NPDE – Topical Lecture Series

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Rate-Optimal Adaptive Finite Element Method

Part 1: Introduction & Newest Vertex Bisection

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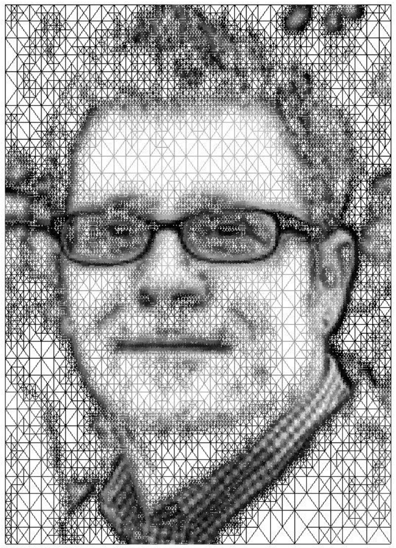
Outline

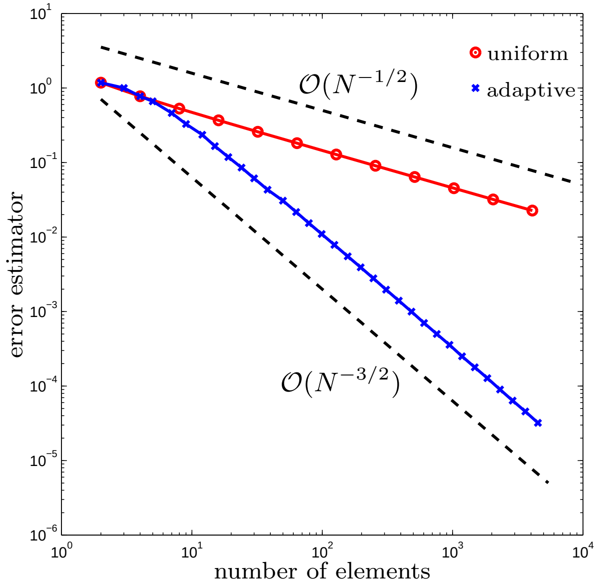
http://www.asc.tuwien.ac.at/~praetorius/afem-lecture/

Introduction

Adaptive Mesh-Refinement?



What is all about?



Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \le 1$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate

1. SOLVE: compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. MARK: find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

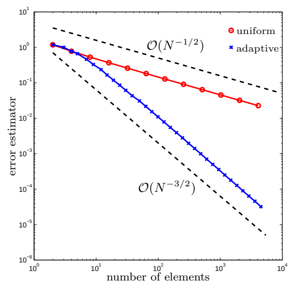
$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

4. REFINE: refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Dörfler: SINUM 33 (1996)

Mathematical Questions?

- can we prove convergence for adaptive mesh-refinement?
- can we prove linear convergence $\eta_{\ell+n} \le C\, q^n\, \eta_\ell$?
  - constant $C > 0$ reflects pre-asymptotic convergence rates
- can we prove optimal convergence rates?
  - at least asymptotically
- what can be said about computational costs?
  - at least asymptotically
- which set of problems can be covered?



Some References

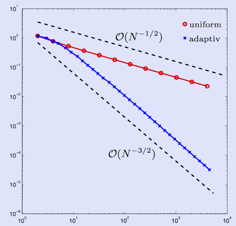
- AFEM: analysis well developed
  - Dörfler: SINUM 33 (1996) — 461 citations
  - Morin, Nochetto, Siebert: SINUM 38 (2000) — 394 citations
  - Binev, Dahmen, DeVore '04: Numer. Math. 97 (2004) — 248 citations
  - Stevenson: Found. Comput. Math. 7 (2007) — 191 citations
  - Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008) — 207 citations
  - Feischl, Führer, Praetorius: SINUM 52 (2014) — 17 citations
- ABEM: analysis for Laplace 2D/3D
  - Gantumur: Numer. Math. 124 (2013) — 18 citations
  - Feischl, Karkulik, Melenk, Praetorius: SINUM 51 (2013) — 26 citations

Main Theorem on Adaptive Algorithms

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- properties of mesh-refinement & estimator axioms (A1)–(A4)
- $0 < \theta \le 1$

$\Longrightarrow \quad \exists C > 0 \; \exists 0 < q < 1 \; \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$



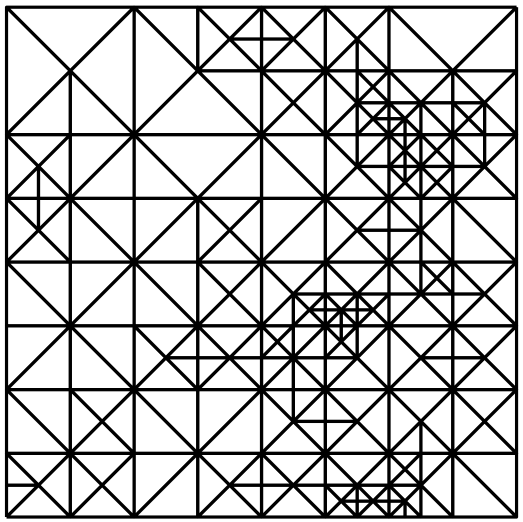
- $\mathbb{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \big(N^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N} \eta_{\mathrm{opt}}\big) =: \|\eta\|_{\mathbb{A}_s}$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Regular Triangulations

Triangulation of $\Omega \subset \mathbb{R}^2$



Triangulation of $\Omega \subset \mathbb{R}^2$

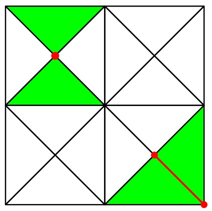
$\mathcal{T}$ is triangulation of $\Omega \subset \mathbb{R}^2$, if

- $\mathcal{T}$ is finite set of compact triangles T with $|T| > 0$
- $\mathcal{T}$ covers $\overline{\Omega}$, i.e., $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} T$.
- for all $T, T' \in \mathcal{T}$ with $T \ne T'$ holds $|T \cap T'| = 0$
  - i.e., the overlap of two elements is trivial

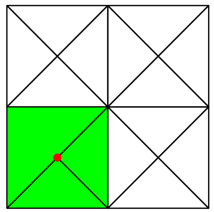
Regular triangulation

Triangulation $\mathcal{T}$ is regular or conforming, if

- for all $T, T' \in \mathcal{T}$ with $T \neq T'$, the intersection $T \cap T'$ is
  - either empty
  - or a joint node $T \cap T' = \{z\}$
  - or a joint edge $T \cap T' = E$



regular



not regular, hanging node

Shape regularity

Triangulation $\mathcal{T}$ is γ-shape regular or (locally) γ-quasi uniform

- $\max_{T \in \mathcal{T}} \frac{\mathtt{diam}(T)^2}{|T|} \leq \gamma < \infty$

- geometrically: minimal angles of all $T \in \mathcal{T}$ are bounded from below
- analytically: enters inverse estimates and approximation estimates
  - e.g., $\|\nabla U\|_{L^2(T)} \leq C(\gamma, q)\,\mathtt{diam}(T)^{-1}\|U\|_{L^2(T)}$ for all $U \in \mathcal{P}^q(T)$
- sloppy formulation in papers:
  - ... where the constants depend on the shape of the elements.
  - If the triangulation is (locally) quasi uniform, it holds ...
- better: ... where the constants depend on γ.

Axioms on mesh-refinement

- given initial mesh $\mathcal{T}_0$
- given mesh-refinement strategy $\mathtt{refine}(\cdot)$

Axioms of Adaptivity require

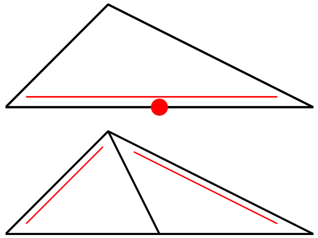
1. all constants in (A1)–(A4) are bounded $\rightsquigarrow$ γ-shape regularity
2. refined elements are split in ≥ 2 and $\leq n$ many sons
3. overlay estimate
4. closure estimate

- aim of first lecture: all is satisfied for Newest Vertex Bisection (NVB)

Newest Vertex Bisection

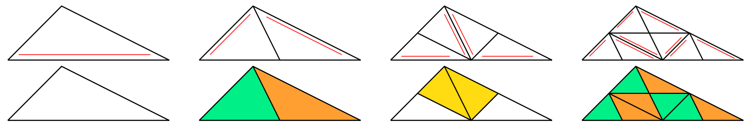
Newest Vertex Bisection (NVB)

- each element has reference edge
- refinement by bisection
  - T' son of T $\implies$ $|T'| = |T|/2$
- new reference edges are opposite to newest vertex
- for $T \in \mathcal{T}$, obtain unique binary tree with possible successors T' with $T' \subsetneqq T$



NVB guarantees finitely many shapes

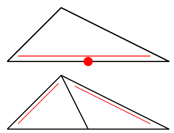
- NVB leads to only finitely many similarity classes of triangles
  - depends only on T and its reference edge



- in particular, uniform shape regularity
  - $\dfrac{\mathtt{diam}(T')^2}{|T'|} \le \gamma(T) < \infty$ for all NVB successors T' of T

Element level function

- suppose initial triangulation $\mathcal{T}_0$ with fixed reference edges
- each NVB successor T of some $T_0 \in \mathcal{T}_0$ has refinement level:
  - $\mathtt{lev}(T) := 0$ if $T = T_0$
  - $\mathtt{lev}(T) := \mathtt{lev}(\widehat{T}) + 1$ if T son of $\widehat{T}$



- note that $\mathtt{diam}(T)^2 \simeq |T| = |T_0|\, 2^{-\mathtt{lev}(T)}$

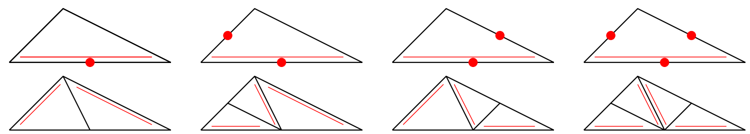
Lemma

- $T, T' \subseteq T_0 \in \mathcal{T}_0$ with $\mathtt{lev}(T) = \mathtt{lev}(T')$
  $\implies$ $T = T'$ or $|T \cap T'| = 0$

- proof by induction on $\mathtt{lev}(T)$

$\mathcal{T}' = \mathtt{refine}(\mathcal{T}, \mathcal{M})$

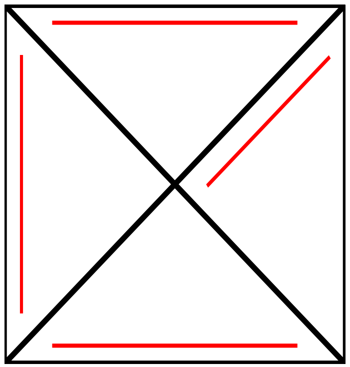
- initialization
  - for all marked elements $T \in \mathcal{T} \cap \mathcal{M}$, mark its reference edge
- recursion
  - if element's edge is marked, mark at least its reference edge
  - terminates, since each triangulation has only finitely many edges
- refinement according to scheme



- note that each case is iterated NVB
- each refined element has 2, 3, or 4 sons

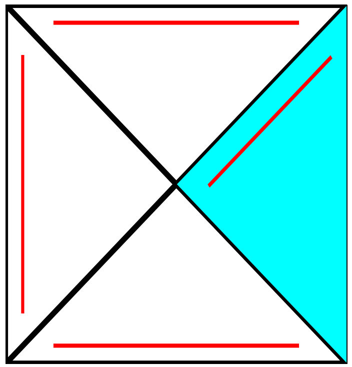
Example

- given reference edges



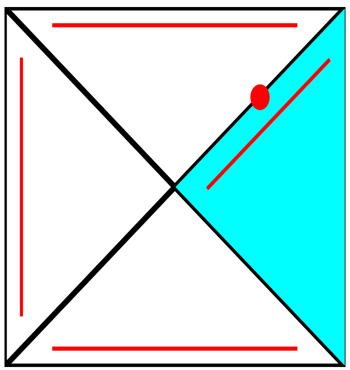
Example

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$



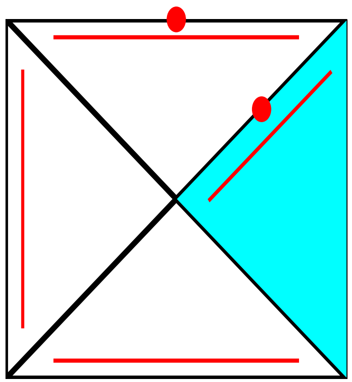
Example

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges



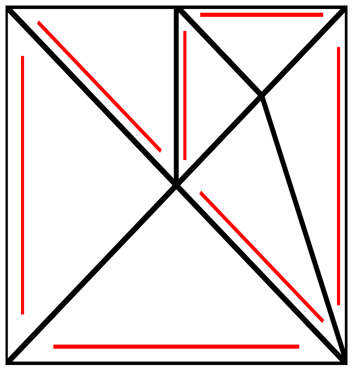
Example

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges
- proceed recursively



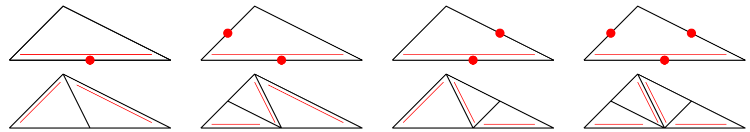
Example

- given reference edges
- marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- mark reference edges
- proceed recursively
- mesh refinement



NVB preserves regularity

- recall refinement scheme



- $\mathcal{T}$ regular $\implies$ $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \mathcal{M})$ regular
  - since new nodes are always edge midpoints
  - marked edge $E = T \cap T'$ is refined for both elements
  - i.e., refinement creates no new hanging nodes
- $\texttt{refine}(\mathcal{T}, \mathcal{M})$ is coarsest conforming refinement of $\mathcal{T}$ s.t.
  - all $T \in \mathcal{M}$ are bisected by NVB

