August 25, 2022 – Innsbruck – BEAF Workshop 2022

Adaptive FEM with quasi-optimal cost for nonlinear PDEs

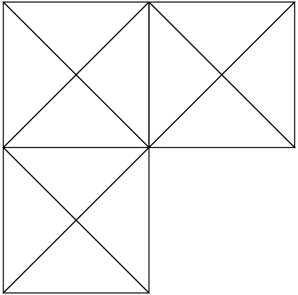
Dirk Praetorius

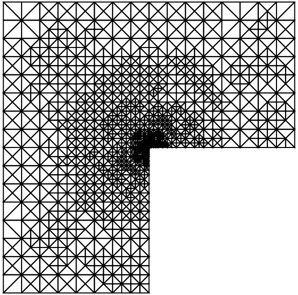
TU Wien
Institute of Analysis and Scientific Computing

What is all about?

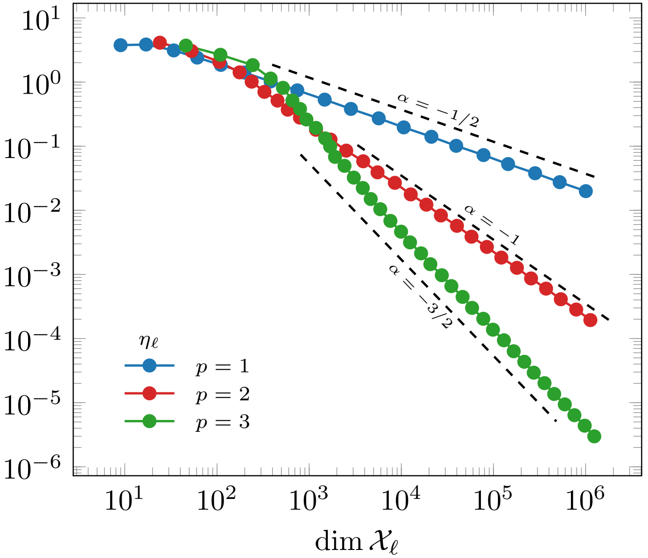
Example: AFEM for Laplace model problem

- solve $-\Delta u = 0$ in $\Omega \subset \mathbb{R}^2$ subject to appropriate boundary conditions
- prescribed exact solution $u(x) = r^{2/3} \sin(2\varphi/3)$
- adaptivity driven by standard residual error estimator

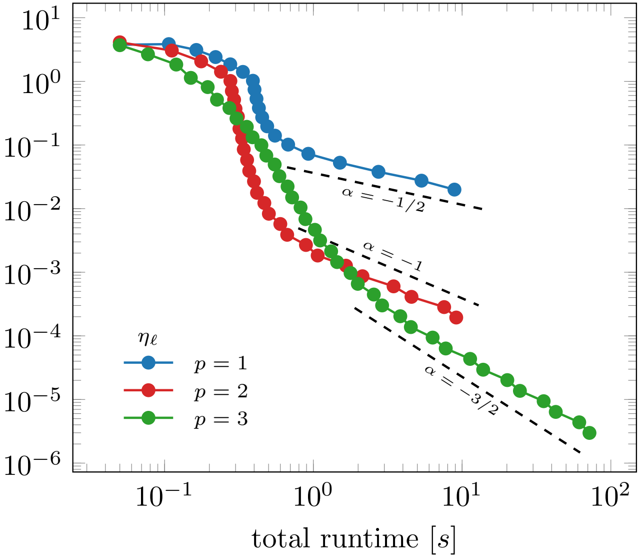




Optimal convergence wrt. dofs



Optimal convergence wrt. time



Optimal complexity = quasi-optimal cost

- $k = 1, \dots, K(\ell)$ solver steps $u_\ell^k \approx u_\ell^\star$ per mesh $\mathcal{T}_\ell$
- index set $(\ell, k) \in \mathcal{Q} \subset \mathbb{N}_0^2$ of discrete approximations u_ℓ^k
- suppose cost $\mathcal{O}(\#\mathcal{T}_\ell)$ for solver step / estimate / mark / refine on mesh $\mathcal{T}_\ell$

Aim for thorough proof of

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N \ge \#\mathcal{T}_0} \Big(N^s \min_{\#\mathcal{T}_{\rm opt} \le N} \big[|||u^\star - u_{\rm opt}^\star||| + \eta_{\rm opt}(u_{\rm opt}^\star) \big] \Big) \qquad \text{for all } s > 0$$

$$\overset{\checkmark}{\simeq} \sup_{(\ell,0) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s \big[|||u^\star - u_\ell^\star||| + \eta_\ell(u_\ell^\star) \big]$$

$$\overset{?}{\simeq} \sup_{(\ell,k) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ (\ell',k') \le (\ell,k)}} \#\mathcal{T}_{\ell'} \Big)^s \big[|||u^\star - u_\ell^k||| + \eta_\ell(u_\ell^k) \big]$$

Some(!) literature...

