Central Workshop
November 10, 2017

AFEM with inhomogeneous Dirichlet data

Dirk Praetorius

joint work with

Carsten Carstensen (HU Berlin), Michael Feischl (UNSW Sydney)

TU Wien
Institute for Analysis and Scientific Computing

Outline

Model problem

Strong Form

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = g \quad \text{on } \Gamma = \partial\Omega$$

Weak Form

- Find $u \in H^1(\Omega)$ s.t. $u|_\Gamma = g$ and

$$\langle \nabla u \,, \nabla v \rangle_\Omega = \langle f \,, v \rangle_\Omega \quad \text{for all } v \in H^1_0(\Omega)$$

- suppose g is known as $g \in H^1(\Omega)$
- $u_0 := u - g \in H^1_0(\Omega) \quad \Longrightarrow \quad \langle \nabla u_0 \,, \nabla v \rangle_\Omega = \langle f \,, v \rangle_\Omega - \langle \nabla g \,, \nabla v \rangle_\Omega$

$\Longrightarrow$ ex. unique $u_0 \in H^1_0(\Omega) \quad \Longrightarrow \quad$ ex. unique $u = g + u_0 \in H^1(\Omega)$

Remarks 1/2

Strong Form

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^d \\ u &= g \quad \text{on } \Gamma = \partial\Omega \end{aligned}$$

- Laplace operator is only for ease of presentation

$\Longrightarrow$ possible: general second-order linear elliptic PDEs

- pure Dirichlet problem is only for ease of presentation

$\Longrightarrow$ possible: general inhom. mixed Dirichlet-Neumann-Robin conditions

- Neumann / Robin boundary conditions are part of variational form

$\Longrightarrow$ numerical analysis is much simpler

Remarks 2/2

Weak Form

- Find $u \in H^1(\Omega)$ s.t. $u|_\Gamma = g$ and

$$\langle \nabla u \,, \nabla v \rangle_\Omega = \langle f \,, v \rangle_\Omega \quad \text{for all } v \in H^1_0(\Omega)$$

- if $g \in H^1(\Omega)$ is known

$\Longrightarrow$ AFEM for $u_0 := u - g \in H^1_0(\Omega)$ yields $U_{0\ell} \in \mathcal{S}^p_0(\mathcal{T}_\ell)$

$\Longrightarrow$ $u_\ell := g + U_{0\ell}$ approximation of u

- but: in practice, only $g \in H^{1/2}(\Gamma) := \{ v|_\Gamma \,:\, v \in H^1(\Omega) \}$ is known
- or: postprocessing requires $u \approx U_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$

$\Longrightarrow$ discretize $g \approx G_\ell \in \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma) := \{ V_\ell|_\Gamma \,:\, V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell) \}$

FEM discretization

FEM discretization

- approximate $g \approx G_\ell \in \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$
- find $U_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$ s.t. $U_\ell|_\Gamma = G_\ell$ and

$$\langle \nabla U_\ell \,, \nabla V_\ell \rangle_\Omega = \langle f \,, V_\ell \rangle_\Omega \quad \text{for all } V_\ell \in \mathcal{S}^p_0(\mathcal{T}_\ell)$$

- discrete lifting $G_\ell \in \mathcal{S}^p(\mathcal{T}_\ell) \quad \rightsquigarrow \quad G_\ell(z) = 0$ for interior nodes
- $U_{0\ell} := U_\ell - G_\ell \in \mathcal{S}^p_0(\mathcal{T}_\ell)$ solves

$$\langle \nabla U_{0\ell} \,, \nabla V_\ell \rangle_\Omega = \langle f \,, V_\ell \rangle_\Omega - \langle \nabla G_\ell \,, \nabla V_\ell \rangle_\Omega \quad \text{for all } V_\ell \in \mathcal{S}^p_0(\mathcal{T}_\ell)$$

$\Longrightarrow$ ex. unique $U_{0\ell} \in \mathcal{S}^p_0(\mathcal{T}_\ell)$

$\Longrightarrow$ ex. unique $U_\ell = G_\ell + U_{0\ell} \in \mathcal{S}^p(\mathcal{T}_\ell)$

Discretization of Dirichlet data

- recall: $\| \cdot \|_{H^1(\Omega)} \simeq \|\nabla(\cdot)\|_{L^2(\Omega)} + \| \cdot \|_{H^{1/2}(\Gamma)}$
- AFEM has to control
  - approximation error $\|u - U_\ell\|_{H^{1/2}(\Gamma)} = \|g - G_\ell\|_{H^{1/2}(\Gamma)}$
  - discretization error $\|\nabla(u - U_\ell)\|_{L^2(\Omega)}$
- usually $G_\ell = \mathbb{P}_\ell g$ for some approximation operator $\mathbb{P}_\ell$
- nodal interpolation $\mathbb{I}_\ell$, if $g \in C(\Gamma)$, e.g., $H^1(\Gamma) \subset C(\Gamma)$ for 2D
  - recall: $H^{1/2}(\Gamma) \not\subset C(\Gamma)$
- Clément-type quasi-interpolation w.r.t. $L^2(\Gamma)$
- $L^2(\Gamma)$ orthogonal projection $\Pi_\ell : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$
- Scott-Zhang projection $\mathbb{S}_\ell : H^1(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$

References

- Morin, Nochetto, Siebert: Math. Comp. 72 (2003)
  - $d = 2$, P1, nodal interpolation
  - a posteriori, convergence of AFEM (extended Dörfler)
- Bartels, Carstensen, Dolzmann: Numer. Math. 99 (2004)
  - $d = 2, 3$, P1, nodal interpolation vs. $L^2(\Gamma)$ orthogonal projection
  - a priori, a posteriori
- Sacchi, Veeser: M3AS 16 (2006)
  - $d = 2, 3$, P1, Scott-Zhang projection
  - a posteriori
- Feischl, Page, Praetorius: JCAM 255 (2014)
  - $d = 2$, P1, nodal interpolation
  - optimal convergence of AFEM (combined Dörfler)
- Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)
  - $d \geq 2$, any $H^{1/2}(\Gamma)$ stable projection
  - a posteriori, optimal convergence of AFEM (separate Dörfler)
- Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)
  - $d \geq 2$, Scott-Zhang projection
  - optimal convergence of AFEM (combined Dörfler)