Order of refinement does not matter

- order of refinement does not matter
  - $T_1, T_2 \in \mathcal{T} \implies$
    $\texttt{refine}(\texttt{refine}(\mathcal{T}, \{T_1\}), \{T_2\})) = \texttt{refine}(\texttt{refine}(\mathcal{T}, \{T_2\}), \{T_1\}))$
  - since only one refinement rule
- $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \mathcal{M})$ can also be realized through single refinements

  $\mathcal{T}' := \mathcal{T}$
  for each $T \in \mathcal{M}$
  $\quad \mathcal{T}' := \texttt{refine}(\mathcal{T}', \{T\})$
  end

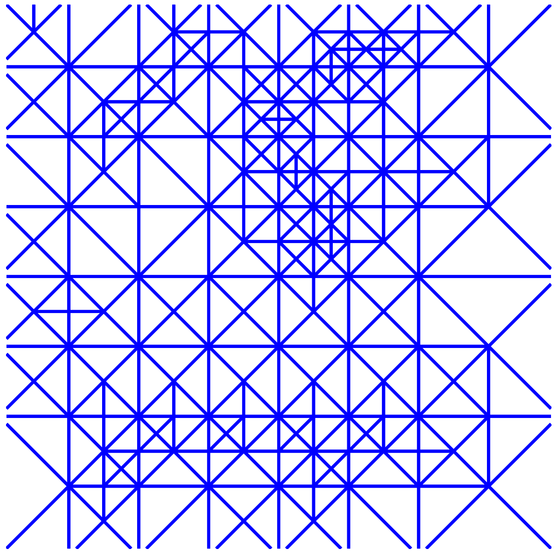
- recall that $\texttt{refine}(\mathcal{T}', \{T\})$ usually refines more than 1 element

Iterated NVB refinement

- $\mathcal{T}$ given regular triangulation
- fixed reference edge for all $T \in \mathcal{T}$
- write $\mathcal{T}' \in \texttt{refine}(\mathcal{T})$ if $\mathcal{T}'$ is obtained by finitely many steps, i.e.,
  - exists $n \in \mathbb{N}_0$,
  - exists $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_n$ with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_n = \mathcal{T}$,
  - exists $\mathcal{M}_j \subseteq \mathcal{T}_j$,
  - such that: $\mathcal{T}_j = \texttt{refine}(\mathcal{T}_{j-1}, \mathcal{M}_{j-1})$ for all $j = 1, \dots, n$.
- initial triangulation $\mathcal{T}_0$ fixes possible $\mathcal{T} \in \mathbb{T} := \texttt{refine}(\mathcal{T}_0)$

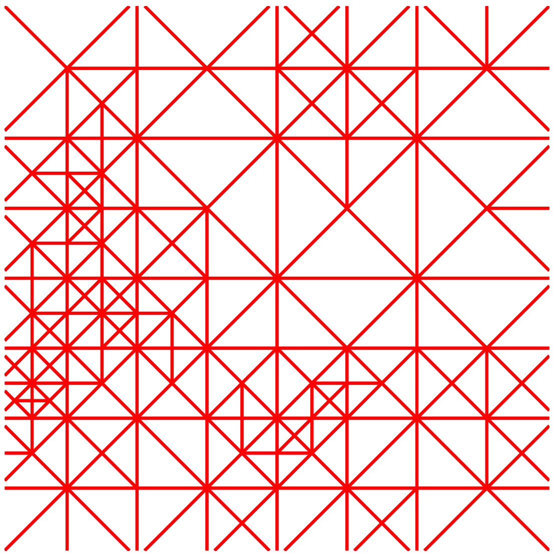
Overlay Estimate

Example



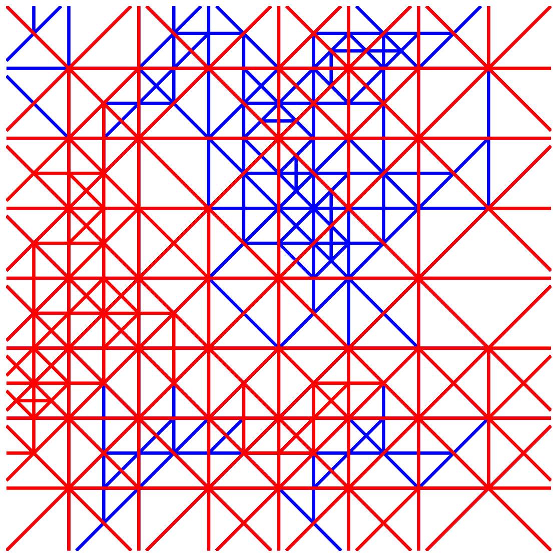
- $\mathcal{T}$ (blue)

Example



- $\mathcal{T}'$ (red)

Example



- $\mathcal{T}$ (blue)
- $\mathcal{T}'$ (red)
- $\mathcal{T} \oplus \mathcal{T}'$

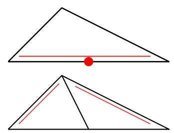
Local compatibility

Lemma

- For all NVB successors $T, T' \subseteq T_0 \in \mathcal{T}_0$ with $|T \cap T'| > 0$, it holds
  - $\mathtt{lev}(T') > \mathtt{lev}(T) \quad \Longrightarrow \quad T' \subsetneqq T$
  - $\mathtt{lev}(T') = \mathtt{lev}(T) \quad \Longrightarrow \quad T' = T$ (already proved)

Proof.

- for all $k \le \mathtt{lev}(T)$ exist unique NVB successors T_k, T_k' of T_0 s.t.
  - $\mathtt{lev}(T_k) = k = \mathtt{lev}(T_k')$
  - $T \subseteq T_k$ and $T' \subseteq T_k'$



- $|T_k \cap T_k'| \ge |T \cap T'| > 0$ implies
  - $T_k = T_k'$ for all $k \le \mathtt{lev}(T)$
  - and $T = T_k$ for $k = \mathtt{lev}(T)$
- i.e., $T' \subseteq T_k' = T_k = T$ for $k = \mathtt{lev}(T)$
  - $T' \subsetneqq T_k'$ for $\mathtt{lev}(T') > k$

■

Overlay 1/2

Proposition (Overlay of NVB meshes)

- $\mathcal{T}, \mathcal{T}' \in \mathtt{refine}(\mathcal{T}_0)$
- $\mathcal{T} \oplus \mathcal{T}' := \{T \cap T' : T \in \mathcal{T}, T' \in \mathcal{T}', |T \cap T'| > 0\}$

$\Longrightarrow \quad \mathcal{T} \oplus \mathcal{T}' \in \mathtt{refine}(\mathcal{T}) \cap \mathtt{refine}(\mathcal{T}')$

Proof.

- for $x \in \Omega$, choose sequence T_k of NVB triangles s.t.
  - $x \in T_k$
  - $\mathtt{lev}(T_k) = k$
  - $T_k \supsetneqq T_{k+1}$

  $\Longrightarrow$ ex. $i, j \in \mathbb{N}$ s.t. $T_i \in \mathcal{T}$, $T_j \in \mathcal{T}'$ and $x \in T_i \cap T_j \in \mathcal{T} \oplus \mathcal{T}'$
- $\widetilde{T} = T \cap T' \in \mathcal{T} \oplus \mathcal{T}' \quad \Longrightarrow \quad \widetilde{T} = T$ or $\widetilde{T} = T'$ (local compatibility)

  $\Longrightarrow \quad \mathcal{T} \oplus \mathcal{T}'$ is triangulation
- remains to show: $\mathcal{T} \oplus \mathcal{T}'$ is regular

Overlay 2/2

- recall: $\mathcal{T} \oplus \mathcal{T}' := \{T \cap T' : T \in \mathcal{T}, T' \in \mathcal{T}', |T \cap T'| > 0\}$
- let $T, T' \in \mathcal{T} \oplus \mathcal{T}'$
- let z be node of T with $z \in T'$
- goal: z is also node of T' (i.e., there are no hanging nodes)
- w.l.o.g. $T \in \mathcal{T}$
- w.l.o.g. $T' \in \mathcal{T}'$ (otherwise claim follows from regularity of $\mathcal{T}$)
- by definition of $\mathcal{T} \oplus \mathcal{T}'$, there exists $\widetilde{T} \in \mathcal{T}$ with $T' \subseteq \widetilde{T}$
- thus, $z \in \widetilde{T} \cap T$
- regularity of $\mathcal{T} \quad \Longrightarrow \quad z$ is node of $\widetilde{T} \quad \Longrightarrow \quad z$ is node of T'
  - T' is NVB successor of $\widetilde{T}$ with $z \in T'$
  - nodes stay nodes under NVB refinement

■

Overlay estimate

Proposition (Overlay estimate)

- Let $\mathcal{T}, \mathcal{T}' \in \mathbb{T} = \mathtt{refine}(\mathcal{T}_0)$

  $\Longrightarrow$ exists $\mathcal{T} \oplus \mathcal{T}' \in \mathtt{refine}(\mathcal{T}) \cap \mathtt{refine}(\mathcal{T}')$ s.t.

$$\#(\mathcal{T} \oplus \mathcal{T}') \le \#\mathcal{T} + \#\mathcal{T}' - \#\mathcal{T}_0$$

Proof.

- recall: $\mathcal{T} \oplus \mathcal{T}' := \{T \cap T' : T \in \mathcal{T}, T' \in \mathcal{T}', |T \cap T'| > 0\}$
  - shown: $\mathcal{T} \oplus \mathcal{T}'$ consists of elements of $\mathcal{T}$ and elements of $\mathcal{T}'$
- too rough estimate: $\#(\mathcal{T} \oplus \mathcal{T}') \le \#\mathcal{T} + \#\mathcal{T}'$
- but: can locally subtract the number of coarser elements
  - e.g., $T = T \cap T' \in \mathcal{T} \oplus \mathcal{T}'$ with $T \in \mathcal{T}$, $T' \in \mathcal{T}'$

    $\Longrightarrow$ do not count T' for $\#(\mathcal{T} \oplus \mathcal{T}')$
- for each $T_0 \in \mathcal{T}_0$, at least one such element in $\mathcal{T} \cup \mathcal{T}'$ can be omitted

  $\Longrightarrow \#(\mathcal{T} \oplus \mathcal{T}') \le \#\mathcal{T} + \#\mathcal{T}' - \#\mathcal{T}_0$ ■

Remarks

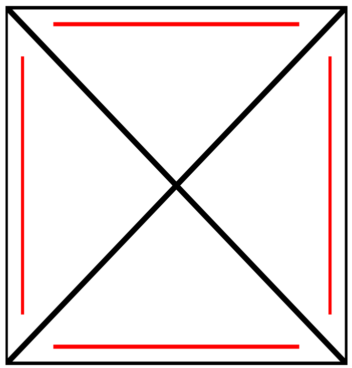
- $\mathcal{T} \oplus \mathcal{T}' := \{T \cap T' : T \in \mathcal{T}, T' \in \mathcal{T}', |T \cap T'| > 0\}$
  - is coarsest common refinement of $\mathcal{T}$ and $\mathcal{T}'$
  - i.e., $\#(\mathcal{T} \oplus \mathcal{T}') = \min\{\#\widehat{\mathcal{T}} : \widehat{\mathcal{T}} \in \mathtt{refine}(\mathcal{T}) \cap \mathtt{refine}(\mathcal{T}')\}$
- same proof works for NVB in $\mathbb{R}^d$ instead of $2D$
- overlay estimate can be proved for "simple" refinement strategies
  - e.g., red refinement with hanging nodes
- overlay estimate fails for 2D red-green-blue refinement

Stevenson: Found. Comp. Math. (2007) NVB in $\mathbb{R}^2$
Cascón, Kreuzer, Nochetto, Siebert: SINUM (2008) NVB in $\mathbb{R}^d$
Bonito, Nochetto: SINUM (2010) red refinement with hanging nodes
Pavlicek: Bakk thesis (TU Wien 2011) RGB in $\mathbb{R}^2$

Closure Estimate

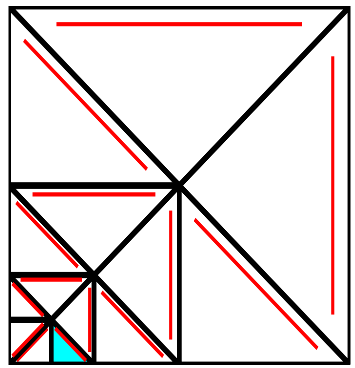
Example

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$



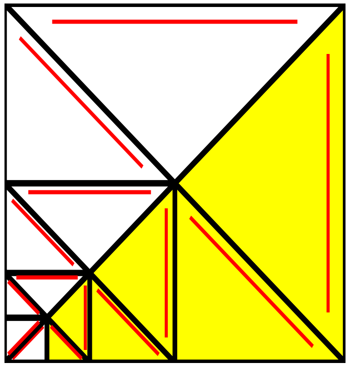
Example

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \mathtt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
- clearly: $\#\mathcal{M}_\ell \le \#\mathcal{R}_\ell$



Example

- $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$
- $\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$
- clearly: $\#\mathcal{M}_\ell \le \#\mathcal{R}_\ell$
- $\#\mathcal{R}_\ell \le C \#\mathcal{M}_\ell$ cannot hold
  - $\#\mathcal{M}_\ell = 1$
  - $\#\mathcal{R}_\ell \sim \ell$

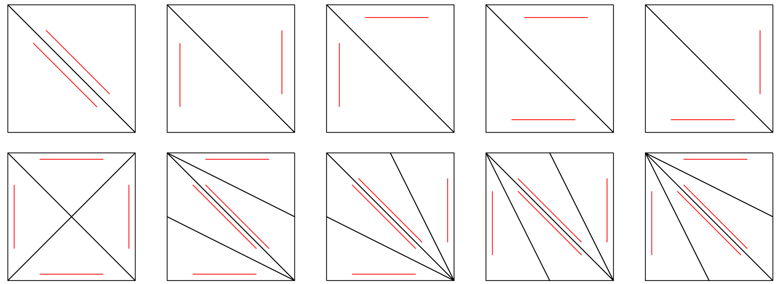


Admissibility

Initial triangulation $\mathcal{T}_0$ is admissible, if

- for all $T, T' \in \mathcal{T}_0$, it holds

  $E := T \cap T'$ is reference edge of T $\iff$ E is reference edge of T'



Closure estimate

Theorem (Binev, Dahmen, DeVore '04, Stevenson '08)

- $\mathcal{T}_0$ admissible
- $\mathcal{T}_j = \texttt{refine}(\mathcal{T}_{j-1}, \mathcal{M}_{j-1})$ with $\mathcal{M}_{j-1} \subseteq \mathcal{T}_{j-1}$ for all $j = 1, 2, 3, \dots$

$$\Longrightarrow \quad \#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$$

- recall: $\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell-1} \not\le C(\mathcal{T}_0)\,\#\mathcal{M}_\ell$
  - by counter example
- remainder of talk: proof of theorem

Uniform refinements are admissible

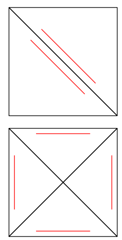
Lemma

- $\mathcal{T}_0^{\rm unif} := \mathcal{T}_0$ admissible
- $\mathcal{T}_k^{\rm unif} := \texttt{refine}(\mathcal{T}_{k-1}^{\rm unif}, \mathcal{T}_{k-1}^{\rm unif})$ uniform refinement for $k = 1, 2, 3, \dots$

$\Longrightarrow$ $\mathcal{T}_k^{\rm unif}$ admissible and $\texttt{lev}(T) = k$ for all $T \in \mathcal{T}_k^{\rm unif}$

Proof by induction on k.

