  - symmetric $\langle\!\langle \mathbb{G}_\ell u\,, v \rangle\!\rangle = \langle\!\langle u\,, \mathbb{G}_\ell v \rangle\!\rangle = \langle\!\langle \mathbb{G}_\ell u\,, \mathbb{G}_\ell v \rangle\!\rangle$
- best approximation property (Céa's lemma)

$$|\!|\!| u - U_\ell |\!|\!| = \min_{V_\ell \in \mathcal{X}_\ell} |\!|\!| u - V_\ell |\!|\!|$$

- $\mathcal{X}_\ell \subset \widehat{\mathcal{X}}_\ell \quad \Longrightarrow \quad \mathbb{G}_\ell \widehat{U}_\ell = U_\ell$
  - since $\langle\!\langle \widehat{U}_\ell - U_\ell\,, V_\ell \rangle\!\rangle = 0$ for all $V_\ell \in \mathcal{X}_\ell$
  - in particular, $|\!|\!| \widehat{U}_\ell - U_\ell |\!|\!| = \min_{V_\ell \in \mathcal{X}_\ell} |\!|\!| \widehat{U}_\ell - V_\ell |\!|\!|$

Galerkin Discretization

- $\mathcal{X}_\ell \subseteq \mathcal{H}$ finite dimensional subspace

Galerkin Formulation

Find solution $U_\ell \in \mathcal{X}_\ell$ of

$$\langle\!\langle U_\ell\,, V_\ell \rangle\!\rangle = F(V_\ell) \quad \text{for all } V_\ell \in \mathcal{X}_\ell$$

- Theorem of Riesz $\Longrightarrow$ unique solution $U_\ell \in \mathcal{X}_\ell$

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \le 1$
- refinement indicators $\mu_\ell(T) \approx |\!|\!| u - U_\ell |\!|\!|_T$

- $\theta \ll 1 \Longrightarrow$ only few marked elts
- $\theta \approx 1 \Longrightarrow$ essentially uniform refinement

Adaptive Algorithm

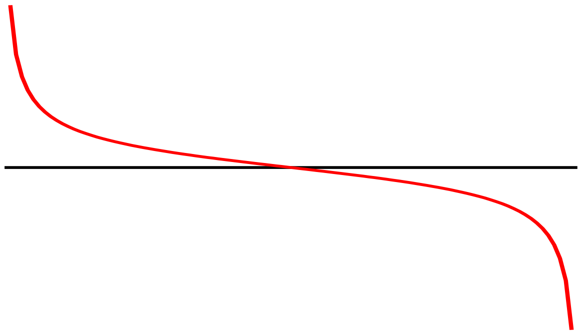
1. compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$, where $\mathcal{X}_\ell = \mathcal{X}_\ell(\mathcal{T}_\ell)$
2. for all $T \in \mathcal{T}_\ell$, compute $\mu_\ell(T)$
3. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \mu_\ell(T)^2$$

4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
5. increase counter $\ell \mapsto \ell + 1$ and iterate

h-h/2 Error Estimator 1/8

- 1D example with singular solution



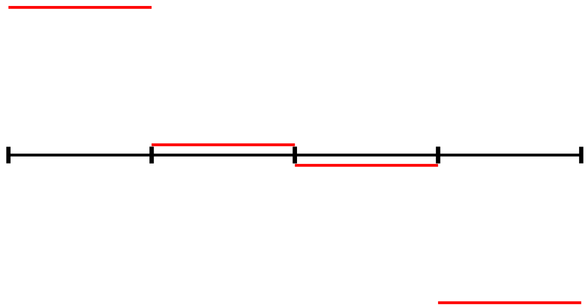
h-h/2 Error Estimator 2/8

- start with mesh $\mathcal{T}_\ell$

- refine $\mathcal{T}_\ell$ uniformly to obtain $\widehat{\mathcal{T}}_\ell$

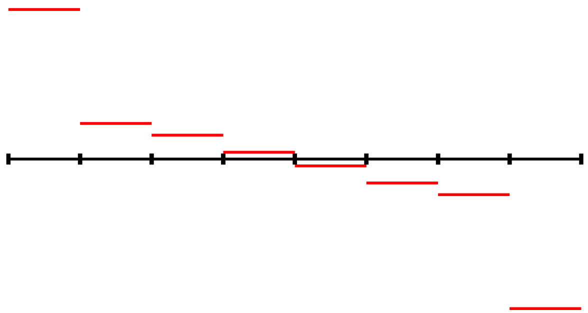
h-h/2 Error Estimator 3/8

- compute Galerkin solution $U_\ell \in \mathcal{X}_\ell := \mathcal{P}^0(\mathcal{T}_\ell)$



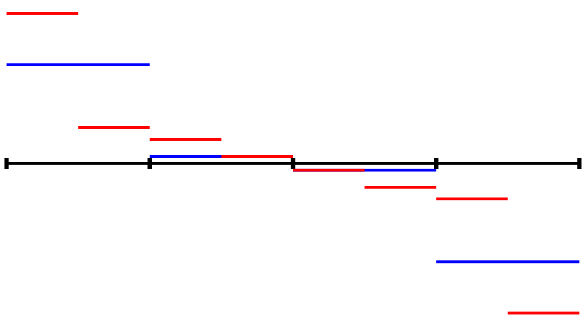
h-h/2 Error Estimator 4/8

- compute Galerkin solution $\widehat{U}_\ell \in \widehat{\mathcal{X}}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$



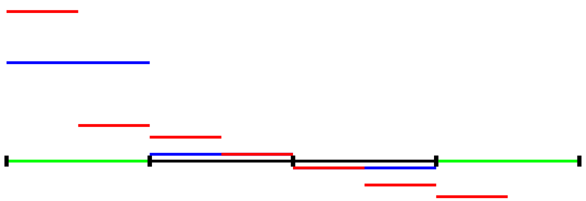
h-h/2 Error Estimator 5/8

- compare $U_\ell \in \mathcal{X}_\ell$ and $\widehat{U}_\ell \in \widehat{\mathcal{X}}_\ell$



h-h/2 Error Estimator 6/8

- mark $\mathcal{T}_\ell$ locally, where $\mu_\ell(T) = \|\widehat{U}_\ell - U_\ell\|_T$ is large

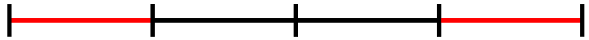


- $\displaystyle \frac{1}{4} \sum_{T \in \mathcal{T}_\ell} \mu_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \mu_\ell(T)^2$

- $\|\widehat{U}_\ell - U_\ell\|_T$ not specified yet

h-h/2 Error Estimator 7/8

- got marking of old mesh $\mathcal{T}_\ell$

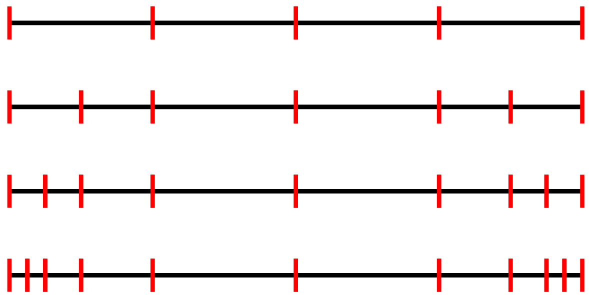


- refine marked elements to obtain new mesh $\mathcal{T}_{\ell+1}$

- refine $\mathcal{T}_{\ell+1}$ uniformly to obtain $\widehat{\mathcal{T}}_{\ell+1}$

h-h/2 Error Estimator 8/8

- creates sequence of meshes $\mathcal{T}_\ell$



- etc.

Efficiency

- $\widehat{\mathcal{T}}_\ell$ uniform refinement of $\mathcal{T}_\ell$
- $\widehat{U}_\ell \in \widehat{\mathcal{X}}_\ell := \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ Galerkin solution
- $\eta_\ell := \| \widehat{U}_\ell - U_\ell \|$ canonical h-h/2 estimator
  - heuristics: $\| u - \widehat{U}_\ell \| \ll \| u - U_\ell \|$

Efficiency

There always holds $\quad \eta_\ell \le \| u - U_\ell \|$

- proof by Galerkin orthogonality $\langle\!\langle u - \widehat{U}_\ell\,, \widehat{U}_\ell - U_\ell \rangle\!\rangle = 0$

$$\begin{aligned} \| u - U_\ell \|^2 &= \| (u - \widehat{U}_\ell) + (\widehat{U}_\ell - U_\ell) \|^2 \\ &= \| u - \widehat{U}_\ell \|^2 + \eta_\ell^2 \end{aligned}$$

Asymptotic Behavior $\implies$ Saturation Assumption

- suppose: asymptotics $\| u - U_\ell \| = C N^{-\alpha}$ with $N = \#\mathcal{T}_\ell$
- uniform refinement satisfies $\#\widehat{\mathcal{T}}_\ell = k \cdot \#\mathcal{T}_\ell$ with $k \ge 2$

$\implies$ $\| u - \widehat{U}_\ell \| = C(k\,\#\mathcal{T}_\ell)^{-\alpha} = k^{-\alpha} \, \| u - U_\ell \|$

$\implies$ $q_{\rm sat} = k^{-\alpha}$

Reliability

Reliability $\iff$ Saturation Assumption

Saturation assumption

$$\| u - \widehat{U}_\ell \| \le q_{\rm sat} \, \| u - U_\ell \| \qquad \text{for some} \quad q_{\rm sat} \in (0,1)$$

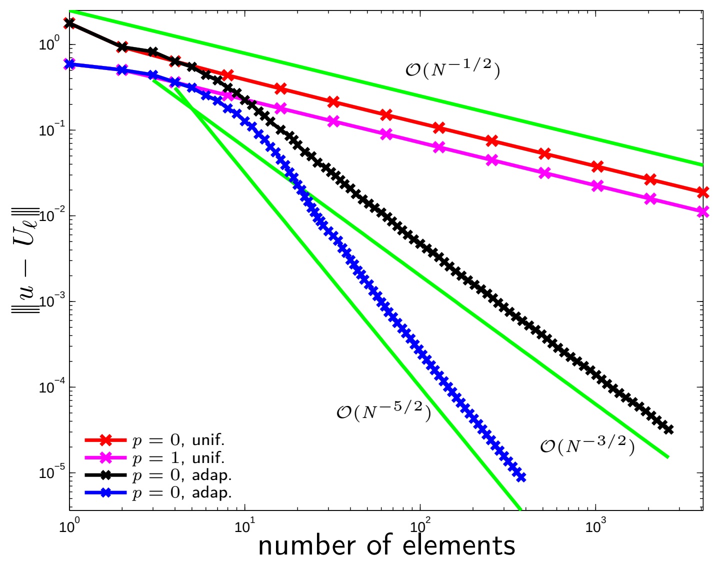
is equivalent to reliability

$$\| u - U_\ell \| \le C_{\rm rel} \, \eta_\ell$$

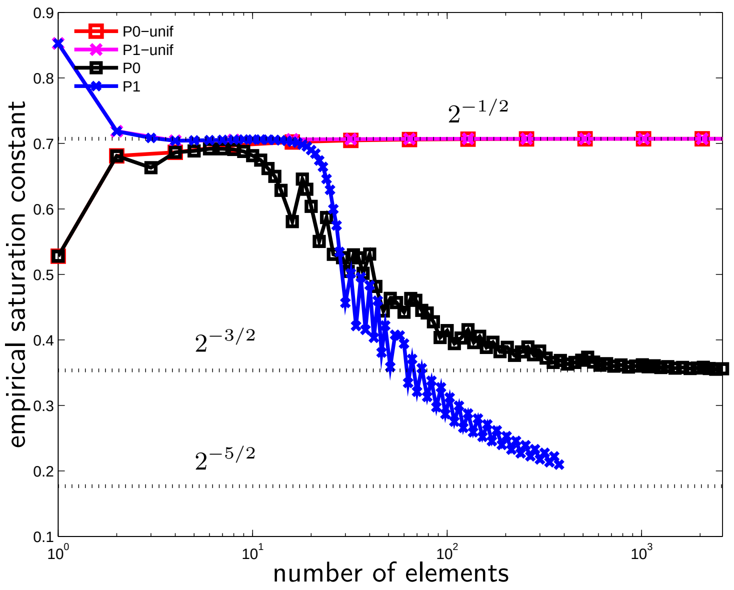
with $C_{\rm rel} = (1 - q_{\rm sat}^2)^{-1/2} \ge 1$

- proof again by Galerkin orth. $\| u - U_\ell \|^2 = \| u - \widehat{U}_\ell \|^2 + \eta_\ell^2$
  - $\| u - U_\ell \|^2 \le q_{\rm sat}^2 \, \| u - U_\ell \|^2 + \eta_\ell^2$
  - $\| u - \widehat{U}_\ell \|^2 = \| u - U_\ell \|^2 - \eta_\ell^2 \le (1 - C_{\rm rel}^2) \, \| u - U_\ell \|^2$

Saturation Assumption for 2D Model Problem 1/2



Saturation Assumption for 2D Model Problem 2/2



Difficulties & Tasks

- $\|\cdot\|_{L^2(\Gamma)}$ is local in the sense of

$$\|v\|^2_{L^2(\Gamma)} = \sum_{T\in\mathcal{T}_\ell} \|v\|^2_{L^2(T)}$$

- $|\!|\!|\cdot|\!|\!| \simeq \|\cdot\|_{\widetilde{H}^{-1/2}(\Gamma)}$ is non-local

$$|\!|\!|v|\!|\!|^2 = \int_\Gamma\int_\Gamma v(x)\,G(x-y)\,v(y)\,dy\,dx$$

- What is appropriate local contribution $|\!|\!|\widehat{U}_\ell - U_\ell|\!|\!|_T$?

- Avoid computation of U_ℓ, since $\widehat{U}_\ell$ is computed and

$$|\!|\!|u-\widehat{U}_\ell|\!|\!| = \min_{\widehat{V}_\ell\in\widehat{\mathcal{X}}_\ell} |\!|\!|u-\widehat{V}_\ell|\!|\!| \le |\!|\!|u-U_\ell|\!|\!|$$

h-h/2 Error Estimation for BEM

Inverse Estimates

- bound stronger (semi-)norms by weaker norms
  - up to negative powers of mesh-size
- e.g. (FEM): $\|\varrho_\ell^{1+\alpha}\nabla W_\ell\|_{L^2(\Omega)} \lesssim \|\varrho_\ell^\alpha W_\ell\|_{L^2(\Omega)}$
  - for all $W_\ell\in\mathcal{P}^q(\mathcal{T}_\ell)$
  - for all $\alpha\in\mathbb{R}$
- can only hold for discrete functions
  - based on norm equivalence on finite dimensional spaces

Local Inverse Estimate in $\widetilde{H}^{-1/2}$

Theorem (Graham, Hackbusch, Sauter '05)

- $\|\varrho_\ell^{1/2+\alpha} V_\ell\|_{L^2(\Gamma)} \le C_{\rm inv}\, \|\varrho_\ell^{\alpha} V_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}$ for all $V_\ell \in \mathcal{P}^p(\mathcal{T}_\ell)$
  - $C_{\rm inv}$ depends only on Γ, p, and K-mesh property of $\mathcal{T}_\ell$
  - for all $\alpha \in \mathbb{R}$

- Corollary: $\mu_\ell := \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} \lesssim |\!|\!| \widehat{U}_\ell - U_\ell |\!|\!| = \eta_\ell$
  - $\alpha = 0$, $|\!|\!| \cdot |\!|\!| \simeq \| \cdot \|_{\widetilde{H}^{-1/2}(\Gamma)}$

Local Approximation Estimate Between $\widetilde{H}^{-1/2}$ and L^2

Theorem (e.g., Carstensen, P. '06)

- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^p(\mathcal{T}_\ell)$ L^2-orthogonal projection
- $v \in L^2(\Gamma)$

$\Longrightarrow$ $C_{\rm apx}^{-1}\, \|(1-\Pi_\ell)v\|_{\widetilde{H}^{-1/2}(\Gamma)} \le \|h_\ell^{1/2}\,(1-\Pi_\ell)v\|_{L^2(\Gamma)} \le \|h_\ell^{1/2}\, v\|_{L^2(\Gamma)}$

- $C_{\rm apx} > 0$ depends only on Γ

- first prove $\|(1-\Pi_\ell)v\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2} v\|_{L^2(\Gamma)}$
- apply this to $v := (1-\Pi_\ell)w$, i.e., $(1-\Pi_\ell)v = (1-\Pi_\ell)w$

$\Longrightarrow$ $\|(1-\Pi_\ell)w\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(1-\Pi_\ell)w\|_{L^2(\Gamma)}$

- Π_ℓ is even $\mathcal{T}_\ell$-piecewise best approximation

$\Longrightarrow$ $\|h_\ell^{1/2}(1-\Pi_\ell)w\|_{L^2(T)} \lesssim \|h_\ell^{1/2} w\|_{L^2(T)}$

$\Longrightarrow$ $\|h_\ell^{1/2}(1-\Pi_\ell)w\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2} w\|_{L^2(\Gamma)}$

Approximation Estimates

- bound weaker norms by stronger (semi-)norms
  - up to positive powers of mesh-size

- require certain "orthogonality" properties

- e.g. (FEM): $\|h_\ell^\alpha \nabla(1-\mathcal{I}_\ell)v\|_{L^2(\Omega)} \lesssim \|h_\ell^{1+\alpha} D^2 v\|_{L^2(\Omega)}$
  - $v \in H^2(\Omega)$
  - $\mathcal{I}_\ell : H^2(\Omega) \to \mathcal{S}^1(\mathcal{T}_\ell)$ nodal interpolant
  - for all $\alpha \in \mathbb{R}$

- e.g. (FEM): $\|h_\ell^\alpha (1-\mathcal{I}_\ell)v\|_{L^2(\Omega)} \lesssim \|h_\ell^{2+\alpha} D^2 v\|_{L^2(\Omega)}$

Equivalent h-h/2 Estimators

Corollary (Ferraz-Leite, P. '08)

- $\eta_\ell = |\!|\!| \widehat{U}_\ell - U_\ell |\!|\!|$
- $\widetilde{\eta}_\ell = |\!|\!| (1-\Pi_\ell)\widehat{U}_\ell |\!|\!|$
- $\mu_\ell = \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}$
- $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)}$

$\Longrightarrow$ $\widetilde{\mu}_\ell \le \mu_\ell \lesssim \eta_\ell \le \widetilde{\eta}_\ell \lesssim \|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}^{1/2}\, \widetilde{\mu}_\ell$

- $\eta_\ell = |\!|\!| \widehat{U}_\ell - U_\ell |\!|\!| = \min_{V_\ell \in \mathcal{X}_\ell} |\!|\!| \widehat{U}_\ell - V_\ell |\!|\!| \le |\!|\!| (1-\Pi_\ell)\widehat{U}_\ell |\!|\!| = \widetilde{\eta}_\ell$
- $\widetilde{\eta}_\ell \simeq \|(1-\Pi_\ell)\widehat{U}_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)} \rightsquigarrow \widetilde{\mu}_\ell$
- Π_ℓ is even $\mathcal{T}_\ell$-piecewise best approximation

$\Longrightarrow$ $\|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(T)} \lesssim \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}$

$\Longrightarrow$ $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)} \lesssim \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} = \mu_\ell$

Remarks

Practical h-h/2 Estimator $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)}$

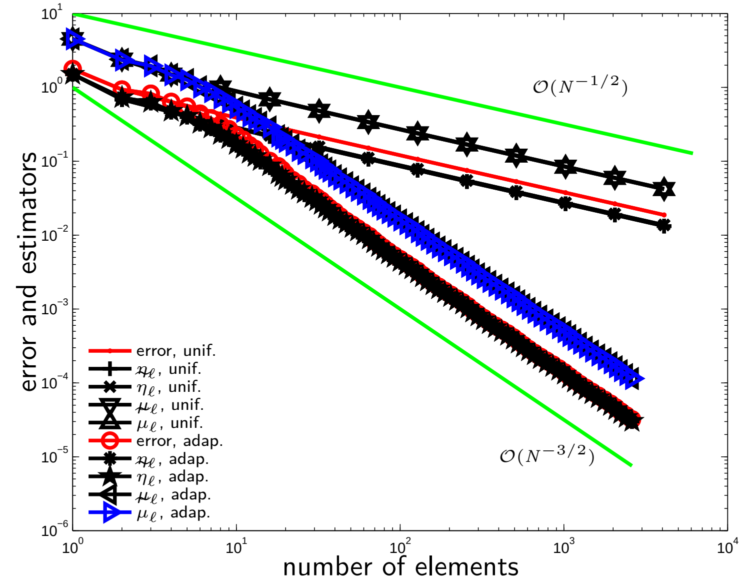
- requires only $\widehat{U}_\ell$, but avoids computation of U_ℓ
- local contributions $\mu_\ell(T) := \|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(T)}$ satisfy

$$\mu_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \mu_\ell(T)^2$$

- $\widetilde{\mu}_\ell \lesssim \eta_\ell = \|\widehat{U}_\ell - U_\ell\| \lesssim \|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}^{1/2}\,\widetilde{\mu}_\ell$

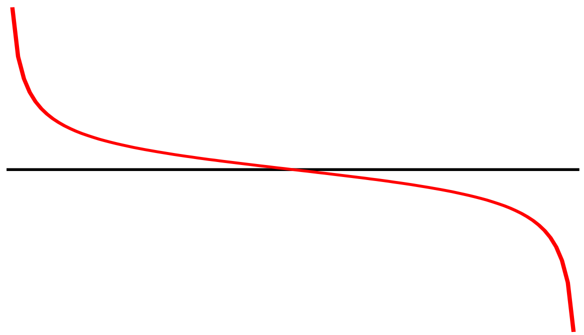
- theoretically, $\|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}$ measures "how isotropic" $\mathcal{T}_\ell$ is
  - bounded for 2D and isotropic mesh-refinement in 3D
  - theoretically unbounded for anisotropic mesh-refinement in 3D
- practically, non visible if $\mathcal{T}_\ell$ fits to singularities of u

