February 03, 2020 – MFO workshop on Boundary Element Methods

Functional a-posteriori error estimates for BEM

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joint work with

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Introduction

Model problem

PDE model problem

- $\Delta u = 0$ in $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$
- $u = g$ on $\Gamma = \partial\Omega$

indirect BEM ansatz

- $u = \widetilde{V}\phi$ in Ω with unknown $\phi \in H^{-1/2}(\Gamma)$

direct BEM ansatz

- $u = \widetilde{V}\phi - \widetilde{K}g$ in Ω with unknown $\phi \in H^{-1/2}(\Gamma)$

Boundary element method

indirect BEM

- solve $V\phi = g$ on Γ
- compute $\phi_h \approx \phi$ by Galerkin / collocation / ...
- obtain approximation $u \approx u_h := \widetilde{V}\phi_h$

direct BEM

- solve $V\phi = (K + 1/2)g$ on Γ
- compute $\phi_h \approx \phi$ by Galerkin / collocation / ...
- obtain approximation $u \approx u_h := \widetilde{V}\phi_h - \widetilde{K}g$

- sought: $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- known: $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

- note: This observation is independent of how ϕ_h is computed!

Quantity of interest?

- integral density ϕ?
  - makes sense for direct BEM, where $\phi_h \approx \phi = \partial_n u$

  $\Longrightarrow$ $\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)}$ should be controlled / small
  - Stephan, Carstensen '95ff., Heuer, Maischak, Funken, Praetorius, ... / Faermann '00, '02 / 'Steinbach '00 / Dahmen, Harbrecht, Schneider '06 / Bakry, Pernet, Collino '17 / ...

- point value $u(x)$?
  - particular strength of BEM!

  $\Longrightarrow$ error control by product of primal and dual errors (resp. estimators)
  - Feischl, Praetorius et al. '16 / Bakry '17 / Harbrecht, Moor '19

- potential u?
  - e.g., in exterior domains

  $\Longrightarrow$ $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ should be controlled / small
  - focus of the present talk!

Kurz, Pauly, Praetorius, Repin, Sebastian: Preprint arXiv:1912.05789, 2019

Functional error estimates

Functional error identity

Theorem

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \boxed{\max_{\substack{\boldsymbol{\tau}\in \boldsymbol{L}^2(\Omega)\\ \nabla\cdot\boldsymbol{\tau}=0}} \underline{\mathfrak{M}}(\boldsymbol{\tau}; u_h|_\Gamma, g) = \|\nabla(u-u_h)\|_\Omega^2 = \min_{\substack{w\in H^1(\Omega)\\ w|_\Gamma = g - u_h|_\Gamma}} \overline{\mathfrak{M}}(\nabla w)}$$

- $\underline{\mathfrak{M}}(\boldsymbol{\tau}; u_h|_\Gamma, g) := 2\,\langle g - u_h|_\Gamma\,,\, \boldsymbol{\tau}|_\Gamma \cdot \boldsymbol{n}\rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2$
- $\overline{\mathfrak{M}}(\nabla w) := \|\nabla w\|_\Omega^2$

- obtain lower bound by choosing $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)$ with $\nabla\cdot\boldsymbol{\tau} = 0$
- obtain upper bound by choosing $w \in H^1(\Omega)$ with $w|_\Gamma = g - u_h|_\Gamma$
- note: bounds come with known constant 1

Repin: De Gruyter 2008

Kurz, Pauly, Praetorius, Repin, Sebastian: Preprint arXiv:1912.05789, 2019

Upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega = \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = g - u_h|_\Gamma}} \|\nabla w\|_\Omega$$

- $v \in H^1(\Omega)$ with $v|_\Gamma = g = u|_\Gamma$

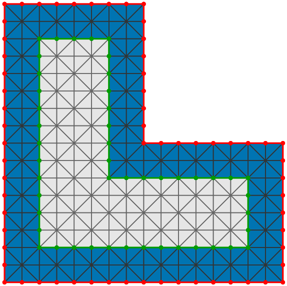
- $\|\nabla(u - u_h)\|_\Omega^2 = \underbrace{\langle \nabla(u - v)\,,\, \nabla(u - u_h)\rangle_\Omega}_{=0} + \langle \nabla(v - u_h)\,,\, \nabla(u - u_h)\rangle_\Omega$

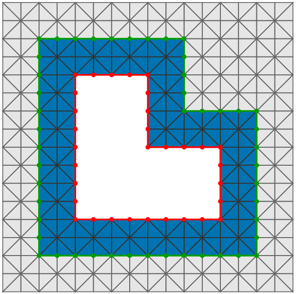
$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \|\nabla(v - u_h)\|_\Omega$$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega = \min_{\substack{v \in H^1(\Omega) \\ v|_\Gamma = g}} \|\nabla(v - u_h)\|_\Omega = \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = g - u_h|_\Gamma}} \|\nabla w\|_\Omega$$

Computable upper bound 1/3

- goal: cheap computation of good $w \in H^1(\Omega)$ with $w|_\Gamma = g - u_h|_\Gamma$
- approach: employ FEM on boundary layer (to approx. $u - u_h \in H^1$)





- problem: FEM functions cannot satisfy continuous trace $g - u_h|_\Gamma$
- solution: allow for data oscillation terms

Perturbed upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{S}_h \subset H^1(\omega)$ FEM space on boundary layer $\omega \subseteq \Omega$
- $J_h : H^{1/2}(\Gamma) \to \{v_h|_\Gamma \,:\, v_h \in \mathcal{S}_h\}$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \min_{\substack{w \in H^1(\Omega) \\ w|_\Gamma = J_h(g - u_h|_\Gamma)}} \|\nabla w\|_\Omega + \|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$$

- known: $\|\nabla(u - u_h)\|_\Omega = \min_{\substack{\overline{w} \in H^1(\Omega) \\ \overline{w}|_\Gamma = g - u_h|_\Gamma}} \|\nabla \overline{w}\|_\Omega$
- choose $v \in H^1(\Omega)$ with $\Delta v = 0$ and $v|_\Gamma = (1 - J_h)(g - u_h|_\Gamma)$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \le \min_{\substack{\overline{w} \in H^1(\Omega) \\ \overline{w}|_\Gamma = g - u_h|_\Gamma}} \|\nabla(\overline{w} - v)\|_\Omega + \|\nabla v\|_\Omega$$