- start: $k = 0$ ✓
- hypothesis: claim OK for k
- goal: claim for $k+1$
- $\mathcal{T}_k^{\rm unif}$ is admissible $\Longrightarrow$ only reference edges are refined
  - no recursive marking needed
- all non-reference edges become reference edges of $\mathcal{T}_{k+1}^{\rm unif}$ ∎



Neighbors differ by level ≤ 1

Lemma

- $\mathcal{T} \in \texttt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T, N \in \mathcal{T}$ with $E := T \cap N$ edge

$\implies$ $|\texttt{lev}(T) - \texttt{lev}(N)| \leq 1$

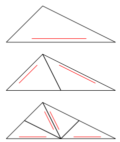
Proof.

- w.l.o.g.: $\texttt{lev}(T) = \texttt{lev}(N) + n$ with $n \geq 1$
- consider uniform refinement of $\mathcal{T}_0$ with $\texttt{lev}(T)$

$\implies$ n levels of bisection provide successor N' of N s.t.

  - $\texttt{lev}(N') = \texttt{lev}(N) + n = \texttt{lev}(T)$
  - $E = T \cap N'$, since uniform refinement is regular

$\implies$ $n \leq 1$, since 2 levels of bisection half all edges ■



Configurations of reference edge 1/5

Lemma

- $\mathcal{T} \in \texttt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T, N \in \mathcal{T}$ with $E := T \cap N$ reference edge of T

$\implies$ Then,

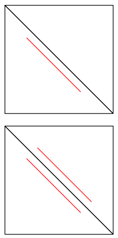
- either $\texttt{lev}(T) = \texttt{lev}(N)$ and E is also reference edge of N
- or $\texttt{lev}(T) = \texttt{lev}(N) + 1$ and E is reference edge of son of N

Proof by consideration of three cases.

- since $|\texttt{lev}(T) - \texttt{lev}(N)| \leq 1$:
  1. $\texttt{lev}(T) = \texttt{lev}(N)$
  2. $\texttt{lev}(T) = \texttt{lev}(N) + 1$
  3. $\texttt{lev}(T) + 1 = \texttt{lev}(N)$

Configurations of reference edge 2/5

1. **Case.** $k := \texttt{lev}(T) = \texttt{lev}(N)$
   - goal: E is reference edge of T and N

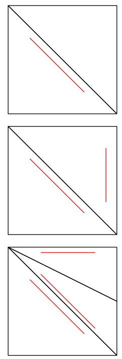


- $T, N \in \mathcal{T}_k^{\rm unif}$
- $\mathcal{T}_k^{\rm unif}$ is admissible
- $E = T \cap N$ is reference edge of T

$\implies$ E is also reference edge of N

Configurations of reference edge 3/5

2. **Case.** $k := \texttt{lev}(T) = \texttt{lev}(N) + 1$
   - goal: E is reference edge of T and son of N

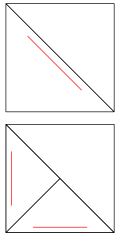


- $T \in \mathcal{T}_k^{\rm unif}$
- Let $N', N'' \in \mathcal{T}_k^{\rm unif}$ be sons of N (one bisection!)
- $\mathcal{T}_k^{\rm unif}$ is admissible
- $E = T \cap N'$ is reference edge of T

$\implies$ E is also reference edge of N'

Configurations of reference edge 4/5

3. **Case.** $\mathtt{lev}(T) + 1 = \mathtt{lev}(N)$
   - goal: this case is never met!



- bisect T into T', T''

  $\Longrightarrow$ $\mathtt{lev}(T') = \mathtt{lev}(T'') = \mathtt{lev}(N)$

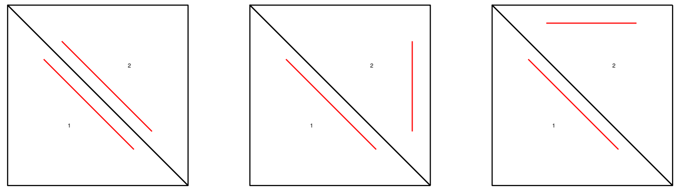
  $\Longrightarrow$ $\{T', T'', N\}$ regular triangulation of $T \cup N$
- E is split, since E is reference edge of T

  $\Longrightarrow$ contradicts regularity

Configurations of reference edge 5/5

Lemma
- $\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T, N \in \mathcal{T}$ with $E := T \cap N$ reference edge of T

$\Longrightarrow$ Then,
- either $\mathtt{lev}(T) = \mathtt{lev}(N)$ and E also reference edge of N
- or $\mathtt{lev}(T) = \mathtt{lev}(N) + 1$ and E is reference edge of son of N



Recursive refinement 1/6

Lemma
- $\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T, N \in \mathcal{T}$ with $E := T \cap N$ reference edge of T

$\Longrightarrow$ $\big(T \notin \mathtt{refine}(\mathcal{T}, \{N\}) \iff E$ also reference edge of $N\big)$

Proof of $\Longrightarrow$ (since $\Longleftarrow$ is clear).
- ex. minimal $n \in \mathbb{N}$ and $T_1, \dots, T_{n+1} \in \mathcal{T}$ s.t.
  - $T_1 = N$, $T_{n+1} = T$
  - $E_j := T_j \cap T_{j+1}$ is reference edge of T_j for all $j = 1, \dots, n$
- **Case 1.** $n = 1$, i.e., $T_2 = T$

  $\Longrightarrow$ E also reference edge of $N = T_1$
- **Case 2.** $n > 1$, i.e., E_j not ref. edge of T_{j+1} for all $j = 1, \dots, n-1$

  $\Longrightarrow$ $\mathtt{lev}(T_j) = \mathtt{lev}(T_{j+1}) + 1$

  $\Longrightarrow$ $\mathtt{lev}(N) = \mathtt{lev}(T_1) = \dots = \mathtt{lev}(T_{n+1}) + n = \mathtt{lev}(T) + n$

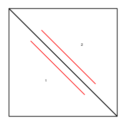
  $\Longrightarrow$ $n \le 1$ $\Longrightarrow$ contradicts $n > 1$ ∎

Recursive refinement 2/6

Lemma (Level of generated elements)
- $T \in \mathcal{T} \in \mathtt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T' \in \mathtt{refine}(\mathcal{T}, \{T\}) \setminus \mathcal{T}$

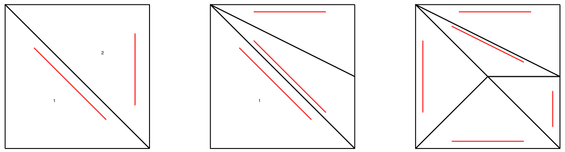
$\Longrightarrow$ $\mathtt{lev}(T') \le \mathtt{lev}(T) + 1$

Proof by induction on $\mathtt{lev}(T)$.
- start: $k = 0$ is clear, since $\mathcal{T}_0$ is admissible ✓
- hypothesis: claim OK for all T with $\mathtt{lev}(T) \le k$
- suppose $\mathtt{lev}(T) = k + 1$
- w.l.o.g.: $\#(\mathtt{refine}(\mathcal{T}, \{T\}) \setminus \mathcal{T}) > 2$, i.e., more than T is refined
- let $N \in \mathcal{T}$ s.t. $E := N \cap T$ is reference edge of T
- **Case 1.** $\mathtt{lev}(T) = \mathtt{lev}(N)$ ✓



Recursive refinement 3/6

- **Case 2.** $\mathtt{lev}(T) = \mathtt{lev}(N) + 1$
  - $T' \in \mathtt{refine}(\mathcal{T}, \{T\}) \setminus \mathcal{T}$
  - $N \in \mathcal{T}$ with $E := T \cap N$ is reference edge of T
- $\mathtt{lev}(T) = k + 1 \quad \Longrightarrow \quad \mathtt{lev}(N) = k$, i.e., hypothesis applies to N
- $\mathtt{refine}(\mathcal{T}, \{T\}) = \mathtt{refine}(\mathcal{T}', \{T\})$ with $\mathcal{T}' := \mathtt{refine}(\mathcal{T}, \{N\})$
- $T' \in \mathcal{T}' \backslash \mathcal{T} \quad \Longrightarrow \quad \mathtt{lev}(T') \le \mathtt{lev}(N) + 1 = \mathtt{lev}(T)$
- $T' \in \mathtt{refine}(\mathcal{T}', \{T\}) \setminus \mathcal{T}'$ is
  - either son of T, i.e., $\mathtt{lev}(T') = \mathtt{lev}(T) + 1$
  - or grandson of N, i.e., $\mathtt{lev}(T') = \mathtt{lev}(N) + 2 = \mathtt{lev}(T) + 1$. ■



Recursive refinement 4/6

- recall $\mathtt{diam}(T)^2 \simeq |T| = |T_0|\, 2^{-\mathtt{lev}(T)}$ with $T \subseteq T_0 \in \mathcal{T}_0$
  $\Longrightarrow$ Ex. $d, D > 0$ s.t. $d\, 2^{-\mathtt{lev}(T)} \le |T| \le \mathtt{diam}(T)^2 \le D^2\, 2^{-\mathtt{lev}(T)}$

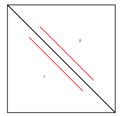
Lemma (Distance of generated elements)

- $T \in \mathcal{T} \in \mathtt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T' \in \mathtt{refine}(\mathcal{T}, \{T\}) \setminus \mathcal{T}$

$$\Longrightarrow \quad \mathbb{d}(T, T') := \min_{\substack{x \in T \\ x' \in T'}} |x - x'| \le \sqrt{2}\, D \sum_{j=\mathtt{lev}(T')}^{\mathtt{lev}(T)} 2^{-j/2}$$

Proof.

- **Step 1.** $T' \subsetneqq T$ son of T
  $\Longrightarrow \quad \mathbb{d}(T, T') = 0$ ✓
- **Step 2.** $E := T \cap N$ is reference edge of $T, N \in \mathcal{T}$
  $\Longrightarrow \quad T$ and N are bisected once $\quad \Longrightarrow \quad \mathbb{d}(T, T') = 0$ ✓



Recursive refinement 5/6

Step 3. Proceed by induction on $k = \mathtt{lev}(T)$

- start: $k = 0$ follows from Step 1–2, since $\mathcal{T}_0$ is admissible ✓
- hypothesis: estimate is true up to level k.
- suppose $\mathtt{lev}(T) = k + 1$ and $T' \not\subseteq T$ (covered by Step 1)
- $N \in \mathcal{T}$ with $E := T \cap N$ reference edge of T, but not of N (Step 2)
  $\Longrightarrow \quad \mathtt{lev}(T) = \mathtt{lev}(N) + 1$ and $T' \in \mathtt{refine}(\mathcal{T}, \{N\}) \setminus \mathcal{T}$

  $$\Longrightarrow \quad \mathtt{lev}(N) = k \quad \text{and} \quad \mathbb{d}(N, T') \le \sqrt{2}\, D \sum_{j=\mathtt{lev}(T')}^{\mathtt{lev}(N)} 2^{-j/2}$$
- triangle inequality $\quad \Longrightarrow \quad \mathbb{d}(T, T') \le \mathtt{diam}(N) + \mathbb{d}(N, T')$
- $\mathtt{lev}(T) = \mathtt{lev}(N) + 1 \Longrightarrow \mathtt{diam}(N) \le D\, 2^{-\mathtt{lev}(N)/2} = \sqrt{2}\, D\, 2^{-\mathtt{lev}(T)/2}$

  $$\Longrightarrow \quad \mathbb{d}(T, T') \le \sqrt{2}\, D \sum_{j=\mathtt{lev}(T')}^{\mathtt{lev}(T)} 2^{-j/2}$$ ■

Recursive refinement 6/6

Proposition (Control of generated elements)

- $T \in \mathcal{T} \in \mathtt{refine}(\mathcal{T}_0)$ with $\mathcal{T}_0$ admissible
- $T' \in \mathtt{refine}(\mathcal{T}, \{T\}) \setminus \mathcal{T}$

$\Longrightarrow \quad \mathtt{lev}(T') \le \mathtt{lev}(T) + 1$ and $\mathbb{d}(T, T') \le B(\mathcal{T}_0)\, 2^{-\mathtt{lev}(T')/2}$

Proof.

- $$\mathbb{d}(T, T') \le \sqrt{2}\, D \sum_{j=\mathtt{lev}(T')}^{\mathtt{lev}(T)} 2^{-j/2} = \sum_{k=0}^{\mathtt{lev}(T)-\mathtt{lev}(T')} 2^{-(\mathtt{lev}(T')+k)/2}$$
- $$\sum_{k=0}^{\infty} 2^{-(\mathtt{lev}(T')+k)/2} \le 2^{-\mathtt{lev}(T')/2}\, \frac{1}{1 - 2^{-1/2}}$$

  $$\Longrightarrow \quad B = \frac{\sqrt{2}\, D}{1 - 2^{-1/2}} \quad \text{and} \quad B = B(\mathcal{T}_0).$$ ■

Closure estimate

Theorem (Binev, Dahmen, DeVore '04, Stevenson '08)

- $\mathcal{T}_0$ admissible
- $\mathcal{T}_j = \texttt{refine}(\mathcal{T}_{j-1}, \mathcal{M}_{j-1})$ with $\mathcal{M}_{j-1} \subseteq \mathcal{T}_{j-1}$ for all $j = 1, 2, 3, \dots$

$$\Longrightarrow \quad \#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j$$

- use these assumptions in the following!