Numerical Experiment in 2D (p=0)



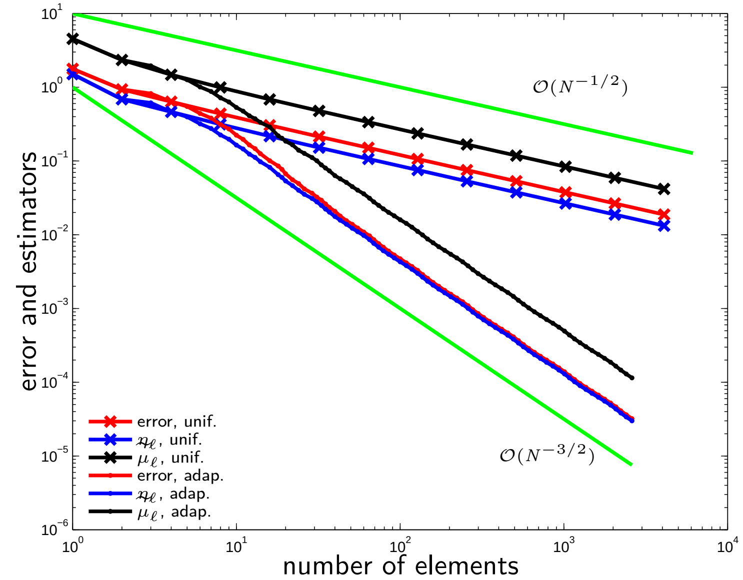
Numerical Experiment in 2D

- 1D boundary piece $\Gamma = (-1,1)\times\{0\}$
- $u(x,y) = -2x/\sqrt{1-x^2}$ solution to $f(x,y) = -x$

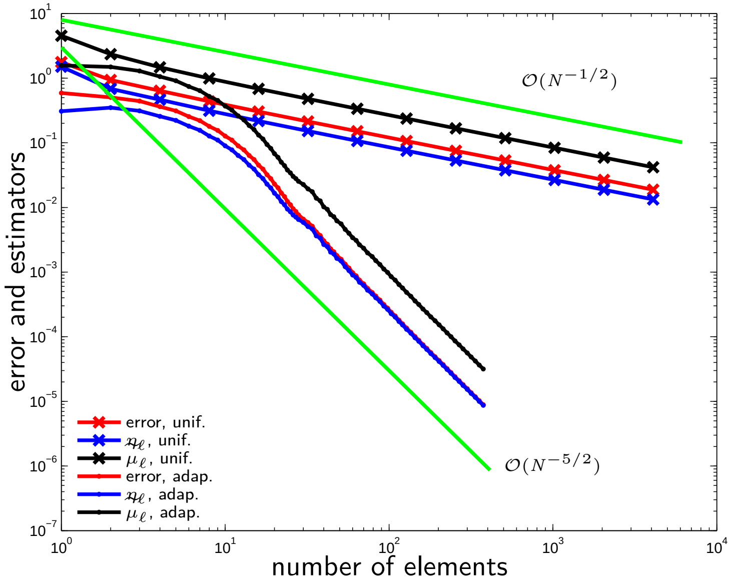


- singularities at $\pm(1,0)$

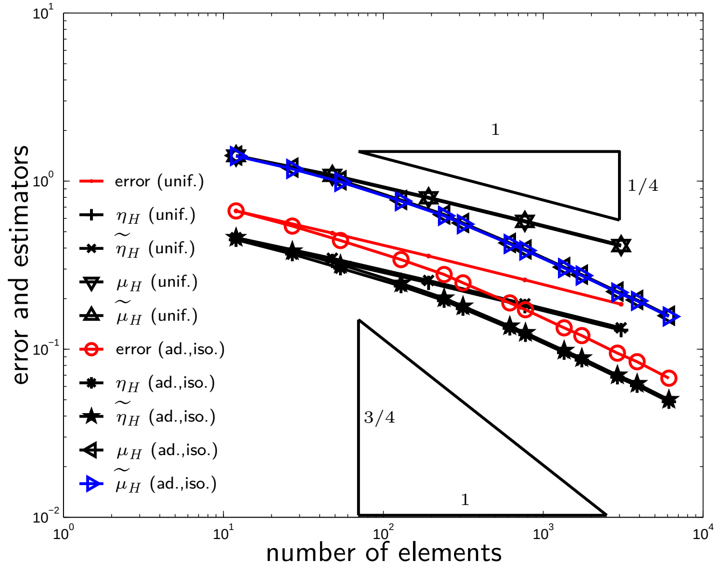
Numerical Experiment in 2D (p=0)



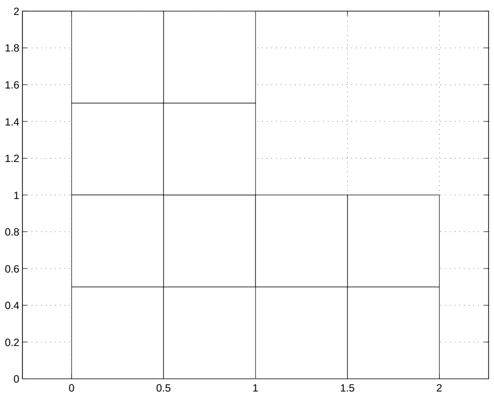
Numerical Experiment in 2D (p=1)



Numerical Experiment in 3D (isotropic)

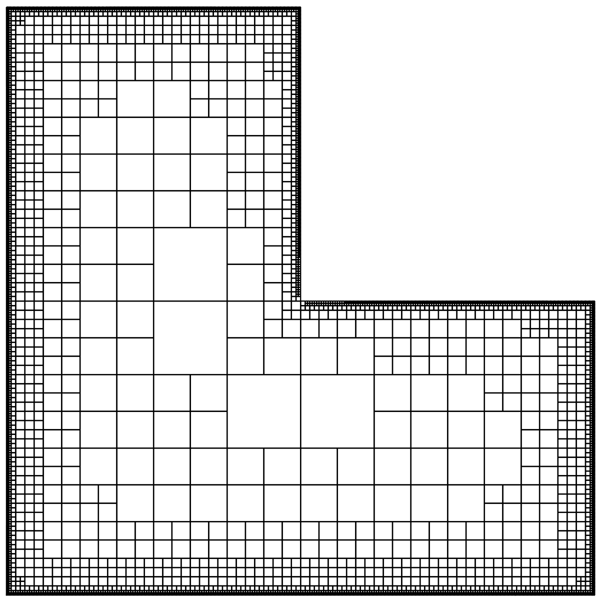


Numerical Experiment in 3D



- consider $Vu = 1$ on L-shaped screen $L \times \{0\} \subset \mathbb{R}^3$
- initial mesh $\mathcal{T}_0$ with $N = 12$ elements
- lowest-order BEM $p = 0$

Adaptive Isotropic Mesh

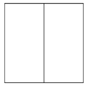


Steering of Anisotropic Mesh-Refinement

- suppose $T \in \mathcal{M}_\ell$
- L^2-orthogonal projection $\Pi_{\texttt{horiz}}$ if T is horizontally refined
- L^2-orthogonal projection $\Pi_{\texttt{vert}}$ if T is vertically refined
- fix parameter $0 < \tau < 1$

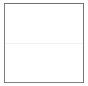
- if: $\|(1-\Pi_{\texttt{horiz}})\,\widehat{U}_\ell\|_{L^2(T)} \le \tau\,\|(1-\Pi_\ell)\,\widehat{U}_\ell\|_{L^2(T)}$

$\Longrightarrow$ refine T horizontally

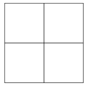


- if: $\|(1-\Pi_{\texttt{vert}})\,\widehat{U}_\ell\|_{L^2(T)} \le \tau\,\|(1-\Pi_\ell)\,\widehat{U}_\ell\|_{L^2(T)}$

$\Longrightarrow$ refine T vertically

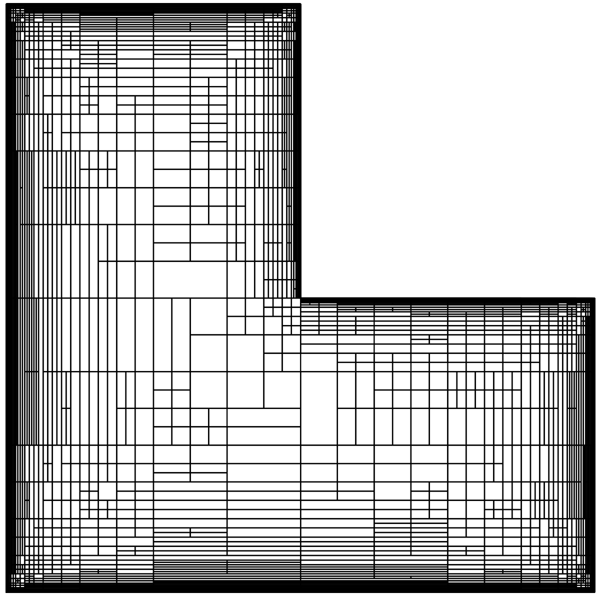


- else: refine T isotropically

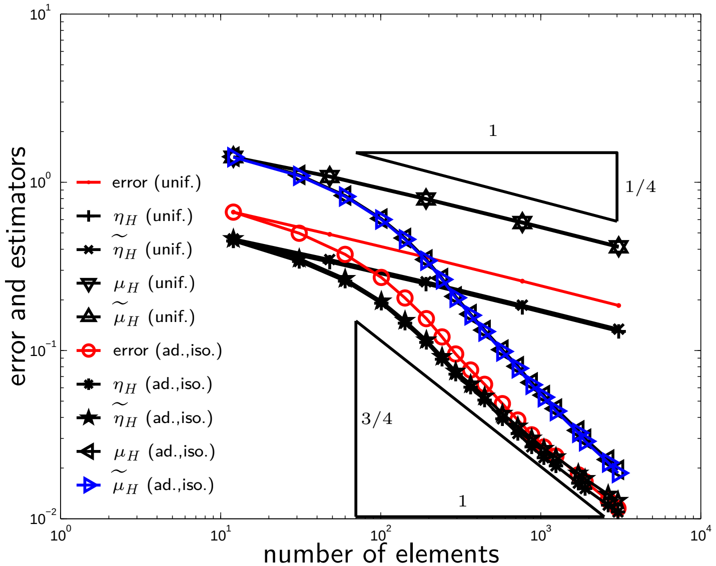


- altogether: $\|\rho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\,\widehat{U}_\ell\|_{L^2(T)} \le \tau\,\widetilde{\mu}_\ell(T)$

Adaptive Anisotropic Mesh-Refinement



Numerical Experiment in 3D (anisotropic)



Conclusions

Conclusions 1/2

- use $\eta_\ell = \|\widehat{U}_\ell - U_\ell\|$ to estimate $\|u - U_\ell\|$

$\oplus$ $\eta_\ell \le \|u - U_\ell\|$ with known constant 1
$\oplus$ most general & most simple strategy
$\oplus$ almost no implementational overhead
$\oplus$ can be computed exactly (up to Galerkin system)

$\ominus$ $\|u - U_\ell\| \lesssim \eta_\ell$ hinges on saturation assumption
$\ominus$ saturation assumption is hard to guarantee in practice
$\ominus$ requires Galerkin solution $\widehat{U}_\ell$ for uniformly refined mesh $\widehat{\mathcal{T}}_\ell$
$\ominus$ cannot steer adaptive mesh-refinement

Extensions

- h-h/2 strategy applies to all problems in Lax-Milgram setting
  - BEM for mixed boundary value problems
  - FEM-BEM coupling
- hyper-singular integral equations with $\|\cdot\| \simeq \|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)}$
  - localization of $\|\cdot\|_{\widetilde{H}^{1/2}(\Gamma)}$ via weighted H^1-seminorm

$$\eta_\ell = \|\widehat{U}_\ell - U_\ell\| \simeq \|\varrho_\ell^{1/2}(1 - \Pi_\ell)\nabla \widehat{U}_\ell\|_{L^2(\Gamma)} = \widetilde{\mu}_\ell$$

  - analysis only for isotropic meshes
- localization analysis works for fixed polynomial degree $p \ge 0$
  - weakly-singular as well as hyper-singular integral equation

Conclusions 2/2

- cheaper local variant $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1 - \Pi_\ell)\widehat{U}_\ell\|_{L^2(\Gamma)}$

$\oplus$ $\widetilde{\mu}_\ell \simeq \eta_\ell$ for 2D and isotropic mesh-refinement in 3D
$\oplus$ only requires $\widehat{U}_\ell$, but avoids U_ℓ
$\oplus$ capable to steer anisotropic mesh-refinement in 3D
$\oplus$ good performance in numerical experiments

$\ominus$ $\widetilde{\mu}_\ell \simeq \eta_\ell$ only empirically for anisotropic mesh-refinement in 3D

- good reasons for (h-h/2)-type estimators
  - check stability of code before implementing other estimators
  - any other estimator should (try to) beat h-h/2
  - each adaptive strategy must satisfy $\widetilde{\mu}_\ell \lesssim \eta_\ell \to 0$

References

Ferraz-Leite, P. (Computing 83, 2008)

Erath, Ferraz-Leite, Funken, P. (Appl. Numer. Math. 59, 2009)
- weakly-singular integral equations in 2D / 3D
- (global) equivalence to other estimators (averaging, two-level)

Aurada, Ferraz-Leite, et al. (Appl. Numer. Math. 62, 2012)
- lowest-order BEM for mixed BVPs in 2D

Erath, Funken, Goldenits, P. (Appl. Anal. 92, 2013)

Aurada, Feischl, Führer, et al. (Appl. Numer. Math, 2014)
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Open Problems

- prove $\eta_\ell \le C\,\widetilde{\mu}_\ell$ with $C = C(\|h_\ell/\varrho_\ell\|_{L^\infty(\Gamma)}\,,\,u) \le M < \infty$
  - as long as $\mathcal{T}_\ell$ is aligned with anisotropic behavior of u
- h-h/2 estimators for hyper-singular IE on anisotropic meshes
  - so far, localization of $\widetilde{H}^{1/2}$-norm requires isotropic meshes
- prove saturation assumption for smooth RHS
  - e.g., if $\mathcal{T}_\ell$ is sufficiently fine

Zurich Summer School 2014

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Adaptive Boundary Element Methods

Part 2: Estimator Reduction & Convergence

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Vienna University of Technology
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Der Wissenschaftsfonds.

Some References on adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

1. Dörfler '96
2. Morin, Nochetto, Siebert '00
3. Morin, Siebert, Veeser '08
4. Siebert '11

- Dörfler marking
- convergence up to Tolerance $\simeq \|h_0 f\|_{L^2(\Omega)}$

- h-h/2 ABEM: Ferraz-Leite, Ortner, Praetorius '10
- simpler argument: Aurada, Ferraz-Leite, Praetorius '12
  - combines MSV '08 + Cascon, Kreuzer, Nochetto, Siebert '08
  - so-called estimator reduction principle

History: Convergence of Adaptive Algs

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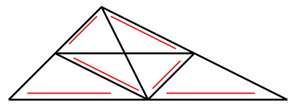
- $\mathrm{err}_{\ell+1} \le q\,\mathrm{err}_\ell + \mathrm{osc}_\ell$
- $\mathrm{osc}_\ell := \|h_\ell(f - f_\ell)\|_{L^2}$
- bisec5-refinement



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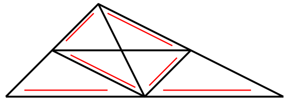
- general theory
- general marking
- bisec5-refinement



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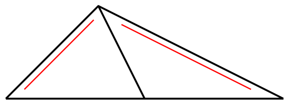

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  - so-called estimator reduction principle

A Priori Convergence

Abstract Setting

- Hilbert space $\mathcal{H}$ with scalar product $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$
- dual space $\mathcal{H}^*$ of $\mathcal{H}$
- right-hand side $f \in \mathcal{H}^*$

Variational Formulation

Find solution $u \in \mathcal{H}$ of

$$\langle\!\langle u\,, v \rangle\!\rangle = F(v) \quad \text{for all } v \in \mathcal{H}$$

- Theorem of Riesz $\implies$ unique solution $u \in \mathcal{H}$

Uniform Mesh-Refinement $\implies$ Convergence 1/2

- Let $\mathcal{T}_\ell$ sequence of uniformly refined meshes, i.e., $\|h_\ell\|_{L^\infty} \to 0$
- suppose: $\mathcal{X}_\ell = \mathcal{X}_\ell(\mathcal{T}_\ell)$
- usual setting: Exists $\mathcal{D} \subseteq \mathcal{H}$ dense with

$$\min_{V_\ell \in \mathcal{X}_\ell} \|\!| v - V_\ell |\!\| \xrightarrow{\ell\to\infty} 0 \quad \text{for all } v \in \mathcal{D}$$

- e.g.: $\mathcal{D} = H^2(\Omega) \subseteq H^1(\Omega)$

$$\min_{V_\ell \in \mathcal{X}_\ell} \|v - V_\ell\|_{H^1(\Omega)} \le \|(1-\mathcal{I}_\ell)v\|_{H^1(\Omega)} \lesssim \|h_\ell D^2 v\|_{L^2(\Omega)}$$

Galerkin Discretization

- $\mathcal{X}_\ell \subseteq \mathcal{H}$ finite dimensional subspace

Galerkin Formulation

Find solution $U_\ell \in \mathcal{X}_\ell$ of

$$\langle\!\langle U_\ell\,, V_\ell \rangle\!\rangle = F(V_\ell) \quad \text{for all } V_\ell \in \mathcal{X}_\ell$$

- Theorem of Riesz $\implies$ unique solution $U_\ell \in \mathcal{X}_\ell$

Uniform Mesh-Refinement $\implies$ Convergence 2/2

Proposition

- Suppose $\mathcal{D} \subseteq \mathcal{H}$ dense with

$$\min_{V_\ell \in \mathcal{X}_\ell} \|\!| v - V_\ell |\!\| \xrightarrow{\ell\to\infty} 0 \quad \text{for all } v \in \mathcal{D}$$

$\implies$ $\|\!| u - U_\ell |\!\| \xrightarrow{\ell\to\infty} 0$

- $\|\!| u - U_\ell |\!\| = \min_{V_\ell \in \mathcal{X}_\ell} \|\!| u - V_\ell |\!\| \le \|\!| u - v |\!\| + \min_{V_\ell \in \mathcal{X}_\ell} \|\!| v - V_\ell |\!\|$
- choose $v \in \mathcal{D}$ with $\|\!| u - v |\!\| \le \varepsilon$

Difficulty with Adaptive Mesh-Refinement

- adaptivity might lead to $\|h_\ell\|_{L^\infty} \not\to 0$ as $\ell \to \infty$
- cannot expect $\min_{V_\ell \in \mathcal{X}_\ell} \|\!|v - V_\ell|\!\| \to 0$ for all $v \in \mathcal{D}$
- cannot expect $\|\!|u - U_\ell|\!\| \to 0$ for all u
- but: adaptivity is driven by concrete estimator
- and: adaptivity provides structure on discrete spaces

Adaptive Schemes Converge!

- $\mathcal{T}_{\ell+1}$ obtained from $\mathcal{T}_\ell$ by local mesh-refinement
- thus $\mathcal{X}_\ell := \mathcal{P}^0(\mathcal{T}_\ell) \implies$ nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$

A Priori Convergence of Adaptive Galerkin Schemes

- there holds $U_\ell \xrightarrow{\ell\to\infty} U_\infty$
- but $u = U_\infty$?