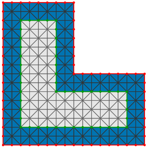
- choose $w := \overline{w} - v$

Computable upper bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{S}_h \subset H^1(\omega)$ FEM space on boundary layer $\omega \subseteq \Omega$
- $J_h : H^{1/2}(\Gamma) \to \{v_h|_\Gamma \,:\, v_h \in \mathcal{S}_h\}$
- compute FEM solution $w_h \in \mathcal{S}_h^\star := \{v_h \in \mathcal{S}_h \,:\, v_h|_{\partial\omega\backslash\Gamma} = 0\}$ s.t.
  - $w_h|_\Gamma = J_h(g - u_h|_\Gamma)$
  - $\langle \nabla w_h \,,\, \nabla v_h \rangle_\omega = 0 \quad \forall w_h \in \mathcal{S}_h^\star$

$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega \leq \|\nabla w_h\|_\omega + \|(1 - J_h)(g - u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$$

- $w_h|_{\partial\omega\backslash\Gamma} = 0$ allows to extend to $w_h \in H^1(\Omega)$
- later: $\mathcal{S}_h = \mathcal{S}^1(\mathcal{T}_h|_\omega)$ p/w affine, globally continuous



Proof of lower bound

Lower bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$

$$\Longrightarrow \quad \max_{\substack{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \\ \nabla \cdot \boldsymbol{\tau} = 0}} \Big(2 \, \langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2 \Big) \;=\; \|\nabla(u - u_h)\|_\Omega^2$$

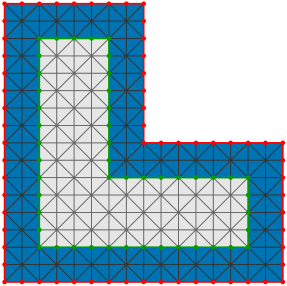
- $\|x\|_{\mathcal{H}}^2 = \max_{y \in \mathcal{H}} \big(2\langle x \,,\, y \rangle_{\mathcal{H}} - \|y\|_{\mathcal{H}}^2 \big)$ in any Hilbert space $\mathcal{H}$
- $\nabla(u - u_h) \in \mathcal{H} := \big\{ \boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega) \;:\; \nabla \cdot \boldsymbol{\sigma} = 0 \big\}$

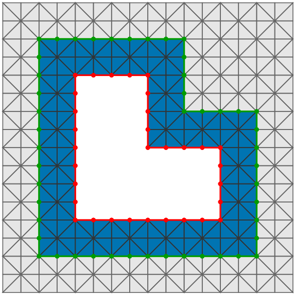
$$\Longrightarrow \quad \|\nabla(u - u_h)\|_\Omega^2 = \max_{\boldsymbol{\tau} \in \mathcal{H}} \Big(2 \, \langle \nabla(u - u_h) \,,\, \boldsymbol{\tau} \rangle_\Omega - \|\boldsymbol{\tau}\|_\Omega^2 \Big)$$

$+$ integration by parts

Computable lower bound 1/2

- goal: cheap computation of good $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\tau} = 0$
- approach: mixed FEM on boundary layer (for $\nabla(u - u_h) \in \boldsymbol{H}(\mathrm{div})$)



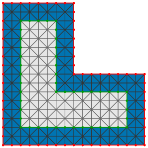


Computable lower bound

- $u \in H^1(\Omega)$ with $\Delta u = 0$ and $u|_\Gamma = g$
- $u_h \in H^1(\Omega)$ with $\Delta u_h = 0$
- $\mathcal{RT}^q(\mathcal{T}_h) \subset \boldsymbol{H}(\mathrm{div}, \omega)$ Raviart–Thomas space on boundary layer $\omega \subseteq \Omega$
- $\mathcal{RT}^q_\star(\mathcal{T}_h) := \big\{ \boldsymbol{\sigma}_h \in \mathcal{RT}^q \,:\, \boldsymbol{\sigma}_h|_{\partial\omega\setminus\Gamma} \cdot \boldsymbol{n} = 0 \big\}$
- compute FEM solution $(\boldsymbol{\tau}_h, p_h) \in \mathcal{RT}^q_\star(\mathcal{T}_h) \times \mathcal{P}^q(\mathcal{T}_h)$ s.t.
  - $\langle \boldsymbol{\tau}_h \,,\, \boldsymbol{\sigma}_h \rangle_\omega + \langle \nabla \cdot \boldsymbol{\sigma}_h \,,\, p_h \rangle_\omega = \langle g - u_h|_\Gamma \,,\, \boldsymbol{\sigma}_h|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma \qquad \forall \boldsymbol{\sigma}_h \in \mathcal{RT}^q_\star(\mathcal{T}_h)$
  - $\langle \nabla \cdot \boldsymbol{\tau}_h \,,\, q_h \rangle_\omega = 0 \qquad \forall q_h \in \mathcal{P}^q(\mathcal{T}_h)$

$$\Longrightarrow \quad \Big(2 \, \langle g - u_h|_\Gamma \,,\, \boldsymbol{\tau}_h|_\Gamma \cdot \boldsymbol{n} \rangle_\Gamma - \|\boldsymbol{\tau}\|_\omega{}^2 \Big) \quad \leq \quad \|\nabla (u - u_h)\|_\Omega^2$$

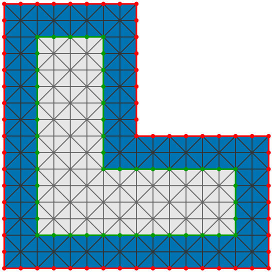
- $\boldsymbol{\tau}_h|_{\partial\omega\setminus\Gamma} \cdot \boldsymbol{n} = 0$ allows to extend to $\boldsymbol{\tau}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$
- note: unique solution, but unclear if $0 \leq \mathrm{LHS}$