Progress

Scott-Zhang projection

$\mathbb{S}_\ell : H^1(\Omega) \to \mathcal{S}^p(\mathcal{T}_\ell)$ linear projection, continuous w.r.t. $H^1(\Omega)$

- locally defined in Ω: $(\mathbb{S}_\ell v)|_T$ depends only on $v|_{\Omega_\ell(T)}$
- locally defined on Γ: $(\mathbb{S}_\ell v)|_E$ depends only on $v|_{\Gamma_\ell(E)}$
- locally H^1 stable: $\|\nabla \mathbb{S}_\ell v\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\Omega_\ell(T))}$
- local approximation property: $\|(1-\mathbb{S}_\ell)v\|_{L^2(T)} \lesssim h_T \, \|\nabla v\|_{L^2(\Omega_\ell(T))}$
- preserves discrete data in Ω: $(\mathbb{S}_\ell v)|_T = v|_T \quad$ if $\quad v \in \mathcal{S}^p(\mathcal{T}_\ell|_{\Omega_\ell(T)})$
- preserves discrete data on Γ: $(\mathbb{S}_\ell v)|_E = v|_E \quad$ if $\quad v \in \mathcal{S}^p(\mathcal{T}_\ell|_{\Gamma_\ell(T)})$

$\Longrightarrow$ induces $\mathbb{S}_\ell : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ linear projection

- continuous w.r.t. $L^2(\Gamma)$, $H^1(\Gamma)$, and hence $H^s(\Gamma)$ for all $0 \le s \le 1$

Scott, Zhang: Math. Comp. 54 (1990)

Why are stable projections interesting?

- suppose $\mathbb{P}_\ell : X \to X_\ell$ stable projection

$\Longrightarrow$ $\mathbb{P}_\ell$ satisfies a Céa lemma

$$\|x - \mathbb{P}_\ell x\|_X = \min_{x_\ell \in X_\ell} \|(1 - \mathbb{P}_\ell)(x - x_\ell)\|_X \lesssim \min_{x_\ell \in X_\ell} \|x - x_\ell\|_X$$

$\Longrightarrow$ Scott-Zhang projection satisfies

- $\|v - \mathbb{S}_\ell v\|_{H^1(\Omega)} \lesssim \min_{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|v - V_\ell\|_{H^1(\Omega)}$
- $\|\nabla(v - \mathbb{S}_\ell v)\|_{L^2(\Omega)} \lesssim \min_{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|\nabla(v - V_\ell)\|_{L^2(\Omega)}$
- $\|v - \mathbb{S}_\ell v\|_{H^s(\Gamma)} \lesssim \min_{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|v - V_\ell\|_{H^s(\Gamma)}$
  $\lesssim \min_{V_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|h_\ell^{1-s} \nabla_\Gamma (v - V_\ell)\|_{L^2(\Gamma)}$ for all $0 \le s \le 1$
- local weights h_ℓ^s for L^2 and H^1 estimates are possible

Local optimality of Scott-Zhang projector

Lemma (Aurada et al. '13; Veeser '16)

- $\mathbb{S}_\ell : H^1(\Omega) \to \mathcal{S}^p(\mathcal{T}_\ell)$ Scott-Zhang
- $\Pi_\ell : L^2(\Omega) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell)$ orth. projection

$\Longrightarrow \quad \|(1-\Pi_\ell)\nabla v\|_{L^2(T)} \le \|\nabla(1-\mathbb{S}_\ell)v\|_{L^2(T)} \lesssim \|(1-\Pi_\ell)\nabla v\|_{L^2(\Omega_\ell(T))}$

$\Longrightarrow \quad \|h_\ell^s(1-\Pi_\ell)\nabla v\|_{L^2(\Omega)} \simeq \|h_\ell^s\nabla(1-\mathbb{S}_\ell)v\|_{L^2(\Omega)}$ for all $s \in \mathbb{R}$

- same result is true on Γ (instead of Ω)
  - our proof: on Γ, exploits NVB refinement
  - Veeser's proof: on Ω, exploits only shape regularity
- also applicable for comparison results

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Veeser: FoCM 16 (2016)

Discrete lifting operator

- let $\mathbb{L} : H^{1/2}(\Gamma) \to H^1(\Omega)$ lifting operator
  - $\mathbb{L}$ is linear and continuous
  - $(\mathbb{L}v)|_\Gamma = v$ for all $v \in H^{1/2}(\Gamma)$

$\Longrightarrow$ $\mathbb{L}_\ell := \mathbb{S}_\ell \mathbb{L} : H^{1/2}(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ discrete lifting operator

  - $\mathbb{L}_\ell$ is linear and continuous
  - $(\mathbb{L}_\ell v)|_\Gamma = v$ for all $v \in \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$

- later application: $\mathbb{S}_\ell u - \mathbb{L}_\ell g \in \mathcal{S}_0^p(\mathcal{T}_\ell)$

Progress

Céa lemma (first version)

- exploit Galerkin orthogonality

$$\langle \nabla(u - U_\ell)\,, \nabla V_\ell \rangle = 0 \quad \text{for all } V_\ell \in \mathcal{S}_0^p(\mathcal{T}_\ell)$$

Céa lemma (with affine discrete space)

- $U_\ell|_\Gamma = G_\ell \in \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$

$$\Longrightarrow \quad \|\nabla(u - U_\ell)\|_{L^2(\Omega)} = \min_{\substack{W_\ell \in \mathcal{S}^p(\mathcal{T}_\ell) \\ W_\ell|_\Gamma = G_\ell}} \|\nabla(u - W_\ell)\|_{L^2(\Omega)}$$

$$\Longrightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \lesssim \min_{\substack{W_\ell \in \mathcal{S}^p(\mathcal{T}_\ell) \\ W_\ell|_\Gamma = G_\ell}} \|\nabla(u - W_\ell)\|_{L^2(\Omega)} + \|g - G_\ell\|_{H^{1/2}(\Gamma)}$$