NumPDEs
ASC ▸ TUWIEN

- optimal complexity with adaptive wavelet FEM: Cohen, Dahmen, DeVore (2001, 2003), ...
- plain convergence: Dörfler (1996), Morin, Nochetto, Siebert (2000), Veeser (2005), Mekchay, Nochetto (2005), Morin, Siebert, Veeser (2008), Siebert (2011), Garau, Morin, Zuppa (2011), Aurada, Ferraz-Leite, Praetorius (2012), Gantner, Praetorius (2021), ...
- linear convergence: Diening, Kreuzer (2008), Cascon, Kreuzer, Nochetto, Siebert (2008), Becker, Mao (2009), Kreuzer, Siebert (2011), Belenki, Diening, Kreuzer (2012), Garau, Morin, Zuppa (2012), Cascon, Nochetto (2012), Feischl, Führer, Praetorius (2014), ...
- optimal convergence rates: Binev, Dahmen, DeVore (2004), Stevenson (2007), Cascon, Kreuzer, Nochetto, Siebert (2008), Belenki, Diening, Kreuzer (2012), Garau, Morin, Zuppa (2012), Cascon, Nochetto (2012), Feischl, Führer, Praetorius (2014), Carstensen, Feischl, Page, Praetorius (2014), ...
- optimal complexity: Stevenson (2007), Carstensen, Gedicke (2012), Gantner, Haberl, Praetorius, Schimanko (2021), Haberl, Praetorius, Schimanko, Vohralik (2021), ...

A nonlinear model problem

Abstract setting

- $\mathcal{H}$ separable Hilbert space with norm $\|\cdot\|$, $\mathcal{X}_\ell$ closed subspace
- nonlinear operator $\boldsymbol{A}\colon \mathcal{H} \to \mathcal{H}^*$ such that, for all $u, v, w \in \mathcal{H}$,
  (O1) strongly monotone $\alpha\,\|u-v\|^2 \le \langle \boldsymbol{A}u - \boldsymbol{A}v\,,\,u-v\rangle$
  (O2) Lipschitz continuous $\langle \boldsymbol{A}u - \boldsymbol{A}v\,,\,w\rangle \le L\,\|u-v\|\,\|w\|$

Main Theorem on Strongly Monotone Operators (Zarantonello '60)

- exists unique $u^\star \in \mathcal{H}$ such that $\langle \boldsymbol{A}u^\star\,,\,v\rangle = 0$ for all $v \in \mathcal{H}$

Corollary

- exists unique $u_\ell^\star \in \mathcal{X}_\ell$ such that $\langle \boldsymbol{A}u_\ell^\star\,,\,v_\ell\rangle = 0$ for all $v_\ell \in \mathcal{X}_\ell$
- $\|u^\star - u_\ell^\star\| \le \dfrac{L}{\alpha} \min\limits_{u_\ell \in \mathcal{X}_\ell} \|u^\star - u_\ell\|$

Zarantonello: U.S. Army Math. Res. Center, Tech. Report, 160 (1960)

Zarantonello iteration

- constructive proof by Banach fixpoint theorem
- $I_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^*$ Riesz mapping $\quad \langle I_{\mathcal{H}} u \,,\, v \rangle = (u \,,\, v)_{\mathcal{H}}$
- $\Phi(u) := u - \delta\, I_{\mathcal{H}}^{-1} \boldsymbol{A} u$ with $0 < \delta < 2\alpha / L^2$

$\Longrightarrow \quad \|\Phi(u) - \Phi(v)\| \le \kappa \, \|u - v\|$ with $\kappa := \big[1 - \delta(2\alpha - \delta L^2) \big]^{1/2}$