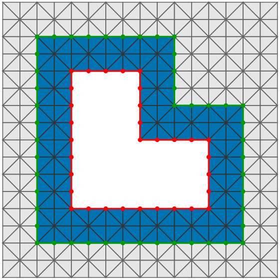


Adaptive algorithm

Comments

- error bounds are independent of ω and Ω
- no reason to fix boundary layer $\omega \subseteq \Omega$
- ω should be shrunken to preserve dimension reduction of BEM
- for numerics:
  - start with triangulation $\mathcal{T}_h$ of some boundary layer $\widehat{\omega} \subset \Omega$
  - always extract $\mathcal{T}_h^\Gamma := \mathcal{T}_h|_\Gamma$
  - always extract $\mathcal{T}_h^\omega := \mathcal{T}_h|_\omega$ as, e.g., 2nd order patch of Γ





Adaptive algorithm

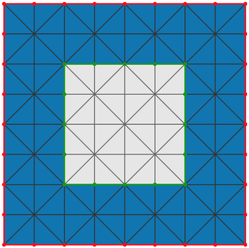
Iterate the following loop, until $\|w_h\|_\omega \le \text{tolerance}$

1. extract BEM triangulation $\mathcal{T}_h^\Gamma := \mathcal{T}_h|_\Gamma$
2. extract $\mathcal{T}_h^\omega := \{T \in \mathcal{T}_h \,:\, \exists T' \in \mathcal{T}_h, \quad T' \cap \Gamma \neq \emptyset \neq T \cap T'\}$
3. define boundary layer $\omega := \bigcup \mathcal{T}_h^\omega$
4. compute (Galerkin) BEM approximation $\phi \approx \phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
5. compute $J_h(g - u_h|_\Gamma)$ and data oscillations $\mathrm{osc}_h(T)$ for all $T \in \mathcal{T}_h^\omega$
6. compute FEM solution $w_h \in \mathcal{S}^1(\mathcal{T}_h^\omega)$ for the majorant $\|w_h\|_\omega$
7. assemble error estimator $\eta_h(T)^2 = \|w_h\|_T^2 + \mathrm{osc}_h(T)^2$ for $T \in \mathcal{T}_h^\omega$
8. choose $\mathcal{M}_h \subseteq \mathcal{T}_h^\omega$ s.t. $\theta \sum_{T \in \mathcal{T}_h^\omega} \eta_h(T)^2 \le \sum_{T \in \mathcal{M}_h} \eta_h(T)^2$
9. generate refined triangulation $\mathcal{T}_h \hookleftarrow \mathrm{refine}(\mathcal{T}_h, \mathcal{M}_h)$

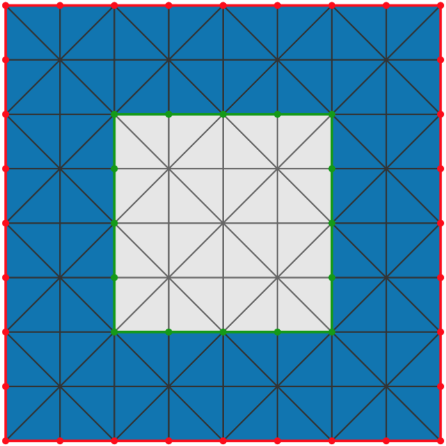
Indirect BEM: smooth potential, square domain

Smooth potential

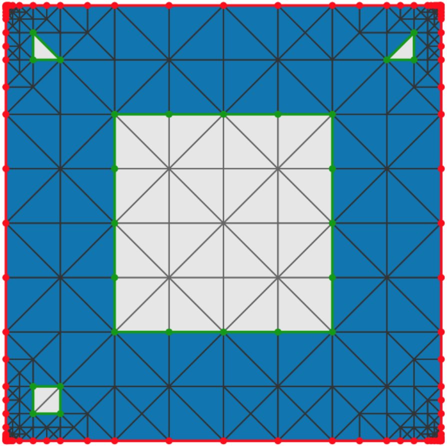
- $\Omega = [0,1/2]^2$
- $u(x) = \cosh(x_1)\cos(x_2)$
- indirect BEM formulation $V\phi = u|_\Gamma$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is smooth, but ϕ is non-smooth