Bartels, Carstensen, Dolzmann: Numer. Math. 99 (2004)

Some magic for P1 nodal interpolation in 2D

- suppose: $g \in H^1(\Gamma) \subset C(\Gamma)$
- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell|_\Gamma)$ orth. projection
- $\mathbb{I}_\ell : C(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ nodal interpoland

$\Longrightarrow \quad (\mathbb{I}_\ell g)' = \Pi_\ell(g')$

$\Longrightarrow \quad \|h_\ell^{1/2}(g - \mathbb{I}_\ell g)'\|_{L^2(\Gamma)} = \|h_\ell^{1/2}(1 - \Pi_\ell)g'\|_{L^2(\Gamma)}$

- $\|(1 - \mathbb{I}_\ell)g\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2} g'\|_{L^2(\Gamma)}$

$\Longrightarrow \quad \|(1 - \mathbb{I}_\ell)g\|_{H^{1/2}(\Gamma)} = \|(1 - \mathbb{I}_\ell)^2 g\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(g - \mathbb{I}_\ell g)'\|_{L^2(\Gamma)}$

$\Longrightarrow \quad \|(1 - \mathbb{I}_\ell)g\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(1 - \Pi_\ell)g'\|_{L^2(\Gamma)}$

Carstensen: Math. Comp. 66 (1997)

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Céa lemma (2D, P1, nodal interpolation)

Céa lemma (with Dirichlet oscillations)

- $g \in H^1(\Gamma)$
- $U_\ell|_\Gamma = \mathbb{I}_\ell g$

$$\Longrightarrow \quad \|\nabla(u - U_\ell)\|_{L^2(\Omega)} + \|h_\ell^{1/2}(u - U_\ell)'\|_{L^2(\Gamma)}$$
$$\lesssim \min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \big(\|\nabla(u - W_\ell)\|_{L^2(\Omega)} + \|h_\ell^{1/2}(u - W_\ell)'\|_{L^2(\Gamma)}\big)$$

- $$\begin{aligned}\|\nabla(u - U_\ell)\|_{L^2(\Omega)} &= \min_{\substack{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)\\ V_\ell|_\Gamma = \mathbb{I}_\ell g}} \|\nabla(u - V_\ell)\|_{L^2(\Omega)} \\ &\le \|\nabla\big\{(u - \mathbb{S}_\ell u) + \mathbb{L}_\ell(g - \mathbb{I}_\ell g)\big\}\|_{L^2(\Omega)} \\ &\le \|\nabla(u - \mathbb{S}_\ell u)\|_{L^2(\Omega)} + \|\nabla \mathbb{L}_\ell(g - \mathbb{I}_\ell g)\|_{L^2(\Omega)}\end{aligned}$$
- $\|g - \mathbb{I}_\ell g\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(u - U_\ell)'\|_{L^2(\Gamma)} = \min_{W_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)} \|h_\ell^{1/2}(u - W_\ell)'\|_{L^2(\Gamma)}$

Feischl, Page, Praetorius: JCAM 255 (2014)

Optimal Céa lemma 1/2

Céa lemma (for $H^{1/2}$ stable projections)

- $\mathbb{P}_\ell : H^{1/2}(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ stable projection
- $U_\ell|_\Gamma = \mathbb{P}_\ell g$

$$\Longrightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \lesssim \min_{W_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|u - W_\ell\|_{H^1(\Omega)}$$

- $\|\nabla(u - U_\ell)\|_{L^2(\Omega)} = \min_{\substack{V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell) \\ V_\ell|_\Gamma = \mathbb{P}_\ell g}} \|\nabla(u - V_\ell)\|_{L^2(\Omega)}$
- $\|(1 - \mathbb{P}_\ell)u\|_{H^{1/2}(\Gamma)} \lesssim \min_{W_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|u - W_\ell\|_{H^{1/2}(\Gamma)}$
- plus: playing around with discrete lifting operator $\mathbb{L}_\ell = \mathbb{S}_\ell \mathbb{L}$

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Optimal Céa lemma 2/2

Céa lemma (for $H^{1/2}$ stable projections)

- $\mathbb{P}_\ell : H^{1/2}(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ stable projection
- $U_\ell|_\Gamma = \mathbb{P}_\ell g$

$$\Longrightarrow \quad \|u - U_\ell\|_{H^1(\Omega)} \lesssim \min_{W_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)} \|u - W_\ell\|_{H^1(\Omega)}$$

- note: RHS is independent of $\mathbb{P}_\ell$
- possible choices of $\mathbb{P}_\ell$:
  - $\mathbb{S}_\ell : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ Scott-Zhang projection
  - $H^{1/2}(\Gamma)$ orthogonal projection (only for theory!)
  - $\Pi_\ell : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ orthogonal projection

---

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Karkulik, Pavlicek, Praetorius: Constr. Approx. 38 (2013)

Gaspoz, Heine, Siebert: IMANUM 36 (2016)

Progress

1. Model problem
2. Scott-Zhang projection
3. A priori error analysis
4. A posteriori error analysis
5. Axioms of Adaptivity
6. Separate Dörfler marking
7. Numerical experiment

Residual error estimator

Residual error estimator

- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell|_\Gamma)$ orth. projection
- $\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2$
- $$\eta_\ell(T)^2 = h_T^2\,\|f+\Delta U_\ell\|_{L^2(T)}^2 + h_T\,\|[\partial_n U_\ell]\|_{L^2(\partial T\setminus\Gamma)}^2 + h_T\,\|(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\partial T\cap\Gamma)}^2$$

- note: estimator is independent of $g \approx U_\ell|_\Gamma$
- P1 & 2D $\quad\Longrightarrow\quad \|h_\ell^{1/2}(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\Gamma)} = \|h_\ell^{1/2}(g-\mathbb{I}_\ell g)'\|_{L^2(\Gamma)}$
- $\mathbb{P}_\ell$ stable projection & $U_\ell|_\Gamma = \mathbb{P}_\ell g$ implies that

$$\|(1-\mathbb{P}_\ell)g\|_{H^{1/2}(\Gamma)} \simeq \|(1-\mathbb{S}_\ell)g\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\Gamma)}$$