A Priori Convergence

- $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ sequence of nested discrete spaces
- $U_\ell \in \mathcal{X}_\ell$ Galerkin solution

Convergence Theorem (Babuška, Vogelius '84)

There holds convergence

$$U_\ell \xrightarrow{\ell\to\infty} U_\infty \in \mathcal{H}$$

with some limit $U_\infty \in \mathcal{H}$

- $\mathcal{X}_\infty := \mathrm{closure}(\bigcup_{\ell=0}^\infty \mathcal{X}_\ell) \subseteq \mathcal{H}$ with Galerkin sol. $U_\infty \in \mathcal{X}_\infty$
- For $\varepsilon > 0$, exists $V_k \in \mathcal{X}_k$ s.t. $\|\!|U_\infty - V_k|\!\| \le \varepsilon$

$\implies$ $\|\!|U_\infty - U_\ell|\!\| = \min_{V_\ell \in \mathcal{X}_\ell} \|\!|U_\infty - V_\ell|\!\| \le \|\!|U_\infty - V_k|\!\| \le \varepsilon$ for $\ell \ge k$

- i.e. $\lim_\ell U_\ell = U_\infty$

Estimator Reduction

Estimator Convergence

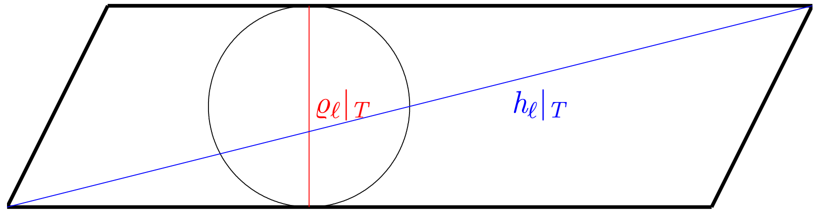
Estimator Convergence $\rho_\ell \xrightarrow{\ell\to\infty} 0$

- conceptually, algorithm only sees estimator, *not* error
- conceptually, estimator convergence $\rho_\ell \to 0$ is necessary
- consequently, rather ask for $\rho_\ell \to 0$ than for $\|u - U_\ell\| \to 0$

- idea: [Morin, Veeser, Siebert '08], [Siebert '11] for AFEM
  - quite general marking strategies (instead of Dörfler marking)
  - but: based on local properties of estimator
  - but: analysis based on Lebesgue theorem
- here: simpler argument from [Aurada, Ferraz-Leite, P. '12]
  - requires Dörfler marking
  - arguments only based on basic calculus

Model Problem

- weakly-singular integral equation in 2D or 3D
- bisection in 2D / isotropic mesh-refinement in 3D
- adaptivity steered by $\mu_\ell = \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}$
- recall mesh-size function $\varrho_\ell \in L^\infty(\Gamma)$



Estimator Reduction Principle

Estimator Reduction $\Longrightarrow$ Estimator Convergence

Let $0 < \kappa < 1$ and $\rho_\ell, \alpha_\ell \geq 0$ with

- $\rho_{\ell+1} \leq \kappa\, \rho_\ell + \alpha_\ell$
- $\alpha_\ell \xrightarrow{\ell\to\infty} 0$

Then, $\rho_\ell \xrightarrow{\ell\to\infty} 0$

- $\rho_{\ell+1} \leq \kappa^{\ell+1}\rho_0 + \sum_{j=0}^{\ell} \kappa^{\ell-j}\alpha_j \leq \kappa^{\ell+1}\rho_0 + \|(\alpha_n)\|_\infty \sum_{k=0}^{\ell} \kappa^\ell$
- (ρ_n) bounded $\Longrightarrow$ $M := \limsup_\ell \rho_\ell \geq \liminf_\ell \rho_\ell \geq 0$ exist
- next, $M = \limsup_\ell \rho_{\ell+1} \leq \kappa \limsup_\ell \rho_\ell + \limsup_\ell \alpha_\ell = \kappa M$
- thus, $M = 0$
- finally, $0 \leq \liminf_\ell \rho_\ell \leq \limsup_\ell \rho_\ell = 0 \Longrightarrow \lim_\ell \rho_\ell = 0$

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

Adaptive Algorithm

1. compute discrete solution Φ_ℓ for mesh $\mathcal{T}_\ell$
2. compute refinement indicators $\mu_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.
$$\theta \sum_{T\in\mathcal{T}_\ell} \mu_\ell(T)^2 \leq \sum_{T\in\mathcal{M}_\ell} \mu_\ell(T)^2$$
4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
5. increase counter $\ell \mapsto \ell + 1$ and iterate

Estimator Reduction (isotropic)

Proposition (Ferraz-Leite, Ortner, P. '10; Aurada, F.-L., P. '12)

- Isotropic refinement steered by $\mu_\ell = \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}$

$\Longrightarrow$ $\mu_{\ell+1} \le \kappa\, \mu_\ell + C\, |||(\widehat{U}_{\ell+1} - \widehat{U}_\ell) - (U_{\ell+1} - U_\ell)|||$

- with $\kappa = \sqrt{1 - \theta/2}$

$\Longrightarrow$ This already implies $\mu_\ell \xrightarrow{\ell\to\infty} 0$

- nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ $\Longrightarrow$ a priori convergence $U_\ell \xrightarrow{\ell\to\infty} U_\infty$
- nestedness $\widehat{\mathcal{X}}_\ell \subseteq \widehat{\mathcal{X}}_{\ell+1}$ $\Longrightarrow$ a priori convergence $\widehat{U}_\ell \xrightarrow{\ell\to\infty} \widehat{U}_\infty$
- thus, $\alpha_\ell := |||(\widehat{U}_{\ell+1} - \widehat{U}_\ell) - (U_{\ell+1} - U_\ell)||| \xrightarrow{\ell\to\infty} 0$
- estimator reduction for $\mu_\ell \Longrightarrow \lim_\ell \mu_\ell = 0$
- in particular, $\eta_\ell = |||\widehat{U}_\ell - U_\ell||| \simeq \mu_\ell \Longrightarrow \widehat{U}_\infty = U_\infty$

Proof of Estimator Reduction 1/4 (isotropic)

1. triangle inequality & inverse estimate prove

$$\begin{aligned}
\mu_{\ell+1} &= \|\varrho_{\ell+1}^{1/2}(\widehat{U}_{\ell+1} - U_{\ell+1})\|_{L^2(\Gamma)}\\
&\le \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2} + \|\varrho_{\ell+1}^{1/2}[(\widehat{U}_{\ell+1} - U_{\ell+1}) - (\widehat{U}_\ell - U_\ell)]\|_{L^2}\\
&\le \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2} + C_{\rm inv}\, |||(\widehat{U}_{\ell+1} - \widehat{U}_\ell) - (U_{\ell+1} - U_\ell)|||
\end{aligned}$$

Proof of Estimator Reduction 2/4 (isotropic)

1. $\mu_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} + C_{\rm inv}\, |||(\widehat{U}_{\ell+1} - \widehat{U}_\ell) - (U_{\ell+1} - U_\ell)|||$
2. use isotropic mesh-refinement to see

$$\begin{aligned}
&\|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}^2\\
&= \sum_{T\in\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{M}_\ell} \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2\\
&\le \frac{1}{2}\sum_{T\in\mathcal{M}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{M}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2\\
&= -\frac{1}{2}\sum_{T\in\mathcal{M}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2
\end{aligned}$$

Proof of Estimator Reduction 3/4 (isotropic)

1. $\mu_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} + C_{\rm inv}\, |||(\widehat{U}_{\ell+1} - \widehat{U}_\ell) - (U_{\ell+1} - U_\ell)|||$
2. $\|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}^2$
$$\le -\frac{1}{2}\sum_{T\in\mathcal{M}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2$$
3. use Dörfler marking strategy

$$\theta \sum_{T\in\mathcal{T}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 \le \sum_{T\in\mathcal{M}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2$$

to see

$$\le \Big(-\frac{\theta}{2} + 1\Big) \sum_{T\in\mathcal{T}_\ell} \|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2$$
$$= (1 - \theta/2)\mu_\ell^2$$

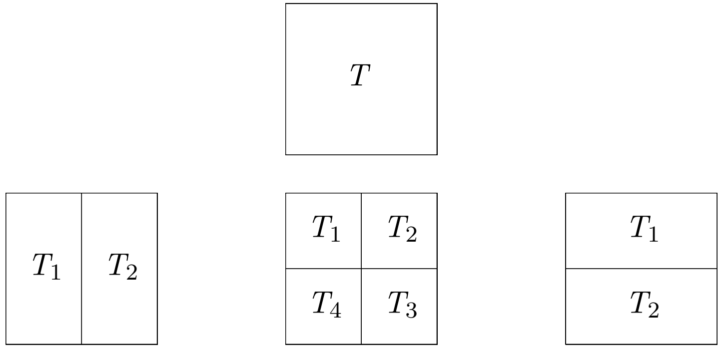
Proof of Estimator Reduction 4/4 (isotropic)

1. $\mu_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)} + C_{\rm inv}\,|\!|\!|(\widehat{U}_{\ell+1}-\widehat{U}_\ell)-(U_{\ell+1}-U_\ell)|\!|\!|$
2. $\|\varrho_{\ell+1}^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(\Gamma)}^2$
   $\le -\dfrac{1}{2}\displaystyle\sum_{T\in\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell}\|\varrho_\ell^{1/2}(\widehat{U}_\ell - U_\ell)\|_{L^2(T)}^2$
3. $\le (1-\theta/2)\,\mu_\ell^2$
4. obtain estimator reduction with $\kappa = \sqrt{1-\theta/2} \in (0,1)$

$$\mu_{\ell+1} \le \kappa\,\mu_\ell + C_{\rm inv}\,|\!|\!|(\widehat{U}_{\ell+1}-\widehat{U}_\ell)-(U_{\ell+1}-U_\ell)|\!|\!|$$

Model Problem

- weakly-singular integral equation in 3D
- anisotropic mesh-refinement



- difficulty: mesh-size ϱ_ℓ not necessarily contracted on $T\in\mathcal{M}_\ell$

Anisotropic Mesh-Refinement in 3D

Remedy: Two-Step Marking Criterion

- use simplified estimator $\widetilde{\mu}_\ell = \|\varrho_\ell^{1/2}(1-\Pi_\ell)\,\widehat{U}_\ell\|_{L^2(\Gamma)}$
- first: single out marked elements by Dörfler marking

$$\theta\sum_{T\in\mathcal{T}_\ell}\widetilde{\mu}_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell}\widetilde{\mu}_\ell(T)^2$$

- second: favor anisotropic refinement for $T\in\mathcal{M}_\ell$, but ensure

$$\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\,\widehat{U}_\ell\|_{L^2(T)} \le \tau\,\|\varrho_{\ell+1}^{1/2}(1-\Pi_\ell)\,\widehat{U}_\ell\|_{L^2(T)} = \tau\,\widetilde{\mu}_\ell(T)$$

  - isotropic ref. $\Longrightarrow \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\,\widehat{U}_\ell\|_{L^2(T)} = 0 \le \tau\,\widetilde{\mu}_\ell(T)$

Estimator Reduction (anisotropic)

- Dörfler marking with $0 < \theta \le 1$
- direction control with $0 < \tau < 1$

Proposition (Aurada, Ferraz-Leite, P. '12)

- Anisotropic refinement steered by $\widetilde{\mu}_\ell = \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(\Gamma)}$

$\Longrightarrow$ $\widetilde{\mu}_{\ell+1} \le \kappa \, \widetilde{\mu}_\ell + C \, \| \widehat{U}_{\ell+1} - \widehat{U}_\ell \|$

- with $\kappa = \sqrt{1 - \theta(1-\tau)}$

$\Longrightarrow$ This already implies $\widetilde{\mu}_\ell \xrightarrow{\ell \to \infty} 0$

Proof of Estimator Reduction 1/4 (anisotropic)

1. triangle ineq. & local orthogonality & inverse estimate prove

$$
\begin{aligned}
\widetilde{\mu}_{\ell+1} &= \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_{\ell+1} \|_{L^2(\Gamma)} \\
&\le \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2} + \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) (\widehat{U}_{\ell+1} - \widehat{U}_\ell) \|_{L^2} \\
&\le \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2} + \| \varrho_{\ell+1}^{1/2} (\widehat{U}_{\ell+1} - \widehat{U}_\ell) \|_{L^2} \\
&\le \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2} + C_{\rm inv} \, \| \widehat{U}_{\ell+1} - \widehat{U}_\ell \|
\end{aligned}
$$

Proof of Estimator Reduction 2/4 (anisotropic)

1. $\widetilde{\mu}_{\ell+1} \le \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(\Gamma)} + C_{\rm inv} \, \| \widehat{U}_{\ell+1} - \widehat{U}_\ell \|$
2. use mesh-refinement to see

$$
\begin{aligned}
&\| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(\Gamma)}^2 \\
&= \sum_{T \in \mathcal{M}_\ell} \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell} \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(T)}^2 \\
&\le \tau \sum_{T \in \mathcal{M}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 \\
&= (\tau - 1) \sum_{T \in \mathcal{M}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2
\end{aligned}
$$

Proof of Estimator Reduction 3/4 (anisotropic)

1. $\widetilde{\mu}_{\ell+1} \le \| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(\Gamma)} + C_{\rm inv} \, \| \widehat{U}_{\ell+1} - \widehat{U}_\ell \|$
2. $\| \varrho_{\ell+1}^{1/2} (1 - \Pi_{\ell+1}) \widehat{U}_\ell \|_{L^2(\Gamma)}^2$
   $\le (\tau - 1) \sum_{T \in \mathcal{M}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 + \sum_{T \in \mathcal{T}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2$
3. use marking strategy

$$
\theta \sum_{T \in \mathcal{T}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 \le \sum_{T \in \mathcal{M}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2
$$

to see

$$
\begin{aligned}
&\le (\theta(\tau - 1) + 1) \sum_{T \in \mathcal{T}_\ell} \| \varrho_\ell^{1/2} (1 - \Pi_\ell) \widehat{U}_\ell \|_{L^2(T)}^2 \\
&= (1 - \theta(1 - \tau)) \, \widetilde{\mu}_\ell^2
\end{aligned}
$$

Proof of Estimator Reduction 4/4 (anisotropic)

1. $\widetilde{\mu}_{\ell+1} \le \|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{U}_\ell\|_{L^2(\Gamma)} + C_{\rm inv}\,|\!|\!|\widehat{U}_{\ell+1}-\widehat{U}_\ell|\!|\!|$
2. $\|\varrho_{\ell+1}^{1/2}(1-\Pi_{\ell+1})\widehat{U}_\ell\|_{L^2(\Gamma)}^2$
   $\le (\tau-1)\sum_{T\in\mathcal{M}_\ell}\|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell}\|\varrho_\ell^{1/2}(1-\Pi_\ell)\widehat{U}_\ell\|_{L^2(T)}^2$
3. $\le (1-\theta(1-\tau))\,\widetilde{\mu}_\ell^2$
4. obtain estimator reduction with $\kappa = \sqrt{1-\theta(1-\tau)} \in (0,1)$

$$\widetilde{\mu}_{\ell+1} \le \kappa\,\widetilde{\mu}_\ell + C_{\rm inv}\,|\!|\!|\widehat{U}_{\ell+1}-\widehat{U}_\ell|\!|\!|$$

Conclusions

- useful adaptive algorithms lead to $\rho_\ell \xrightarrow{\ell\to\infty} 0$
  - concept: ask for $\rho_\ell \xrightarrow{\ell\to\infty} 0$, *not* for error
- estimator reduction $\Longrightarrow$ estimator convergence $\rho_\ell \xrightarrow{\ell\to\infty} 0$
  - Dörfler marking
  - triangle inequality & inverse estimates
  - a priori convergence
- estimator reduction is independent of reliability/efficiency
  - decouples convergence and saturation assumption
  - does not need Galerkin orthogonality
  - relies on contraction of estimator on marked elements
    - e.g.: local contributions $\rho_\ell(T)$ weighted by mesh-size

Conclusion

Extensions

- various problem settings
  - in particular, BEM for hyper-singular integral equations
  - independent of FEM, BEM, FEM-BEM
  - allows certain nonlinearities in FEM domain
- various error estimators
  - h-h/2 error estimators for FEM, BEM, FEM-BEM
  - averaging error estimators for BEM
  - weighted-residual error estimators for FEM, BEM, FEM-BEM
- adaptive BEM with data approximation of RHS
  - further consistency terms to error estimator

References

Aurada, Ferraz-Leite, P. (Appl. Numer. Math. 62, 2012)

- weakly-singular integral equations in 2D / 3D
- isotropic / anisotropic mesh-refinement
- different estimators (h-h/2, averaging)

Aurada, Ferraz-Leite, et al. (Appl. Numer. Math. 62, 2012)

- BEM for mixed BVPs in 2D
- including approximation of RHS

Feischl, Führer, et al. (Arch. Comput. Meth. Engrg., 2014)

- 60-page review article on ABEM
- a priori convergence of Clément-type operators for data approx.

Zurich Summer School 2014

July 18–22, 2014

Adaptive Boundary Element Methods

Part 3: Linear Convergence of ABEM

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Institute for Analysis and Scientific Computing

FWF Der Wissenschaftsfonds.

2D Model Problem

- weakly-singular integral equation

$$Vu(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\,u(y)\,dy = f(x) \quad \text{for } x\in\Gamma$$

- $\Gamma \subseteq \partial\Omega$ with $\Omega \subset \mathbb{R}^2$ bounded and Lipschitz

Variational Formulation

Find solution $u \in \mathcal{H}$ of

$$\langle\!\langle u\,,v\rangle\!\rangle := \langle Vu\,,v\rangle_{L^2(\Gamma)} = \langle f\,,v\rangle_{L^2(\Gamma)} \quad \text{for all } v\in\mathcal{H}$$

- $\langle\!\langle\cdot\,,\cdot\rangle\!\rangle$ is scalar product
- induced norm $|\!|\!|v|\!|\!| := \langle\!\langle v\,,v\rangle\!\rangle^{1/2} \simeq \|v\|_{\widetilde{H}^{-1/2}(\Gamma)}$

Introduction

Galerkin BEM

- $\mathcal{T}_\ell$ partition of Γ into affine line segments
- local mesh-width $h_\ell \in L^\infty(\Gamma)$, $h_\ell|_T := \mathrm{diam}(T)$ for $T\in\mathcal{T}_\ell$
- $\mathcal{X}_\ell := \mathcal{P}^p(\mathcal{T}_\ell)$ discrete subspace of $\mathcal{H} = \widetilde{H}^{-1/2}(\Gamma)$

Galerkin Formulation

Find solution $U_\ell \in \mathcal{X}_\ell$ of

$$\langle\!\langle U_\ell\,,V_\ell\rangle\!\rangle = \langle f\,,V_\ell\rangle_{L^2(\Gamma)} \quad \text{for all } V_\ell\in\mathcal{X}_\ell$$

Weighted-Residual Error Estimator

Poincaré inequality

$w \in H^1(\Gamma)$ with $w \perp \mathcal{P}^0(\mathcal{T}_\ell) \quad \Longrightarrow \quad \|w\|_{L^2(\Gamma)} \lesssim \|h_\ell w'\|_{L^2(\Gamma)}$

- $w := f - VU_\ell \perp \mathcal{P}^p(\mathcal{T}_\ell) \supseteq \mathcal{P}^0(\mathcal{T}_\ell)$

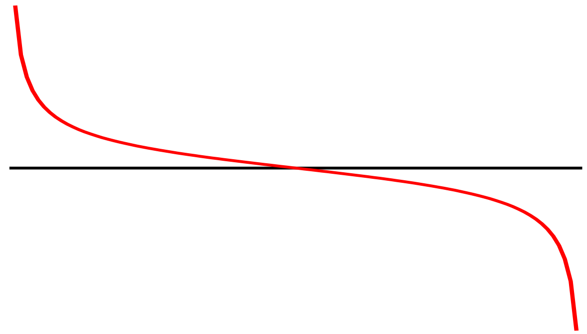
Reliability (Carstensen, Stephan '95, '96, & Maischak '01)

$\|u - U_\ell\| \simeq \|f - VU_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(f - VU_\ell)'\|_{L^2(\Gamma)} =: \eta_\ell$

- proof (later!) requires regularity of $\mathcal{T}_\ell$ for localization
  - 2D: uniformly bounded local mesh-ratio
  - 3D: uniform γ-shape regularity

1D Example

- 1D boundary piece $\Gamma = (-1, 1) \times \{0\}$
- $u(x, y) = -2x/\sqrt{1 - x^2}$ solution to $f(x, y) = -x$



- singularities at $\pm(1, 0)$

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

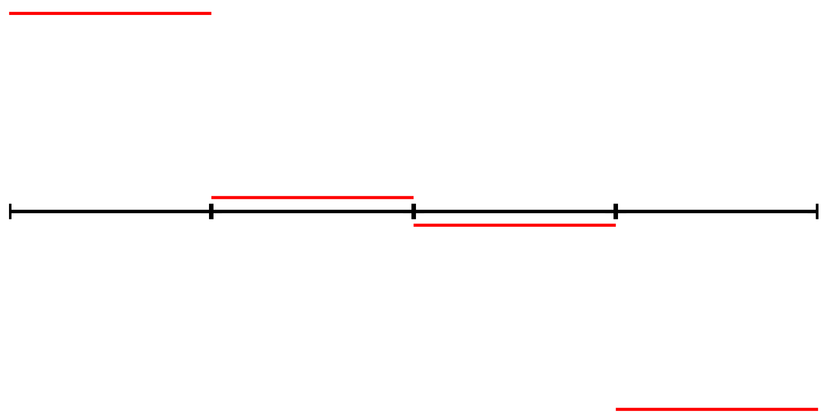
Adaptive Algorithm

1. compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. compute refinement indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. find (essentially minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

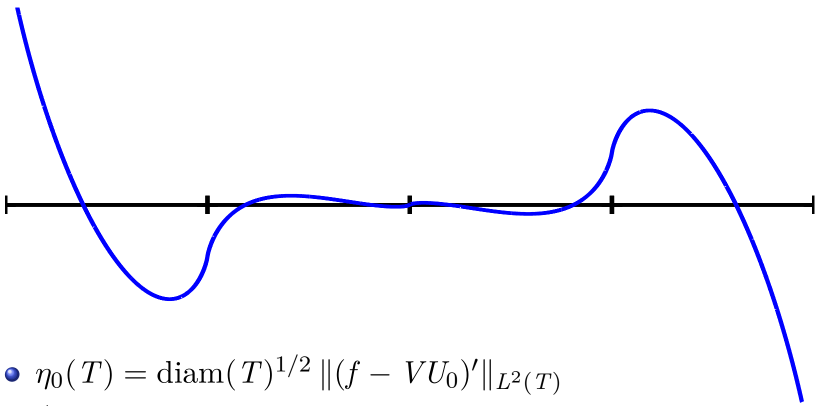
$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
5. increase counter $\ell \mapsto \ell + 1$ and iterate

Solve on Coarse Mesh $\mathcal{T}_0$

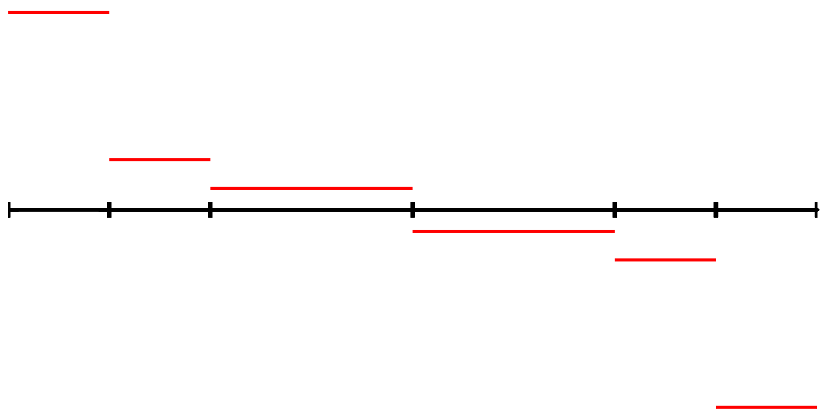


Compute Residual on $\mathcal{T}_0$

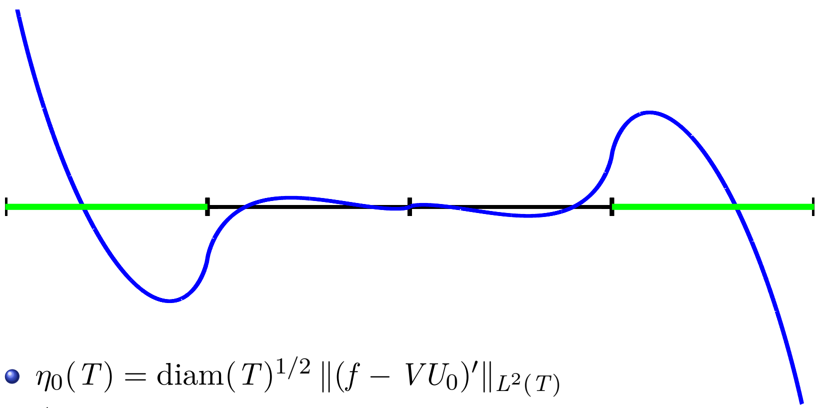


- $\eta_0(T) = \mathrm{diam}(T)^{1/2}\,\|(f - VU_0)'\|_{L^2(T)}$
- $\displaystyle\frac{1}{4}\sum_{T\in\mathcal{T}_0}\eta_0(T)^2 \le \sum_{T\in\mathcal{M}_0}\eta_0(T)^2$