Adaptively generated meshes 1/2

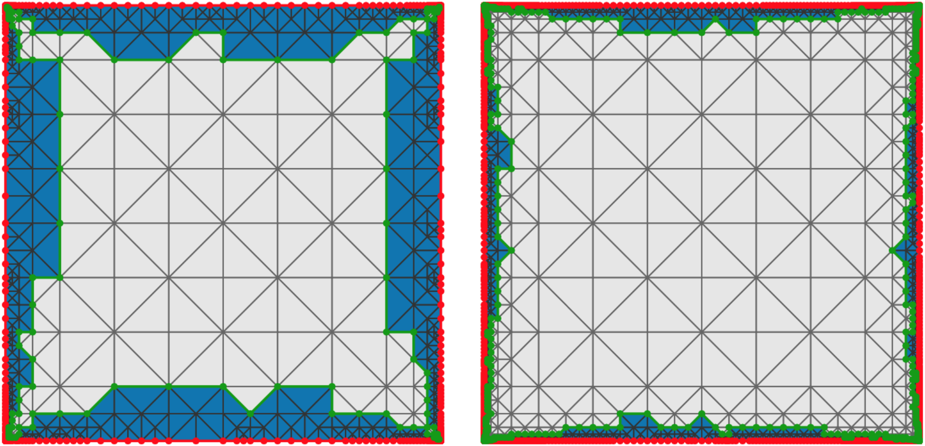


a) $\#\mathcal{T}_h^S = 72$, $\#\mathcal{F}_h^\Gamma = 32$, $\ell = 0$

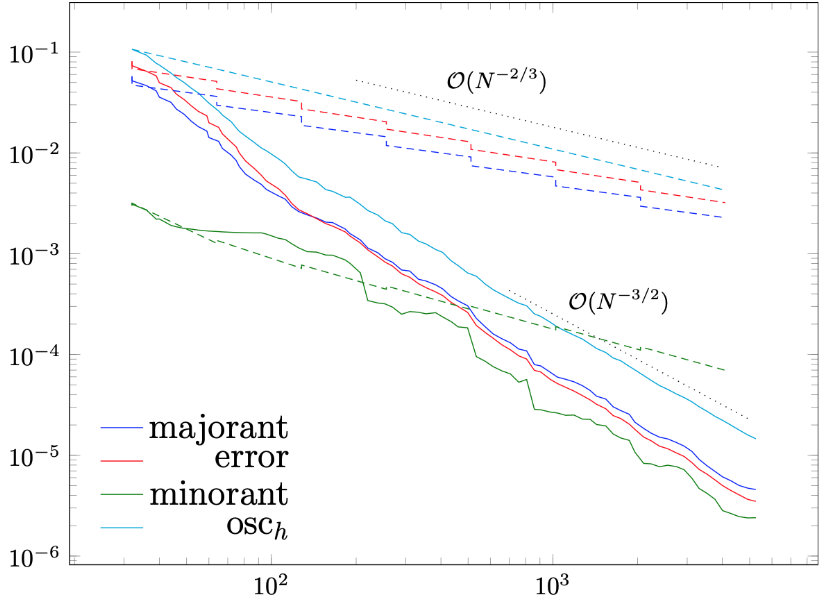


b) $\#\mathcal{T}_h^S = 314$, $\#\mathcal{F}_h^\Gamma = 119$, $\ell = 17$

Adaptively generated meshes 2/2

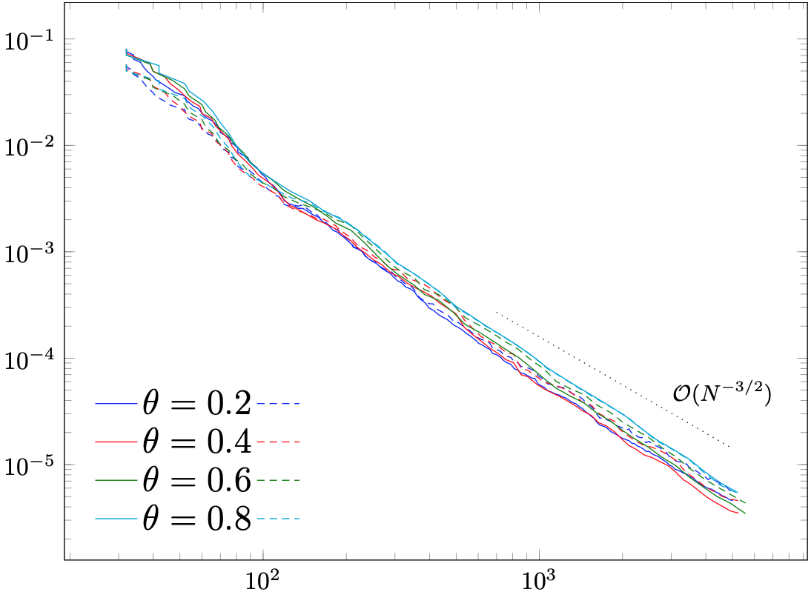


c) $\#\mathcal{T}_h^S = 923$, $\#\mathcal{F}_h^\Gamma = 352$, $\ell = 27$ **d)** $\#\mathcal{T}_h^S = 3176$, $\#\mathcal{F}_h^\Gamma = 1148$, $\ell = 38$



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{2}
10^{3}
$\mathcal{O}(N^{-2/3})$
$\mathcal{O}(N^{-3/2})$
majorant
error
minorant
osc_h

Numerical results 2/3



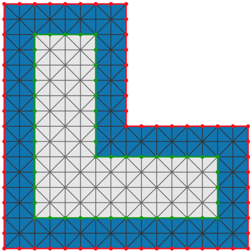
Numerical results 3/3

ℓ	$\#\mathcal{F}_h^\Gamma$	$\frac{\#\mathcal{T}_h^S}{\#\mathcal{F}_h^\Gamma}$	$\mathrm{dof}(\mathcal{T}_h^S)$	$\|\nabla(u-u_h)\|_{L^2(\Omega)}$	$\|\nabla w_h\|_{L^2(S)}$	$\frac{\|\nabla w_h\|_{L^2(S)}}{\|\nabla(u-u_h)\|_{L^2(\Omega)}}$
0	32	2.25	15	$8.01e-2$	$5.75e-2$	0.71
4	40	2.33	28	$4.97e-2$	$3.58e-2$	0.72
10	59	2.44	60	$2.33e-2$	$1.65e-2$	0.71
16	77	2.66	103	$9.95e-3$	$7.29e-3$	0.73
22	112	2.60	148	$3.81e-3$	$3.48e-3$	0.91
28	165	2.82	234	$1.88e-3$	$2.03e-3$	1.08
34	253	2.83	343	$8.27e-4$	$8.92e-4$	1.08
40	383	2.81	512	$4.18e-4$	$4.91e-4$	1.18
46	575	2.70	707	$1.66e-4$	$1.89e-4$	1.14
52	860	2.63	978	$6.96e-5$	$7.94e-5$	1.14
58	1072	2.61	1389	$3.92e-5$	$4.92e-5$	1.25
64	1869	2.61	2008	$2.04e-5$	$2.55e-5$	1.25
70	2748	2.58	2803	$1.06e-5$	$1.34e-5$	1.27
76	4007	2.55	3976	$5.00e-6$	$6.12e-6$	1.22
80	5259	2.53	5077	$3.50e-6$	$4.58e-6$	1.31