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Reliability: $\|u - U_\ell\|_{H^1(\Omega)} \lesssim \eta_\ell$

Residual error estimator

- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell|_\Gamma)$ orth. projection
- $\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2$
- $$\eta_\ell(T)^2 = h_T^2\,\|f + \Delta U_\ell\|_{L^2(T)}^2 + h_T\,\|[\partial_n U_\ell]\|_{L^2(\partial T\setminus\Gamma)}^2 + h_T\,\|(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\partial T\cap\Gamma)}^2$$

- assume: $\mathbb{P}_\ell$ is nodal interpolation (P1 & 2D) or stable projection
- it only remains to control $\|\nabla(u - U_\ell)\|_{L^2(\Omega)}$
- auxiliary problem: $-\Delta w = 0$ in Ω and $w|_\Gamma = (1-\mathbb{P}_\ell)g$

$\Longrightarrow$ $\|\nabla(u-U_\ell)\|_{L^2(\Omega)} \le \|\nabla(u-U_\ell-w)\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}$

- $\|\nabla w\|_{L^2(\Omega)} \lesssim \|(1-\mathbb{P}_\ell)g\|_{H^{1/2}(\Gamma)}$ & $u - U_\ell - w \in H^1_0(\Omega)$

Bartels, Carstensen, Dolzmann: Numer. Math. 99 (2004)

Efficiency: $\eta_\ell \lesssim \|u - U_\ell\|_{H^1(\Omega)} + \mathrm{osc}_\ell$

Residual error estimator

- $\Pi_\ell : L^2(\Omega) \to \mathcal{P}^{\max\{p-2,0\}}(\mathcal{T}_\ell)$ orth. projection
- $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^{p-1}(\mathcal{T}_\ell|_\Gamma)$ orth. projection
- $\eta_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2$
- $$\eta_\ell(T)^2 = h_T^2 \,\|f + \Delta U_\ell\|_{L^2(T)}^2 + h_T \,\|[\partial_n U_\ell]\|_{L^2(\partial T\backslash\Gamma)}^2 + h_T \,\|(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\partial T\cap\Gamma)}^2$$

- efficiency only relies on variational formulation & inverse estimates

$$\eta_\ell \le \|\nabla(u - U_\ell)\|_{L^2(\Omega)} + \|h_\ell(1-\Pi_\ell)f\|_{L^2(\Omega)} + \|h_\ell^{1/2}(1-\Pi_\ell)\nabla_\Gamma g\|_{L^2(\Gamma)}$$

Bartels, Carstensen, Dolzmann: Numer. Math. 99 (2004)

Progress

1. Model problem
2. Scott-Zhang projection
3. A priori error analysis
4. A posteriori error analysis
5. Axioms of Adaptivity
6. Separate Dörfler marking
7. Numerical experiment

Adaptive algorithm with combined Dörfler marking

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

Adaptive Algorithm

1. compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. for all $T \in \mathcal{T}_\ell$, compute $\eta_\ell(T)$
3. find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

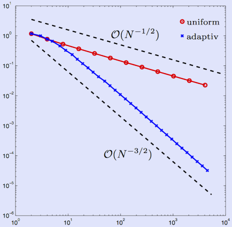
4. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
5. increase counter $\ell \mapsto \ell + 1$ and iterate

Main theorem on rate-optimal AFEM

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- validity of axioms (A1)–(A4)
- $0 < \theta \le 1$

$\Longrightarrow \quad \exists C > 0\ \exists 0 < q < 1\ \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$



- $\mathbb{T}_N := \{\mathcal{T} \in \mathrm{refine}(\mathcal{T}_0) : \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N} \eta_{\mathrm{opt}}\Big) =: \|\eta\|_{\mathbb{A}_s}$$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Axioms of Adaptivity

$\forall \mathcal{T}_\bullet \quad \forall \mathcal{T}_\circ \in \text{refine}(\mathcal{T}_\bullet)$

(A1) $$\Big|\Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} \eta_\circ(T)^2\Big)^{1/2} - \Big(\sum_{T\in\mathcal{T}_\bullet\cap\mathcal{T}_\circ} \eta_\bullet(T)^2\Big)^{1/2}\Big| \le C_{\text{stab}}\,\|U_\circ - U_\bullet\|$$

(A2) $$\sum_{T\in\mathcal{T}_\circ\setminus\mathcal{T}_\bullet} \eta_\circ(T)^2 \le q_{\text{red}} \sum_{T\in\mathcal{T}_\bullet\setminus\mathcal{T}_\circ} \eta_\bullet(T)^2 + C_{\text{red}}\,\|U_\circ - U_\bullet\|^2$$

(A3) $$\|U_\circ - U_\bullet\|^2 \le C_{\text{rel}}^2 \sum_{T\in\mathcal{R}_{\bullet\circ}} \eta_\bullet(T)^2$$

where $\mathcal{T}_\bullet\setminus\mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet\circ} \subseteq \mathcal{T}_\bullet, \quad \#\mathcal{R}_{\bullet\circ} \le C_{\text{rel}}\,\#(\mathcal{T}_\bullet\setminus\mathcal{T}_\circ)$

$\forall \ell, N \ge 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(\|U_{k+1} - U_k\|^2 - \varepsilon\eta_k^2\big) \le C_{\text{orth}}(\varepsilon)\,\eta_\ell^2$$

Axiom (A1): stability on non-refined elements

$\forall \mathcal{T}_\bullet \quad \forall \mathcal{T}_\circ \in \mathrm{refine}(\mathcal{T}_\bullet)$

$$\Big| \Big(\sum_{T \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ} \eta_\circ(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ} \eta_\bullet(T)^2 \Big)^{1/2} \Big| \le C_{\mathrm{stab}} \, |||U_\circ - U_\bullet|||$$