1. Φ has unique fixpoint $u^\star \in \mathcal{H}$ $\quad (\iff \boldsymbol{A} u^\star = 0 \;)$
2. $u^0 \in \mathcal{H}$ arbitrary, $\quad u^k := \Phi(u^{k-1}) \quad \Longrightarrow \quad \|u^\star - u^k\| \to 0$

Zarantonello: U.S. Army Math. Res. Center, Tech. Report, 160 (1960)

Discrete Zarantonello iteration / linearization

- $u_\ell^0 \in \mathcal{X}_\ell$ arbitrary, $\quad u_\ell^k := \Phi_\ell(u_\ell^{k-1})$

$\Longrightarrow \quad \|u_\ell^\star - u_\ell^k\| \le \kappa \, \|u_\ell^\star - u_\ell^{k-1}\|$ for all $k \in \mathbb{N}$

- triangle inequality $\quad \Longrightarrow \quad \dfrac{1-\kappa}{\kappa} \, \|u_\ell^\star - u_\ell^k\| \le \|u_\ell^k - u_\ell^{k-1}\| \le (1+\kappa) \, \|u_\ell^\star - u_\ell^{k-1}\|$

Discrete Zarantonello iteration

- solve $\quad (u_\ell^k \, , \, v_\ell)_{\mathcal{H}} = (u_\ell^{k-1} \, , \, v_\ell)_{\mathcal{H}} - \delta \, \langle \boldsymbol{A} u_\ell^{k-1} \, , \, v_\ell \rangle$ for all $v_\ell \in \mathcal{X}_\ell$

- i.e., each step of Zarantonello iteration solves one Laplace problem

Congreve, Wihler: J. Comput. Appl. Math., 311 (2017)

AFEM with exact solver

Well-known AFEM algorithm

Input: $\mathcal{T}_0$, $0 < \theta \leq 1$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE:** compute $u_\ell^\star$
- **ESTIMATE:** compute $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^\star)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^\star)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete solutions $u_\ell^\star$ and corresponding estimators $\eta_\ell(u_\ell^\star)$

- in practice: $\mathcal{M}_\ell$ is always chosen with quasi-minimal cardinality

Dörfler: SIAM J. Numer. Anal., 33 (1996)

Stevenson: Found. Comput. Math., 7 (2007)

Plain convergence

- recall: (O1) strongly monotone & (O2) Lipschitz continuous

Proposition

- (O1) & (O2) & residual error estimator
- arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q < 1$ and $C > 0$ such that

$$\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star) \le q\,\eta_{\ell-1}(u_{\ell-1}^\star) + C\,\|u_\ell^\star - u_{\ell-1}^\star\| \xrightarrow{\ell\to\infty} 0$$

Gantner, Haberl, Praetorius, Stiftner: IMA J. Numer. Anal., 38 (2018)
Gantner, Praetorius: IMA J. Numer. Anal., 42 (2018)

Energy minimization

- suppose that $\boldsymbol{A}$ has a potential, i.e.,

(O3) $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ such that $\langle \boldsymbol{A}u\,,\,v\rangle = \lim_{r\to 0} \dfrac{\mathcal{J}(u+rv) - \mathcal{J}(u)}{r}$ for all $u, v \in \mathcal{H}$

Lemma

- (O1) & (O2) & (O3)
- $v_\ell \in \mathcal{X}_\ell$

$$\Longrightarrow \quad \frac{\alpha}{2}\,\|u_\ell^\star - v_\ell\|^2 \le \mathcal{J}(v_\ell) - \mathcal{J}(u_\ell^\star) \le \frac{L}{2}\,\|u_\ell^\star - v_\ell\|^2$$

- i.e., Galerkin formulation is equivalent to minimization of $\mathcal{J}$ over $\mathcal{X}_\ell$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

Linear convergence with optimal rates

Theorem (Diening, Kreuzer '08; ... ; Carstensen, Feischl, Page, P. '14)

- (O1) & (O2) & (O3) & residual error estimator

1. arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q < 1$ s.t. $\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star) \lesssim q^{\ell-\ell'} \eta_{\ell'}(u_{\ell'}^\star)$ for all $\ell' \le \ell$

2. sufficiently small $0 < \theta \le 1$

$\Longrightarrow$ $\sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^s \, \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_s}$ for all $s > 0$