Solve on Mesh $\mathcal{T}_1$

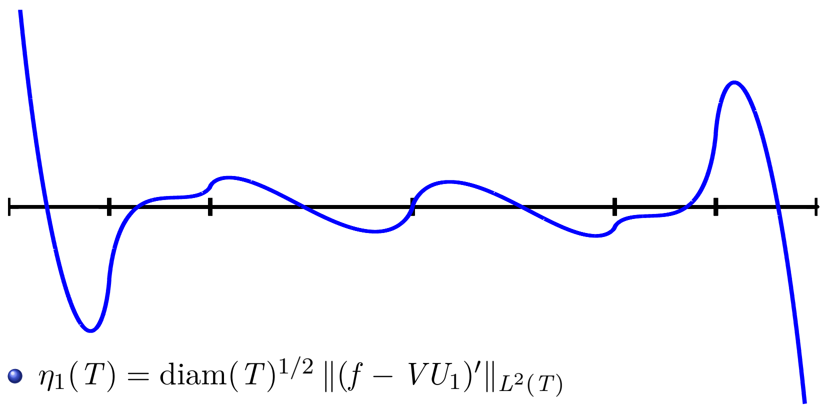


Mark Elements $\mathcal{M}_0 \subseteq \mathcal{T}_0$ for Refinement



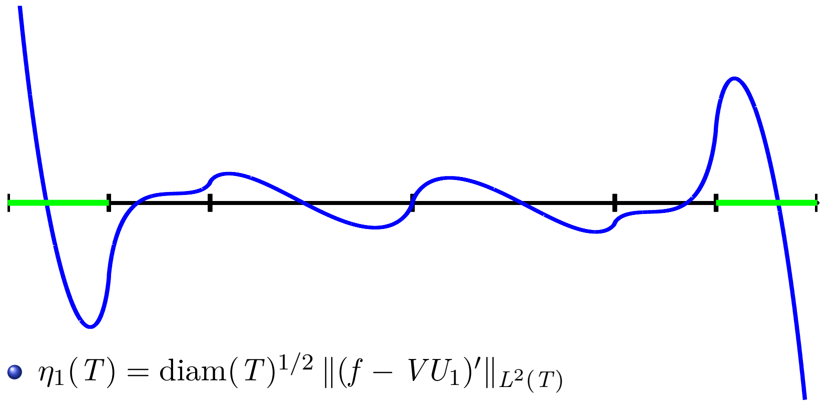
- $\eta_0(T) = \mathrm{diam}(T)^{1/2}\,\|(f - VU_0)'\|_{L^2(T)}$
- $\displaystyle\frac{1}{4}\sum_{T\in\mathcal{T}_0}\eta_0(T)^2 \le \sum_{T\in\mathcal{M}_0}\eta_0(T)^2$

Compute Residual on $\mathcal{T}_1$



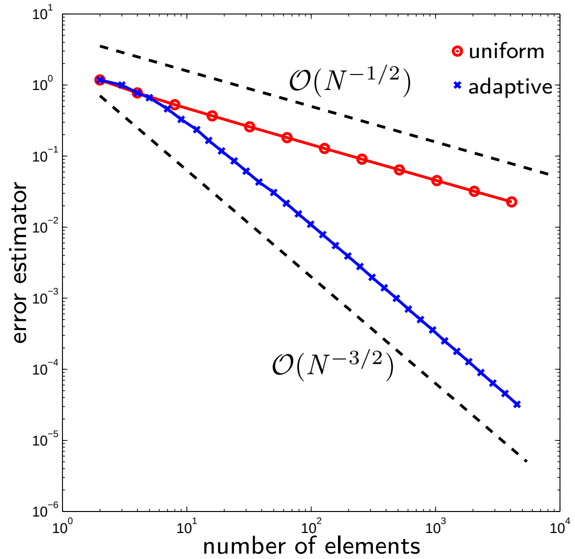
- $\eta_1(T) = \mathrm{diam}(T)^{1/2}\,\|(f - VU_1)'\|_{L^2(T)}$
- $\displaystyle\frac{1}{4}\sum_{T\in\mathcal{T}_1}\eta_1(T)^2 \le \sum_{T\in\mathcal{M}_1}\eta_1(T)^2$

Mark Elements $\mathcal{M}_1 \subseteq \mathcal{T}_1$ for Refinement

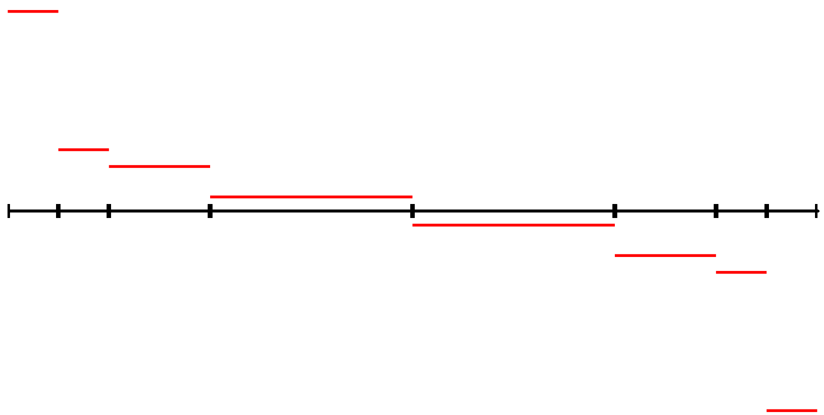


- $\eta_1(T) = \operatorname{diam}(T)^{1/2}\,\|(f - VU_1)'\|_{L^2(T)}$
- $\displaystyle \frac{1}{4} \sum_{T\in\mathcal{T}_1} \eta_1(T)^2 \le \sum_{T\in\mathcal{M}_1} \eta_1(T)^2$

ABEM results in...



Solve on Mesh $\mathcal{T}_2$ etc.



Mathematical Questions?

- can we prove convergence for adaptive mesh-refinement?
- can we prove linear convergence $\eta_{\ell+n}^2 \le C\, q^n\, \eta_\ell^2$?
  - constant $C > 0$ reflects pre-asymptotic convergence rates
- can we prove optimal convergence rates?
  - at least asymptotically
- which set of problems can be covered?
  - AFEM for 2nd order elliptic PDEs?
  - ABEM for 2nd order elliptic PDEs?
  - adaptive FEM-BEM coupling?
  - linear/nonlinear problems?

Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

1. Dörfler '96
2. Morin, Nochetto, Siebert '00
3. Binev, Dahmen, DeVore '04
4. Stevenson '07
5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

- Dörfler marking
- convergence up to $\varepsilon \simeq \|h_0 f\|_{L^2(\Omega)}$

Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

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5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

- MNS algorithm with coarsening
- link to approximation theory
- optimal rates

Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

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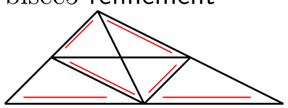
3. Binev, Dahmen, DeVore '04
4. Stevenson '07
5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

- $\mathrm{err}_{\ell+1} \le q\,\mathrm{err}_\ell + \mathrm{osc}_\ell$
- $\mathrm{osc}_\ell := \|h_\ell(f - f_\ell)\|_{L^2}$
- bisec5-refinement



Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

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5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

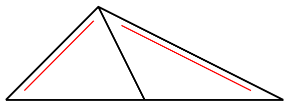
- MNS algorithm without coarsening
- optimal rates

Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

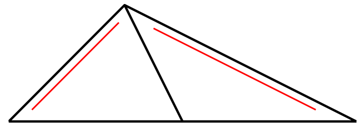
1. Dörfler '96
2. Morin, Nochetto, Siebert '00
3. Binev, Dahmen, DeVore '04
4. Stevenson '07
5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

- symmetric PDEs
- optimal rates
- NVB refinement



Optimal Rates in Adaptive BEM

- Tsogtorel '13
  - general integral kernels
  - smooth boundaries
  - lowest-order BEM



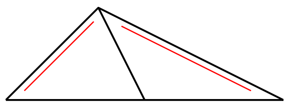
- Feischl, Karkulik, Melenk, P. '13
  - weakly-singular integral equation for 2D / 3D Laplace
  - polygonal boundaries
  - lowest-order BEM
- Feischl, Führer, Karkulik, Melenk, P. '14
  - weakly-singular / hyper-singular IE for 2D / 3D Laplace
  - polygonal boundaries
  - fixed-order BEM with approximation of RHS
- all results: restricted to isotropic mesh-refinement

Optimal Rates in Adaptive FEM

- $-\Delta u = f$ in Ω with $u|_\Gamma = 0$
- AFEM with residual error estimator

1. Dörfler '96
2. Morin, Nochetto, Siebert '00
3. Binev, Dahmen, DeVore '04
4. Stevenson '07
5. Cascón, Kreuzer, Nochetto, Siebert '08
6. Feischl, Führer, P. '14

- non-symmetric PDEs
- optimal rates
- NVB refinement



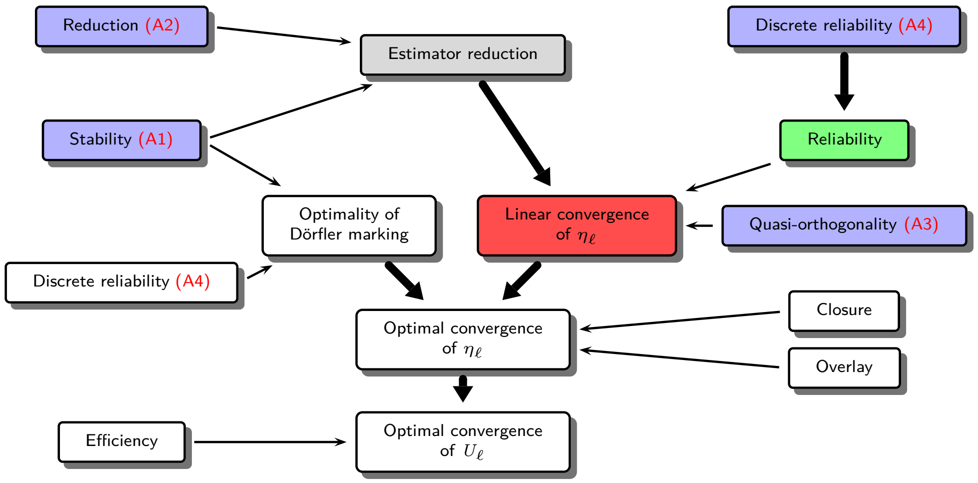
Main Results on Rate Optimality

1. linear convergence $\eta_{\ell+n} \le C\, q^n\, \eta_\ell$
   - for $0 < \theta \le 1$ arbitrary
   - with $0 < q = q(\theta) < 1,\ C = C(\theta) > 0$
   - requires: appropriate orthogonality
2. optimal convergence $\eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$
   - for $0 < \theta \ll 1$ sufficiently small
   - for each possible $s > 0$
3. optimality constrained by estimator and isotropic refinement!
   - ok for 2D
   - suboptimal in 3D w.r.t. error!

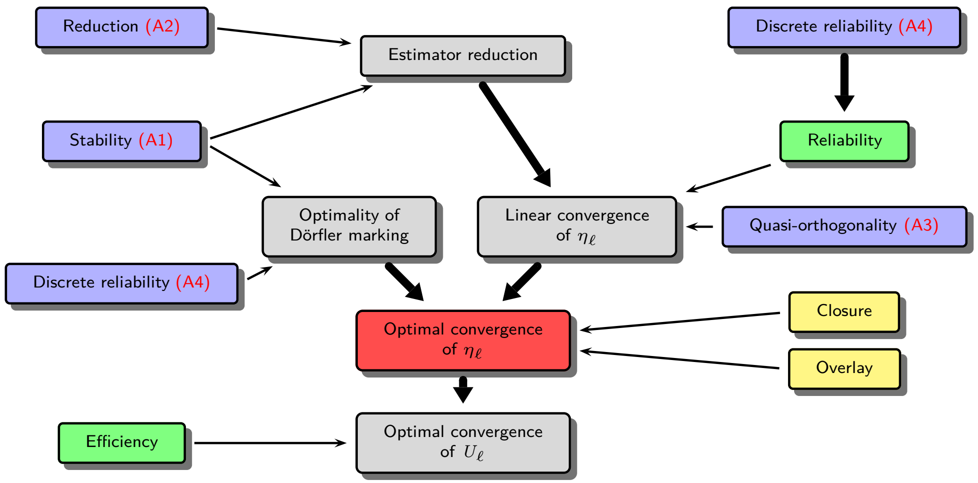
What are the Overall Topics?

- abstract framework for convergence with optimal rates
- formulated in terms of estimator only
- boils down to four properties of error estimator
- reproduces all available results from the literature
  - independent of FEM, BEM, nonconforming FEM, mixed FEM
- sharp characterization of linear convergence

Road Map (Part 3)



Road Map (Part 3 + Part 4)



Setting

Continuous and Discrete Setting

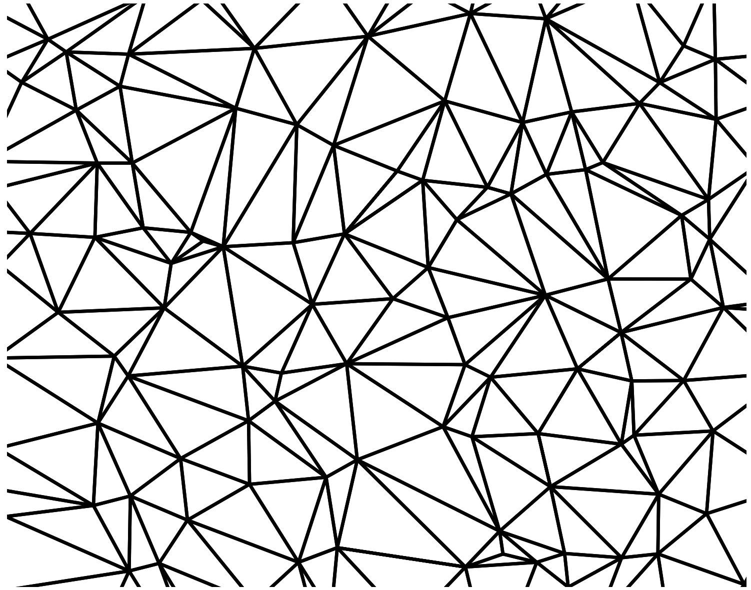
- $\mathcal{H}$ vector space
- $u \in \mathcal{H}$ unknown solution

- $\mathcal{T}_0$ initial mesh
- fixed mesh-refinement strategy $\mathtt{refine}(\cdot)$

- $\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_0)$
- $\mathcal{X}_\star$ finite dimensional space
- $U_\star \in \mathcal{X}_\star$ computable discrete solution

- $\|\cdot\|$ norm to measure $\|u - U_\star\|$

Regular Triangulation



Model Problem

- weakly-singular integral equation $Vu = f$ in 2D/3D
- $u \in \mathcal{H} := \widetilde{H}^{-1/2}(\Gamma)$

$$\langle\!\langle u\,, v \rangle\!\rangle := \langle Vu\,, v\rangle_{L^2} = \langle f\,, v\rangle_{L^2} \quad \text{for all } v \in \mathcal{H}$$

- $\|\cdot\| = \langle\!\langle \cdot\,, \cdot \rangle\!\rangle^{1/2}$
- $\mathcal{T}_\star$ regular and γ-shape regular triangulation
  - 2D: partition into affine line segments, local mesh ratio $\le \gamma$
  - 3D: partition into affine surface triangles, no hanging nodes

$$\max_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)^2}{|T|} \le \gamma$$

- $U_\star \in \mathcal{P}^p(\mathcal{T}_\star) =: \mathcal{X}_\star$ unique solution of

$$\langle\!\langle U_\star\,, V_\star \rangle\!\rangle = \langle f\,, V_\star\rangle_{L^2} \quad \text{for all } V_\star \in \mathcal{P}^p(\mathcal{T}_\star)$$

Approximation Property

Assumption: Uniform refinement yields convergence

$\forall \mathcal{T}_\ell \in \mathtt{refine}(\mathcal{T}_0)\ \forall \varepsilon > 0\ \exists \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell): \quad \|u - U_\star\| \le \varepsilon$

- Verification for BEM model problem:
- Céa lemma + L^2-projection + density of smooth fcts

$$\begin{aligned}
\|u - U_\star\| &\simeq \|u - U_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\lesssim \|u - \Pi_\star v\|_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\le \|u - v\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|v - \Pi_\star v\|_{\widetilde{H}^{-1/2}(\Gamma)} \\
&\lesssim \varepsilon + \|h_\star^{1/2} v\|_{L^2(\Gamma)} \\
&\lesssim \varepsilon
\end{aligned}$$

Residual Error Estimator for Model Problem

Reliability and Efficiency

- $\|u - U_\star\| \overset{\text{OK}}{\lesssim} \eta_\star \overset{???}{\lesssim} \|u - U_\star\| + \text{osc}_\star$

- $\eta_\star = \Big(\sum_{T\in\mathcal{T}_\star} \eta_\star(T)^2 \Big)^{1/2}$
- $\eta_\star(T)^2 = |T|^{1/(d-1)} \, \|\nabla(f - VU_\star)\|^2_{L^2(T)}$
- $h_\star|_T := |T|^{1/(d-1)} \simeq \text{diam}(T)$

- reliability 2D: [Carstensen, Stephan '95,'96]
- reliability 3D: [Carstensen, Maischak, Stephan '01]
- efficiency 2D: [Carstensen '96], [Aurada, Feischl, et al. '13]
  - only direct BEM & closed boundaries & smooth data

Local Inverse Estimate for Non-Local Operators

Theorem (Feischl, Karkulik, Melenk, P. '13; Tsogtorel '13)

- $\|h_\star^{1/2}\nabla VW_\star\|_{L^2(\Gamma)} \le C_{\text{inv}} \, \|W_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}$ for all $W_\star \in \mathcal{P}^p(\mathcal{T}_\star)$
- $C_{\text{inv}} = C_{\text{inv}}(\Gamma, p, \gamma)$

- sketch of proof later (part 5)
- is "classical" inv.est. for "non-classical" space $V\mathcal{P}^p(\mathcal{T}_\star) \ni \Psi_\star$
  - LHS $= \|h_\star^{1/2}\nabla\Psi_\star\|_{L^2}$ weighted H^1-seminorm (stronger)
  - RHS $\simeq \|\Psi_\star\|_{H^{1/2}(\Gamma)}$ (weaker norm)

- lowest-order case $W_\star \in \mathcal{P}^0(\mathcal{T}_\ell)$
  - Feischl, Karkulik, Melenk, P. '13
  - Tsogtorel '13 for smooth boundary, but general kernels

Stability & Reduction

Local Inverse Estimate: Uniform Meshes

- aim to prove: $\|h_\star^{1/2}\nabla VW_\star\|_{L^2(\Gamma)} \lesssim \|W_\star\|$ for all $W_\star \in \mathcal{P}^0(\mathcal{T}_\star)$

- stability $V : L^2(\Gamma) \to H^1(\Gamma)$ provides

$$\begin{aligned}\|h_\star^{1/2}\nabla Vw\|_{L^2(\Gamma)} \le h_\star^{1/2}\|Vw\|_{H^1(\Gamma)} &\lesssim h_\star^{1/2}\|w\|_{L^2(\Gamma)} \\ &= \|h_\star^{1/2}w\|_{L^2(\Gamma)}\end{aligned}$$

- local inverse estimate of Graham, Hackbusch, Sauter '05

$$\|h_\star^{1/2}\,W_\star\|_{L^2(\Gamma)} \lesssim \|W_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \|W_\star\|$$

Axiom (A1): Stability on Non-Refined Elements

(A1) Stability on non-refined elements, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$

$$\Big|\Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\star(T)^2\Big)^{1/2}-\Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\ell(T)^2\Big)^{1/2}\Big|\lesssim |||\,U_\star-U_\ell\,|||$$

- Verification for BEM model problem:
- $\eta_\star(T)^2=|T|^{1/(d-1)}\,\|\nabla(f-VU_\star)\|^2_{L^2(T)}$
- $\displaystyle\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\star(T)^2=\|h_\star^{1/2}\nabla(f-VU_\star)\|^2_{L^2(\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star))}$
- inverse triangle inequality + novel inverse estimate

$$\begin{aligned}\mathrm{LHS}&\le\|h_\star^{1/2}\nabla(f-VU_\star)-h_\ell^{1/2}\nabla(f-VU_\star)\|_{L^2(\bigcup(\mathcal{T}_\ell\cap\mathcal{T}_\star))}\\&\le\|h_\star^{1/2}\nabla V(U_\ell-U_\star)\|_{L^2(\Gamma)}\\&\lesssim\|U_\ell-U_\star\|_{\widetilde{H}^{-1/2}(\Gamma)}\end{aligned}$$