- $$\eta_\circ(T)^2 = h_T^2 \, \|f + \Delta U_\circ\|_{L^2(T)}^2 + h_T \, \|[\partial_n U_\circ]\|_{L^2(\partial T \setminus \Gamma)}^2 + h_T \, \|(1 - \Pi_\circ)\nabla_\Gamma g\|_{L^2(\partial T \cap \Gamma)}^2$$
- inverse triangle inequality + scaling arguments

$$\mathrm{LHS} \le \Big(\sum_{T \in \mathcal{T}_\bullet \cap \mathcal{T}_\circ} h_T \, \|[\partial_n (U_\circ - U_\bullet)]\|_{L^2(\partial T \cap \Omega)}^2 \Big)^{1/2} \lesssim \|\nabla (U_\circ - U_\bullet)\|_{L^2(\Omega)} =: |||U_\circ - U_\bullet|||$$

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Axiom (A2): reduction on refined elements

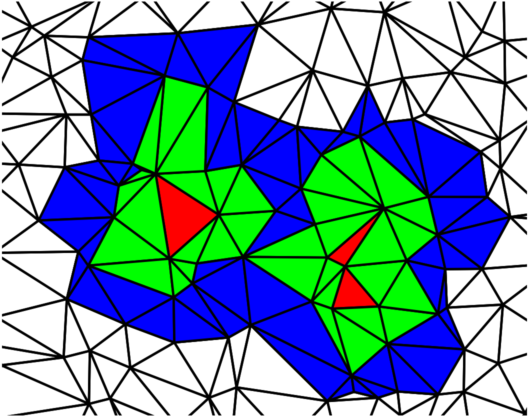
$\forall \mathcal{T}_\bullet \quad \forall \mathcal{T}_\circ \in \mathrm{refine}(\mathcal{T}_\bullet)$

$$\sum_{T \in \mathcal{T}_\circ \setminus \mathcal{T}_\bullet} \eta_\circ(T)^2 \le q_{\mathrm{red}} \sum_{T \in \mathcal{T}_\bullet \setminus \mathcal{T}_\circ} \eta_\bullet(T)^2 + C_{\mathrm{red}} \, \| U_\circ - U_\bullet \|^2$$

- $\eta_\circ(T)^2 = h_T^2 \, \| f + \Delta U_\circ \|_{L^2(T)}^2 + h_T \, \| [\partial_n U_\circ] \|_{L^2(\partial T \setminus \Gamma)}^2 + h_T \, \| (1 - \Pi_\circ) \nabla_\Gamma g \|_{L^2(\partial T \cap \Gamma)}^2$
- $\bigcup(\mathcal{T}_\circ \setminus \mathcal{T}_\bullet) = \bigcup(\mathcal{T}_\bullet \setminus \mathcal{T}_\circ)$
- $h_{T'} \le \frac{1}{2} h_T$ for $\mathcal{T}_\circ \ni T' \subsetneqq T \in \mathcal{T}_\bullet$
- $\| (1 - \Pi_\circ) \nabla_\Gamma g \|_{L^2(\partial T' \cap \Gamma)} \le \| (1 - \Pi_\bullet) \nabla_\Gamma g \|_{L^2(\partial T' \cap \Gamma)}$

Cascón, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Patches



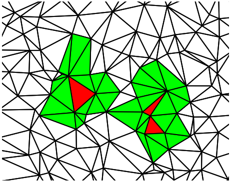
- $\mathcal{S} \subseteq \mathcal{T}_\bullet$ set of elements
- 1st-order patch $\Omega^1_\bullet(\mathcal{S})$
- 2nd-order patch $\Omega^2_\bullet(\mathcal{S})$
- etc.

Axiom (A3): discrete reliability

$\forall \mathcal{T}_\bullet \quad \forall \mathcal{T}_\circ \in \mathrm{refine}(\mathcal{T}_\bullet)$

- suppose: $\mathbb{P}_\ell$ is nodal interpoland (P1 & 2D) or Scott-Zhang projection
- exists $\mathcal{R}_{\bullet\circ} \subseteq \mathcal{T}_\bullet$ with
  - $\mathcal{T}_\bullet \backslash \mathcal{T}_\circ \subseteq \mathcal{R}_{\bullet\circ}$
  - $\#\mathcal{R}_{\bullet\circ} \le C_{\mathrm{rel}} \#(\mathcal{T}_\bullet \backslash \mathcal{T}_\circ)$
  - $\|U_\circ - U_\bullet\|^2 \le C_{\mathrm{rel}}^2 \sum_{T \in \mathcal{R}_{\bullet\circ}} \eta_\bullet(T)^2$

- discrete reliability $\Longrightarrow$ reliability
- $\mathcal{R}_{\bullet\circ} = \mathcal{T}_\bullet \backslash \mathcal{T}_\circ$ for FEM (hom. Dirichlet prbl.)
- $\mathcal{R}_{\bullet\circ} = \Omega_\bullet^1(\mathcal{T}_\bullet \backslash \mathcal{T}_\circ)$ for BEM / FVM
- $\mathcal{R}_{\bullet\circ} = \Omega_\bullet^5(\mathcal{T}_\bullet \backslash \mathcal{T}_\circ)$ here!



Stevenson: Found. Comput. Math. 7 (2007)

Axiom (A4): quasi-orthogonality

$\forall \varepsilon > 0 \quad \forall \ell, N \geq 0$

$$\sum_{k=\ell}^{N} \big(|||U_{k+1} - U_k|||^2 - \varepsilon \eta_k^2 \big) \leq C_{\mathrm{orth}}(\varepsilon)\, \eta_\ell^2$$

- Galerkin orthogonality + symmetry $\Longrightarrow$ Pythagoras theorem

$$|||u - U_{k+1}|||^2 + |||U_{k+1} - U_k|||^2 = |||u - U_k|||^2$$

- telescoping series $\Longrightarrow$ quasi-orth. with $C_{\mathrm{orth}}(\varepsilon) = C_{\mathrm{rel}}^2$, $\varepsilon = 0$

$$\sum_{k=\ell}^{N} |||U_{k+1} - U_k|||^2 = \sum_{k=\ell}^{N} \big(|||u - U_k|||^2 - |||u - U_{k+1}|||^2 \big) \leq |||u - U_\ell|||^2$$

- problem: $U_{k+1} - U_k \notin \mathcal{S}_0^p(\mathcal{T}_{k+1})$ $\Longrightarrow$ no Pythagoras theorem now!