- required: certain Pythagoras-type quasi-orthogonality for linear convergence
- here: energy $\mathcal{J}(u_\ell^\star) - \mathcal{J}(u_{\ell+1}^\star) = [\mathcal{J}(u_\ell^\star) - \mathcal{J}(u^\star)] - [\mathcal{J}(u_{\ell+1}^\star) - \mathcal{J}(u^\star)]$

Diening, Kreuzer: SIAM J. Numer. Anal., 46 (2008)
Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

AFEM with iterative solver

Stopping criterion for linearization

- reliability & stability of residual error estimator

$$\|u^\star - u_\ell^k\| \le \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| \overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\|$$

$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$

$$\overset{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$$

$\Longrightarrow$ contraction allows for a-posteriori error control

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

$\Longrightarrow$ stop linearization for $K = k$ with $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ a-posteriori error control for all u_ℓ^k but u_0^0

AFEM with contractive linearization

Input: $\mathcal{T}_0$, u_0^0, $0 < \theta \le 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \dots, K$, **repeat**
  - compute u_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

  until $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$

- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$

Output: Discrete solutions u_ℓ^k and corresponding estimator $\eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$

- note: number of solver steps $K = K(\ell)$ might vary with ℓ

Gantner, Haberl, Praetorius, Stiftner: IMA J. Numer. Anal., 38 (2018)
Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Main result 1: Full linear convergence

Theorem (Gantner, Haberl, P., Schimanko '21)

- (O1) & (O2) & (O3) & residual error estimator
- $|\ell, k| := \#\{(\ell', k') \in \mathcal{Q} : u_{\ell'}^{k'}$ computed earlier than $u_\ell^k\}$
- quasi-error $\Delta_\ell^k := \big[|||u^\star - u_\ell^k||| + \eta_\ell(u_\ell^k) \big]$
- arbitrary u_0^0, arbitrary $0 < \theta \leq 1$, sufficiently small $\lambda > 0$ (later arbitrary!)

$\Longrightarrow$ exists $0 < q < 1$ such that $|||u^\star - u_\ell^k||| \leq \Delta_\ell^k \lesssim q^{|\ell,k| - |\ell',k'|} \Delta_{\ell'}^{k'}$

- note from [CFPP14]: linear convergence $\iff \displaystyle\sum_{\substack{(\ell,k)\in\mathcal{Q} \\ |\ell,k|>|\ell',k'|}} \Delta_\ell^k \lesssim \Delta_{\ell'}^{k'}$

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)
Führer, Haberl, Praetorius, Schimanko: Numer. Math., 141 (2019)
Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Perturbation argument = Why sufficiently small λ?

- $\eta_\ell(u_\ell^K) \overset{\text{stability}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,\|u_\ell^\star - u_\ell^K\| \overset{\text{solver}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\,\|u_\ell^K - u_\ell^{K-1}\|$

  $\overset{\text{terminate}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$

- similarly: $\eta_\ell(u_\ell^\star) \le \eta_\ell(u_\ell^K) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$

$\Longrightarrow \quad (1 - C\lambda)\,\eta_\ell(u_\ell^K) \le \eta_\ell(u_\ell^\star) \le (1 + C\lambda)\,\eta_\ell(u_\ell^K)$

- moreover: Dörfler marking for $(\theta, u_\ell^K) \iff$ Dörfler marking for $(\widetilde\theta, u_\ell^\star)$

$\Longrightarrow$ linear convergence of $\eta_\ell(u_\ell^K) \simeq \eta_\ell(u_\ell^\star)$

$\Longrightarrow$ summability of $\Delta_\ell^k = \|u^\star - u_\ell^k\| + \eta_\ell(u_\ell^k) \simeq \|u_\ell^\star - u_\ell^k\| + \eta_\ell(u_\ell^k)$ by geometric series

Main result 2: Rates = complexity

Corollary (Gantner, Haberl, P., Schimanko '21)

- suppose full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k$$

- $M := \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \, \Delta_\ell^k \le \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \overset{?!}{\lesssim} M$

- definition of M $\quad\Longrightarrow\quad \#\mathcal{T}_{\ell'} \le M^{1/s} \, (\Delta_{\ell'}^{k'})^{-1/s}$

- full linear convergence $\quad\Longrightarrow\quad \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \le M^{1/s} \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} (\Delta_{\ell'}^{k'})^{-1/s} \lesssim M^{1/s} \, (\Delta_\ell^k)^{-1/s}$

Main result 3: Quasi-optimal computational cost

Theorem (Gantner, Haberl, P., Schimanko '21)

- arbitrary $s > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} N^s \, \Delta_{\rm opt}^\star < \infty$
- sufficiently small $\quad 0 < \theta \leq 1 \quad$ and $\quad \lambda > 0$
- full linear convergence of Δ_ℓ^k

$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_s} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_s} \, , \Delta_0^0 \big\}$$

Why sufficiently small λ and θ?