Sketch of Proof

Find $\lambda(T, M) \ge 0$ such that

- $M \in \mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j \quad \Longrightarrow \quad \sum_{T \in \mathcal{T}_\ell} \lambda(T, M) \le C(\mathcal{T}_0)$
- $T \in \mathcal{T}_\ell \backslash \mathcal{T}_0 \quad \Longrightarrow \quad \sum_{M \in \mathcal{M}} \lambda(T, M) \ge 1$

This implies closure estimate.

- $\#\mathcal{T}_\ell - \#\mathcal{T}_0 \le \#(\mathcal{T}_\ell \backslash \mathcal{T}_0) \le \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_0} \sum_{M \in \mathcal{M}} \lambda(T, M)$ (lower bound)
- $\sum_{M \in \mathcal{M}} \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_0} \lambda(T, M) \le C(\mathcal{T}_0)\, \#\mathcal{M}$ (upper bound)

$$\Longrightarrow \#\mathcal{T}_\ell - \#\mathcal{T}_0 \le C(\mathcal{T}_0)\, \#\mathcal{M} = C(\mathcal{T}_0) \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \qquad \blacksquare$$

Proof of closure estimate 1/8

- $A = A(\mathcal{T}_0) := (D + B) \sum_{p=0}^\infty (p+2)^3\, 2^{-p/2} < \infty$

Lemma

- $M \in \mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j$
- $0 \le k \le \texttt{lev}(M) + 1$
- $\mathcal{U}_k(M) := \{T \in \mathcal{T}_\ell \,:\, \texttt{lev}(T) = k \text{ and } \mathrm{d}(T, M) \le A\, 2^{-k/2}\}$

$\Longrightarrow \quad \#\mathcal{U}_k(M) \le \widetilde{A}(\mathcal{T}_0) < \infty$

Proof.

- $\texttt{diam}(M) \le D\, 2^{-\texttt{lev}(M)/2} \le D\, 2^{-(k-1)/2} \le \sqrt{2}\, D\, 2^{-k/2}$
- $x \in T \in \mathcal{U}_k(M),\ y \in M$

  $\Longrightarrow \quad |x - y| \le \texttt{diam}(T) + \mathrm{d}(T, M) + \texttt{diam}(M) \lesssim 2^{-k/2}$

  $\Longrightarrow \quad \left|\bigcup \mathcal{U}_k(M)\right| \lesssim 2^{-k}$
- $|T| \simeq 2^{-k}$ for $T \in \mathcal{U}_k(M) \quad \Longrightarrow \quad \#\mathcal{U}_k(M) \lesssim 1.$ $\blacksquare$

Proof of closure estimate 2/8

- for $T \in \mathcal{T}$ and $M \in \mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j$ define

$$\lambda(T, M) := \begin{cases} (\texttt{lev}(M) - \texttt{lev}(T) + 2)^{-2} & \text{if } \mathrm{d}(T, M) \le A\, 2^{-\texttt{lev}(T)/2} \\ & \text{and } \texttt{lev}(T) \le \texttt{lev}(M) + 1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma (upper bound)

- $M \in \mathcal{M} \quad \Longrightarrow \quad \sum_{T \in \mathcal{T}_\ell} \lambda(T, M) \le \frac{\pi^2}{6} \widetilde{A}(\mathcal{T}_0)$

Proof.

- recall: $\mathcal{U}_k(M) := \{T \in \mathcal{T}_\ell \,:\, \texttt{lev}(T) = k \text{ and } \mathrm{d}(T, M) \le A\, 2^{-k/2}\}$
- $\sum_{T \in \mathcal{T}_\ell} \lambda(T, M) = \sum_{k=0}^{\texttt{lev}(M)+1} \sum_{T \in \mathcal{U}_k(M)} \lambda(T, M)$
- observe: $\lambda(T, M) = (\texttt{lev}(M) - k + 2)^{-2}$.

Proof of closure estimate 3/8

Lemma (upper bound)

- $M \in \mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j \implies \sum_{T \in \mathcal{T}_\ell} \lambda(T, M) \le \frac{\pi^2}{6} \widetilde{A}(\mathcal{T}_0)$

Proof (cont'd).

- $\sum_{T \in \mathcal{T}_\ell} \lambda(T, M) = \sum_{k=0}^{\mathtt{lev}(M)+1} \sum_{T \in \mathcal{U}_k(M)} \lambda(T, M)$
- observe: $\lambda(T, M) = (\mathtt{lev}(M) - k + 2)^{-2}$.

$\implies \sum_{T \in \mathcal{T}_\ell} \lambda(T, M) \le \widetilde{A}(\mathcal{T}_0) \sum_{k=0}^{\mathtt{lev}(M)+1} (\mathtt{lev}(M) - k + 2)^{-2}$

- $\sum_{k=0}^{\mathtt{lev}(M)+1} (\mathtt{lev}(M) - k + 2)^{-2} \le \sum_{p=1}^{\infty} \frac{1}{p^2} = \frac{\pi^2}{6}$ ■

Proof of closure estimate 4/8

Lemma (lower bound)

- $T \in \mathcal{T}_\ell \backslash \mathcal{T}_0 \implies \sum_{M \in \mathcal{M}} \lambda(T, M) \ge 1$, where $\mathcal{M} := \bigcup_{j=0}^{\ell-1} \mathcal{M}_j$

Proof.

- ex. $M_1, \dots, M_n \in \mathcal{M}$ and $\ell_1 \ge \ell_2 \ge \dots \ge \ell_n = 0$ s.t.
  - with $M_0 := T$
  - $M_j \in \mathtt{refine}(\mathcal{T}_{\ell_j}, \{M_{j+1}\}) \setminus \mathcal{T}_{\ell_j}$

  $\implies \mathtt{lev}(M_j) \le \mathtt{lev}(M_{j+1}) + 1$, i.e., level decrease at most 1 and $\mathrm{d}(M_{j+1}, M_j) \le B\, 2^{-\mathtt{lev}(M_j)/2}$
- clearly, $\mathtt{lev}(T) \ge 1$

  $\implies$ ex. minimal index $t \in \{1, \dots, n\}$ s.t. $\mathtt{lev}(M_t) = \mathtt{lev}(T) - 1$

  $\implies$ in particular, $\mathtt{lev}(T) \le \mathtt{lev}(M_j) + 1$ for all $1 \le j \le t$
- next aim: $\mathrm{d}(T, M_j) \le (D + B) \sum_{k=0}^{j-1} 2^{-\mathtt{lev}(M_k)/2}$

Proof of closure estimate 5/8

- $M_0 := T,\ M_1, \dots, M_t \in \mathcal{M}$ chosen s.t.
  - $\mathtt{lev}(T) \le \mathtt{lev}(M_j) + 1$
  - $\mathtt{lev}(M_{j-1}) \le \mathtt{lev}(M_j) + 1$
  - $\mathrm{d}(M_j, M_{j-1}) \le B\, 2^{-\mathtt{lev}(M_{j-1})/2}$
- triangle ineq. $\implies \mathrm{d}(T, M_j) \le \mathrm{d}(T, M_1) + \mathtt{diam}(M_1) + \mathrm{d}(M_1, M_j)$
- inductively $\implies \mathrm{d}(T, M_j) \le \sum_{k=1}^{j-1} \mathtt{diam}(M_k) + \sum_{k=0}^{j-1} \mathrm{d}(M_k, M_{k+1})$

  $\implies \mathrm{d}(T, M_j) \le (D + B) \sum_{k=0}^{j-1} 2^{-\mathtt{lev}(M_k)/2}$
- define $m(p, j) := \#\{k = 1, \dots, j-1 \,:\, \mathtt{lev}(M_k) = \mathtt{lev}(T) + p\}$

  $\implies \mathrm{d}(T, M_j) \le (D + B) \sum_{p=0}^{\infty} m(p, j)\, 2^{-p/2}\, 2^{-\mathtt{lev}(T)/2}$

Proof of closure estimate 6/8

- shown: $\mathrm{d}(T, M_j) \le (D + B) \sum_{p=0}^{\infty} m(p, j)\, 2^{-p/2}\, 2^{-\mathtt{lev}(T)/2}$ for $j = 1, \dots, t$
- **Case 1.** $m(p, t) \le (p + 2)^3$ for all $p = 0, 1, 2, \dots$

  $\implies \mathrm{d}(T, M_t) \le (D + B) \sum_{p=0}^{\infty} (p + 2)^3\, 2^{-p/2}\, 2^{-\mathtt{lev}(T)/2} =: A\, 2^{-\mathtt{lev}(T)/2}$
- recall: $\mathtt{lev}(M_t) = \mathtt{lev}(T) - 1$
- $\lambda(T, M) := \begin{cases} (\mathtt{lev}(M) - \mathtt{lev}(T) + 2)^{-2} & \text{if } \mathrm{d}(T, M) \le A\, 2^{-\mathtt{lev}(T)/2} \text{ and } \mathtt{lev}(T) \le \mathtt{lev}(M) + 1 \\ 0 & \text{otherwise} \end{cases}$

  $\implies \lambda(T, M_t) = 1$

  $\implies \sum_{M \in \mathcal{M}} \lambda(T, M) \ge \lambda(T, M_t) = 1$ ✓

Proof of closure estimate 7/8

- recall: $m(p,j) = \#\{k = 1,\dots,j-1 \,:\, \mathtt{lev}(M_k) = \mathtt{lev}(T)+p\}$
- **Case 2.** $m(p,t) > (p+2)^3$ for some $p = 0,1,2,\dots$
- for all those p, choose minimal $t_p = 2,\dots,t$ s.t. $m(p,t_p) > (p+2)^3$
  - note: $m(p,1) = 0 < (p+2)^3$
- choose $p_\star$ with minimal $t_\star := t_{p_\star}$

  $\Longrightarrow \quad m(p,k) \le (p+2)^3$ for all p and all $k = 1,\dots,t_\star - 1$
- as in Case 1 $\Longrightarrow$ $\mathrm{d}(T,M_k) \le A 2^{-\mathtt{lev}(T)/2}$
- recall: $\mathtt{lev}(T) \le \mathtt{lev}(M_k)+1$

  $\Longrightarrow \quad \lambda(T,M_k) = (\mathtt{lev}(M_k) - \mathtt{lev}(T)+2)^{-2}$

Proof of closure estimate 8/8

- $M_1,\dots,M_{t_\star-1} \in \mathcal{M}$ chosen s.t.
  - $\lambda(T,M_k) = (\mathtt{lev}(M_k) - \mathtt{lev}(T)+2)^{-2}$
  - $t_\star = 2,\dots,t$ minimal s.t. $m(p_\star,t_\star) > (p_\star+2)^3$
- recall: $m(p,j) = \#\{k = 1,\dots,j-1 \,:\, \mathtt{lev}(M_k) = \mathtt{lev}(T)+p\}$

  $\Longrightarrow \quad m(p_\star,t_\star) - 1 \le m(p_\star,t_\star-1) \le (p_\star+2)^3 < m(p_\star,t_\star)$
- inequality in $\mathbb{N}$ $\Longrightarrow$ $m(p_\star,t_\star-1) = (p_\star+2)^3$

  $$\Longrightarrow \sum_{M\in\mathcal{M}} \lambda(T,M) \ge \sum_{\substack{k=0\\ \mathtt{lev}(M_k)=\mathtt{lev}(T)+p_\star}}^{t_\star-1} \lambda(T,M_k) = \frac{m(p_\star,t_\star-1)}{(p_\star+2)^2}$$

  $$\Longrightarrow \sum_{M\in\mathcal{M}} \lambda(T,M) \ge (p_\star+2) \ge 2 \ \checkmark$$
- in both cases $\Longrightarrow$ $\sum_{M\in\mathcal{M}} \lambda(T,M) \ge 1$ ∎

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Thanks for listening!

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NPDE – Topical Lecture Series

January 08, 2019

Rate-Optimal Adaptive Finite Element Method

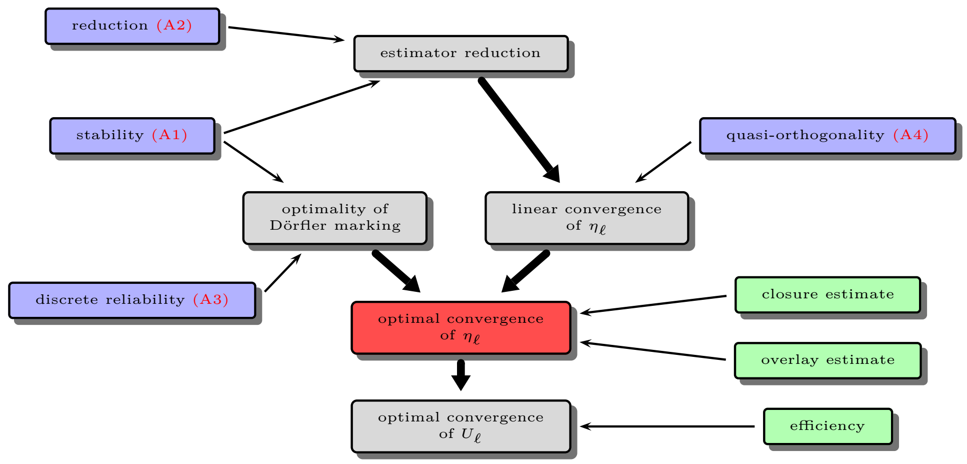
Part 2: Axioms of Adaptivity

Dirk Praetorius

TU Wien
Institute for Analysis and Scientific Computing

FWF
Der Wissenschaftsfonds.