Feischl, Führer, Praetorius: SINUM 52 (2014)

Quasi-Pythagoras theorem

- goal: $\sum_{k=\ell}^{N} \left(\|U_{k+1} - U_k\|^2 - \varepsilon \eta_k^2 \right) \leq C_{\rm orth}(\varepsilon)\, \eta_\ell^2$
- hom. Dirichlet condition: $\|U_{k+1} - U_k\|^2 = \|u - U_k\|^2 - \|u - U_{k+1}\|^2$

Quasi-Pythagoras theorem

- $U_\bullet|_\Gamma = \mathbb{P}_\bullet g$
- $\exists\, C > 0 \quad \forall\, \alpha > 0 \quad \forall\, k \in \mathbb{N}_0$

$$\begin{aligned} \|U_{k+1} - U_k\|^2 &\leq \|u - U_k\|^2 - (1-\alpha)\, \|u - U_{k+1}\|^2 \\ &\quad + \alpha^{-1} C\, \|(\mathbb{P}_{k+1} - \mathbb{P}_k) g\|_{H^{1/2}(\Gamma)}^2 \end{aligned}$$

- proof follows with discrete lifting operator

Feischl, Page, Praetorius: JCAM 255 (2014)

(A4) for P1 nodal interpolation in 2D (simple case)

- abbreviate: $\mathsf{dir}_\bullet := \|h_\bullet^{1/2}(1-\Pi_\bullet)\nabla_\Gamma g\|_{L^2(\Gamma)} \le \eta_\bullet$

- (A4) remains to show: $\|(\mathbb{P}_{k+1}-\mathbb{P}_k)g\|_{H^{1/2}(\Gamma)}^2 \lesssim \mathsf{dir}_k^2 - \mathsf{dir}_{k+1}^2$

- $\mathbb{P}_\bullet = \mathbb{I}_\bullet \quad \Longrightarrow \quad (\mathbb{P}_{k+1}-\mathbb{P}_k)g = (1-\mathbb{P}_k)\mathbb{P}_{k+1}g$

$$
\begin{aligned}
\Longrightarrow \quad \|(\mathbb{P}_{k+1}-\mathbb{P}_k)g\|_{H^{1/2}(\Gamma)}^2 &\lesssim \|h_k^{1/2}(\Pi_{k+1}-\Pi_k)g'\|_{L^2(\Gamma)}^2 \\
&= \|h_k^{1/2}(1-\Pi_k)g'\|_{L^2(\Gamma)}^2 - \|h_k^{1/2}(1-\Pi_{k+1})g'\|_{L^2(\Gamma)}^2 \\
&\le \mathsf{dir}_k^2 - \mathsf{dir}_{k+1}^2
\end{aligned}
$$

Modified mesh-size function

- $\mathrm{diam}(T)$ does not necessarily shrink if T is refined
- remedy: consider $h_\ell|_T := |T|^{1/d}$
  1. $h_\ell|_T \simeq \mathrm{diam}(T)$
  2. $h_{\ell+1} \le h_\ell$ pointwise
  3. $h_{\ell+1} \le q\, h_\ell$ on refined elements $T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$
- sometimes, one needs reduction of h_ℓ on patch $\Omega_\ell^k(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1})$

Proposition (Carstensen, Feischl, Page, P. 14)

- Fix arbitrary $k \in \mathbb{N}$
- For $\mathcal{T}_\ell$ exists mesh-size $h_\ell \in L^\infty(\Omega)$ s.t.
  1. $\mathrm{diam}(T) \lesssim h_\ell|_T \le \mathrm{diam}(T)$
  2. $h_{\ell+1} \le h_\ell$ pointwise
  3. $h_{\ell+1} \le q\, h_\ell$ on patch $\Omega_\ell^k(\mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1})$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

(A4) for Scott-Zhang projection (more advanced)

- (A4) remains to show: $\|(\mathbb{P}_{k+1} - \mathbb{P}_k)g\|_{H^{1/2}(\Gamma)}^2 \lesssim \mathrm{dir}_k^2 - \mathrm{dir}_{k+1}^2$
  - $\mathrm{dir}_\bullet := \|h_\bullet^{1/2}(1 - \Pi_\bullet)\nabla_\Gamma g\|_{L^2(\Gamma)} \leq \eta_\bullet$

- proof of discrete reliability

$$\Longrightarrow \quad \|(\mathbb{S}_{k+1} - \mathbb{S}_k)g\|_{H^{1/2}(\Gamma)}^2 \lesssim \mathrm{dir}_k(\Omega_k^5(\mathcal{T}_k \backslash \mathcal{T}_{k+1}))^2$$

- modified mesh-size function

$$\Longrightarrow \quad h_{k+1} \leq h_k \quad \text{and} \quad h_{k+1} \leq q\, h_k \text{ on } \Omega_k^5(\mathcal{T}_k \backslash \mathcal{T}_{k+1})$$

$$\Longrightarrow \quad (1-q)\, h_k \leq h_k - h_{k+1} \text{ on } \Omega_k^5(\mathcal{T}_k \backslash \mathcal{T}_{k+1})$$

$$\Longrightarrow \quad \mathrm{dir}_k(\Omega_k^5(\mathcal{T}_k \backslash \mathcal{T}_{k+1}))^2 \lesssim \mathrm{dir}_k^2 - \mathrm{dir}_{k+1}^2$$