- full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k$$

and $\quad \sup_{(0,k)\in\mathcal{Q}} (\#\mathcal{T}_0)^s \Delta_0^k \lesssim \Delta_0^0$

- sufficiently small $\lambda > 0$

$$\Longrightarrow \quad \sup_{\substack{(\ell,k)\in\mathcal{Q} \\ \ell>0}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k \lesssim \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^K \simeq \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^\star \simeq \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star)$$

- sufficiently small $0 < \theta \le 1$

$$\Longrightarrow \quad \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_s}$$

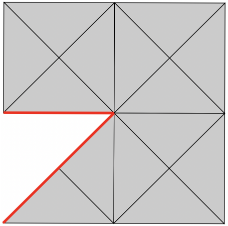
Numerical experiment

mixed BVP

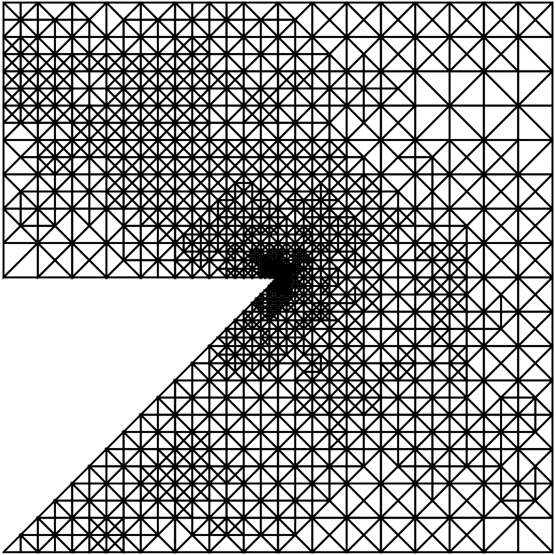
$$
\begin{aligned}
-\mathrm{div}\big[\,\mu(|\nabla u^\star|)\,\nabla u^\star\,\big] &= f \quad \text{in } \Omega\\
\mu(|\nabla u^\star|)\,\nabla u^\star\cdot\boldsymbol{n} &= g \quad \text{on } \Gamma_N\\
u^\star &= 0 \quad \text{on } \Gamma_D
\end{aligned}
$$

- $\mu(t) := 2 + \dfrac{1}{\sqrt{1+t^2}} \implies \alpha = 2, \quad L = 3$ wrt. $\|\!|\cdot|\!\| = \|\nabla(\cdot)\|_{L^2(\Omega)}$

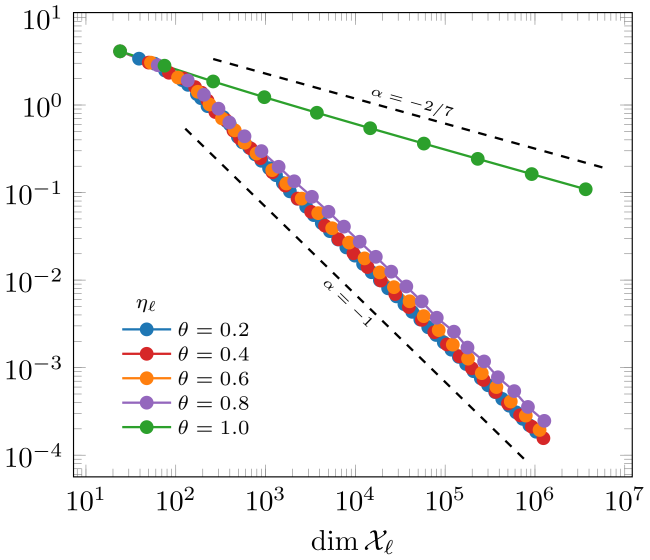
- $$
\begin{aligned}
\eta_\ell(T,u_\ell) = h_T^2\,\|f + \mathrm{div}\big[\,\mu(|\nabla u_\ell|)\,\nabla u_\ell\,\big]\|_{L^2(T)}^2\\
+ h_T\,\|g - \mu(|\nabla u_\ell|)\,\nabla u_\ell\cdot\boldsymbol{n}\|_{L^2(\partial T\cap\Gamma_N)}^2\\
+ h_T\,\|\,[\![\,\mu(|\nabla u_\ell|)\,\nabla u_\ell\cdot\boldsymbol{n}\,]\!]\,\|_{L^2(\partial T\cap\Omega)}^2
\end{aligned}
$$