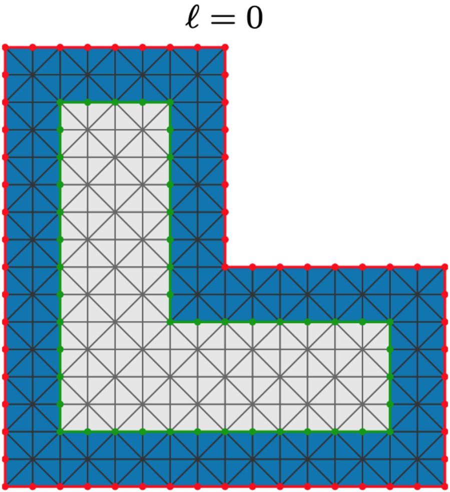
Indirect BEM: non-smooth potential

Non-smooth potential

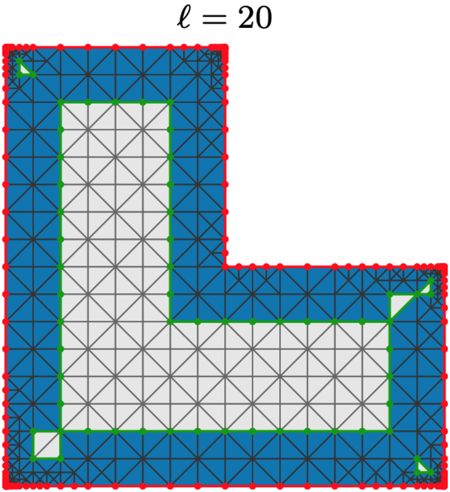
- $\Omega \subsetneqq [0, 1/2]^2$ L-shaped domain
- $u(x) = r^{2/3} \cos(2\varphi/3)$ w.r.t. re-entrant corner
- indirect BEM formulation $V\phi = u|_\Gamma$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is non-smooth and ϕ is non-smooth



Adaptively generated meshes 1/2



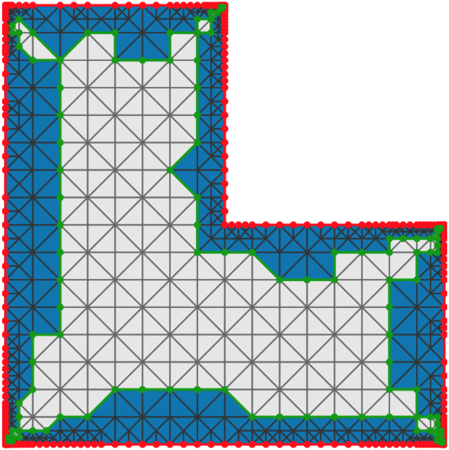
$\#\mathcal{T}_h^S = 168,\ \#\mathcal{F}_h^\Gamma = 64$



$\#\mathcal{T}_h^S = 514,\ \#\mathcal{F}_h^\Gamma = 201$

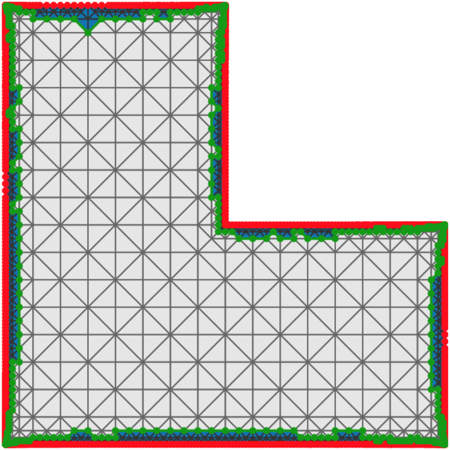
Adaptively generated meshes 2/2

$\ell = 26$



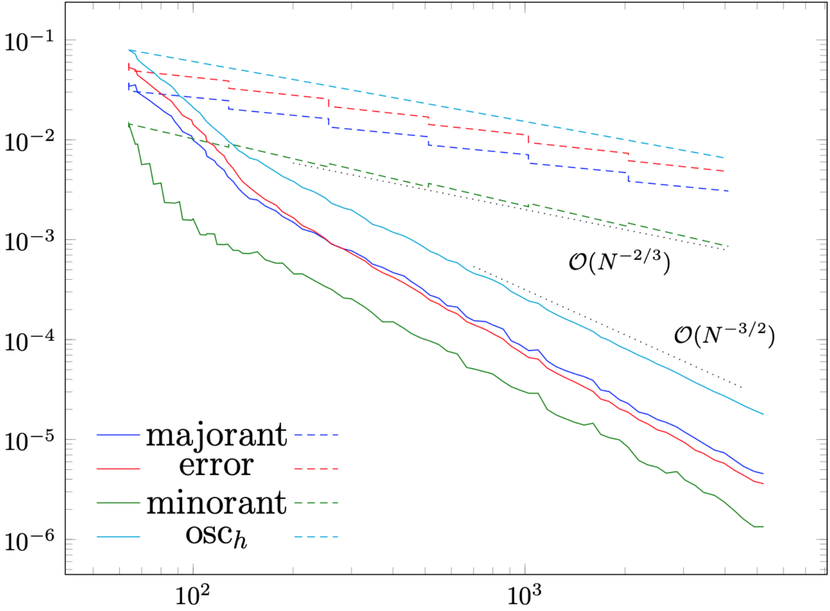
$\#\mathcal{T}_h^S = 919, \ \#\mathcal{F}_h^\Gamma = 368$

$\ell = 38$



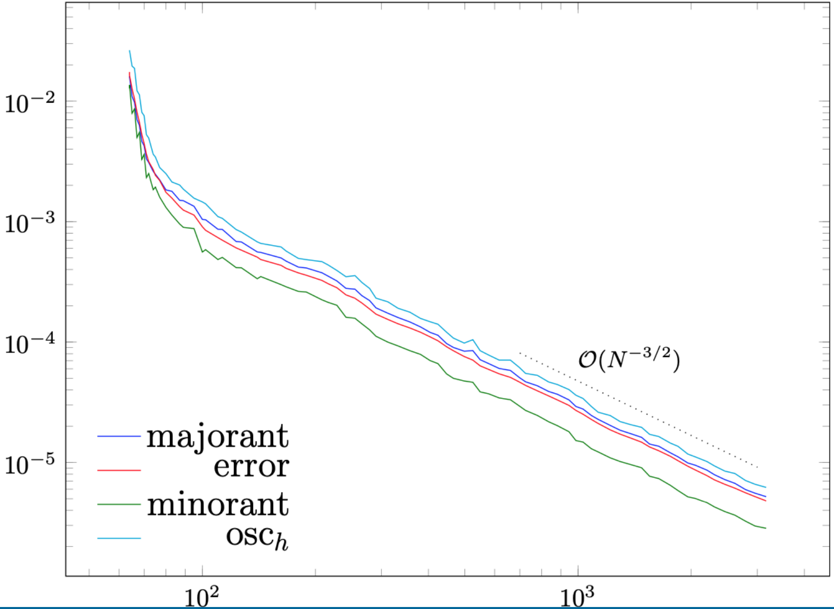
$\#\mathcal{T}_h^S = 3177, \ \#\mathcal{F}_h^\Gamma = 1216$