Introduction Setting Stability & Reduction Discrete Reliability Linear Convergence Conclusions

Estimator Reduction

Stability (A1) + Reduction (A2) $\Longrightarrow$ Estimator Reduction

- $\forall 0<\theta\le 1\;\exists 0<q_{\rm est}<1\;\exists C_{\rm est}>0\;\forall\ell\in\mathbb{N}_0:$

$$\eta_{\ell+1}^2\le q_{\rm est}\,\eta_\ell^2+C_{\rm est}\,|||\,U_{\ell+1}-U_\ell\,|||^2$$

- sketch: Young inequality + (A1) + (A2)
  - variable parameter $\delta>0$ sufficiently small
- $q_{\rm est}=(1+\delta)-\theta(1+\delta-q_{\rm red})\approx 1-\theta(1-q_{\rm red})$
- $C_{\rm est}=C_{\rm stab}^2(1+\delta^{-1})+C_{\rm red}$

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Axiom (A2): Reduction on Refined Elements

(A2) Reduction on refined elements, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$

$$\sum_{T\in\mathcal{T}_\star\backslash\mathcal{T}_\ell}\eta_\star(T)^2\le q_{\rm red}\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star}\eta_\ell(T)^2+C_{\rm red}\,|||\,U_\star-U_\ell\,|||^2$$

- Verification for BEM model problem:
- $\eta_\star(T)^2=|T|^{1/(d-1)}\,\|\nabla(f-VU_\star)\|^2_{L^2(T)}$
- $\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_\ell)=\bigcup(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$
- $|T'|\le\frac{1}{2}|T|$ for $\mathcal{T}_\star\ni T'\subsetneqq T\in\mathcal{T}_\ell$
- proof as for estimator reduction for h-h/2 estimators
- triangle inequality + Young ineq. + novel inverse estimate
- $q_{\rm red}\approx 2^{-1/(d-1)}$

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Proof of Estimator Reduction 1/3

1. (A1) + (A2) + Young ineq. $(a+b)^2\le(1+\delta)a^2+(1+\delta^{-1})b^2$

$$\begin{aligned}\eta_{\ell+1}^2&=\sum_{T\in\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell}\eta_{\ell+1}(T)^2+\sum_{T\in\mathcal{T}_{\ell+1}\backslash\mathcal{T}_\ell}\eta_{\ell+1}(T)^2\\
&\le(1+\delta)\sum_{T\in\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell}\eta_\ell(T)^2+q_{\rm red}\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}}\eta_\ell(T)^2\\
&\quad+\big(C_{\rm stab}^2(1+\delta^{-1})+C_{\rm red}\big)\,|||\,U_{\ell+1}-U_\ell\,|||^2\\
&=(1+\delta)\,\eta_\ell^2+\big(q_{\rm red}-(1+\delta)\big)\sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1}}\eta_\ell(T)^2\\
&\quad+\underbrace{\big(C_{\rm stab}^2(1+\delta^{-1})+C_{\rm red}\big)}_{=C_{\rm est}}\,|||\,U_{\ell+1}-U_\ell\,|||^2\end{aligned}$$

Proof of Estimator Reduction 2/3

1. (A1) + (A2) + Young inequality

$$\eta_{\ell+1}^2 \le (1+\delta)\,\eta_\ell^2 + (q_{\rm red} - (1+\delta)) \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}} \eta_\ell(T)^2$$
$$+ C_{\rm est}\,|\!|\!|\, U_{\ell+1} - U_\ell |\!|\!|^2$$

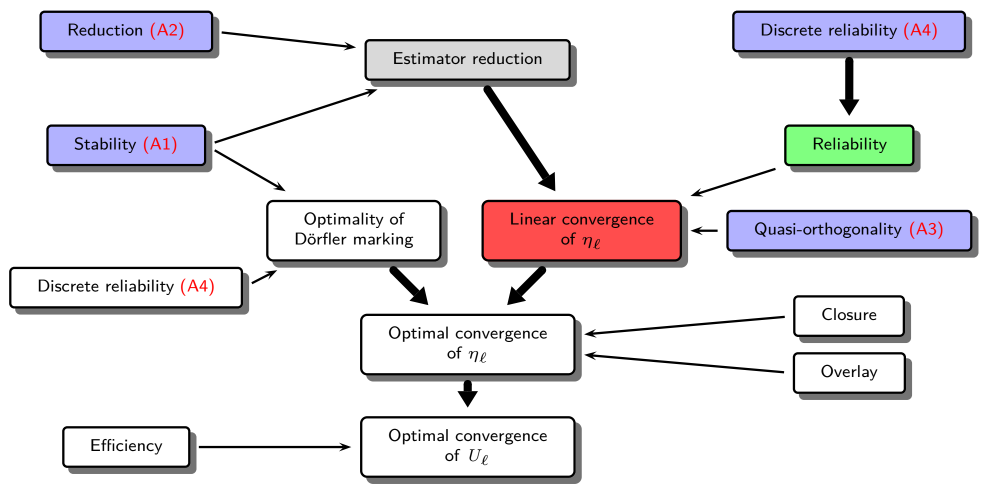
2. $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$

$$\eta_{\ell+1}^2 \le (1+\delta)\,\eta_\ell^2 - (1+\delta-q_{\rm red}) \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2$$
$$+ C_{\rm est}\,|\!|\!|\, U_{\ell+1} - U_\ell |\!|\!|^2$$

3. Dörfler marking

$$\le ((1+\delta) - \theta(1+\delta-q_{\rm red}))\,\eta_\ell^2$$
$$+ C_{\rm est}\,|\!|\!|\, U_{\ell+1} - U_\ell |\!|\!|^2$$

Road Map



Proof of Estimator Reduction 3/3

- proved: for all $\delta > 0$

$$\eta_{\ell+1}^2 \le ((1+\delta) - \theta(1+\delta-q_{\rm red}))\,\eta_\ell^2 + C_{\rm est}\,|\!|\!|\, U_{\ell+1} - U_\ell |\!|\!|^2$$

- $q_{\rm est} = (1+\delta) - \theta(1+\delta-q_{\rm red}) \xrightarrow{\delta\to 0} 1 - \theta(1-q_{\rm red}) < 1$
- small $\delta > 0$ $\Longrightarrow$ estimator reduction estimate

Discrete Reliability

Necessity of Discrete Reliability

- Suppose BEM model problem
- Let $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_{\ell+1})$

$\Longrightarrow$ $\|U_\star - U_\ell\|^2 \leq \|u - U_\ell\|^2$

- Suppose knowledge that η_ℓ is reliable + Dörfler marking

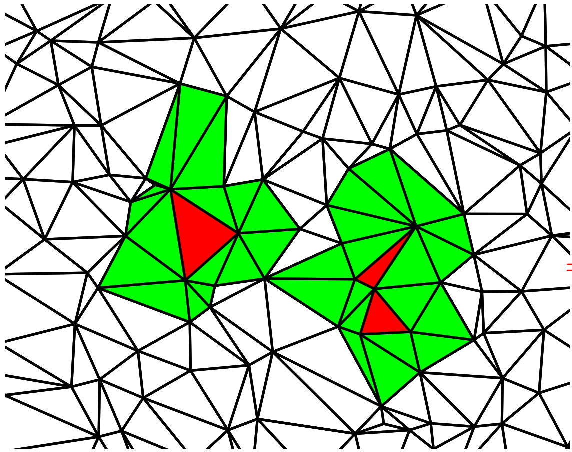
$\Longrightarrow$ $\|u - U_\ell\|^2 \lesssim \eta_\ell^2 \leq \theta^{-1} \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$

- marked elements are refined, i.e., $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1} \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_\star$

$\Longrightarrow$ obtain discrete reliability

$$\|U_\star - U_\ell\|^2 \lesssim \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star} \eta_\ell(T)^2$$

Patch



- $\mathcal{S} \subseteq \mathcal{T}_\ell$ set of elements
- $\mathrm{patch}(\mathcal{S})$
- γ-shape regularity

$\Longrightarrow$ $\#\mathcal{S} \simeq \#\mathrm{patch}(\mathcal{S})$

Axiom (A4): Discrete Reliability

(A4) Discrete reliability, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$

- exists $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ with
  - $\mathcal{T}_\ell \backslash \mathcal{T}_\star \subseteq \mathcal{R}_\ell$
  - $\#\mathcal{R}_\ell \simeq \#(\mathcal{T}_\ell \backslash \mathcal{T}_\star)$
  - $\|U_\star - U_\ell\|^2 \leq C_{\mathrm{rel}}^2 \sum_{T \in \mathcal{R}_\ell} \eta_\ell(T)^2$

- introduced by Stevenson '07
- proof refines usual reliability proof by choice of clever test fct
- $\mathcal{R}_\ell = \mathcal{T}_\ell \backslash \mathcal{T}_\star$ for FEM
- $\mathcal{R}_\ell = \mathrm{patch}(\mathcal{T}_\ell \backslash \mathcal{T}_\star)$ for BEM

Discrete Reliability $\Longrightarrow$ Reliability

- triangle inequality for $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$

$$\|u - U_\ell\| \leq \|u - U_\star\| + \|U_\star - U_\ell\| \leq \|u - U_\star\| + C_{\mathrm{rel}}\,\eta_\ell$$

- approximation property

$$\forall \varepsilon > 0 \ \exists \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell): \quad \|u - U_\star\| \leq \varepsilon$$

$\Longrightarrow$ $\|u - U_\ell\| \leq C_{\mathrm{rel}}\,\eta_\ell$

Proof of Discrete Reliability 1/5

- $W_\star = U_\star - U_\ell$
- $\mathbb{P}_\ell W_\star \in \mathcal{P}^0(\mathcal{T}_\ell)$, $\quad \mathbb{P}_\ell W_\star = \begin{cases} W_\star & \text{on } T \in \mathcal{T}_\ell \cap \mathcal{T}_\star \\ 0 & \text{on } T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star \end{cases}$
- $R_\ell := \{z \in \mathcal{N}_\ell \,:\, z \in \bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)\}$

1. use Galerkin orthogonality

$$|\!|\!| W_\star |\!|\!|^2 = \langle VU_\star - VU_\ell\,,\, W_\star \rangle = \Big\langle \sum_{z \in R_\ell} \varphi_z(f - VU_\ell)\,,\, (1 - \mathbb{P}_\ell)\, W_\star \Big\rangle$$

- $(1 - \mathbb{P}_\ell)\, W_\star = 0$ on $T \in \mathcal{T}_\ell \cap \mathcal{T}_\star$
- $\sum_{z \in R_\ell} \varphi_z = 1$ on $T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star$

Proof of Discrete Reliability 2/5

- $W_\star = U_\star - U_\ell$
- $\mathbb{P}_\ell W_\star \in \mathcal{P}^0(\mathcal{T}_\ell)$, $\quad \mathbb{P}_\ell W_\star = \begin{cases} W_\star & \text{on } T \in \mathcal{T}_\ell \cap \mathcal{T}_\star \\ 0 & \text{on } T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star \end{cases}$
- $R_\ell := \{z \in \mathcal{N}_\ell \,:\, z \in \bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)\}$

1. $|\!|\!| W_\star |\!|\!|^2 = \Big\langle \sum_{z \in R_\ell} \varphi_z(f - VU_\ell)\,,\, W_\star - \mathbb{P}_\ell W_\star \Big\rangle$

- $\langle \mathtt{sum}\,,\, W_\star \rangle \lesssim \|\mathtt{sum}\|_{H^{1/2}(\Gamma)} \|W_\star\|_{\widetilde{H}^{-1/2}(\Gamma)} \simeq \|\mathtt{sum}\|_{H^{1/2}(\Gamma)} |\!|\!| W_\star |\!|\!|$
- $\langle \mathtt{sum}\,, \mathbb{P}_\ell W_\star \rangle \leq \|h_\ell^{-1/2}\mathtt{sum}\|_{L^2(\Gamma)} \|h_\ell^{1/2}\mathbb{P}_\ell W_\star\|_{L^2(\Gamma)}$
- $\|h_\ell^{1/2}\mathbb{P}_\ell W_\star\|_{L^2(\Gamma)} = \|h_\star^{1/2}\mathbb{P}_\ell W_\star\|_{L^2(\Gamma)} \leq \|h_\star^{1/2}\, W_\star\|_{L^2(\Gamma)} \lesssim |\!|\!| W_\star |\!|\!|$

2. Obtain $|\!|\!| W_\star |\!|\!|^2 \lesssim \|\mathtt{sum}\|^2_{H^{1/2}(\Gamma)} + \|h_\ell^{-1/2}\mathtt{sum}\|^2_{L^2(\Gamma)}$

Proof of Discrete Reliability 3/5

- $R_\ell := \{z \in \mathcal{N}_\ell \,:\, z \in \bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)\}$
- $\mathtt{sum} = \sum_{z \in R_\ell} \varphi_z(f - VU_\ell)$

2. $|\!|\!| U_\star - U_\ell |\!|\!|^2 \lesssim \|\mathtt{sum}\|^2_{H^{1/2}(\Gamma)} + \|h_\ell^{-1/2}\mathtt{sum}\|^2_{L^2(\Gamma)}$
3. elementwise Poincaré inequality, since $f - VU_\ell \perp \mathcal{P}^0(\mathcal{T}_\ell)$

$$\begin{aligned}\|h_\ell^{-1/2}\mathtt{sum}\|^2_{L^2(\Gamma)} &\simeq \sum_{T \in \mathcal{T}_\ell} \mathrm{diam}(T)^{-1}\|\mathtt{sum}\|^2_{L^2(T)} \\ &\leq \sum_{T \cap R_\ell \neq \emptyset} \mathrm{diam}(T)^{-1}\|f - VU_\ell\|^2_{L^2(T)} \\ &\lesssim \sum_{T \cap R_\ell \neq \emptyset} \mathrm{diam}(T)\|\nabla(f - VU_\ell)\|^2_{L^2(T)} \\ &\simeq \sum_{T \cap R_\ell \neq \emptyset} \eta_\ell(T)^2\end{aligned}$$

Proof of Discrete Reliability 4/5

- $R_\ell := \{z \in \mathcal{N}_\ell \,:\, z \in \bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)\}$
- $\mathtt{sum} = \sum_{z \in R_\ell} \varphi_z(f - VU_\ell)$

2. $|\!|\!| U_\star - U_\ell |\!|\!|^2 \lesssim \|\mathtt{sum}\|^2_{H^{1/2}(\Gamma)} + \|h_\ell^{-1/2}\mathtt{sum}\|^2_{L^2(\Gamma)}$
3. $\|h_\ell^{-1/2}\mathtt{sum}\|^2_{L^2(\Gamma)} \lesssim \sum_{T \cap R_\ell \neq \emptyset} \eta_\ell(T)^2$
4. coloring argument as in Carstensen, Maischak, Stephan '01

$$\|\mathtt{sum}\|^2_{H^{1/2}(\Gamma)} \lesssim \sum_{z \in R_\ell} \|\varphi_z(f - VU_\ell)\|^2_{H^{1/2}(\Gamma)}$$

5. interpolation + Poincaré estimate + scaling arguments

$$\begin{aligned}\|\varphi_z(f - VU_\ell)\|^2_{H^{1/2}(\Gamma)} &\lesssim \|\varphi_z(f - VU_\ell)\|_{L^2(\Gamma)}\|\varphi_z(f - VU_\ell)\|_{H^1(\Gamma)} \\ &\lesssim \|h_\ell^{1/2}\nabla(f - VU_\ell)\|^2_{L^2(\mathrm{supp}(\varphi_z))}\end{aligned}$$

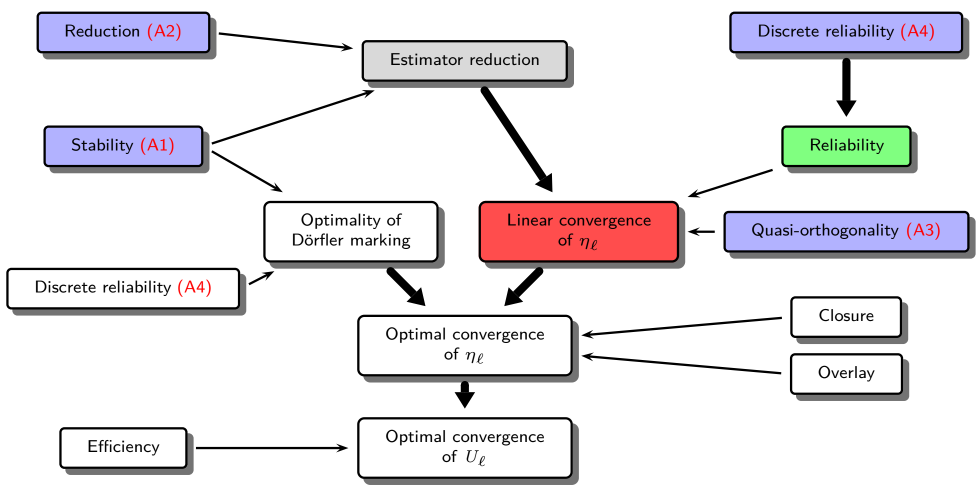
Proof of Discrete Reliability 5/5

- $R_\ell := \{z \in \mathcal{N}_\ell \,:\, z \in \bigcup(\mathcal{T}_\ell \backslash \mathcal{T}_\star)\}$
- $\texttt{sum} = \sum_{z\in R_\ell} \varphi_z(f - VU_\ell)$

2. $|\!|\!| U_\star - U_\ell |\!|\!|^2 \lesssim \|\texttt{sum}\|^2_{H^{1/2}(\Gamma)} + \|h_\ell^{-1/2}\texttt{sum}\|^2_{L^2(\Gamma)}$
3. $\|h_\ell^{-1/2}\texttt{sum}\|^2_{L^2(\Gamma)} \lesssim \sum_{T\cap R_\ell \neq \emptyset} \eta_\ell(T)^2$
4. $\|\texttt{sum}\|^2_{H^{1/2}(\Gamma)} \lesssim \sum_{z\in R_\ell} \|\varphi_z(f - VU_\ell)\|^2_{H^{1/2}(\Gamma)}$
5. $\|\varphi_z(f - VU_\ell)\|^2_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}\nabla(f - VU_\ell)\|^2_{L^2(\mathrm{supp}(\varphi_z))}$

$$\Longrightarrow \; |\!|\!| U_\star - U_\ell |\!|\!|^2 \lesssim \|\texttt{sum}\|^2_{H^{1/2}(\Gamma)} + \|h_\ell^{-1/2}\texttt{sum}\|^2_{L^2(\Gamma)} \lesssim \sum_{T\cap R_\ell \neq \emptyset} \eta_\ell(T)^2$$

Road Map



Linear Convergence

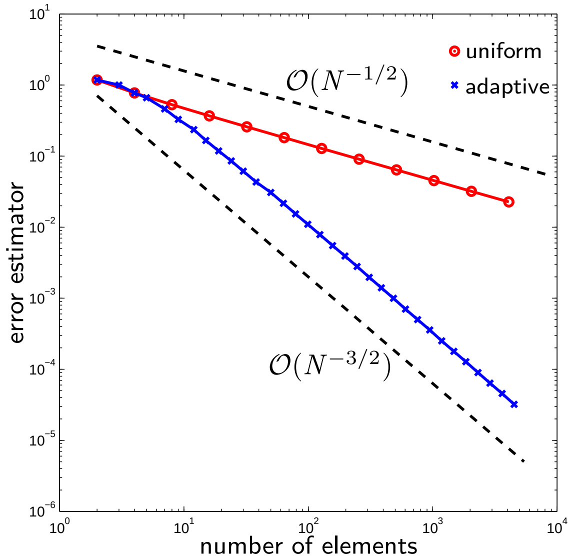
Slow Convergence?

Current State

1. (A1) + (A2) $\Longrightarrow$ $\eta_{\ell+1}^2 \le q_{\rm est}\eta_\ell^2 + C_{\rm est}\,|\!|\!| U_{\ell+1} - U_\ell |\!|\!|^2$
2. Céa lemma $U_\ell \to U_\infty$ as $\ell \to \infty$
3. estimator reduction principle $\Longrightarrow$ $\eta_\ell \to 0$
4. reliability $\Longrightarrow$ $|\!|\!| u - U_\ell |\!|\!| \lesssim \eta_\ell \to 0$
   - in particular, $u = U_\infty$

- question: Can convergence of $\eta_\ell \to 0$ be slow?
  - $|\!|\!| U_{\ell+1} - U_\ell |\!|\!|^2 \to 0$ could be slow?
- goal: $\eta_{\ell+n}^2 \lesssim q^n\,\eta_\ell^2$

Slow Convergence? No!