Axioms of Adaptivity



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Outline

http://www.asc.tuwien.ac.at/~praetorius/afem-lecture/

Axioms of adaptivity

Adaptive Algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \le 1$

For all $\ell = 0, 1, 2, 3, \dots$ iterate

1. SOLVE: compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. MARK: find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

4. REFINE: refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

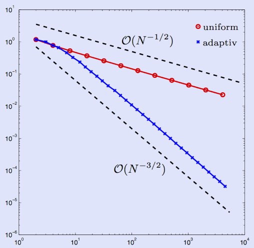
Dörfler: SINUM 33 (1996)

Main Theorem on Adaptive Algorithms

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- properties of mesh-refinement & estimator axioms (A1)–(A4)
- $0 < \theta \le 1$

$\Longrightarrow \quad \exists C > 0 \; \exists 0 < q < 1 \; \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$



- $\mathbb{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} - \#\mathcal{T}_0 \le N\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N \ge 0} \big((N+1)^s \min_{\mathcal{T}_{\text{opt}} \in \mathbb{T}_N} \eta_{\text{opt}}\big) =: \|\eta\|_{\mathbb{A}_s}$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

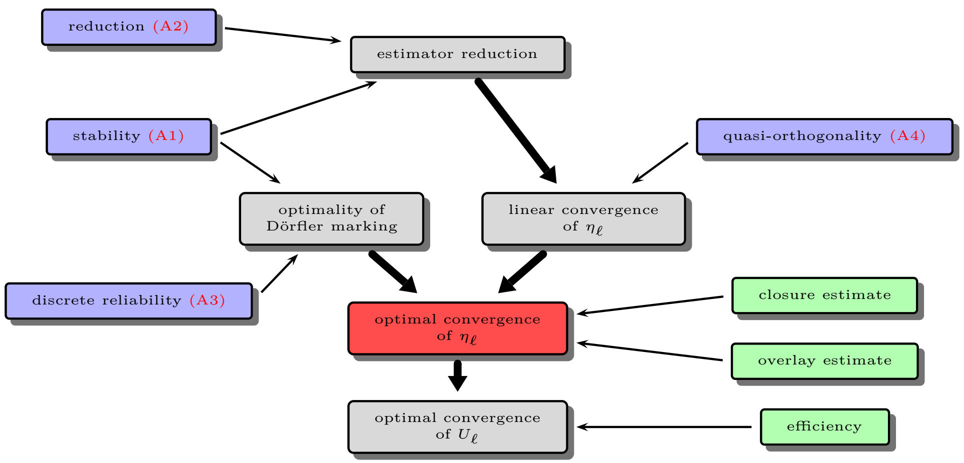
Elementary Exercise

- last lecture:
  - $\mathbb{T}'_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
  - $\sup_{N > 0} \big(N^s \min_{\mathcal{T}_{\text{opt}} \in \mathbb{T}'_N} \eta_{\text{opt}}\big) =: \|\eta\|_{\mathbb{A}'_s}$
- this lecture:
  - $\mathbb{T}_N := \{\mathcal{T} \in \texttt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} - \#\mathcal{T}_0 \le N\}$
  - $\sup_{N \ge 0} \big((N+1)^s \min_{\mathcal{T}_{\text{opt}} \in \mathbb{T}_N} \eta_{\text{opt}}\big) =: \|\eta\|_{\mathbb{A}_s}$

Exercise: prove that

- $\sup_{N > 0} \big(N^s \min_{\mathcal{T}_{\text{opt}} \in \mathbb{T}'_N} \eta_{\text{opt}}\big) \simeq \sup_{N \ge 0} \big((N+1)^s \min_{\mathcal{T}_{\text{opt}} \in \mathbb{T}_N} \eta_{\text{opt}}\big)$

Axioms of Adaptivity



Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Framework

- $\mathcal{H}$ Hilbert space with norm $\|\!|\cdot|\!\|$
- $\mathcal{T}_\star$ corresponds to discrete space $\mathcal{X}_\star$
- $u \in \mathcal{H}$ sought solution
- $U_\star \in \mathcal{X}_\star$ computed discrete approximation
- $\eta_\star(T)$ refinement indicator for $T \in \mathcal{T}_\star$

Sometimes: Lax–Milgram setting

- $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle$ elliptic bilinear form on $\mathcal{H}$
- $F \in \mathcal{H}^*$ functional
- $u \in \mathcal{H}$ solves $\langle\!\langle u\,,v \rangle\!\rangle = F(v)$ for all $v \in \mathcal{H}$
- $U_\star \in \mathcal{X}_\star$ solves $\langle\!\langle U_\star\,,V_\star \rangle\!\rangle = F(v)$ for all $V_\star \in \mathcal{X}_\star$

The Axioms

$\forall \mathcal{T}_+ \quad \forall \mathcal{T}_\star \in \mathtt{refine}(\mathcal{T}_+)$

(A1) $$\Big| \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_\star(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_+ \cap \mathcal{T}_\star} \eta_+(T)^2 \Big)^{1/2} \Big| \le C_{\rm stab}\, \|\!|U_\star - U_+|\!\|$$

(A2) $$\sum_{T \in \mathcal{T}_\star \backslash \mathcal{T}_+} \eta_\star(T)^2 \le q_{\rm red} \sum_{T \in \mathcal{T}_+ \backslash \mathcal{T}_\star} \eta_+(T)^2 + C_{\rm red}\, \|\!|U_\star - U_+|\!\|^2$$

(A3) $$\|\!|U_\star - U_+|\!\|^2 \le C_{\rm rel}^2 \sum_{T \in \mathcal{R}_+} \eta_+(T)^2$$

where $\mathcal{T}_+ \backslash \mathcal{T}_\star \subseteq \mathcal{R}_+ \subseteq \mathcal{T}_+, \quad \#\mathcal{R}_+ \le C_{\rm rel}\, \#(\mathcal{T}_+ \backslash \mathcal{T}_\star)$

$\forall \ell, N \ge 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(\|\!|U_{k+1} - U_k|\!\|^2 - \varepsilon \eta_k^2 \big) \le C_{\rm orth}(\varepsilon)\, \eta_\ell^2$$

Poisson Model Problem

Strong formulation

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega$$

Weak formulation

- find $u \in H^1_0(\Omega)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \text{for all } v \in H^1_0(\Omega)$$

Poisson Model Problem

Strong formulation

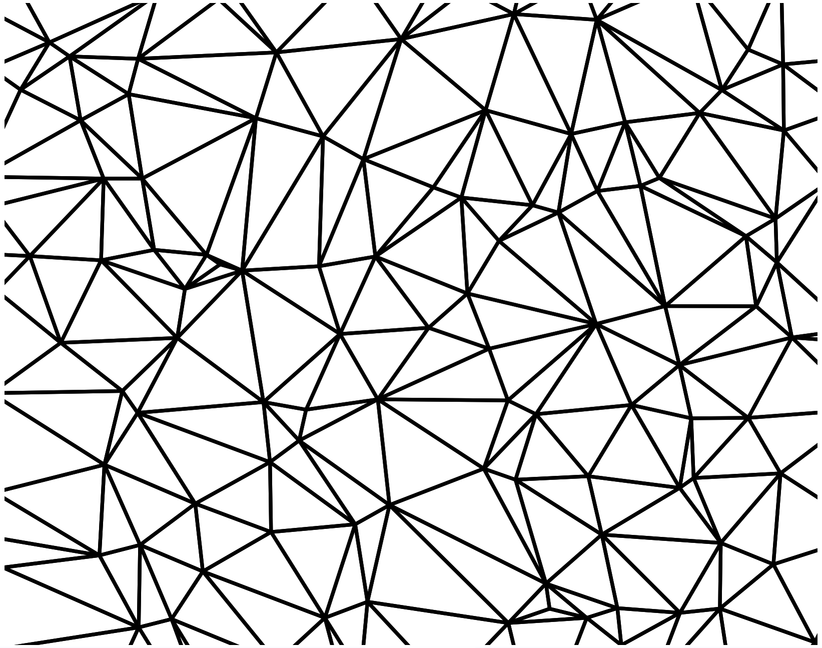
$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega$$

FEM formulation

- find $U_\ell \in \mathcal{S}^1_0(\mathcal{T}_\ell)$ s.t.

$$\int_\Omega \nabla U_\ell \cdot \nabla V_\ell = \int_\Omega f V_\ell \quad \text{for all } V_\ell \in \mathcal{S}^1_0(\mathcal{T}_\ell)$$

Regular Triangulation $\mathcal{T}_\ell$

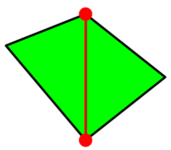


Residual Error Estimator for Poisson Model Problem

Reliability and efficiency

- $\|u - U_\star\| \lesssim \eta_\star \lesssim \|u - U_\star\| + \mathrm{osc}_\star \simeq \min_{V_\star \in \mathcal{S}^1_0(\mathcal{T}_\star)} \|u - V_\star\| + \mathrm{osc}_\star$

- $\|\cdot\| := \|\nabla(\cdot)\|_{L^2(\Omega)}$
- $\eta_\star := \Big(\sum_{T\in\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2}$
- $\eta_\star(T)^2 := h_T^2\,\|f\|^2_{L^2(T)} + h_T\,\|[\partial_n U_\star]\|^2_{L^2(\partial T\cap\Omega)}$
- $\mathrm{osc}_\star := \Big(\sum_{T\in\mathcal{T}_\star} h_T^2\,\|f - f_T\|^2_{L^2(T)}\Big)^{1/2}$



Axiom (A1): Stability on Non-Refined Elements

(A1) Stability on non-refined elements $\forall \mathcal{T}_+ \quad \forall \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

$$\Big|\Big(\sum_{T\in\mathcal{T}_+\cap\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_+\cap\mathcal{T}_\star} \eta_+(T)^2\Big)^{1/2}\Big| \le C_{\mathrm{stab}}\,\|U_\star - U_+\|$$

- verification for Poisson model problem:
- $\eta_\star(T)^2 = h_T^2\,\|f\|^2_{L^2(T)} + h_T\,\|[\partial_n U_\star]\|^2_{L^2(\partial T\cap\Omega)}$
- inverse triangle inequality + scaling arguments

$$\mathrm{LHS} \le \Big(\sum_{T\in\mathcal{T}_+\cap\mathcal{T}_\star} h_T\,\|[\partial_n(U_\star - U_+)]\|^2_{L^2(\partial T\cap\Omega)}\Big)^{1/2}$$
$$\lesssim \|\nabla(U_\star - U_+)\|_{L^2(\Omega)}$$

- reminder: $\big|\|x\| - \|y\|\big| \le \|x - y\|$.

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A2): Reduction on Refined Elements

(A2) Reduction on refined elements $\forall \mathcal{T}_+ \quad \forall \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

$$\sum_{T\in\mathcal{T}_\star\backslash\mathcal{T}_+} \eta_\star(T)^2 \le q_{\mathrm{red}} \sum_{T\in\mathcal{T}_+\backslash\mathcal{T}_\star} \eta_+(T)^2 + C_{\mathrm{red}}\,\|U_\star - U_+\|^2$$

- verification for Poisson model problem:
- $\eta_\star(T)^2 = h_T^2\,\|f\|^2_{L^2(T)} + h_T\,\|[\partial_n U_\star]\|^2_{L^2(\partial T\cap\Omega)}$
- $\bigcup(\mathcal{T}_\star\backslash\mathcal{T}_+) = \bigcup(\mathcal{T}_+\backslash\mathcal{T}_\star)$
- $h_{T'} \le 2^{-1/2}\,h_T$ for $\mathcal{T}_\star \ni T' \subsetneqq T \in \mathcal{T}_+$ with $h_T := |T|^{1/2} \simeq \mathrm{diam}(T)$
- triangle inequality + scaling arguments

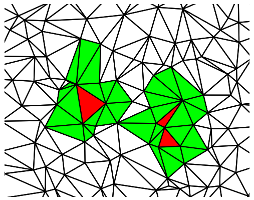
Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A3): Discrete Reliability

(A3) Discrete reliability $\forall \mathcal{T}_+ \quad \forall \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_+)$

- exists $\mathcal{R}_+ \subseteq \mathcal{T}_+$ with
  - $\mathcal{T}_+ \backslash \mathcal{T}_\star \subseteq \mathcal{R}_+$
  - $\#\mathcal{R}_+ \leq C_{\rm rel} \#(\mathcal{T}_+ \backslash \mathcal{T}_\star)$
  - $\|U_\star - U_+\|^2 \leq C_{\rm rel}^2 \sum_{T \in \mathcal{R}_+} \eta_+(T)^2$

- $\mathcal{R}_+ = \mathcal{T}_+ \backslash \mathcal{T}_\star$ for FEM
- $\mathcal{R}_+ = \mathrm{patch}(\mathcal{T}_+ \backslash \mathcal{T}_\star)$ for BEM / FVM
- discrete reliability $\implies$ reliability



Stevenson: Found. Comput. Math. 7 (2007)

Discrete Reliability $\implies$ Reliability

- $\eta_\ell := \Big(\sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \Big)^{1/2}$
- triangle inequality for $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ and (A3)

$$\|u - U_\ell\| \leq \|u - U_\star\| + \|U_\star - U_\ell\| \leq \|u - U_\star\| + C_{\rm rel}\, \eta_\ell$$

- approximation property

$$\forall \varepsilon > 0 \;\; \exists \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell) \quad \|u - U_\star\| \leq \varepsilon$$

$\implies \quad \|u - U_\ell\| \leq C_{\rm rel}\, \eta_\ell$

Approximation Property

- suppose: Céa lemma $\|u - U_\star\| \simeq \min_{V_\star \in \mathcal{X}_\star} \|u - V_\star\|$
- usual setting: exists $\mathcal{D} \subseteq \mathcal{H}$ dense with

$$\forall\, \delta > 0\, \exists\, \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)\, \forall\, v \in \mathcal{D} \quad \min_{V_\star \in \mathcal{X}_\star} \|v - V_\star\| \leq \delta$$