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Open Problems

- consider $\Pi_\bullet : L^2(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\bullet|_\Gamma)$ orth. projection
- clear: AFEM leads to convergence (by estimator reduction principle)
  - (A1) + (A2) $\implies \eta_{\ell+1}^2 \le q\,\eta_\ell^2 + C\,\|U_{\ell+1} - U_\ell\|_{H^1(\Omega)}^2$
  - nested boundary meshes $\implies$ a priori convergence $\Pi_\ell g \to g_\infty \in L^2(\Gamma)$
  - boundedness in $H^1(\Gamma)$ $\implies$ a priori convergence $\Pi_\ell g \rightharpoonup g_\infty \in H^1(\Gamma)$
  - nested volume meshes $\implies$ a priori convergence $U_\ell \to u_\infty \in H^1(\Omega)$

  $\implies \eta_\ell \to 0$
- (A3) discrete reliability and (A4) quasi-orthogonality are open!
- only needed: $\|(\Pi_{k+1} - \Pi_k)g\|_{H^{1/2}(\Gamma)}^2 \lesssim \mathrm{dir}_k(\Omega_k^n(\mathcal{T}_k \backslash \mathcal{T}_{k+1}))^2$

Progress

Adaptive algorithm with separate Dörfler marking

- adaptivity parameter $0 < \theta_1, \theta_2, \vartheta \leq 1$
- initial triangulation $\mathcal{T}_0$
- split error estimator $\eta_\bullet^2 = \varrho_\bullet^2 + \mathrm{osc}_{D,\ell}^2$

Adaptive Algorithm

1. compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. for all $T \in \mathcal{T}_\ell$, compute $\varrho_\ell(T)$ and $\mathrm{osc}_{D,\ell}(T)$
3. if $\mathrm{osc}_{D,\ell}^2 \leq \vartheta \varrho_\ell^2$, find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.
$$\theta_1 \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \varrho_\ell(T)^2$$
4. if $\mathrm{osc}_{D,\ell}^2 > \vartheta \varrho_\ell^2$, find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.
$$\theta_2 \sum_{T \in \mathcal{T}_\ell} \mathrm{osc}_{D,\ell}(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \mathrm{osc}_{D,\ell}(T)^2$$
5. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$
6. increase counter $\ell \mapsto \ell + 1$ and iterate

All Dirichlet discretizations are equivalent

- $\mathbb{P}_\ell, \widetilde{\mathbb{P}}_\ell : H^{1/2}(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ stable projections

$\Longrightarrow$ $\Big| \sum_{T\in\mathcal{U}_\ell} \varrho_\ell(T)^2 - \sum_{T\in\mathcal{U}_\ell} \widetilde{\varrho}_\ell(T)^2 \Big| \lesssim \mathrm{osc}_{D,\ell}^2$ for all $\mathcal{U}_\ell \subseteq \mathcal{T}_\ell$

$\Longrightarrow$ $\eta_\ell \simeq \widetilde{\eta}_\ell$

- suppose: $0 < \vartheta \ll 1$ sufficiently small

$\Longrightarrow$ separate Dörfler vor η_ℓ implies combined Dörfler for $\widetilde{\eta}_\ell$

- choose: $\widetilde{\mathbb{P}}_\ell := \mathbb{S}_\ell$ Scott-Zhang projection

$\Longrightarrow$ linear convergence of $\widetilde{\eta}_\ell \simeq \eta_\ell$

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

Rate optimality

Theorem (Aurada et al. '13)

- $\mathbb{P}_\ell : H^{1/2}(\Gamma) \to \mathcal{S}^p(\mathcal{T}_\ell|_\Gamma)$ stable projection
- $0 < \theta_1, \theta_2 \le 1$
- $0 < \vartheta \ll 1$ sufficiently small

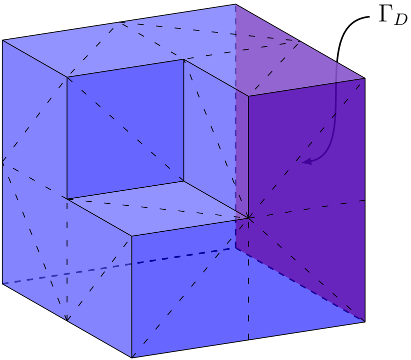
$\Longrightarrow \quad \exists C > 0 \ \exists 0 < q < 1 \ \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$

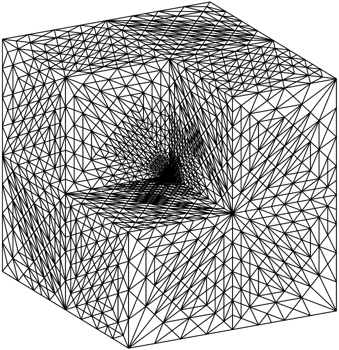
- $\mathbb{T}_N := \{\mathcal{T} \in \mathrm{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta_1, \theta_2, \vartheta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \eta_{\rm opt}\big) =: \|\eta\|_{\mathbb{A}_s}$

Aurada, Feischl, Kemetmüller, Page, Praetorius: M2AN 47 (2013)