Numerical results



Direct BEM: non-smooth potential

Numerical results 1/2



Numerical results 2/2

ℓ	$\#\mathcal{F}_h^\Gamma$	$\frac{\#\mathcal{T}_h^S}{\#\mathcal{F}_h^\Gamma}$	$\mathrm{dof}(\mathcal{T}_h^S)$	$\Vert\nabla(u-u_h)\Vert_{L^2(\Omega)}$	$\Vert\nabla w_h\Vert_{L^2(S)}$	$\frac{\Vert\nabla w_h\Vert_{L^2(S)}}{\Vert\nabla(u-u_h)\Vert_{L^2(\Omega)}}$
0	64	2.63	33	$1.74e-2$	$1.61e-2$	0.93
6	69	2.62	37	$5.30e-3$	$4.64e-3$	0.88
12	77	2.69	42	$2.20e-3$	$2.22e-3$	1.01
18	100	2.57	46	$9.06e-4$	$1.04e-3$	1.15
24	140	2.50	79	$5.07e-4$	$5.60e-4$	1.10
30	208	2.58	159	$3.25e-4$	$3.76e-4$	1.16
36	279	2.61	248	$1.88e-4$	$2.21e-4$	1.18
42	404	2.59	381	$1.10e-4$	$1.20e-4$	1.09
48	549	2.62	531	$6.35e-5$	$7.14e-5$	1.12
54	780	2.67	790	$3.93e-5$	$4.33e-5$	1.10
60	1085	2.67	1131	$2.28e-5$	$2.46e-5$	1.08
66	1550	2.73	1695	$1.35e-5$	$1.42e-5$	1.05
72	2203	2.71	2372	$7.78e-6$	$7.92e-6$	1.02
78	3166	2.74	3442	$4.80e-6$	$5.20e-6$	1.08

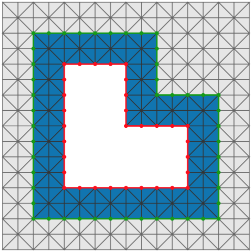
Direct exterior BEM: non-smooth potential

Non-smooth potential

- $\mathbb{R}^2 \backslash \Omega \subsetneq [0, 1/2]^2$ L-shaped domain
- $g \equiv 1 = (1/2 - K)1$ on Γ

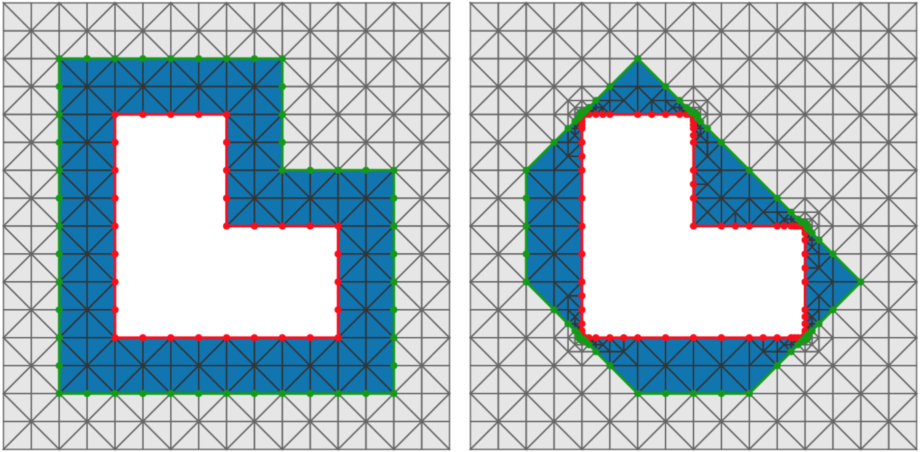
$\Longrightarrow$ indirect BEM = direct BEM

- BEM formulation $V\phi = 1$
- Galerkin BEM with $\phi_h \in \mathcal{P}^0(\mathcal{T}_h^\Gamma)$
- note: u is non-smooth and ϕ is non-smooth



Adaptively generated meshes 1/2

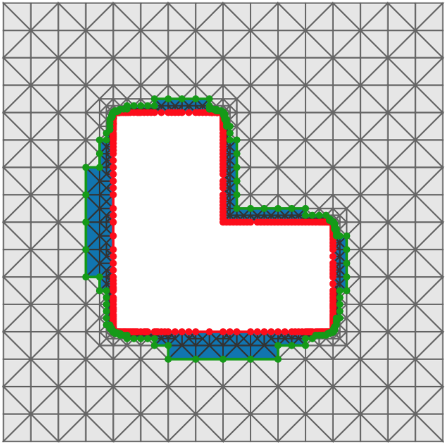
$\ell = 0$ $\qquad$ $\ell = 13$



$\#\mathcal{T}_h^S = 120,\ \#\mathcal{F}_h^\Gamma = 32$ $\qquad$ $\#\mathcal{T}_h^S = 347,\ \#\mathcal{F}_h^\Gamma = 113$