- e.g.: $\mathcal{D} = H^2(\Omega) \subseteq H^1(\Omega)$

$$\min_{V_\star \in \mathcal{X}_\star} \|v - V_\star\|_{H^1(\Omega)} \leq \|(1 - \mathcal{I}_\star)v\|_{H^1(\Omega)} \lesssim \|h_\star D^2 v\|_{L^2(\Omega)} \leq \|h_\star\|_{L^\infty(\Omega)} \|D^2 v\|_{L^2(\Omega)}$$

$\implies \quad \|u - U_\star\| \simeq \min_{V_\star \in \mathcal{X}_\star} \|u - V_\star\| \leq \|u - v\| + \min_{V_\star \in \mathcal{X}_\star} \|v - V_\star\|$

$\implies$ approximation property $\forall\, \varepsilon > 0\, \exists\, \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell) \quad \|u - U_\star\| \leq \varepsilon$

Axiom (A4): Quasi-Orthogonality

(A4) Quasi-orthogonality $\forall \varepsilon > 0 \quad \forall \ell, N \in \mathbb{N}_0$

$$\sum_{k=\ell}^{N} \big(\|U_{k+1} - U_k\|^2 - \varepsilon \eta_k^2 \big) \leq C_{\rm orth}(\varepsilon)\, \eta_\ell^2$$

- verification for Poisson model problem
- Galerkin orthogonality + symmetry $\implies$ Pythagoras theorem

$$\|u - U_{k+1}\|^2 + \|U_{k+1} - U_k\|^2 = \|u - U_k\|^2$$

- telescoping series $\implies$ quasi-orth. with $C_{\rm orth}(\varepsilon) = C_{\rm rel}^2$, $\varepsilon = 0$

$$\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 = \sum_{k=\ell}^{N} \big(\|u - U_k\|^2 - \|u - U_{k+1}\|^2 \big) \leq \|u - U_\ell\|^2$$

Feischl, Führer, Praetorius: SINUM 52 (2014)

Validity of Axioms

Theorem (Feischl, Führer, P. '14; Bespalov, Haberl, P. '17)

- $\boldsymbol{A} \in W^{1,\infty}$ symmetric, $\boldsymbol{b} \in L^\infty$, $c \in L^\infty$ piecewise on $\mathcal{T}_0$
- $\mathcal{L}u := -\nabla \cdot \boldsymbol{A}\nabla u + \boldsymbol{b} \cdot \nabla u + cu$
- induced bilinear form $a(\cdot,\cdot)$ is elliptic (plus compact perturbation)
- nestedness $\mathcal{S}_0^p(\mathcal{T}_\ell) \subset \mathcal{S}_0^p(\mathcal{T}_{\ell+1})$

$\Longrightarrow$ weighted-residual error estimator satisfies (A1)–(A4)

Sketch of quasi-orthogonality (A3):

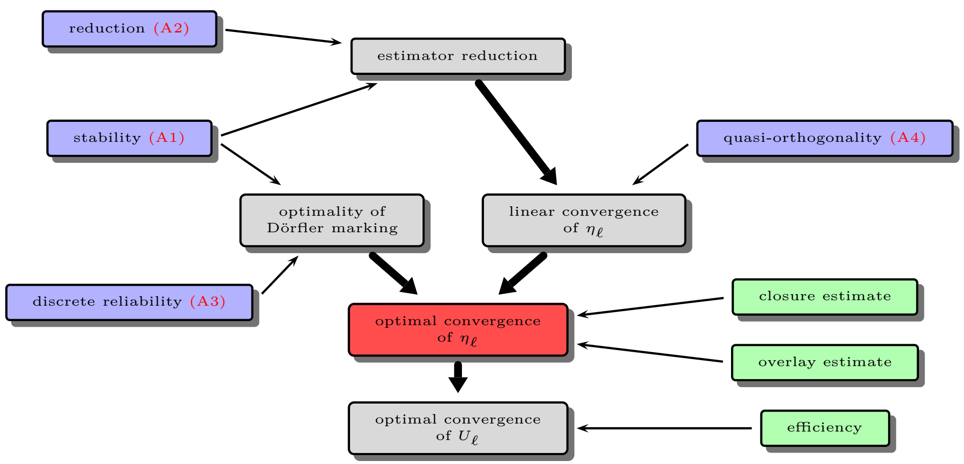
- $\mathcal{L}$ is compact perturbation of symmetric + elliptic operator
- $\|U_\infty - U_\ell\|_{H^1(\Omega)} \to 0$ for some $U_\infty \in H^1_0(\Omega)$
- moreover, $(U_\infty - U_\ell)/\|U_\infty - U_\ell\|_{H^1(\Omega)} \rightharpoonup 0$ weakly in $H^1(\Omega)$

Feischl, Führer Praetorius: SINUM 52 (2014)

Bespalov, Haberl, Praetorius: CMAME 317 (2017)

Estimator Reduction & Convergence

Axioms of Adaptivity



Estimator Reduction

Stability (A1) + reduction (A2) $\Longrightarrow$ estimator reduction

- $\forall 0 < \theta \le 1 \ \exists 0 < q_{\rm est} < 1 \ \exists C_{\rm est} > 0 \ \forall \ell \in \mathbb{N}_0$:

$$\eta_{\ell+1}^2 \le q_{\rm est}\, \eta_\ell^2 + C_{\rm est}\, |||U_{\ell+1} - U_\ell|||^2$$

- sketch: Young inequality + (A1) + (A2)
  - variable parameter $\delta > 0$ sufficiently small
- $q_{\rm est} = (1+\delta) - \theta(1+\delta - q_{\rm red}) \approx 1 - \theta(1 - q_{\rm red})$
- $C_{\rm est} = C_{\rm stab}^2(1+\delta^{-1}) + C_{\rm red}$

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Proof of Estimator Reduction 1/3

1. (A1) + (A2) + Young ineq. $(a+b)^2 \le (1+\delta)a^2 + (1+\delta^{-1})b^2$

$$\eta_{\ell+1}^2 = \sum_{T\in\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell} \eta_{\ell+1}(T)^2 + \sum_{T\in\mathcal{T}_{\ell+1}\setminus\mathcal{T}_\ell} \eta_{\ell+1}(T)^2$$

$$\le (1+\delta) \sum_{T\in\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell} \eta_\ell(T)^2 + q_{\rm red} \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + (C_{\rm stab}^2(1+\delta^{-1}) + C_{\rm red})\,|||U_{\ell+1}-U_\ell|||^2$$

$$= (1+\delta)\,\eta_\ell^2 + (q_{\rm red}-(1+\delta)) \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + \underbrace{(C_{\rm stab}^2(1+\delta^{-1}) + C_{\rm red})}_{=C_{\rm est}}\,|||U_{\ell+1}-U_\ell|||^2$$

Proof of Estimator Reduction 2/3

1. (A1) + (A2) + Young inequality

$$\eta_{\ell+1}^2 \le (1+\delta)\,\eta_\ell^2 + (q_{\rm red}-(1+\delta)) \sum_{T\in\mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}} \eta_\ell(T)^2 + C_{\rm est}\,|||U_{\ell+1}-U_\ell|||^2$$

2. $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell\setminus\mathcal{T}_{\ell+1}$

$$\eta_{\ell+1}^2 \le (1+\delta)\,\eta_\ell^2 - (1+\delta-q_{\rm red}) \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2 + C_{\rm est}\,|||U_{\ell+1}-U_\ell|||^2$$

3. Dörfler marking

$$\le \big((1+\delta) - \theta(1+\delta-q_{\rm red})\big)\,\eta_\ell^2 + C_{\rm est}\,|||U_{\ell+1}-U_\ell|||^2$$

Proof of Estimator Reduction 3/3

- proved: for all $\delta > 0$

$$\eta_{\ell+1}^2 \le \big((1+\delta) - \theta(1+\delta-q_{\rm red})\big)\,\eta_\ell^2 + C_{\rm est}\,|||U_{\ell+1}-U_\ell|||^2$$

- $q_{\rm est} = (1+\delta) - \theta(1+\delta-q_{\rm red}) \xrightarrow{\delta\to 0} 1-\theta(1-q_{\rm red}) < 1$
- $C_{\rm est} = C_{\rm stab}^2(1+\delta^{-1}) + C_{\rm red}$
- small $\delta > 0$ $\Longrightarrow$ estimator reduction estimate

$$\eta_{\ell+1}^2 \le q_{\rm est}\,\eta_\ell^2 + C_{\rm est}\,|||U_{\ell+1}-U_\ell|||^2$$

Estimator Reduction $\Longrightarrow$ Convergence 1/2

- $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1} \subseteq \mathcal{H}$ sequence of nested discrete spaces
- suppose: Lax–Milgram setting

Convergence Theorem (Babuška, Vogelius '84)

- ex. limit $U_\infty \in \mathcal{H}$ such that $|||U_\infty - U_\ell||| \to 0$ as $\ell\to\infty$
- in particular, $|||U_{\ell+1}-U_\ell||| \to 0$

- $\mathcal{X}_\infty := \mathrm{closure}(\bigcup_{\ell=0}^\infty \mathcal{X}_\ell) \subseteq \mathcal{H}$ with Galerkin solution $U_\infty \in \mathcal{X}_\infty$

$\Longrightarrow$ $U_\ell \in \mathcal{X}_\ell \subseteq \mathcal{X}_\infty$ is Galerkin approx. of U_∞

- For all $\varepsilon > 0$, exists $k\in\mathbb{N}_0$ and $V_k \in \mathcal{X}_k$ s.t. $|||U_\infty - V_k||| \le \varepsilon$

$\Longrightarrow$ $|||U_\infty - U_\ell||| \simeq \min_{V_\ell\in\mathcal{X}_\ell} |||U_\infty - V_\ell||| \le |||U_\infty - V_k||| \le \varepsilon$ for $\ell \ge k$

Babuska, Vogelius: Numer. Math. 44 (1984)

Estimator Reduction $\Longrightarrow$ Convergence 2/2

Let $0 < \kappa < 1$ and $\rho_\ell, \alpha_\ell \geq 0$ with

- $\rho_{\ell+1} \leq \kappa\, \rho_\ell + \alpha_\ell$
- $\alpha_\ell \xrightarrow{\ell\to\infty} 0$

Then, $\rho_\ell \xrightarrow{\ell\to\infty} 0$

- $\rho_{\ell+1} \leq \kappa^{\ell+1}\rho_0 + \sum_{j=0}^{\ell} \kappa^{\ell-j}\alpha_j \leq \kappa^{\ell+1}\rho_0 + \|(\alpha_n)\|_\infty \sum_{k=0}^{\ell} \kappa^k$
- (ρ_n) bounded $\Longrightarrow$ $M := \limsup_\ell \rho_\ell \geq \liminf_\ell \rho_\ell \geq 0$ exist
- next, $M = \limsup_\ell \rho_{\ell+1} \leq \kappa \limsup_\ell \rho_\ell + \limsup_\ell \alpha_\ell = \kappa M$
- thus, $M = 0$
- finally, $0 \leq \liminf_\ell \rho_\ell \leq \limsup_\ell \rho_\ell = 0 \Longrightarrow \lim_\ell \rho_\ell = 0$

Adaptive algorithm $\Longrightarrow$ convergence

- $\|u - U_\ell\| \lesssim \eta_\ell \to 0$ as $\ell \to \infty$, since $\eta_{\ell+1}^2 \leq q_{\rm est}\eta_\ell^2 + C_{\rm est}|\!|\!|U_{\ell+1} - U_\ell|\!|\!|^2$

Remarks on Estimator Convergence

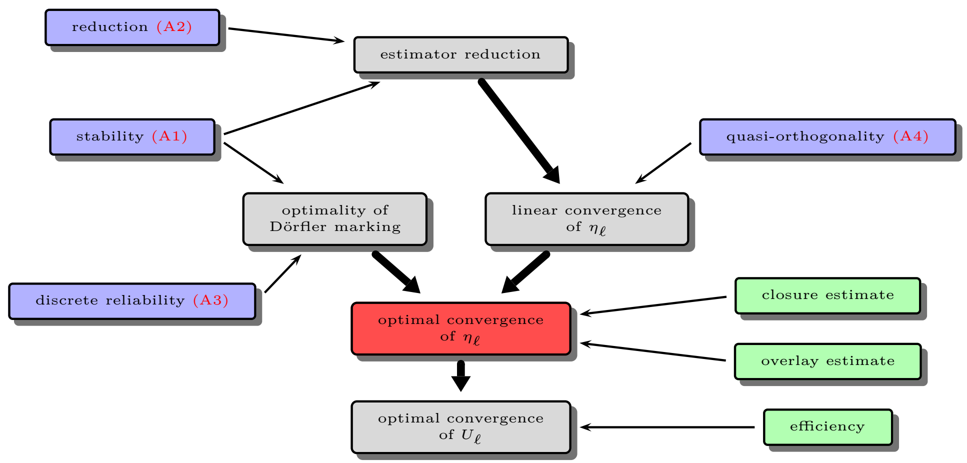
Estimator convergence $\eta_\ell \xrightarrow{\ell\to\infty} 0$

- conceptually, algorithm only sees estimator, *not* error
- conceptually, estimator convergence $\eta_\ell \to 0$ is necessary
- consequently, rather ask for $\eta_\ell \to 0$ than for $\|u - U_\ell\| \to 0$

- basically: general marking strategies possible (instead of Dörfler)
- here: simple argument from own work (tailored to Dörfler)
- note: mesh-refinement strategy (NVB) did not enter yet

Morin, Siebert, Veeser: M3AS 18 (2008)
Siebert: IMA J. Numer. Anal. 31 (2011)
Aurada, Ferraz-Leite, Praetorius: Appl. Numer. Math. 62 (2012)

Axioms of Adaptivity



Linear Convergence

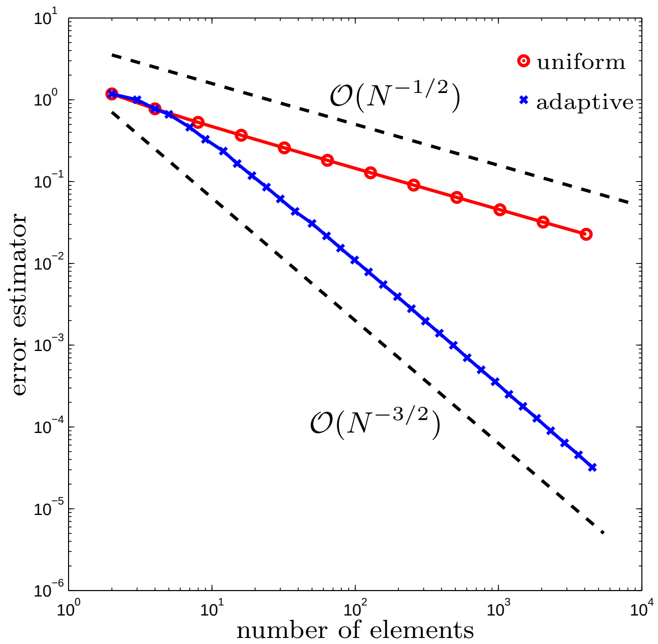
Slow Convergence?