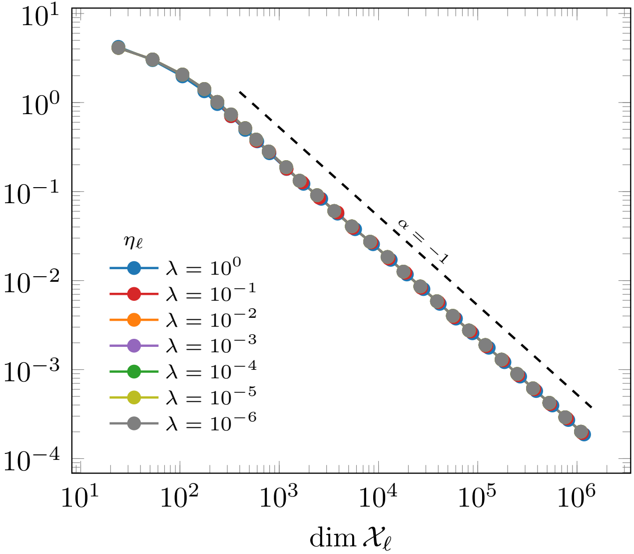
Adaptive mesh $\#\mathcal{T}_7 = 2.873$



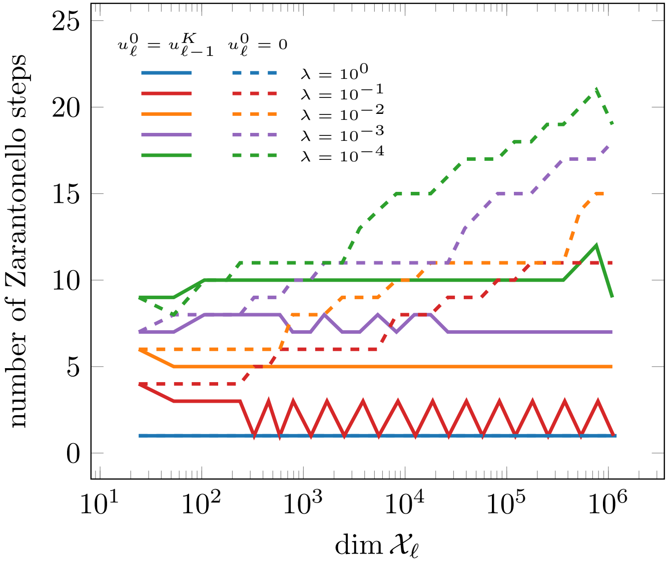
Dependence on θ for $\lambda = 10^{-2}$ $(p = 2)$



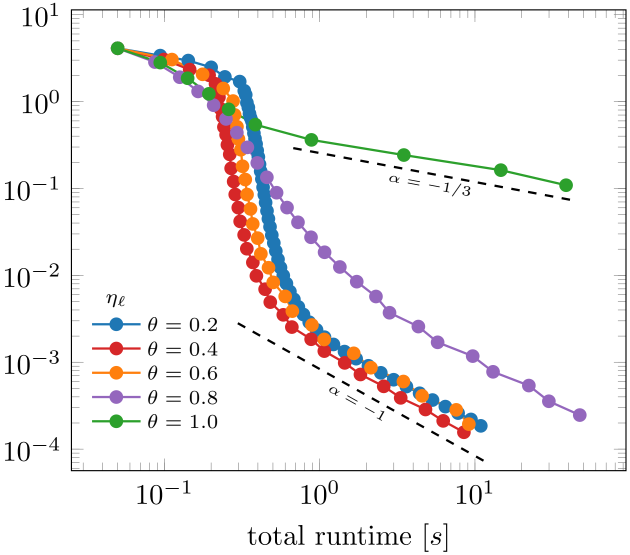
Dependence on λ for $\theta = 0.2 \quad (p = 2)$