Linear Convergence $\Longrightarrow$ Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- reliability $\|u - U_\ell\| \lesssim \eta_\ell$
- linear convergence $\eta_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell$

$\Longrightarrow$ quasi-orthogonality (A3) with $\varepsilon = 0$, $C_{\rm orth}(\varepsilon) = C_{\rm orth}(0) > 0$

- i.e.: $\displaystyle\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 \lesssim \eta_\ell^2$ for all ℓ, N

Axiom (A3): Quasi-Orthogonality

(A3) Quasi-orthogonality, for all $\varepsilon > 0$ and ℓ, N

$$\sum_{k=\ell}^{N} \big(\|U_{k+1} - U_k\|^2 - \varepsilon\|u - U_k\|^2\big) \le C_{\rm orth}(\varepsilon)\, \eta_\ell^2$$

- Verification for model problem
- Galerkin orthogonality + symmetry $\Longrightarrow$ Pythagoras theorem

$$\|u - U_{k+1}\|^2 + \|U_{k+1} - U_k\|^2 = \|u - U_k\|^2$$

- telescoping series $\Longrightarrow$ quasi-orth. with $C_{\rm orth}(\varepsilon) = C_{\rm rel}^2$, $\varepsilon = 0$

$$\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 = \sum_{k=\ell}^{N} \big(\|u - U_k\|^2 - \|u - U_{k+1}\|^2\big) \le \|u - U_\ell\|^2$$

Proof: Linear Convergence $\Longrightarrow$ Quasi-Orthogonality

- triangle inequality + reliability

$$\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 \le 2 \sum_{k=\ell}^{N} \big(\|u - U_{k+1}\|^2 + \|u - U_k\|^2\big) \le 4 \sum_{k=\ell}^{N+1} \|u - U_k\|^2 \lesssim \sum_{k=\ell}^{N+1} \eta_k^2$$

- linear convergence + geometric series

$$\sum_{k=\ell}^{N+1} \eta_k^2 \lesssim \Big(\sum_{k=\ell}^{N+1} q^{k-\ell}\Big)\, \eta_\ell^2 \lesssim \eta_\ell^2$$

Linear Convergence $\Longleftarrow$ General Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- reliability $\|u - U_\ell\| \lesssim \eta_\ell$
- estimator reduction for $0 < \theta \le 1$, e.g., stab. (A1) + red. (A2)
- quasi-orthogonality (A3)

$\Longrightarrow$ linear convergence $\eta_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell$

Proof: Linear Convergence $\Longleftarrow$ Quasi-Orthogonality 1/2

1. estimator reduction and $0 < \delta \ll 1$

$$\sum_{k=\ell+1}^{N} \eta_k^2 \le \sum_{k=\ell+1}^{N} \Big[q\,\eta_{k-1}^2 + C\,\|U_k - U_{k-1}\|^2\Big]$$

$$= \sum_{k=\ell+1}^{N} (q+\delta)\eta_{k-1}^2 + C \sum_{k=\ell+1}^{N} \Big[\|U_k - U_{k-1}\|^2 - C^{-1}\delta\,\eta_{k-1}^2\Big]$$

2. reliability

$$\|U_k - U_{k+1}\|^2 - C^{-1}\delta\eta_{k-1}^2 \le \|U_k - U_{k+1}\|^2 - C_{\rm rel}^{-2}\, C^{-1}\delta\,\|u - U_{k-1}\|^2$$

3. quasi-orthogonality with $\varepsilon = C_{\rm rel}^{-2}\, C^{-1}\delta$

$$\sum_{k=\ell+1}^{N} \Big[\|U_k - U_{k-1}\|^2 - \varepsilon\,\|u - U_{k-1}\|^2\Big] \lesssim \eta_\ell^2$$

Proof: Linear Convergence $\Longleftarrow$ Quasi-Orthogonality 2/2

4. obtain: $\displaystyle \sum_{k=\ell+1}^{N} \eta_k^2 \le \sum_{k=\ell+1}^{N} (q+\delta)\eta_{k-1}^2 + \widetilde{C}\,\eta_\ell^2$

5. $0 < q+\delta < 1$ and $N \to \infty$ $\Longrightarrow$ $\displaystyle \sum_{k=\ell+1}^{\infty} \eta_k^2 \le M\,\eta_\ell^2$

6. $\Longrightarrow$ $\displaystyle (1+M^{-1}) \sum_{k=\ell+1}^{\infty} \eta_k^2 \le \sum_{k=\ell+1}^{\infty} \eta_k^2 + \eta_\ell^2 = \sum_{k=\ell}^{\infty} \eta_k^2$

7. induction $\Longrightarrow$ $\displaystyle (1+M^{-1})^n \sum_{k=\ell+n}^{\infty} \eta_k^2 \le \sum_{k=\ell}^{\infty} \eta_k^2 \le (1+M)\,\eta_\ell^2$

8. finally: $\displaystyle \eta_{\ell+n}^2 \le \sum_{k=\ell+n}^{\infty} \eta_k^2 \le \underbrace{(1+M^{-1})^{-n}}_{=q_{\rm lin}^n}\underbrace{(1+M)}_{=C_{\rm lin}}\,\eta_\ell^2$

Contraction

- axioms (A1)–(A4) $\Longrightarrow$ linear convergence $\eta_{\ell+n}^2 \lesssim q^n \eta_\ell^2$
- for symmetric problems, slightly stronger result available

Theorem (Feischl, Karkulik, Melenk, P. '13)

- $\Delta_\ell = \|\phi - \Phi_\ell\|^2 + \gamma\eta_\ell^2 \simeq \eta_\ell^2$
- with $0 < \gamma < 1$ sufficiently small

$\Longrightarrow$ Exists $0 < \kappa = \kappa(\theta) < 1$ s.t.

$$\Delta_{\ell+1} \le \kappa\,\Delta_\ell \quad \text{hence} \quad \lim_{\ell\to\infty} \eta_\ell = 0 = \lim_{\ell\to\infty} \|\phi - \Phi_\ell\|$$

- concept of proof goes back to CKNS '08

Proof of Contraction Theorem

1. Pythagoras $\|\phi - \Phi_{\ell+1}\|^2 = \|\phi - \Phi_\ell\|^2 - \|\Phi_{\ell+1} - \Phi_\ell\|^2$
2. estimator reduction $\eta_{\ell+1}^2 \le q\,\eta_\ell^2 + C\|\Phi_{\ell+1} - \Phi_\ell\|^2$
3. reliability $\|\phi - \Phi_\ell\|^2 \le C_{\rm rel}^2 \eta_\ell^2$

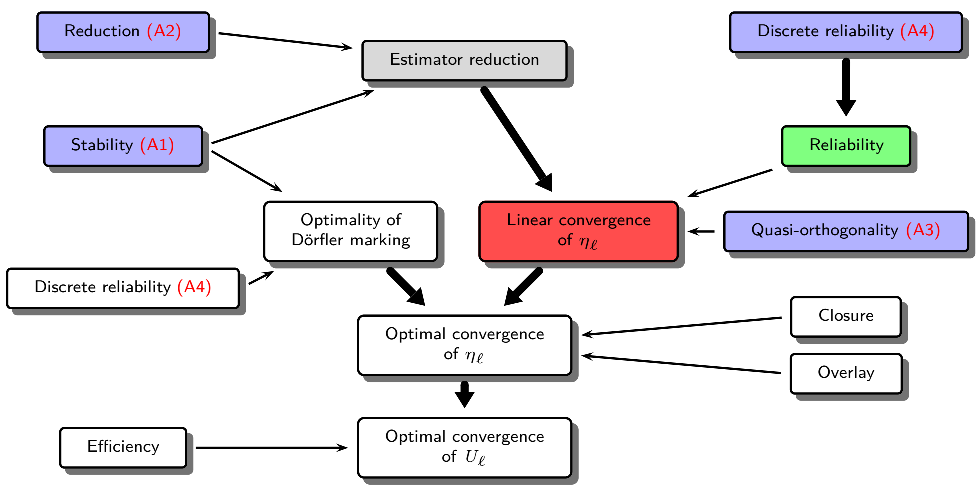
- use small parameters $0 < \gamma, \varepsilon < 1$ to see

$$\begin{aligned}\|\phi - \Phi_{\ell+1}\|^2 + \gamma\eta_{\ell+1}^2 &\le \|\phi - \Phi_\ell\|^2 + \gamma\, q\,\eta_\ell^2 \\ &\le (1 - \varepsilon\gamma)\|\phi - \Phi_\ell\|^2 + \gamma(q + \varepsilon C_{\rm rel}^2)\eta_\ell^2 \\ &\le \kappa\,(\|\phi - \Phi_\ell\|^2 + \gamma\eta_\ell^2)\end{aligned}$$

- obtain: $\Delta_{\ell+1} \le \kappa\,\Delta_\ell$
  - $0 < \kappa < 1$
  - $\Delta_\ell = \|\phi - \Phi_\ell\|^2 + \gamma\eta_\ell^2$

Conclusions

Road Map



Conclusions 1/2

Two simple ways to prove convergence of adaptive scheme:

1. estimator reduction principle
   - quite general approach
   - needs appropriate inverse estimates
   - needs a priori convergence
   - no assumption on reliability / efficiency of estimator
2. linear convergence / contraction theorem
   - based on estimator reduction (only estimate!)
   - needs reliability of estimator
   - needs certain quasi-orthogonality
   - avoids a priori convergence

Conclusions 2/2

- framework with axioms (A1)–(A4) is independent of
  - continuous problem
  - discretization method
  - linearity / nonlinearity of problem
- quasi-orth. (A4) essentially equivalent to linear convergence
  - est. reduction + reliability + quasi-orth. $\Longrightarrow$ linear conv.
  - reliability + linear convergence $\Longrightarrow$ quasi-orth.
- for linear convergence suffices reliability $|||u - U_\ell||| \lesssim \eta_\ell$
- however: discrete reliability (A3) seems to be necessary
  - follows from Dörfler marking and reliability
  - implies reliability

References

- Carstensen, Feischl, et al. (Comput. Math. Appl. 67, 2014)
  - axiomatic framework for optimal adaptivity
  - 59-page review article on available results
- Feischl, Karkulik, et al. (SIAM J. Numer. Anal. 51, 2013)
- Feischl, Führer, et al. (Calcolo, 2014)
  - optimal ABEM for weakly-singular IE (with approx. of RHS)
  - fixed polynomial order $p \geq 0$
- Feischl, Führer, et al. (ASC Preprint 30/2013)
  - optimal ABEM for hyper-singular IE (with approx. of RHS)
  - fixed polynomial order $p \geq 0$

Extensions

- linear convergence also OK for h-h/2 estimators
  - requires: saturation assumption to ensure reliability
- (A1)–(A4) valid for hyper-singular IE with residual estimator

$$\eta_\ell = \|h_\ell^{1/2}(f - WU_\ell)\|_{L^2(\Gamma)}$$

- symmetric IE for mixed BVP on smooth boundaries
  - then: operator is elliptic with compact non-symmetric part
  - use residual error estimator
- abstract analysis does not need norm $|||\cdot|||$
  - can also use error measure, e.g.: difference of energies

Open Problems

- reliability of residual error estimator on anisotropic meshes
  - if mesh is aligned with singularities of u
  - numerical evidence in [Carstensen, Maischak, Stephan '01]
- estimator reduction for estimator from [Faermann '00, '01]
  - recent convergence result by [Feischl, Führer, et al. '14]
- quasi-orth. for symmetric IE for mixed BVP for polygonal Γ
- quasi-orthogonality for FEM-BEM coupling for polygonal Γ
  - then: non-symmetric part of operator is non-compact
- reliable approximation of Galerkin matrix, e.g., by $\mathcal{H}$-matrices

Zurich Summer School 2014
July 18–22, 2014

Adaptive Boundary Element Methods

Part 4: Optimal Convergence of ABEM

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Introduction

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

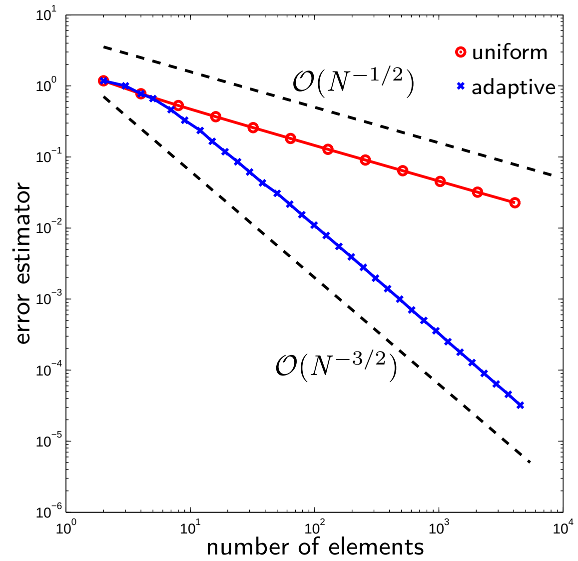
Adaptive Algorithm

1. compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. compute refinement indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. find (essentially) minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2$$

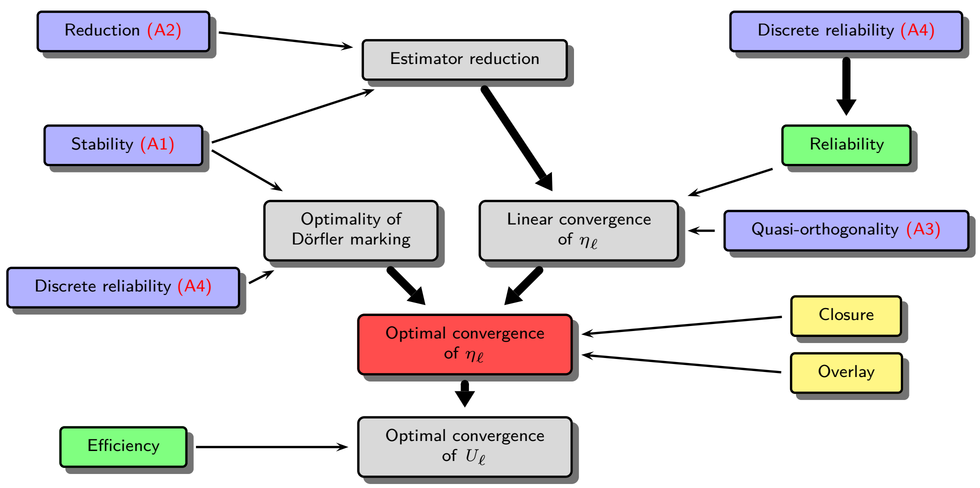
4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
5. increase counter $\ell \mapsto \ell+1$ and iterate

What is all about?



- linear convergence
  $\eta_{\ell+n} \leq C\, q^n\, \eta_\ell$
  - for $0 < \theta \leq 1$ arbitrary
  - with $0 < q = q(\theta) < 1$
  - with $C = C(\theta) > 0$
- optimal convergence
  $\eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$
  - for small $0 < \theta \ll 1$
  - for each possible $s > 0$

Road Map



Optimal Convergence Rates

Theorem (Carstensen, Feischl, Page, P. '14)

- "optimal mesh-refinement"
- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ has (essentially) minimal cardinality
- $0 < \theta < \theta_\star$ sufficiently small

$\Longrightarrow$ $(u,f) \in \mathbb{A}_s \quad \Longleftrightarrow \quad \eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$

- proof analyzes AFEM proof of Stevenson '07
  1. linear convergence $\eta_{\ell+n} \lesssim q_{\rm lin}^n \, \eta_\ell$ (sufficient and required)
  2. $\rightsquigarrow$ optimality of Dörfler marking
  3. $\rightsquigarrow$ optimality of closure & overlay of meshes
  4. efficiency is not needed
- interesting implication is $\Longrightarrow$

What is Optimality?

- $\mathbb{T}$ set of possible meshes for fixed mesh-refinement strategy
- $\mathbb{T}_N = \{\mathcal{T}_\star \in \mathbb{T} \,:\, \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N\}$
- possible convergence order $\eta_\star = \mathcal{O}(N^{-s})$ with $s > 0$?
- $\mathbb{A}_s = \{(u,f) \,:\, \|u,f\|_{\mathbb{A}_s} = \sup\limits_{N>0} \big((N+1)^s \min\limits_{\mathcal{T}_\star \in \mathbb{T}_N} \eta_\star\big) < \infty\}$
  - u (unknown) exact solution
  - f given data

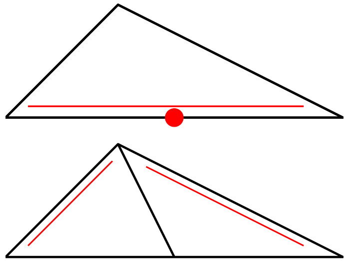
Quasi-Optimality of Adaptive Algorithm

- $(u,f) \in \mathbb{A}_s \quad \Longleftrightarrow \quad$ order $\mathcal{O}(N^{-s})$ is possible
- algorithm is rate optimal $\quad \Longleftrightarrow \quad$ it leads to $\mathcal{O}(N^{-s})$

Newest Vertex Bisection

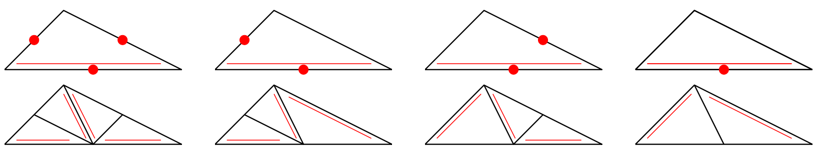
Newest Vertex Bisection

- each element has reference edge
- refinement by bisection
- new reference edges are opposite to newest vertex
- for $T \in \mathcal{T}_0$, obtain unique binary tree with possible sons



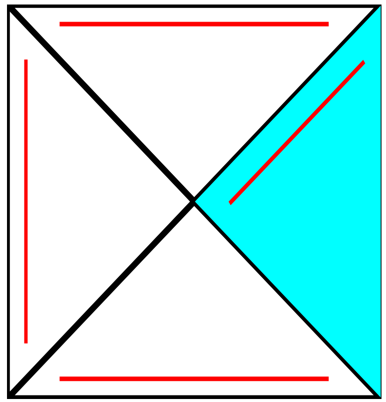
NVB Algorithm 1/2

- initialization
  - if element is marked, mark its reference edge
- recursion
  - if element's edge is marked, mark at least its reference edge
- refinement according to scheme



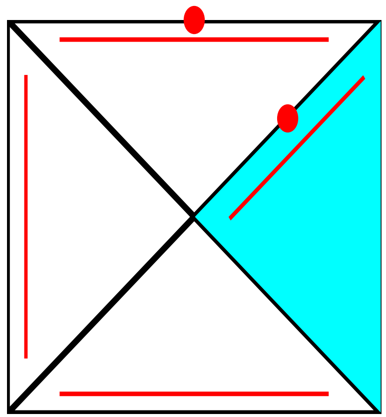
NVB Algorithm 2/2

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges
- proceed recursively
- mesh refinement



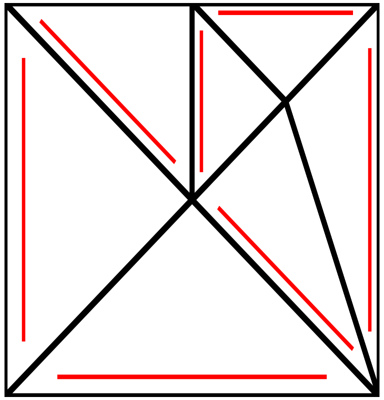
NVB Algorithm 2/2

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges
- proceed recursively
- mesh refinement



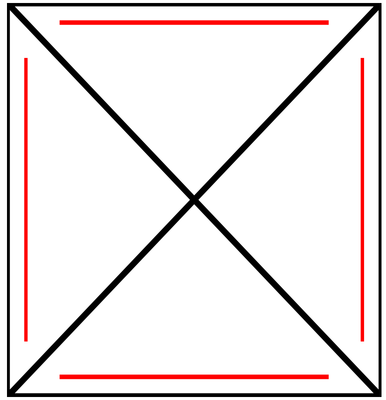
NVB Algorithm 2/2

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges
- proceed recursively
- mesh refinement



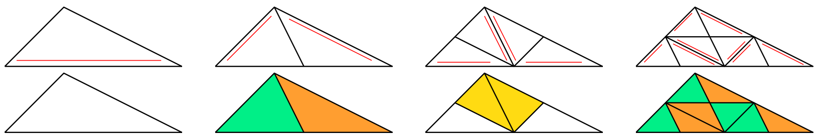
Control of Mesh Closure

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
- clearly: $\#\mathcal{M}_\ell \le \#\mathcal{R}_\ell$
- $\#\mathcal{R}_\ell \le C\#\mathcal{M}_\ell$ cannot hold
  - $\#\mathcal{M}_\ell = 1$
  - $\#\mathcal{R}_\ell \sim \ell$



Similarity Classes

- NVB leads to only finitely many similarity classes of triangles
  - depends only on $\mathcal{T}_0$ and its reference edges



- in particular, only finitely many shapes of patches
  $\Longrightarrow$ convenient for numerical analysis
- but: no anisotropic elements possible

Control of Mesh Closure

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
- clearly: $\#\mathcal{M}_\ell \le \#\mathcal{R}_\ell$
- $\#\mathcal{R}_\ell \le C\#\mathcal{M}_\ell$ cannot hold
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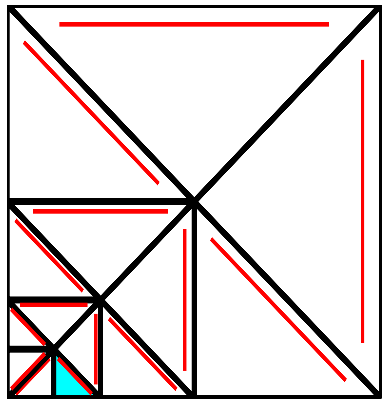

Control of Mesh Closure

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
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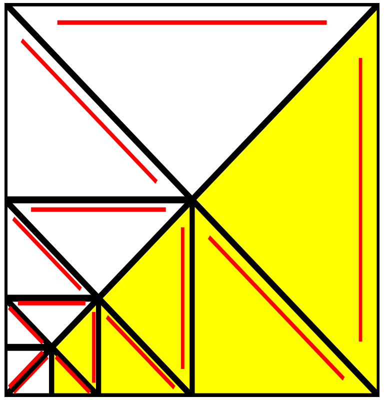
- $\#\mathcal{R}_\ell \le C \#\mathcal{M}_\ell$ cannot hold
  - $\#\mathcal{M}_\ell = 1$
  - $\#\mathcal{R}_\ell \sim \ell$



Avoid BDD Assumption for 2D

Theorem (Karkulik, Pavlicek, P. '13)

- 2-dimensional manifold Ω (e.g. 2D FEM, 3D BEM)
- $\mathcal{T}_0$ regular triangulation into non-degenerate triangles
- arbitrary choice of reference edges

$\Longrightarrow$ NVB guarantees

1. mesh-closure estimate $\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$
2. uniform H^1-stability of L^2-projection onto $\mathcal{S}^1_0(\mathcal{T}_\ell)$, $\mathcal{S}^1(\mathcal{T}_\ell)$

Optimality of Mesh Closure

Mesh closure estimate (Binev, Dahmen, DeVore '04)