Current state

1. (A1) + (A2) $\Longrightarrow$ $\eta_{\ell+1}^2 \le q_{\rm est}\eta_\ell^2 + C_{\rm est}\,\|U_{\ell+1} - U_\ell\|^2$
2. Céa lemma $U_\ell \to U_\infty$ as $\ell \to \infty$
3. estimator reduction principle $\Longrightarrow$ $\eta_\ell \to 0$
4. reliability $\Longrightarrow$ $\|u - U_\ell\| \lesssim \eta_\ell \to 0$
   - in particular, $u = U_\infty$

- question: Can convergence of $\eta_\ell \to 0$ be slow?
  - $\|U_{\ell+1} - U_\ell\|^2 \to 0$ could be slow?
- goal: $\eta_{\ell+n}^2 \lesssim q^n\,\eta_\ell^2$

Slow Convergence? No!



Linear Convergence $\Longrightarrow$ Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- reliability $\|u - U_\ell\| \lesssim \eta_\ell$
- linear convergence $\eta_{\ell+n} \le C_{\rm lin}\,q_{\rm lin}^n\,\eta_\ell$

$\Longrightarrow$ quasi-orthogonality (A3) with $\varepsilon = 0$, $C_{\rm orth}(\varepsilon) = C_{\rm orth}(0) > 0$

- i.e.: $\displaystyle\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 \lesssim \eta_\ell^2$ for all ℓ, N

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Proof: Linear Convergence $\Longrightarrow$ Quasi-Orthogonality

- triangle inequality + reliability

$$\sum_{k=\ell}^{N} \|U_{k+1} - U_k\|^2 \le 2 \sum_{k=\ell}^{N} \big(\|u - U_{k+1}\|^2 + \|u - U_k\|^2\big) \le 4 \sum_{k=\ell}^{N+1} \|u - U_k\|^2 \lesssim \sum_{k=\ell}^{N+1} \eta_k^2$$

- linear convergence + geometric series

$$\sum_{k=\ell}^{N+1} \eta_k^2 \lesssim \Big(\sum_{k=\ell}^{N+1} q^{k-\ell}\Big)\,\eta_\ell^2 \lesssim \eta_\ell^2$$

Linear Convergence $\Longleftarrow$ General Quasi-Orthogonality

Proposition (Carstensen, Feischl, Page, P. 14)

- reliability $\|u - U_\ell\| \lesssim \eta_\ell$
- estimator reduction for $0 < \theta \le 1$, e.g., stab. (A1) + red. (A2)
- quasi-orthogonality (A3)

$\Longrightarrow$ linear convergence $\eta_{\ell+n} \le C_{\rm lin}\, q_{\rm lin}^n\, \eta_\ell$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Proof: Linear Convergence $\Longleftarrow$ Quasi-Orthogonality 1/2

1. estimator reduction and $0 < \delta \ll 1$

$$\sum_{k=\ell+1}^{N} \eta_k^2 \le \sum_{k=\ell+1}^{N} \Big[q\,\eta_{k-1}^2 + C\,\|U_k - U_{k-1}\|^2\Big]$$

$$= \sum_{k=\ell+1}^{N} (q+\delta)\eta_{k-1}^2 + C \sum_{k=\ell+1}^{N} \Big[\|U_k - U_{k-1}\|^2 - C^{-1}\delta\,\eta_{k-1}^2\Big]$$

2. reliability

$$\|U_k - U_{k+1}\|^2 - C^{-1}\delta\eta_{k-1}^2 \le \|U_k - U_{k+1}\|^2 - C_{\rm rel}^{-2}C^{-1}\delta\,\|u - U_{k-1}\|^2$$

3. quasi-orthogonality with $\varepsilon = C_{\rm rel}^{-2}C^{-1}\delta$

$$\sum_{k=\ell+1}^{N} \Big[\|U_k - U_{k-1}\|^2 - \varepsilon\,\|u - U_{k-1}\|^2\Big] \lesssim \eta_\ell^2$$

Proof: Linear Convergence $\Longleftarrow$ Quasi-Orthogonality 2/2

4. obtain: $\displaystyle\sum_{k=\ell+1}^{N} \eta_k^2 \le \sum_{k=\ell+1}^{N} (q+\delta)\eta_{k-1}^2 + \widetilde{C}\,\eta_\ell^2$
5. $0 < q+\delta < 1$ and $N \to \infty$ $\quad\Longrightarrow\quad \displaystyle\sum_{k=\ell+1}^{\infty} \eta_k^2 \le M\,\eta_\ell^2$
6. $\Longrightarrow \quad (1 + M^{-1}) \displaystyle\sum_{k=\ell+1}^{\infty} \eta_k^2 \le \sum_{k=\ell+1}^{\infty} \eta_k^2 + \eta_\ell^2 = \sum_{k=\ell}^{\infty} \eta_k^2$
7. induction $\quad\Longrightarrow\quad (1+M^{-1})^n \displaystyle\sum_{k=\ell+n}^{\infty} \eta_k^2 \le \sum_{k=\ell}^{\infty} \eta_k^2 \le (1+M)\,\eta_\ell^2$
8. finally: $\displaystyle\eta_{\ell+n}^2 \le \sum_{k=\ell+n}^{\infty} \eta_k^2 \le \underbrace{(1+M^{-1})^{-n}}_{=q_{\rm lin}^n}\,\underbrace{(1+M)}_{=C_{\rm lin}}\,\eta_\ell^2$

Remarks on Linear Convergence 1/3

- algorithm leads to convergence rate $s > 0$
  - w.r.t. dofs $\quad \displaystyle\sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell < \infty$
  - w.r.t. work $\quad \displaystyle\sup_{\ell\in\mathbb{N}_0} \Big(\sum_{j=0}^{\ell} \#\mathcal{T}_j\Big)^s \eta_\ell < \infty$

- linear convergence $\quad\Longrightarrow\quad \displaystyle\sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{\ell\in\mathbb{N}_0} \Big(\sum_{j=0}^{\ell} \#\mathcal{T}_j\Big)^s \eta_\ell$

- LHS $< \infty \quad\Longrightarrow\quad \eta_j \lesssim (\#\mathcal{T}_j)^{-s} \quad\Longrightarrow\quad \#\mathcal{T}_j \lesssim \eta_j^{-1/s}$
- linear convergence $\quad\Longrightarrow\quad \eta_\ell \lesssim q^{\ell-j}\eta_j \quad\Longrightarrow\quad \eta_j^{-1} \lesssim q^{(\ell-j)}\eta_\ell^{-1}$

$$\Longrightarrow \sum_{j=0}^{\ell} \#\mathcal{T}_j \lesssim \sum_{j=0}^{\ell} \eta_j^{-1/s} \lesssim \eta_\ell^{-1/s} \sum_{j=0}^{\ell} q^{(\ell-j)/s} \simeq \eta_\ell^{-1/s} \quad\Longrightarrow\quad \text{RHS} \le \text{LHS}$$

Remarks on Linear Convergence 2/3

Our proof also shows equivalence of

- $\exists C_{\rm lin} > 0 \, \exists 0 < q_{\rm lin} < 1 \, \forall \ell, n \in \mathbb{N}_0 \quad \eta_{\ell+n} \le C_{\rm lin} q_{\rm lin}^n \, \eta_\ell$
- $\exists C > 0 \, \exists p > 0 \, \forall \ell, n \in \mathbb{N}_0 \quad \sum_{k=\ell}^{\infty} \eta_k^p \le C_p \, \eta_\ell^p$
- $\exists C > 0 \, \forall p > 0 \, \forall \ell, n \in \mathbb{N}_0 \quad \sum_{k=\ell}^{\infty} \eta_k^p \le C_p \, \eta_\ell^p$

- linear convergence is required for optimal computational costs
- use summability criterion for $p = 1$, if you include iterative solvers

Führer, Haberl, Praetorius, Schimanko: Numer. Math. 2019 (published online first)

Remarks on Linear Convergence 3/3

- analysis relies only on specified axioms!

- analysis is independent of
  - linear / nonlinear problem
  - discretization method (e.g., FEM / BEM / FVM)
  - nestedness / conformity of discrete spaces
  - exact / inexact computation of discrete solution
  - mesh-refinement strategy (e.g., NVB)
- all this enters for proofs of axioms for precise setting

Contraction

- axioms (A1)–(A4) $\implies$ linear convergence $\eta_{\ell+n}^2 \lesssim q^n \eta_\ell^2$
- for symmetric + elliptic problems, slightly stronger result available
- Lax–Milgram setting (with exact solver)

Theorem (Cascon, Kreuzer, Nochetto, Siebert '08)

- $\Delta_\ell = \|\phi - \Phi_\ell\|^2 + \gamma \eta_\ell^2 \simeq \eta_\ell^2$
- with $0 < \gamma < 1$ sufficiently small

$\implies$ Exists $0 < \kappa = \kappa(\theta) < 1$ s.t.

$$\Delta_{\ell+1} \le \kappa \, \Delta_\ell \quad \text{hence} \quad \lim_{\ell\to\infty} \eta_\ell = 0 = \lim_{\ell\to\infty} \|\phi - \Phi_\ell\|$$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Proof of Contraction Theorem

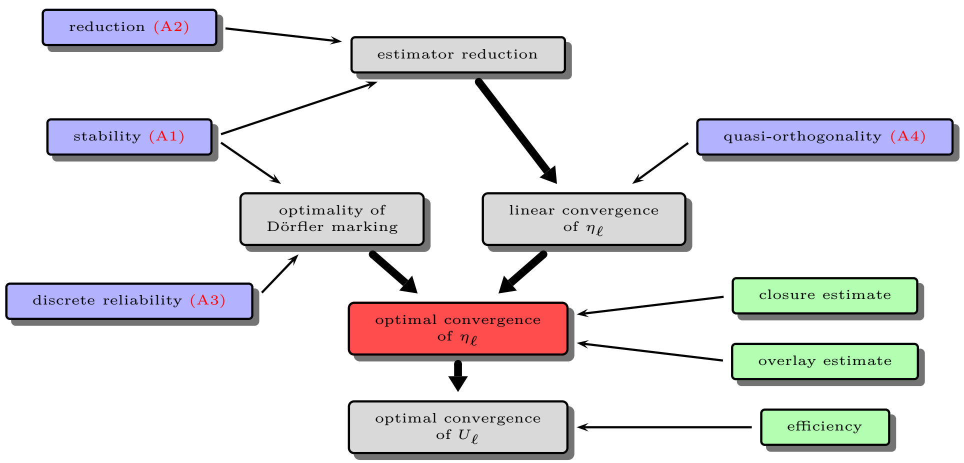
1. Pythagoras $\|\phi - \Phi_{\ell+1}\|^2 = \|\phi - \Phi_\ell\|^2 - \|\Phi_{\ell+1} - \Phi_\ell\|^2$
2. estimator reduction $\eta_{\ell+1}^2 \le q \, \eta_\ell^2 + C \|\Phi_{\ell+1} - \Phi_\ell\|^2$
3. reliability $\|\phi - \Phi_\ell\|^2 \le C_{\rm rel}^2 \eta_\ell^2$

- use small parameters $0 < \gamma, \varepsilon < 1$ to see that

$$\begin{aligned} \|\phi - \Phi_{\ell+1}\|^2 + \gamma \eta_{\ell+1}^2 &\le \|\phi - \Phi_\ell\|^2 + \gamma q \, \eta_\ell^2 \\ &\le (1 - \varepsilon\gamma) \|\phi - \Phi_\ell\|^2 + \gamma (q + \varepsilon C_{\rm rel}^2) \eta_\ell^2 \\ &\le \kappa \, (\|\phi - \Phi_\ell\|^2 + \gamma \eta_\ell^2) \end{aligned}$$

- obtain: $\Delta_{\ell+1} \le \kappa \, \Delta_\ell$
  - $0 < \kappa < 1$
  - $\Delta_\ell = \|\phi - \Phi_\ell\|^2 + \gamma \eta_\ell^2$

Axioms of Adaptivity



Dörfler Marking $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2$

- goal: determine $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with minimal cardinality

$\Longrightarrow$ requires(?) sorting $\eta_\ell(T_1) \ge \dots \ge \eta_\ell(T_N)$ $\rightsquigarrow \mathcal{O}(N \log N)$

- sufficient: $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ has essentially minimal cardinality
  - i.e., $\#\mathcal{M}_\ell \le C\,\#\widetilde{\mathcal{M}}_\ell$ if $\widetilde{\mathcal{M}}_\ell$ has minimal cardinality

- idea: sorting with binning [Stevenson '07] $\rightsquigarrow \mathcal{O}(N)$

$\Longrightarrow$ $\#\mathcal{M}_\ell \le 2\,\#\widetilde{\mathcal{M}}_\ell$

- ongoing: modified Quicksort allows for minimal cardinality $\rightsquigarrow \mathcal{O}(N)$

Dörfler: SINUM 33 (1996)

Stevenson: Found. Comput. Math. 7 (2007)

Pfeiler, Praetorius: work in progress 2019

Optimality of Dörfler Marking

Dörfler Marking $\Longrightarrow$ (Linear) Convergence

- shown: stability (A1) + reduction (A2) + Dörfler marking

$\Longrightarrow$ estimator reduction

- estimator reduction + reliability (A3) + quasi-orth. (A4)

$\Longrightarrow$ linear convergence $\eta_{\ell+n} \lesssim q_{\rm lin}^n \, \eta_\ell$

- i.e.: Dörfler marking sufficient for (linear) convergence

- question: Dörfler marking also necessary?