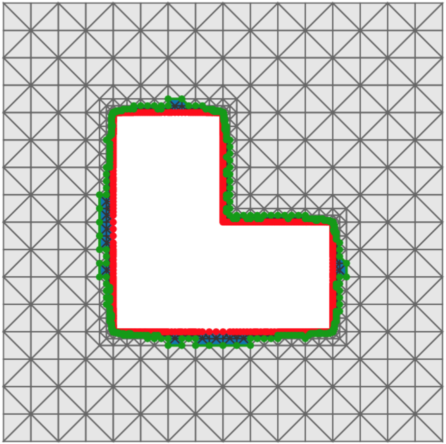
Adaptively generated meshes 2/2

$\ell = 27$

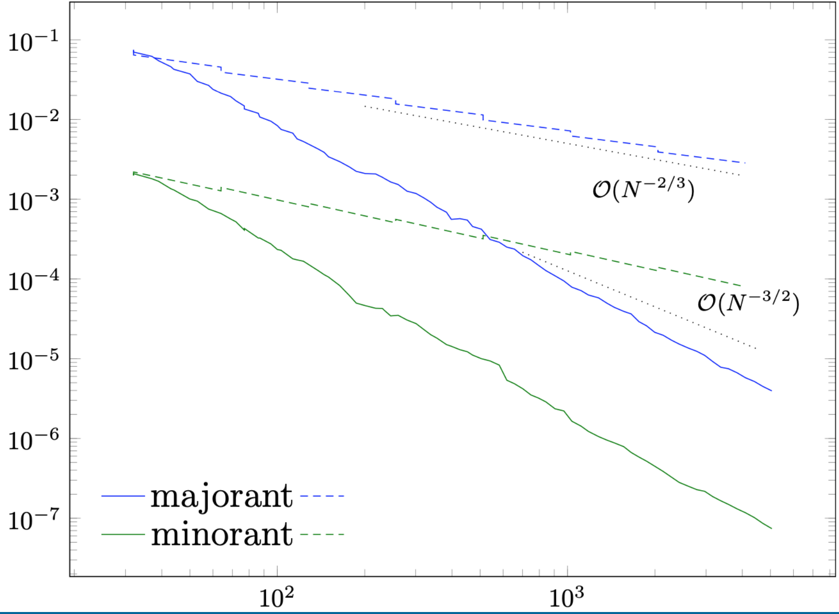


$\#\mathcal{T}_h^S = 1316, \; \#\mathcal{F}_h^\Gamma = 461$

$\ell = 35$



$\#\mathcal{T}_h^S = 2891, \; \#\mathcal{F}_h^\Gamma = 1024$



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^2
10^3
$\mathcal{O}(N^{-2/3})$
$\mathcal{O}(N^{-3/2})$
majorant
minorant

Conclusion

What did I present?

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- new philosophy for a-posteriori error estimation for BEM
  - it is hardly needed that $\|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \approx \text{small}$
  - it is rather needed that $\|\nabla(u - u_h)\|_\Omega \approx \text{small}$
- simple FEM-based strategy for a-posteriori error estimation
  - $2\,\langle g - u_h|_\Gamma\,,\, \boldsymbol{\tau}_h|_\Gamma \cdot \boldsymbol{n}\rangle_\Gamma - \|\boldsymbol{\tau}\|_\omega^2 \leq \|\nabla(u - u_h)\|_\Omega^2 \leq \big(\|\nabla w_h\|_\omega + \mathrm{osc}_h\big)^2$
  - known constants 1 in lower and upper estimate
- upper bound can drive an adaptive algorithm
- empirically $\|\nabla(u - u_h)\|_\Omega \approx \|\nabla w_h\|_\omega$ for lowest-order 2D Galerkin BEM
  - i.e., osc_h only employed for refinement, but negligible for error estimation
- analysis works for arbitrary $\phi_h \approx \phi$ and 2D / 3D
  - Galerkin / collocation
  - exact / inexact solution

What remains open?

Empirical experiments:

- How can we include inexact solvers into the adaptive algorithm?
- How does the new strategy perform for 3D BEM?
  - Galerkin BEM vs. collocation
  - standard discretization vs. isogeometric analysis
  - higher order BEM
  - anisotropic triangulations

Numerical analysis:

- Can we guarantee (optimal) convergence of the adaptive algorithm?
- Can we extend the argument to other relevant PDEs?
- Can we guarantee that $\text{lower bound} \simeq \text{upper bound}$?

CMAM 2020

July 13–17, 2020 – TU Wien – CMAM-9

Computational Methods in Applied Mathematics

https://www.asc.tuwien.ac.at/cmam2020/

The conference is organized under the aegis of the journal Computational Methods in Applied Mathematics (CMAM) and is focused on theoretical numerical analysis and applications to modeling, simulation, and scientific computing. The scientific program will include invited plenary lectures, minisymposia, parallel sessions with contributed talks, and a poster session.

Confirmed plenary speakers

Daniele Boffi (University of Pavia)
Thomas Führer (Pontifical Catholic University of Chile)
Philipp Grohs (University of Vienna)
Jun Hu (Peking University)
Barbara Kaltenbacher (University of Klagenfurt)
Dalibor Lukas (Technical University of Ostrava)
Svetozar Margenov (Bulgarian Academy of Sciences)
Neela Nataraj (IIT Bombay)
Christoph Ortner (University of Warwick)
Amiya Kumar Pani (IIT Bombay)
Sergei Pereverzyev (Austrian Academy of Sciences)
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Venue

Building: TU Wien, Neues EI
Address: Gußhausstraße 27–29, 1040 Vienna

Der Wissenschaftsfonds.

- before: ICOSAHOM, TU Wien
- perfect fit: CMAM, TU Wien
- after: WCCM, Paris

Thank you for your attention!

S. Kurz, D. Pauly, D. Praetorius, S. Repin, D. Sebastian:
Functional a posteriori error estimates for boundary element methods
Preprint arXiv:1912.05789, 2019

Prof. Dr. Dirk Praetorius

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NumPDEs
Work group on Numerics of PDEs

How to localize the data oscillations?