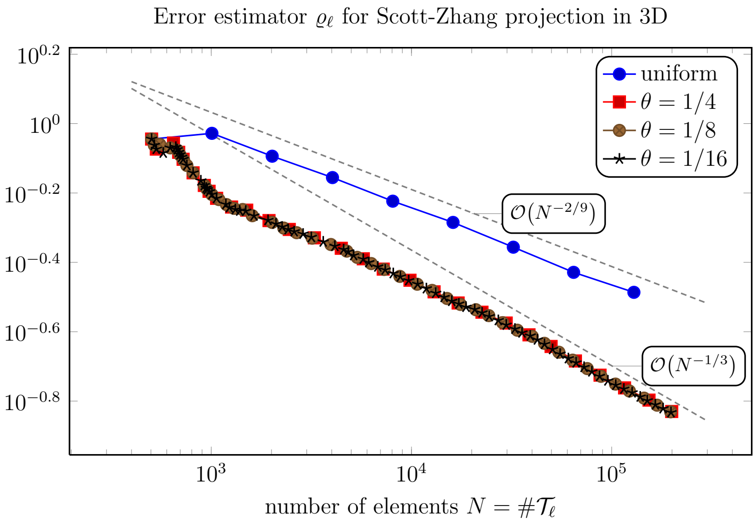
Progress

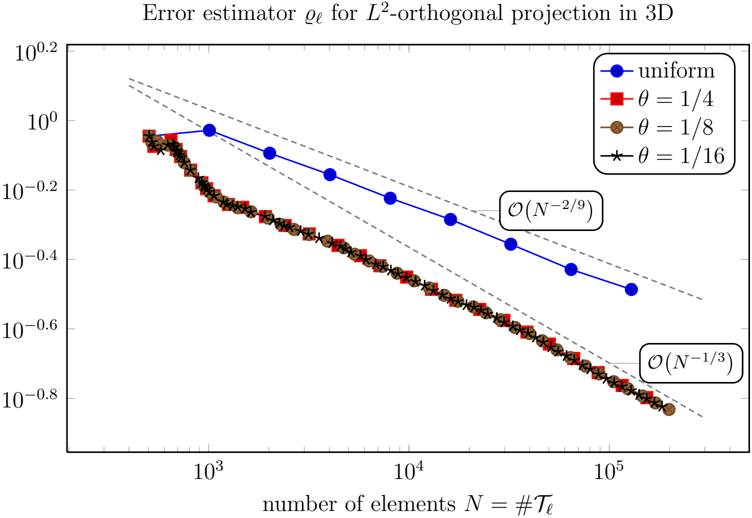




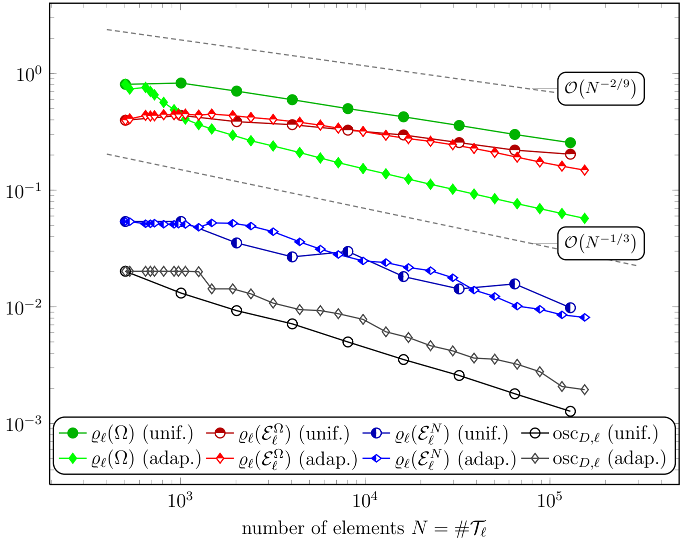
- $\mathcal{T}_0$
- $\#\mathcal{T}_{30} = 200.814$

- prescribed exact solution $u(x, y, z) = (x^2 + y^2 + z^2)^{1/8}$
- mixed Dirichlet-Neumann BVP


Error estimator ϱ_ℓ for Scott-Zhang projection in 3D
$10^{0.2}$
10^{0}
$10^{-0.2}$
$10^{-0.4}$
$10^{-0.6}$
$10^{-0.8}$
10^{3}
10^{4}
10^{5}
number of elements $N = \#\mathcal{T}_\ell$
uniform
$\theta = 1/4$
$\theta = 1/8$
$\theta = 1/16$
$\mathcal{O}(N^{-2/9})$
$\mathcal{O}(N^{-1/3})$


Error estimator ϱ_ℓ for L^2-orthogonal projection in 3D
$10^{0.2}$
10^{0}
$10^{-0.2}$
$10^{-0.4}$
$10^{-0.6}$
$10^{-0.8}$
uniform
$\theta = 1/4$
$\theta = 1/8$
$\theta = 1/16$
$\mathcal{O}(N^{-2/9})$
$\mathcal{O}(N^{-1/3})$
10^3
10^4
10^5
number of elements $N = \#\mathcal{T}_\ell$

Individual contributions of ϱ_ℓ for L^2-orthogonal projection in 3D



Conclusions

- Review of standard FEM with inhomogeneous Dirichlet conditions
- Scott-Zhang projection is important for a priori / a posteriori analysis
  - discrete lifting operator
  - local and global quasioptimal w.r.t. various (semi-) norms
  - Scott-Zhang error $\simeq$ local best approximation error w.r.t. discont. pols.
- equivalent H^1-errors for different $H^{1/2}(\Gamma)$-stable $\mathbb{P}_\ell g \approx g$
- residual error estimator is independent of $H^{1/2}(\Gamma)$-stable $\mathbb{P}_\ell g \approx g$
- discretization with Scott-Zhang satisfies axioms of adaptivity

$\Longrightarrow$ optimal AFEM with combined Dörfler marking

- discretization with other $H^{1/2}(\Gamma)$-stable $\mathbb{P}_\ell g \approx g$

$\Longrightarrow$ optimal AFEM with separate Dörfler marking

- For P1 & 2D, nodal projection leads to optimal AFEM

Open questions for joint research?!

- AFEM with combined Dörfler with $g \approx \mathbb{I}_\ell g$ by $L^2(\Gamma)$-orth. projection
  - discrete reliability (A3) is critical?!
  - quasi-orthogonality (A4) is open, but should follow
- other FEM formulations & AFEM for Dirichlet-Laplace problem
  - mixed FEM
  - Nietsche's method
  - penalization method
- How to incorporate inhom. Dirichlet data & comparison results for
  - non-conforming FEM
  - FVM

Thanks for listening!

M. Feischl, M. Page, D. Praetorius:
Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data
JCAM, 255 (2014), 481-501.

M. Aurada, M. Feischl, J. Kemetmüller, M. Page, D. Praetorius:
Each $H^{1/2}$-stable projection yields convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data in $\mathbb{R}^d$
M2AN, 47 (2013), 1207–1235.

Dirk Praetorius

TU Wien
Institute for Analysis
and Scientific Computing

dirk.praetorius@tuwien.ac.at
http://www.asc.tuwien.ac.at/∼praetorius