Recall Stability (A1) and Discrete Reliability (A4)

(A1) Stability on non-refined elements, $\mathcal{T}_\ell, \mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_0)$

$$\Big|\Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\star(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2\Big)^{1/2}\Big| \lesssim |||U_\star - U_\ell|||$$

(A4) Discrete reliability, $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$

- exists $\mathcal{R}_\ell \subseteq \mathcal{T}_\ell$ with
  - $\mathcal{T}_\ell \backslash \mathcal{T}_\star \subseteq \mathcal{R}_\ell$
  - $\#\mathcal{R}_\ell \le C_{\rm rel}\, \#(\mathcal{T}_\ell\backslash\mathcal{T}_\star)$
  - $|||U_\star - U_\ell|||^2 \le C_{\rm rel}^2 \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

Dörfler Marking $\Longleftarrow$ Convergence

- so far: Dörfler $\Longrightarrow$ linear convergence $\eta_{\ell+n} \lesssim q_{\rm lin}^n\, \eta_\ell$

Stab. (A1) + disc. rel. (A3) $\Longrightarrow$ optimality of Dörfler marking

Exists $0 < \theta_\star < 1$ and $0 < q_\star < 1$ s.t.

- for all $\mathcal{T}_\star = \texttt{refine}(\mathcal{T}_\ell)$ with $\eta_\star^2 \le q_\star\, \eta_\ell^2$
- and all $0 < \theta < \theta_\star$
- and $\mathcal{R}_\ell \supseteq \mathcal{T}_\ell\backslash\mathcal{T}_\star$ from discrete reliability (A4)

holds Dörfler marking $\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

- linear convergence

$\Longrightarrow$ Dörfler marking holds every fixed number n of steps

- independently of how elements are actually marked!

$\Longrightarrow$ if discrete reliability (A3), then Dörfler necessary for linear conv.

Proof: Optimality of Dörfler Marking 1/2

1. stability (A1) + Young inequality + discrete reliability (A4)

$$\begin{aligned}\sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2 &\le (1+\delta^{-1}) \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\star(T)^2 + (1+\delta) C_{\rm stab}^2\, |||U_\star - U_\ell|||^2 \\ &\le (1+\delta^{-1})\eta_\star^2 + (1+\delta) C_{\rm stab}^2 C_{\rm rel}^2 \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2\end{aligned}$$

2. $\eta_\star \le q_\star \eta_\ell$ and $\mathcal{T}_\ell\backslash\mathcal{T}_\star \subseteq \mathcal{R}_\ell$

$$\begin{aligned}\eta_\ell^2 &= \sum_{T\in\mathcal{T}_\ell\backslash\mathcal{T}_\star} \eta_\ell(T)^2 + \sum_{T\in\mathcal{T}_\ell\cap\mathcal{T}_\star} \eta_\ell(T)^2 \\ &\le (1+\delta^{-1}) q_\star^2\, \eta_\ell^2 + [1 + (1+\delta)\, C_{\rm stab}^2 C_{\rm rel}^2] \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2\end{aligned}$$

Proof: Optimality of Dörfler Marking 2/2

2. $\eta_\ell^2 \le (1+\delta^{-1}) q_\star^2\, \eta_\ell^2 + [1 + (1+\delta)\, C_{\rm stab}^2 C_{\rm rel}^2] \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$

$$\Longrightarrow \frac{1 - (1+\delta^{-1}) q_\star^2}{1 + (1+\delta)\, C_{\rm stab}^2 C_{\rm rel}^2}\, \eta_\ell^2 \le \sum_{T\in\mathcal{R}_\ell} \eta_\ell(T)^2$$

3. given: $\theta < \theta_\star := \dfrac{1}{1 + C_{\rm stab}^2 C_{\rm rel}^2}$
   - $\theta_\star > \dfrac{1 - (1+\delta^{-1}) q_\star^2}{1 + (1+\delta)\, C_{\rm stab}^2 C_{\rm rel}^2}\, \eta_\ell^2 \xrightarrow[\delta\to 0]{(1+\delta^{-1})q_\star \to 0} \theta_\star$

4. choose $\delta > 0$ and $0 < q_\star \ll 1$ small such that

$$\theta < \frac{1 - (1+\delta^{-1}) q_\star^2}{1 + (1+\delta)\, C_{\rm stab}^2 C_{\rm rel}^2}\, \eta_\ell^2 < \theta_\star$$

Necessity of Discrete Reliability

- suppose: Céa lemma $\|u - U_\ell\| \simeq \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\|$
- Let $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_{\ell+1})$

$\Longrightarrow$ $\|U_\star - U_\ell\| \leq \|u - U_\star\| + \|u - U_\ell\| \lesssim \|u - U_\ell\|$

- Suppose knowledge that η_ℓ is reliable + Dörfler marking

$\Longrightarrow$ $\|u - U_\ell\|^2 \lesssim \eta_\ell^2 \leq \theta^{-1} \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$

- marked elements are refined, i.e., $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1} \subseteq \mathcal{T}_\ell \backslash \mathcal{T}_\star$

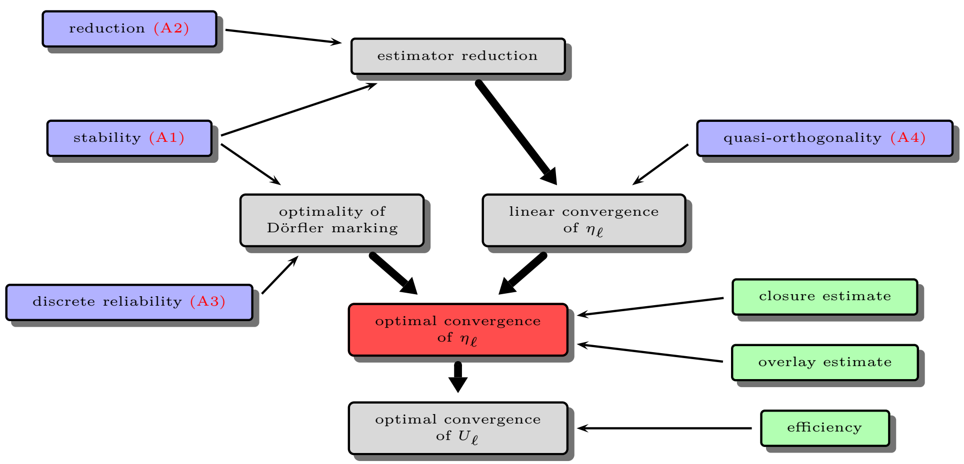
$\Longrightarrow$ obtain discrete reliability

$$\|U_\star - U_\ell\|^2 \lesssim \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star} \eta_\ell(T)^2$$

$\Longrightarrow$ If Dörfler marking is used, then discrete reliability is necessary!

Rate Optimality

Axioms of Adaptivity



Quasi-Monotonicity of Estimator

stab. (A1) + red. (A2) + rel. (A4) $\Longrightarrow$ monotonicity

- $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ $\Longrightarrow$ $\eta_\star \leq C_{\rm mon}\, \eta_\ell$

$$\begin{aligned}\eta_\star^2 &= \sum_{T \in \mathcal{T}_\star \backslash \mathcal{T}_\ell} \eta_\star(T)^2 + \sum_{T \in \mathcal{T}_\star \cap \mathcal{T}_\ell} \eta_\star(T)^2 \\ &\lesssim \sum_{T \in \mathcal{T}_\ell \backslash \mathcal{T}_\star} \eta_\ell(T)^2 + \sum_{T \in \mathcal{T}_\ell \cap \mathcal{T}_\star} \eta_\ell(T)^2 + \|U_\star - U_\ell\|^2 \\ &\lesssim \eta_\ell^2\end{aligned}$$

Proof of Optimality Theorem 1/3

Heart of the Matter: Exists $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ s.t.

- $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\,\eta_\ell^{-1/s}$
- $\eta_\star \le q_\star \eta_\ell$

- recall $\|\eta\|_{\mathbb{A}_s} = \sup_{N\in\mathbb{N}_0} \big((N+1)^s \min_{\mathcal{T}_\star\in\mathbb{T}_N} \eta_\star\big)$
- define $\varepsilon := C_{\rm mon}^{-1} q_\star \eta_\ell < \eta_0 \le \|\eta\|_{\mathbb{A}_s} < \infty$
- choose minimal $N\in\mathbb{N}_0$ with $\|\eta\|_{\mathbb{A}_s} \le \varepsilon\,(N+1)^s$
  - $\|\eta\|_{\mathbb{A}_s} > \varepsilon N^s$
- let $\mathcal{T}_\varepsilon \in \mathbb{T}_N = \{\mathcal{T}\in \texttt{refine}(\mathcal{T}_0) : \#\mathcal{T} - \#\mathcal{T}_0 \le N\}$ s.t. $\eta_\varepsilon = \min_{\mathcal{T}_\star\in\mathbb{T}_N}\eta_\star$
  - $\eta_\varepsilon \le (N+1)^{-s}\|\eta\|_{\mathbb{A}_s} \le \varepsilon$
  - $\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \le N < \|\eta\|_{\mathbb{A}_s}^{1/s}\varepsilon^{-1/s}$
- define $\mathcal{T}_\star := \mathcal{T}_\varepsilon \oplus \mathcal{T}_\ell$ & use overlay estimate
  - $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \le (\#\mathcal{T}_\varepsilon + \#\mathcal{T}_\ell - \#\mathcal{T}_0) - \#\mathcal{T}_\ell \lesssim \|\eta\|_{\mathbb{A}_s}\varepsilon^{-1/s} \simeq \eta_\ell^{-1/s}$
  - $\eta_\star \le C_{\rm mon}\eta_\varepsilon \le C_{\rm mon}\varepsilon = q_\star\eta_\ell$

Proof of Optimality Theorem 2/3

1. exists $\mathcal{T}_\star \in \texttt{refine}(\mathcal{T}_\ell)$ s.t.
   - $\#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\,\eta_\ell^{-1/s}$
   - $\eta_\star \le q_\star \eta_\ell$
2. optimality of Dörfler marking
   $\Longrightarrow$ $\mathcal{R}_\ell \simeq \mathcal{T}_\ell\backslash\mathcal{T}_\star$ satisfies Dörfler marking
3. $\mathcal{M}_\ell$ has (essentially) minimal cardinality
   $$\Longrightarrow \#\mathcal{M}_\ell \lesssim \#\mathcal{R}_\ell \simeq \#(\mathcal{T}_\ell\backslash\mathcal{T}_\star) \lesssim \#\mathcal{T}_\star - \#\mathcal{T}_\ell \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\,\eta_\ell^{-1/s} \qquad \forall \ell\in\mathbb{N}_0$$
4. overlay estimate
   $$\Longrightarrow \#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{\ell-1}\#\mathcal{M}_j \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\eta_j^{-1/s}$$

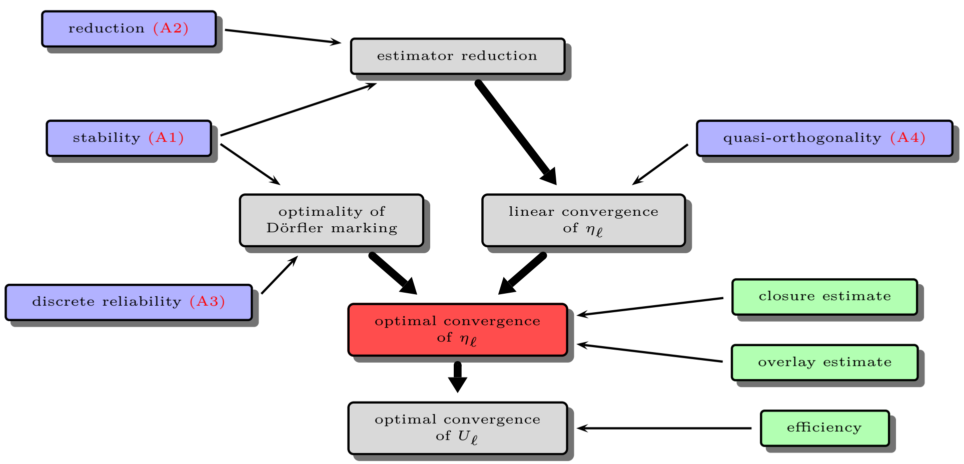
Proof of Optimality Theorem 3/3

4. obtained: $\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\sum_{j=0}^{\ell-1}\eta_j^{-1/s}$
5. linear convergence $\eta_{j+n} \lesssim q^n\,\eta_j$ & geometric series
   $$\Longrightarrow \eta_\ell \lesssim q^{\ell-j}\,\eta_j$$
   $$\Longrightarrow \sum_{j=0}^{\ell-1}\eta_j^{-1/s} \lesssim \Big(\sum_{j=0}^{\ell-1}q^{(\ell-j)/s}\Big)\,\eta_\ell^{-1/s} \lesssim \eta_\ell^{-1/s}$$
6. combining this, we obtain
   $$\Longrightarrow \#\mathcal{T}_\ell \simeq \#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1 \simeq \#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \|\eta\|_{\mathbb{A}_s}^{1/s}\,\eta_\ell^{-1/s} \qquad \forall \ell > 0$$
   - trivial: $\#\mathcal{T}_0 \le \|\eta\|_{\mathbb{A}_s}^{1/s}\eta_0^{-1/s}$

   $\Longrightarrow (\#\mathcal{T}_\ell)^s\,\eta_\ell \lesssim \|\eta\|_{\mathbb{A}_s}$ for all $s>0$ with $\|\eta\|_{\mathbb{A}_s} < \infty$ $\qquad \forall \ell \ge 0$

   $\Longrightarrow \sup_{\ell\in\mathbb{N}_0}(\#\mathcal{T}_\ell)^s\,\eta_\ell \lesssim \|\eta\|_{\mathbb{A}_s}$ $\quad\rightsquigarrow$ interesting part of main thm.

Axioms of Adaptivity



Thanks for listening!

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