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$$

- in 2D first proved by Binev, Dahmen, DeVore '04
  - proof requires BDD assumption on $\mathcal{T}_0$
    - each $E = T \cap T'$ is reference edge of both T, T' or of none
  - each 2D mesh allows for BDD labeling
  - no effective algorithm known to determine BDD labeling
- extension to $\mathbb{R}^d$ by Stevenson '08
  - open whether each $\mathcal{T}_0$ allows for BDD labeling
  - refine $\mathcal{T}_0$ uniformly to ensure BDD assumption

Optimal Mesh Refinement 1/3

$\mathcal{T}_\ell$ regular and uniformly shape regular

- usually required for boundedness of constants in
  1. stability (A1)
  2. reduction (A2)
  3. quasi-orthogonality (A3)
  4. discrete reliability (A4)

- $\mathbb{R}^d$ newest vertex bisection
- 2D red-green-blue refinement
- 1D bisection with bounded local mesh-ratio
- red refinement with first-order hanging nodes in $\mathbb{R}^d$

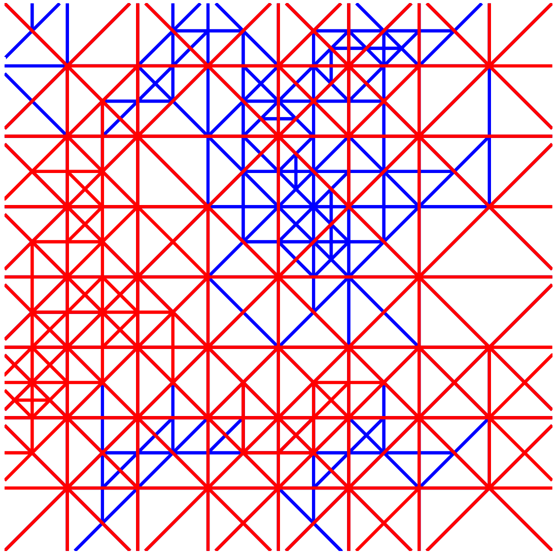
Optimal Mesh Refinement 2/3

Mesh closure estimate

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$$

- $\mathbb{R}^d$ newest vertex bisection
  - Binev, Rahmen, DeVore '04
  - Stevenson '08
  - Karkulik, Pavlicek, P. '13 (2D without BDD assumption)
- $\mathbb{R}^d$ red refinement with first-order hanging nodes
  - Bonito, Nochetto '10
- 2D red-green-blue refinement
  - Pavlicek, P. '11 (bacc thesis)
- 1D bisection with bounded local mesh-ratio
  - Aurada, Feischl, Führer, Karkulik, P. '13

Overlay Estimate: $\#(\mathcal{T}_\star \oplus \mathcal{T}_\ell) \le \#\mathcal{T}_\star + \#\mathcal{T}_\ell - \#\mathcal{T}_0$



- $\mathcal{T}_\star$ (blue)
- $\mathcal{T}_\ell$ (red)
- $\mathcal{T}_\star \oplus \mathcal{T}_\ell$

Optimal Mesh Refinement 3/3

Overlay estimate

For $\mathcal{T}_\star, \mathcal{T}_\ell$ exists common refinement $\mathcal{T}_\star \oplus \mathcal{T}_\ell$ s.t.

$$\#(\mathcal{T}_\star \oplus \mathcal{T}_\ell) \le \#\mathcal{T}_\star + \#\mathcal{T}_\ell - \#\mathcal{T}_0$$

- $\mathbb{R}^d$ newest vertex bisection
  - Stevenson '07
  - Cascón, Kreuzer, Nochetto, Siebert '08
- $\mathbb{R}^d$ red refinement with first-order hanging nodes
  - Bonito, Nochetto '10
- 1D bisection with bounded local mesh-ratio
  - Aurada, Feischl, Führer, Karkulik, P. '13
- wrong for 2D red-green-blue refinement
  - Pavlicek, P. '11 (bacc thesis)

Optimality of Dörfler Marking

Dörfler Marking $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2$

- goal: determine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with minimal cardinality

$\Longrightarrow$ requires sorting $\eta_\ell(T_1) \ge \cdots \ge \eta_\ell(T_N)$ $\rightsquigarrow \mathcal{O}(N \log N)$

- sufficient: $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ has essentially minimal cardinality
  - i.e., $\#\mathcal{M}_\ell \le C\,\#\widetilde{\mathcal{M}}_\ell$ if $\widetilde{\mathcal{M}}_\ell$ has minimal cardinality

- idea: sorting with binning [Stevenson '07] $\rightsquigarrow \mathcal{O}(N)$
  1. exclude all $T \in \mathcal{T}_\ell$ with $\eta_\ell(T)^2 \le (1-\theta)\eta_\ell^2/N$ with $N := \#\mathcal{T}_\ell$
     - the remaining elements will satisfy the Dörfler marking
  2. determine $M := \max_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2$
  3. determine minimal $K \in \mathbb{N}_0$ with $2^{-(K+1)}M \le (1-\theta)\eta_\ell^2/N$
  4. fill $k = 0,\dots,K$ bins with all $2^{-(k+1)}M < \eta_\ell(T)^2 \le 2^{-k}M$
  5. successively take elts from (unsorted) bins until $\mathcal{M}_\ell$ is OK

$\Longrightarrow$ $\#\mathcal{M}_\ell \le 2\,\#\widetilde{\mathcal{M}}_\ell$

Recall Stability (A1) and Discrete Reliability (A4)

(A1) Stability on non-refined elements, $\mathcal{T}_\ell, \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_0)$

$$\Big|\Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2\Big)^{1/2}\Big| \lesssim \|U_\star - U_\ell\|$$

(A4) Discrete reliability, $\mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_\ell)$

- exists $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ with
  - $\mathcal{T}_\ell \backslash \mathcal{T}_\star \subseteq \mathcal{R}_\ell$
  - $\#\mathcal{R}_\ell \simeq \#(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$
  - $\|U_\star - U_\ell\|^2 \le C_{\rm rel}^2 \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

Dörfler Marking $\Longrightarrow$ (Linear) Convergence

- shown: stability (A1) + reduction (A2) + Dörfler marking

$\Longrightarrow$ estimator reduction

- estimator reduction + quasi-orth. (A3) + reliability (A4)

$\Longrightarrow$ linear convergence $\eta_{\ell+n}^2 \lesssim q_{\rm lin}^n\,\eta_\ell^2$

- i.e.: Dörfler marking sufficient for (linear) convergence

- question: Dörfler marking also necessary?

Dörfler Marking $\Longleftarrow$ Convergence

- so far: Dörfler $\Longrightarrow$ linear convergence $\eta_{\ell+n}^2 \lesssim q_{\rm lin}^n\,\eta_\ell^2$

Stab. (A1) + Rel. (A4) $\Longrightarrow$ Optimality of Dörfler Marking

Exists $0 < \theta_\star < 1$ and $0 < q_\star < 1$ s.t.

- for all $\mathcal{T}_\star = \mathtt{refine}(\mathcal{T}_\ell)$ with $\eta_\star^2 \le q_\star\,\eta_\ell^2$
- and all $0 < \theta < \theta_\star$
- and $\mathcal{R}_\ell$ from discrete reliability (A4)

holds Dörfler marking $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

- linear convergence

$\Longrightarrow$ Dörfler marking holds every fixed number n of steps

- independently of how elements are actually marked!

Proof: Optimality of Dörfler Marking 1/2

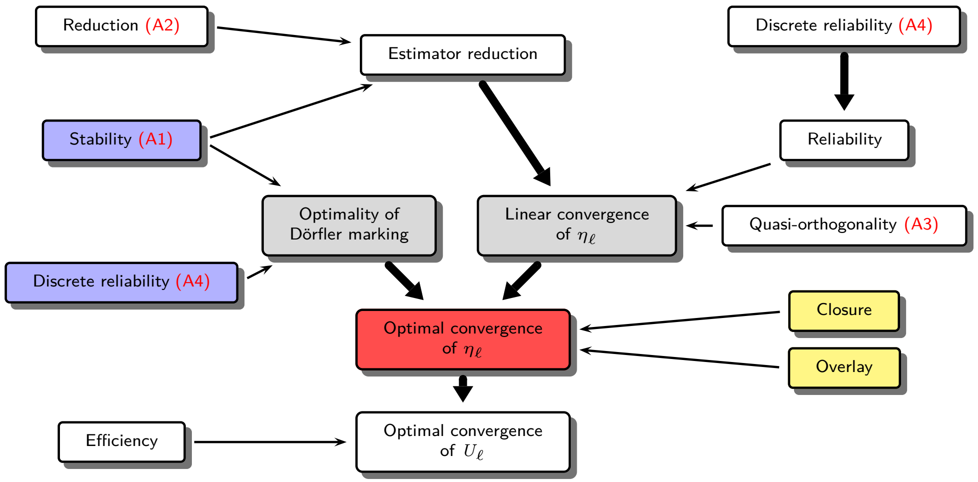
1. stability (A1) + Young inequality + discrete reliability (A4)

$$\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\ell(T)^2 \le (1+\delta)\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\star(T)^2 + (1+\delta^{-1})\,C_{\rm stab}^2\,\|U_\star - U_\ell\|^2 \le (1+\delta)\eta_\star^2 + (1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2\sum_{T\in\mathcal{R}_\ell}\eta_\ell(T)^2$$

2. $\eta_\star^2 \le q_\star\eta_\ell^2$ and $\mathcal{T}_\ell\backslash\mathcal{T}_\star \subseteq \mathcal{R}_\ell$

$$\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star}\eta_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star}\eta_\ell(T)^2 \le (1+\delta)q_\star\,\eta_\ell^2 + [1 + (1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2]\sum_{T\in\mathcal{R}_\ell}\eta_\ell(T)^2$$

Road Map



Proof: Optimality of Dörfler Marking 2/2

2. $\eta_\ell^2 \le (1+\delta)q_\star\,\eta_\ell^2 + [1 + (1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2]\sum_{T\in\mathcal{R}_\ell}\eta_\ell(T)^2$

$$\Longrightarrow \quad \frac{1-(1+\delta)q_\star}{1+(1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2}\,\eta_\ell^2 \le \sum_{T\in\mathcal{R}_\ell}\eta_\ell(T)^2$$

3. given: $\theta < \theta_\star := \dfrac{1}{1+C_{\rm stab}^2\,C_{\rm rel}^2}$
   - $\theta_\star > \dfrac{1-(1+\delta)q_\star}{1+(1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2}\,\eta_\ell^2 \xrightarrow[\delta\to\infty]{(1+\delta)q_\star\to 0} \theta_\star$

4. choose $\delta > 0$ large and $0 < q_\star \ll 1$ small such that

$$\theta < \frac{1-(1+\delta)q_\star}{1+(1+\delta^{-1})\,C_{\rm stab}^2\,C_{\rm rel}^2}\,\eta_\ell^2 < \theta_\star$$

Optimal Convergence of Estimator

Optimal Convergence Rates

- $\mathbb{A}_s = \{(u,f) \,:\, \|u,f\|_{\mathbb{A}_s} = \sup\limits_{N\in\mathbb{N}_0} \big((N+1)^s \min\limits_{\mathcal{T}_\star\in\mathbb{T}_N} \eta_\star\big) < \infty\}$

Theorem (Carstensen, Feischl, Page, P. '14)

- "optimal mesh-refinement"
- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ has (essentially) minimal cardinality
- $0 < \theta < \theta_\star$ sufficiently small

$\Longrightarrow \quad (u,f) \in \mathbb{A}_s \quad \Longleftrightarrow \quad \eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$

Proof of Optimality Theorem 1/3

Heart of the Matter: Exists $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ s.t.

- $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \eta_\ell^{-1/s}$
- $\eta_\star^2 \le q_\star \eta_\ell^2$

- recall $\|u,f\|_{\mathbb{A}_s} = \sup\limits_{N\in\mathbb{N}_0} \big((N+1)^s \min\limits_{\mathcal{T}_\star\in\mathbb{T}_N} \eta_\star\big)$
- define $\varepsilon := C_{\rm mon}^{-1} q_\star^{1/2} \eta_\ell \le \eta_0 \le \|u,f\|_{\mathbb{A}_s} < \infty$
- choose minimal $N \in \mathbb{N}_0$ with $\|u,f\|_{\mathbb{A}_s} \le \varepsilon\,(N+1)^s$
  - $\|u,f\|_{\mathbb{A}_s} > \varepsilon N^s$
- choose $\mathcal{T}_\varepsilon \in \mathbb{T}_N$ with $\eta_\varepsilon = \min_{\mathcal{T}_\star\in\mathbb{T}_N} \eta_\star$
  - $\eta_\varepsilon \le (N+1)^{-s}\|u,f\|_{\mathbb{A}_s} \le \varepsilon$
  - $\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \le N < \|u,f\|_{\mathbb{A}_s}^{1/s}\varepsilon^{-1/s}$
- define $\mathcal{T}_\star := \mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$ & use overlay estimate
  - $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \le (\#\mathcal{T}_\varepsilon + \#\mathcal{T}_\ell - \#\mathcal{T}_0) - \#\mathcal{T}_\ell \lesssim \varepsilon^{-1/s} \simeq \eta_\ell^{-1/s}$
  - $\eta_\star^2 \le C_{\rm mon}^2 \eta_\varepsilon^2 \le C_{\rm mon}^2 \varepsilon^2 = q_\star \eta_\ell^2$

Quasi-Monotonicity of Estimator

stab. (A1) + red. (A2) + rel. (A4) $\Longrightarrow$ monotonicity

- $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell) \quad \Longrightarrow \quad \eta_\star \le C_{\rm mon}\,\eta_\ell$

$$\begin{aligned}
\eta_\star^2 &= \sum_{T\in\mathcal{T}_\star\backslash\mathcal{T}_\ell} \eta_\star(T)^2 + \sum_{T\in\mathcal{T}_\star\cap\mathcal{T}_\ell} \eta_\star(T)^2 \\
&\lesssim \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star} \eta_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2 + |\!|\!| U_\star - U_\ell |\!|\!|^2 \\
&\lesssim \eta_\ell^2
\end{aligned}$$

Proof of Optimality Theorem 2/3

1. exists $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ s.t.
   - $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \eta_\ell^{-1/s}$
   - $\eta_\star^2 \le q_\star \eta_\ell^2$
2. optimality of Dörfler marking
   $\Longrightarrow$ $\mathcal{R}_\ell \simeq \mathcal{T}_\ell\backslash\mathcal{T}_\star$ satisfies Dörfler marking
3. $\mathcal{M}_\ell$ has (essentially) minimal cardinality
   $\Longrightarrow \#\mathcal{M}_\ell \le \#\mathcal{R}_\ell \simeq \#(\mathcal{T}_\ell\backslash\mathcal{T}_\star) \lesssim \#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \eta_\ell^{-1/s} \qquad \forall \ell \in \mathbb{N}_0$
4. overlay estimate
   $\Longrightarrow \#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum\limits_{j=0}^{\ell-1} \#\mathcal{M}_j \lesssim \sum\limits_{j=0}^{\ell-1} \eta_j^{-1/s}$

Proof of Optimality Theorem 3/3

4. obtained: $\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{\ell-1} \eta_j^{-1/s}$
5. linear convergence $\eta_{j+n} \lesssim q^n\,\eta_j$ & geometric series

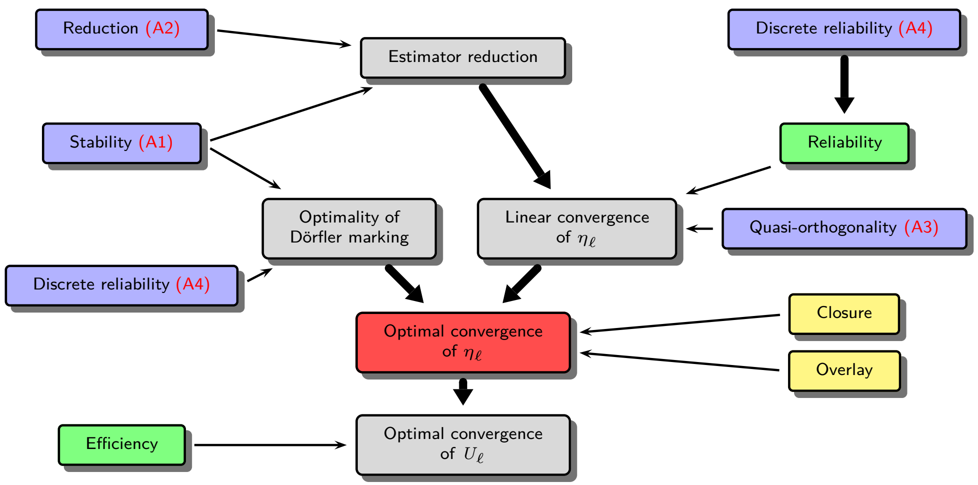
   $\implies \eta_\ell \lesssim q^{\ell-j}\,\eta_j$

   $$\implies \sum_{j=0}^{\ell-1} \eta_j^{-1/s} \lesssim \Big(\sum_{j=0}^{\ell-1} q^{(\ell-j)/s}\Big)\,\eta_\ell^{-1/s} \lesssim \eta_\ell^{-1/s}$$
6. combining this, we obtain

   $\implies \#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \eta_\ell^{-1/s}$

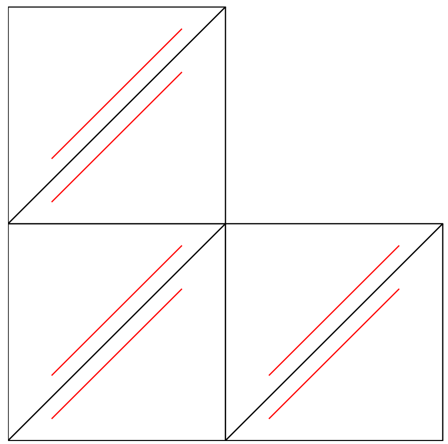
   $\implies \eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s}$ for all $s > 0$ with $\|u,f\|_{\mathbb{A}_s} < \infty$