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- computable upper bound $\|\nabla(u-u_h)\|_\Omega \leq \|\nabla w_h\|_\omega + \mathrm{perturbation}$
- $\mathrm{perturbation} := \|(1-J_h)(g-u_h|_\Gamma)\|_{H^{1/2}(\Gamma)} \lesssim \mathrm{osc}_h$

Known upper bound for $\|(1-J_h)(g-u_h|_\Gamma)\|_{H^{1/2}(\Gamma)}$

- standard-FEM to compute $w_h \in \mathcal{S}^p(\mathcal{T}_h|_\omega)$
- J_h one of the following operators
  - H^1-stable $L^2(\Gamma)$-orthogonal projection
  - L^2-variant of Scott–Zhang projection
  - nodal interpolation for $p=1$ and $d=2$
- $g \in H^1(\Omega)$
- $\phi_h \in L^2(\Gamma)$

$\Longrightarrow \quad g-u_h|_\Gamma \in H^1(\Gamma)$

- $\|(1-J_h)(g-u_h|_\Gamma)\|_{H^{1/2}(\Gamma)} \leq C\,\|h^{1/2}\nabla\big[(1-J_h)(g-u_h|_\Gamma)\big]\|_{L^2(\Gamma)}$

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Theorem

- $u \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \times (\nabla \times \boldsymbol{u}) = 0$, $\nabla \cdot \boldsymbol{u} = 0$, and $\boldsymbol{n} \times \boldsymbol{u}|_\Gamma = \boldsymbol{g}$
- $u_h \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \times (\nabla \times \boldsymbol{u}_h) = 0$, $\nabla \cdot \boldsymbol{u}_h = 0$

$$\Longrightarrow \boxed{\max_{\substack{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega) \\ \nabla \times \boldsymbol{\tau} = 0}} \underline{\mathfrak{M}}(\boldsymbol{\tau}; \boldsymbol{u}_h, \boldsymbol{g}) = \|\nabla \times (\boldsymbol{u} - \boldsymbol{u}_h)\|_\Omega^2 = \min_{\substack{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \\ \boldsymbol{n} \times \boldsymbol{w}|_\Gamma = \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{u}_h|_\Gamma}} \overline{\mathfrak{M}}(\boldsymbol{w})}$$

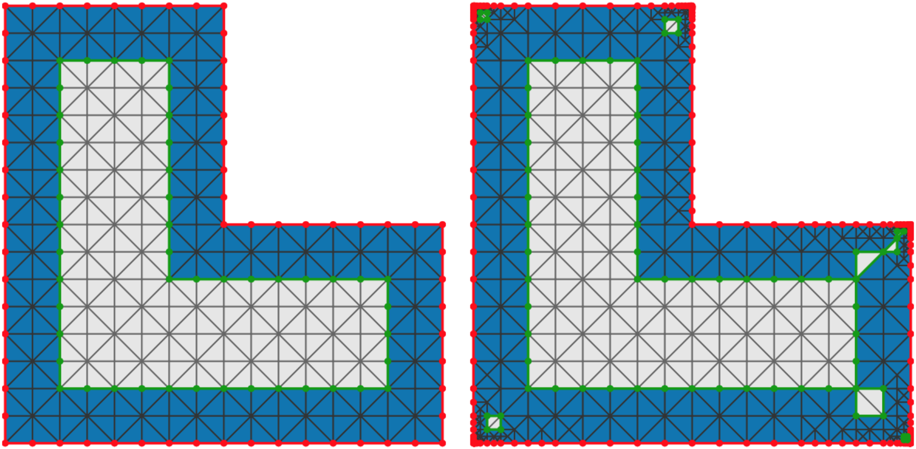
- $\underline{\mathfrak{M}}(\boldsymbol{\tau}; \boldsymbol{u}_h, \boldsymbol{g}) := 2 \, \langle \boldsymbol{g} - \boldsymbol{n} \times \boldsymbol{u}_h|_\Gamma \, , \, \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\tau}|_\Gamma) \rangle_\Gamma - \|\boldsymbol{\tau}\|_\Omega^2$
- $\overline{\mathfrak{M}}(\boldsymbol{w}) := \|\nabla \times \boldsymbol{w}\|_\Omega^2$

Pauly, Praetorius: Work to be started 2020

Indirect BEM: smooth potential, L-shaped domain

Adaptively generated meshes 1/2

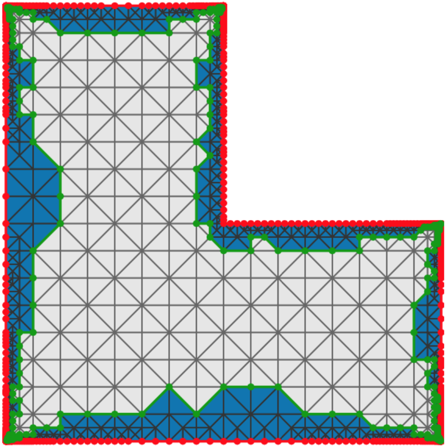
$\ell = 0$ $\qquad$ $\ell = 17$



$\#\mathcal{T}_h^S = 168,\ \#\mathcal{F}_h^\Gamma = 64$ $\qquad$ $\#\mathcal{T}_h^S = 532,\ \#\mathcal{F}_h^\Gamma = 193$

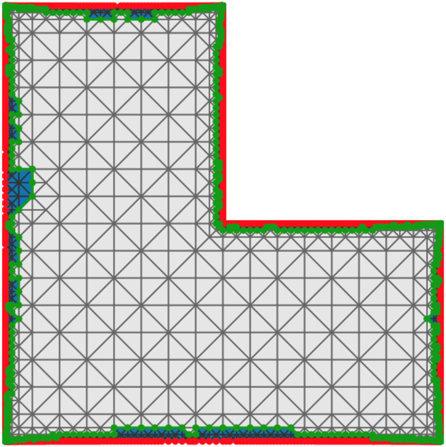
Adaptively generated meshes 2/2

$\ell = 27$



$\#\mathcal{T}_h^S = 1505, \ \#\mathcal{F}_h^\Gamma = 562$

$\ell = 38$



$\#\mathcal{T}_h^S = 4943, \ \#\mathcal{F}_h^\Gamma = 1835$

Numerical results