Nested iteration vs. $u_\ell^0 := 0 \quad (p = 2,\ \theta = 0.2)$

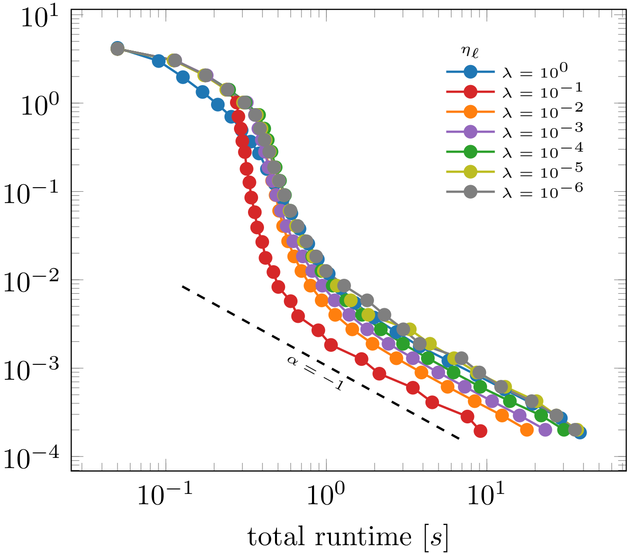


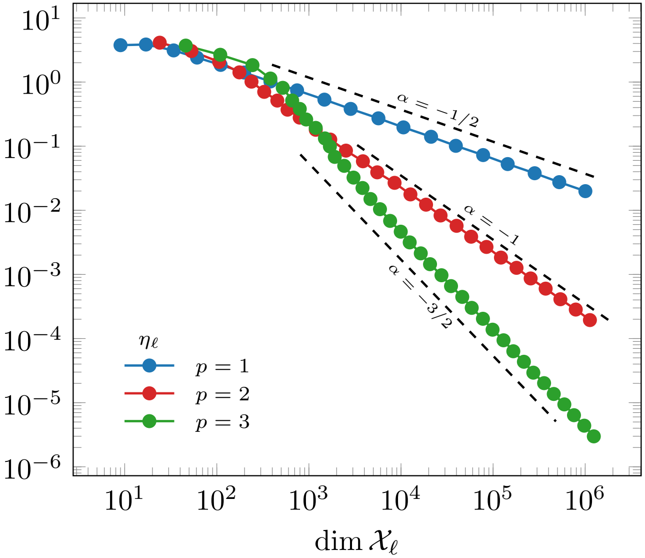
Dependence on θ for $\lambda = 10^{-2}$ wrt. time $(p = 2)$



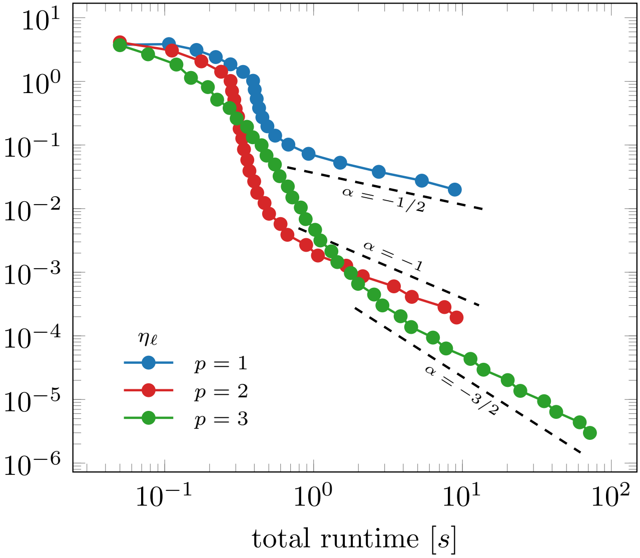
Dependence on λ for $\theta = 0.2$ wrt. time $(p = 2)$

NumPDEs ASC TUWIEN




10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
$\dim \mathcal{X}_\ell$
η_ℓ
$p = 1$
$p = 2$
$p = 3$
$\alpha = -1/2$
$\alpha = -1$
$\alpha = -3/2$