Road Map



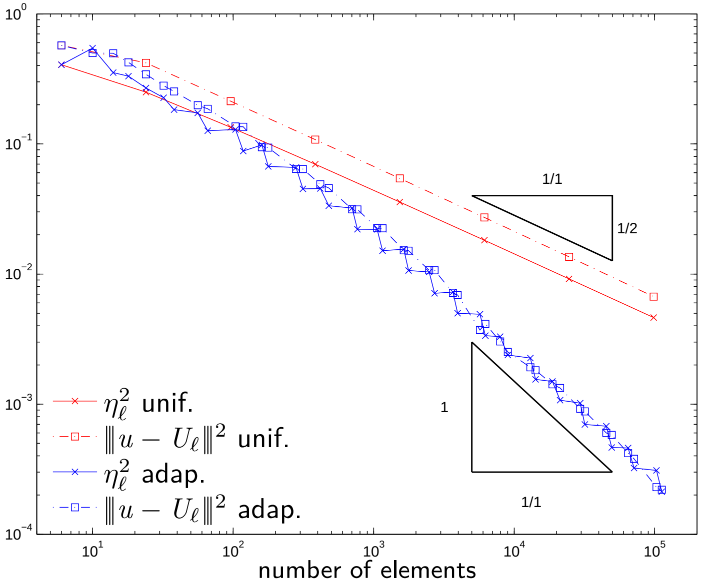
Examples 3D BEM

L-Shaped Screen

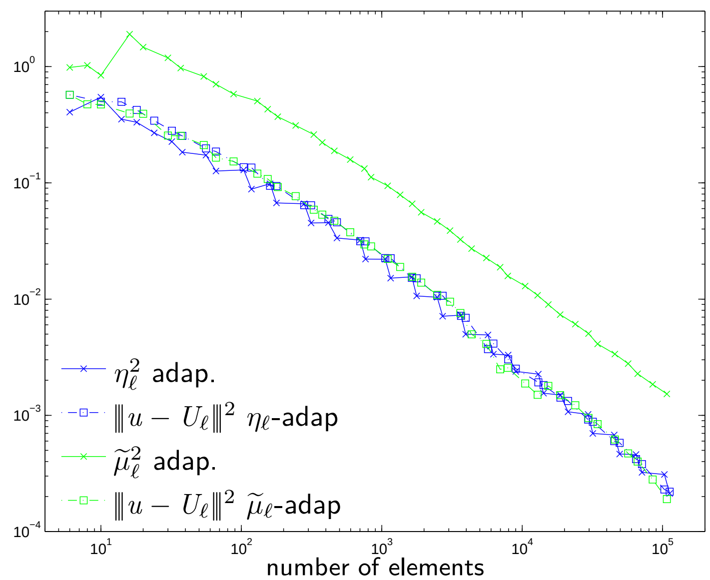


- solve $Vu = 1$ on Γ with lowest-order BEM $p = 0$

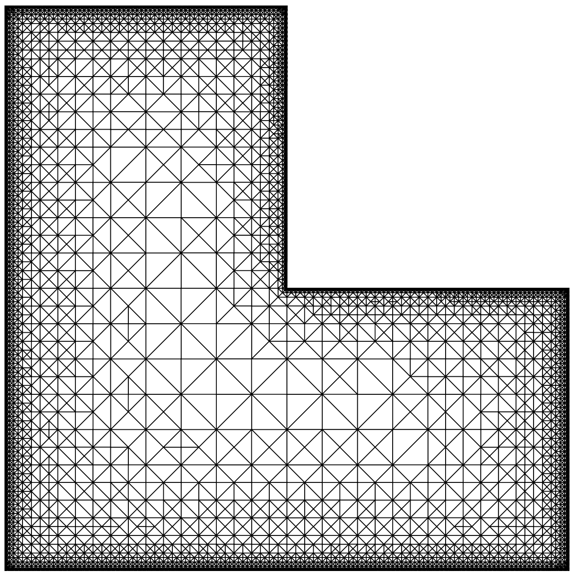
L-Shaped Screen: Errors and Estimators



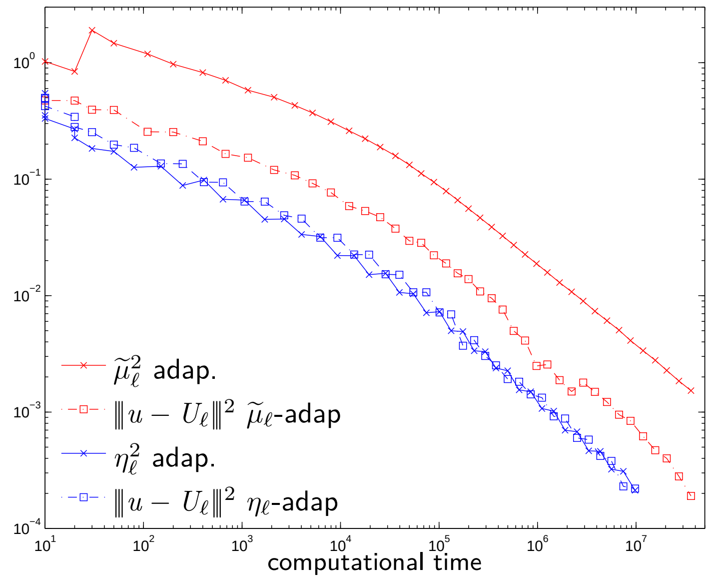
L-Shaped Screen: Estimator Competition



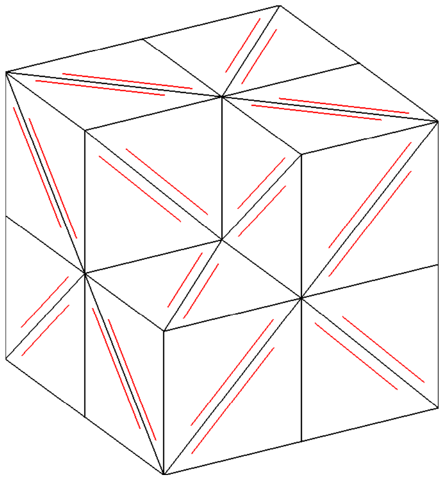
L-Shaped Screen: Adaptive Meshes



L-Shaped Screen: Estimator Competition

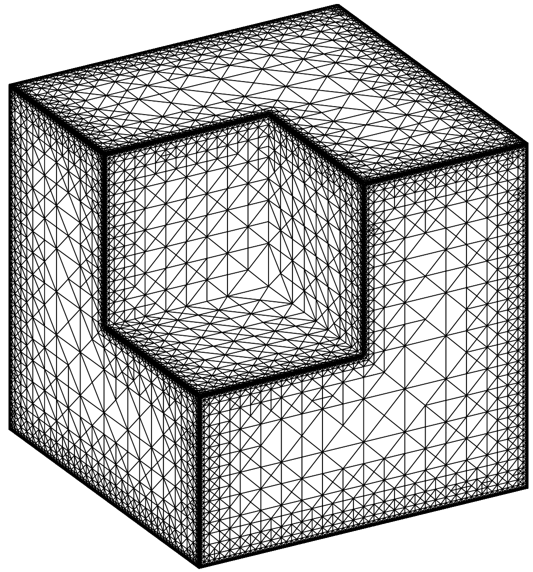


Fichera's Cube

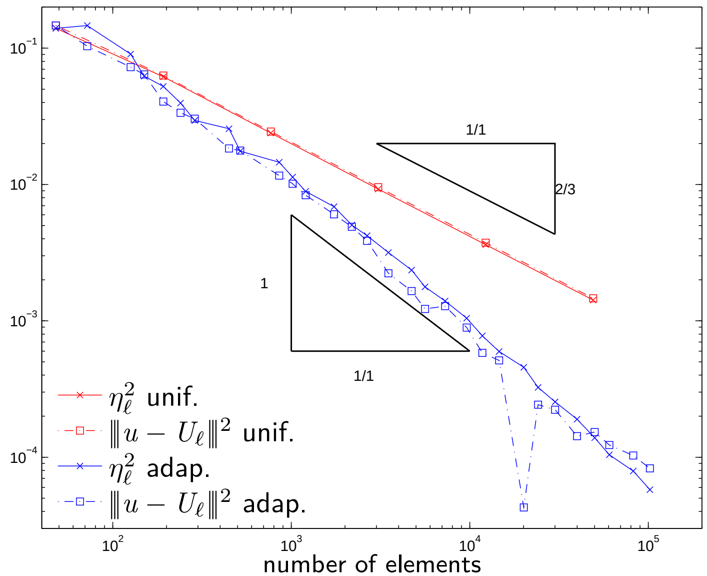


- solve $Vu = 1$ on Γ with lowest-order BEM $p = 0$

Fichera's Cube: Adaptive Meshes



Fichera's Cube: Errors and Estimators



Conclusions

Conclusions

- optimal convergence rates require optimal mesh-refinement
  - mesh-closure estimate
  - overlay estimate
- discrete reliability (A3) $\implies$ Dörfler marking is optimal
  - essentially: Dörfler marking $\implies$ discrete reliability (A3)
  - no good reason to use different marking strategy!
- linear convergence sufficient & necessary for Stevenson's proof
  - see exercises: $\sum_{j=0}^{\ell-1} \eta_j^{-1/s} \lesssim \eta_\ell^{-1/s} \implies \eta_{\ell+n} \lesssim q^n \, \eta_\ell$
- analysis made for weighted-residual error estimators
  - $\eta_\ell = \|h_\ell^{1/2} \nabla (f - V U_\ell)\|_{L^2(\Gamma)}$ for weakly-singular IE
  - $\eta_\ell = \|h_\ell^{1/2} (f - W U_\ell)\|_{L^2(\Gamma)}$ for hyper-singular IE
  - standard residual error estimator for FEM

References

- Carstensen, Feischl, et al. (Comput. Math. Appl. 67, 2014)
  - axiomatic framework for optimal adaptivity
  - 59-page review article on available results
- Feischl, Karkulik, et al. (SIAM J. Numer. Anal. 51, 2013)
- Feischl, Führer, et al. (Calcolo, 2014)
  - optimal ABEM for weakly-singular IE (with approx. of RHS)
  - fixed polynomial order $p \geq 0$
- Feischl, Führer, et al. (ASC Preprint 30/2013)
  - optimal ABEM for hyper-singular IE (with approx. of RHS)
  - fixed polynomial order $p \geq 0$
- Feischl, Führer, P. (SIAM J. Numer. Anal. 52, 2014)
  - optimal AFEM for general 2nd order (linear) elliptic PDE
  - quasi-orth. (A3) for symm. operator + compact perturbation

Extensions

- everything holds for (locally) equivalent error estimators
  - e.g.: ZZ-type error estimators for FEM
- inexact / iterative solvers are OK (convergence + optimality)
  - control discrete residual by error estimator
- discrete reliability for "standard problems" well developed
  - use Scott-Zhang projector for problems in H^s with $s > 0$

Open Problems

- optimal anisotropic mesh-refinement
  - how to deal with overlay estimate?
- discrete reliability (A4) for problems with anisotropies
- quasi-orth. (A3) for FEM-BEM coupling with polygonal Γ
  - weighted-residual error estimator satisfies (A1), (A2), (A4)
- linear convergence for Faermann residual estimator
  - clear: quasi-orthogonality (A3), since weakly-singular IE
  - OK: stability (A1), discrete reliability (A4)
  - problem: reduction (A2), since no h_ℓ-weighting
  - interesting: Faermann estimator satisfies $\eta_\ell \simeq \|u - U_\ell\|$

What is HILBERT ?

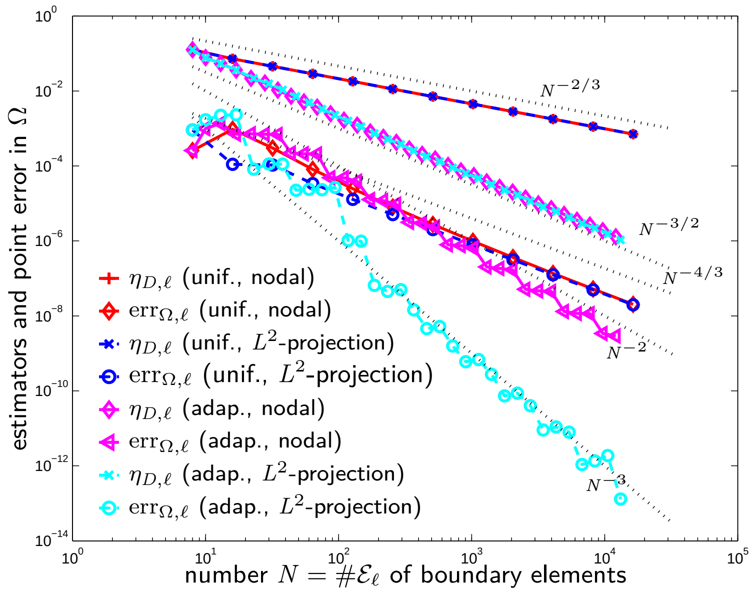
Hilbert Is a Lovely Boundary Element Research Tool

- Matlab library for h-adaptive Galerkin BEM
- lowest-order elements for 2D Laplacian
  - $\mathcal{P}^0$ for normal derivatives
  - $\mathcal{S}^1$ for traces
- research code for FWF project P21732
  - overview on current state of the art
  - starting point for further investigations

Features of HILBERT

- C implementation of integral operators via MEX interface
  - $V(\mathcal{P}^0_\Gamma \times \mathcal{P}^0_\Gamma)$
  - $K(\mathcal{S}^1_\Gamma \times \mathcal{P}^0_\Gamma)$
  - $W(\mathcal{S}^1_\Gamma \times \mathcal{S}^1_\Gamma)$
  - $N(\mathcal{P}^0_\Omega \times \mathcal{P}^0_\Gamma)$
- remaining codes in Matlab fully vectorized
  - different error estimators (h-h/2, 2-Level, residual, Faermann)
  - different marking strategies
  - local mesh-refinement (1D bisection, 2D NVB)
- demo files and adaptive algorithms for
  - weakly-singular integral equation
  - hypersingular integral equation
  - symmetric integral formulation of mixed BVP
  - with/without volume force

Example: Point Errors



Restrictions of HILBERT

- only 2D
- only lowest-order elements
- only canonical bases
  - characteristic functions for $\mathcal{P}^0$
  - hat functions for $\mathcal{S}^1$
- no matrix compression, i.e., dense matrices
- direct solution of linear systems by Matlab backslash operator

Thank You for Listening

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July 18–22, 2014

Adaptive Boundary Element Methods

Part 5: Inverse Estimates

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Inverse Estimates

- bound stronger norm $\|\cdot\|$ by weaker norm $|||\cdot|||$

$$\|x\| \le C\,|||x||| \quad \text{for all } x \in X$$

- requires finite dimensional space X
- naive norm equivalence yields $C = C(X)$
- not sufficient for $X = \mathcal{P}^0(\mathcal{T}_\ell)$ if mesh is refined!

Local Inverse Estimates for Non-Local Operators

Inverse Estimates for FEM

- Lipschitz domain $\Omega \subset \mathbb{R}^d$
- γ-shape regular triangulation $\mathcal{T}_\ell$
- local mesh-size $h_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$
- lowest-order Courant FE space $\mathcal{S}^1(\mathcal{T}_\ell)$

FEM Inverse Estimate

Exists $C = C(\gamma) > 0$ s.t.

$$\|h_\ell \nabla U_\ell\|_{L^2} \le C\,\|U_\ell\|_{L^2} \quad \text{for all } U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$$

- $C = C(\gamma)$ essentially independent of $X = \mathcal{S}^1(\mathcal{T}_\ell)$

Proof of FEM Inverse Estimate

- equivalence of norms on reference situation

$$\|\nabla U\|_{L^2(T_{\rm ref})} \le C\,\|U\|_{L^2(T_{\rm ref})} \quad \text{for all } U \in \mathcal{P}^1(T_{\rm ref})$$

  - constant $C = C(T_{\rm ref}) > 0$

- scaling argument provides powers of h_ℓ

$$\|h_\ell \nabla U_\ell\|_{L^2(T)} \le C\,\|U_\ell\|_{L^2(T)} \quad \text{for all } U_\ell \in \mathcal{P}^1(T)$$

  - constant $C = C(T_{\rm ref}, \gamma)$

- reduction to local configurations & summation

$$\|h_\ell \nabla U_\ell\|_{L^2(\Omega)} \le C\,\|U_\ell\|_{L^2(\Omega)} \quad \text{for all } U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$$

  - constant $C = C(T_{\rm ref}, \gamma)$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\nabla V\psi_T\|_{L^2(T)}^2 + \sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\nabla V\psi_{\Gamma\setminus T}\|_{L^2(T)}^2$$

- $\psi \in L^2(\Gamma) \quad \Longrightarrow \quad$ decomposition $V\psi = V(\psi_T) + V(\psi_{\Gamma\setminus T})$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(T)}^2 \lesssim \|h_\ell^{1/2}\nabla V\psi_T\|_{L^2(T)}^2 + \|h_\ell^{1/2}\nabla V\psi_{\Gamma\setminus T}\|_{L^2(T)}^2$$

- idea: $\psi_T = \psi|_{{\rm patch}(T)} \quad \Longrightarrow \quad V\psi_T$ is local
  - estimate as for uniform mesh-refinement
- idea: $\psi_{\Gamma\setminus T} = \psi - \psi_T \quad \Longrightarrow \quad V\psi_{\Gamma\setminus T}$ smooth potential
  - trace inequality + elliptic regularity theory

Inverse Estimate for Non-Local Operator

- goal: $\|h_\ell^{1/2}\nabla V\Psi_\ell\|_{L^2(\Gamma)} \lesssim \|V\Psi_\ell\|_{H^{1/2}(\Gamma)} \simeq \|\Psi_\ell\|_{H^{-1/2}(\Gamma)}$
- $X = V\mathcal{P}^0(\mathcal{T}_\ell)$ finite dimensional space
- reduction to local configurations

$$\|h_\ell^{1/2}\nabla V\Psi_\ell\|_{L^2(\Gamma)}^2 = \sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\nabla V\Psi_\ell\|_{L^2(T)}^2$$

- V is non-local $\quad \Longrightarrow \quad$ dimension on T depends on $\mathcal{T}_\ell$

$$\|h_\ell^{1/2}\nabla V\psi)\|_{L^2(\Gamma}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\nabla V\psi_{\Gamma\setminus T}\|_{L^2(T)}^2$$

- stability $V : L^2(\Gamma) \to H^1(\Gamma)$ & shape regularity

$$\begin{aligned}\|h_\ell^{1/2}\nabla V\psi_T\|_{L^2(T)}^2 &\le h_T\,\|\nabla V\psi_T\|_{L^2(\Gamma)}^2 \\ &\lesssim h_T\,\|\psi_T\|_{L^2(\Gamma)}^2 \\ &\simeq \|h_\ell^{1/2}\psi\|_{L^2({\rm patch}(T))}^2\end{aligned}$$

- shape regularity

$$\begin{aligned}\sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\nabla V\psi_T\|_{L^2(T)}^2 &\lesssim \sum_{T\in\mathcal{T}_\ell} \|h_\ell^{1/2}\psi\|_{L^2({\rm patch}(T))}^2 \\ &\simeq \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2\end{aligned}$$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}\|h_\ell^{1/2}\nabla V\psi_{\Gamma\setminus T}\|_{L^2(T)}^2$$

- $u_T = V\psi_{\Gamma\setminus T}$ solves transmission problem

$$\begin{aligned} -\Delta u_T &= 0 && \text{in } \mathbb{R}^d\setminus\Gamma\\ [u_T] &= 0 && \text{on } \Gamma\\ [\partial_n u_T] &= -\psi_{\Gamma\setminus T} && \text{on } \Gamma\\ &+ \text{radiation condition} \end{aligned}$$

$\Longrightarrow$ $u_T \in H^2(\widehat{U}_T)$ locally around T

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}\|DV\psi_T\|_{L^2(\widehat{U}_T)}^2$$

- decomposition $V\psi = V(\psi_T) + V(\psi_{\Gamma\setminus T})$

$$\|DV\psi_{\Gamma\setminus T}\|_{L^2(\widehat{U}_T)}^2 \lesssim \|DV\psi\|_{L^2(\widehat{U}_T)}^2 + \|DV\psi_T\|_{L^2(\widehat{U}_T)}^2$$

- shape regularity & stability $V : \widetilde{H}^{-1/2}(\Gamma) \to H^1(U_\Gamma)$

$$\sum_{T\in\mathcal{T}_\ell}\|DV\psi\|_{L^2(\widehat{U}_T)}^2 \lesssim \|V\psi\|_{H^1(U_\Gamma)}^2 \lesssim \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2$$

  - $\Gamma \subset \bigcup_{T\in\mathcal{T}_\ell}\widehat{U}_T \subset U_\Gamma \subset \mathbb{R}^d$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}\|DV\psi_{\Gamma\setminus T}\|_{L^2(\widehat{U}_T)}^2$$

- $u_T = V\psi_{\Gamma\setminus T} \in H^2(\widehat{U}_T)$ locally around T
  - $T \subset U_T \subset \widehat{U}_T$
- trace inequality for $Du_T \in H^1(U_T)$

$$\|\nabla u_T\|_{L^2(T)}^2 \lesssim h_T^{-1}\|Du_T\|_{L^2(U_T)}^2 + h_T\|D^2u_T\|_{L^2(U_T)}^2$$

- Caccioppoli inequality

$$\|D^2u_T\|_{L^2(U_T)}^2 \lesssim h_T^{-2}\|Du_T\|_{L^2(\widehat{U}_T)}^2$$

$\Longrightarrow$ $h_T\|\nabla u_T\|_{L^2(T)}^2 \lesssim \|Du_T\|_{L^2(\widehat{U}_T)}^2$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}\|(1-\Pi_\ell)\psi_T\|_{\widetilde{H}^{-1/2}(\Gamma)}^2$$

- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ L^2-orthogonal projection

$$\|DV\psi_T\|_{L^2(\widehat{U}_T)}^2 \lesssim \|DV\Pi_\ell\psi_T\|_{L^2(\widehat{U}_T)}^2 + \|DV(1-\Pi_\ell)\psi_T\|_{L^2(\widehat{U}_T)}^2$$

- scaling argument

$$\|DV\Pi_\ell\psi_T\|_{L^2(\widehat{U}_T)}^2 \lesssim h_T\|\Pi_\ell\psi_T\|_{L^2(\text{patch}(T))}^2 \lesssim h_T\,\|\psi\|_{L^2(\text{patch}(T))}^2$$

$\Longrightarrow$ $\sum_{T\in\mathcal{T}_\ell}\|DV\Pi_\ell\psi_T\|_{L^2(\widehat{U}_T)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2$

- shape regularity & stability $V : \widetilde{H}^{-1/2}(\Gamma) \to H^1(U_\Gamma)$

$$\sum_{T\in\mathcal{T}_\ell}\|DV(1-\Pi_\ell)\psi_T\|_{L^2(\widehat{U}_T)}^2 \lesssim \sum_{T\in\mathcal{T}_\ell}\|(1-\Pi_\ell)\psi_T\|_{\widetilde{H}^{-1/2}(\Gamma)}^2$$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2 + \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 + \sum_{T\in\mathcal{T}_\ell}\|(1-\Pi_\ell)\psi_T\|_{\widetilde{H}^{-1/2}(\Gamma)}^2$$

- local approximation property of L^2-projection

$$\|(1-\Pi_\ell)\psi_T\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi_T\|_{L^2(\Gamma)}^2 = \|h_\ell^{1/2}\psi\|_{L^2(\mathrm{patch}(T))}^2$$

- shape regularity

$$\sum_{T\in\mathcal{T}_\ell}\|(1-\Pi_\ell)\psi_T\|_{\widetilde{H}^{-1/2}(\Gamma)}^2 \lesssim \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}^2$$

- inverse estimate $\|h_\ell^{1/2}\Psi_\ell\|_{L^2(\Gamma} \lesssim \|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}$

$\Longrightarrow$ $\|h_\ell^{1/2}\nabla V\Psi_\ell\|_{L^2(\Gamma)} \lesssim \|\Psi_\ell\|_{\widetilde{H}^{-1/2}(\Gamma)}$

Conclusions

Inverse Estimates for Non-Local Operators

Thm (Aurada, Feischl, Führer, Karkulik, Melenk, P. '12++)

- $\psi \in L^2(\Gamma)$ $\Longrightarrow$

$$\|h_\ell^{1/2}\nabla V\psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}$$
$$\|h_\ell^{1/2}K^*\psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{\widetilde{H}^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}$$

- $v \in H_0^1(\Gamma)$ $\Longrightarrow$

$$\|h_\ell^{1/2}\nabla Kv\|_{L^2(\Gamma)} \lesssim \|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|h_\ell^{1/2}\nabla v\|_{L^2(\Gamma)}$$
$$\|h_\ell^{1/2}Wv\|_{L^2(\Gamma)} \lesssim \|v\|_{\widetilde{H}^{1/2}(\Gamma)} + \|h_\ell^{1/2}\nabla v\|_{L^2(\Gamma)}$$

Conclusions / Extensions

- non-local operators are less non-local than expected
  - since they are linked to layer potentials
- analysis tailored to shape regular meshes
- such analysis are required
  - to stability + reduction for optimal BEM adaptivity
  - to prove efficiency of weighted-residual BEM error estimator
- inverse estimates can also be proved for hp-BEM
- analysis works on piecewise smooth boundaries

References

- Feischl, Karkulik, et al. (SIAM J. Numer. Anal. 51, 2013)
  - inverse estimate for V and lowest-order BEM $p = 0$
  - polygonal boundary
- Aurada, Feischl, Führer, Karkulik, et al. (Preprint 2012, 2014)
  - inverse estimate for V, W, K, K'
  - formulated for fcts. in L^2 resp. H^1
  - applicable to hp-BEM
  - polygonal boundary (will be: piecewise smooth)

Thank You for Listening

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