Optimal convergence wrt. time $(\theta = 0.2,\ \lambda = 0.1)$



Extensions

Extension 1: Arbitrary λ for linear SPD problems

$$\begin{aligned} -\mathrm{div}\,\boldsymbol{A}\nabla u^\star + cu^\star &= f - \mathrm{div}\,\vec{\boldsymbol{f}} && \text{in } \Omega \subset \mathbb{R}^d \\ u^\star &= 0 && \text{on } \partial\Omega \end{aligned}$$

- contractive iterative solvers: optimal PCG [CNX12] or multigrid [WZ17]

$\Longrightarrow$ [GHPS21] full linear convergence holds for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda > 0$

- proof of [GHPS21] extends the argument of [CKNS08] to $\Lambda_\ell^k := |||u^\star - u_\ell^k|||^2 + \gamma\,\eta_\ell(u_\ell^k)^2$
  - $\Lambda_\ell^{k+1} \le q\,\Lambda_\ell^k$
  - $\Lambda_{\ell+1}^0 \le q\,\Lambda_\ell^{K-1}$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)
Chen, Nochetto, Xu: Numer. Math., 120 (2012)
Wu, Zheng: Appl. Numer. Math., 113 (2017)
Cascon, Kreuzer, Nochetto, Siebert: SIAM J. Numer. Anal. 46 (2008)

Extension 2: Full linear convergence for arbitrary λ!

NumPDEs ASC TUWIEN

- energy minimization problems satisfying (O1) & (O2) & (O3)

- stop linearization if $|\mathcal{J}(u_\ell^K) - \mathcal{J}(u_\ell^{K-1})|^{1/2} \le \lambda\,\eta_\ell(u_\ell^K)$
  - instead of $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\,\eta_\ell(u_\ell^K)$

$\Longrightarrow$ [HPW21] full linear convergence holds for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda > 0$ and holds for Zarantonello iteration & Kacanov iteration & damped Newton method

- proof of [HPW21] extends argument of [GHPS21] to $\Lambda_\ell^k := [\mathcal{J}(u_\ell^k) - \mathcal{J}(^\star)] + \gamma\,\eta_\ell(u_\ell^k)^2$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)

Extension 3: Including an algebraic solver!

NumPDEs ASC TUWIEN

- employ contractive algebraic solver for the linear systems, i.e., $u^\star \approx u_\ell^\star \approx u_\ell^k \approx u_\ell^{k,j}$

$\Longrightarrow$ index set $(\ell, k, j) \in \mathcal{Q} \subset \mathbb{N}_0^3$

- stop algebraic solver if $|||u_\ell^{k,J} - u_\ell^{k,J-1}||| \le \mu \, [\, \eta_\ell(u_\ell^{k,J}) + |||u_\ell^{k,J} - u_\ell^{k-1,J}||| \,]$
- stop Zarantonello linearization if $|||u_\ell^{K,J} - u_\ell^{K-1,J}||| \le \lambda \, \eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ [HPSV21] full linear convergence for sufficiently small λ and μ

- full linear convergence for arbitrary λ and sufficiently small μ [MPV22+] and extension to Zarantonello iteration & Kacanov iteration & damped Newton method

Haberl, Praetorius, Schimanko, Vohralik: Numer. Math., 147 (2021)
Miraci, Praetorius, Vohralik: in progress (2022+)

Conclusion

Conclusion

- analysis of optimal complexity (= quasi-optimal cost) of AFEM
  - core concept: full linear convergence yields rates = complexity
- new: abstract framework for full linear convergence
  - applies to (strongly monotone) energy minimization problems
  - exploits only usual properties of residual error estimators
  - relies only on contractive linearization / solver
- nested linearization + solver for nonlinear problems is possible
- adaptivity is steered via natural equidistribution of estimated error components, e.g.,
  - $\| u_\ell^K - u_\ell^{K-1} \| \le \lambda \eta_\ell(u_\ell^K)$, with arbitraty λ for full linear convergence
- extensions to semilinear PDEs and goal-oriented adaptivity

Thank you for your attention!

Gregor Gantner, Alexander Haberl, Dirk Praetorius, Stefan Schimanko:
Rate optimality of adaptive finite element methods with respect to the overall computational costs
Math. Comp., 90 (2021), 2011–2040

... and Happy 75, Ernst ...

Dirk Praetorius

TU Wien

Institute of Analysis and Scientific Computing

dirk.praetorius@asc.tuwien.ac.at

https://www.asc.tuwien.ac.at/∼praetorius

NumPDEs
Workgroup on Numerics of PDEs