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On optimal computational costs of AFEM

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Outline

Explaining the title

Model problem

- symmetric linear elliptic PDE

$$-\mathrm{div}\,(\boldsymbol{A}\nabla u^\star) = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u^\star = 0 \quad \text{on } \partial\Omega$$

- $u^\star \in \mathbb{V} := H^1_0(\Omega)$ such that

$$a(u^\star, v) := \int_\Omega \boldsymbol{A}\nabla u^\star \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x =: F(v) \quad \text{for all } v \in \mathbb{V}$$

- energy norm $|||v|||^2 := a(v, v)$

FEM discretization

- $\mathcal{T}_0$ conforming initial triangulation
- $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ set of admissible conforming refinements $\mathcal{T}_\ell \in \mathbb{T}$
- $u_\ell^\star \in \mathbb{V}_\ell := \mathcal{S}_0^p(\mathcal{T}_\ell)$ conforming FEM solution

$$a(u_\ell^\star, v_\ell) = F(v_\ell) \quad \text{for all} \quad v_\ell \in \mathbb{V}_\ell$$

- apply: $\boxed{\textbf{SOLVE} \to \textbf{ESTIMATE} \to \textbf{MARK} \to \textbf{REFINE}}$
- goal: $\|u^\star - u_\ell^\star\| = \mathcal{O}((\#\mathcal{T}_\ell)^{-s})$ with $s > 0$ as large as possible

Optimal adaptive algorithms?!

- starting point: an adaptive algorithm leads to optimal rates

$\Longleftrightarrow$ error decays with any possible rate along generated sequence $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}}$

- first question: Which rates are possible?
  - $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} N^s \|\!|u^\star - u^\star_{\rm opt}\|\!|$
  - then: $\|u^\star\|_{\mathbb{A}_s} < \infty \quad \Longleftrightarrow \quad$ rate $s > 0$ is possible

- second question: When does the adaptive algorithm converge with rate $s > 0$?

$$\Longleftrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \|\!|u^\star - u^\star_\ell\|\!| < \infty$$

$\Longrightarrow$ adaptive algorithm guarantees optimal rates if and only if

- $\forall s > 0 : \quad \|u^\star\|_{\mathbb{A}_s} \simeq \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \|\!|u^\star - u^\star_\ell\|\!|$

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl. 67 (2014)

Optimal rates vs. optimal complexity

- optimal rates

$$\forall s > 0 : \quad \|u^\star\|_{\mathbb{A}_s} \simeq \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \|u^\star - u_\ell^\star\|$$

- idealized optimal computational complexity — because of $u_\ell^\star$

$$\forall s > 0 : \quad \|u^\star\|_{\mathbb{A}_s} \simeq \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^s \|u^\star - u_\ell^\star\|$$

- optimal computational complexity — with $u_\ell^k \approx u_\ell^\star$

$$\forall s > 0 : \quad \|u^\star\|_{\mathbb{A}_s} \simeq \sup_{(\ell,k) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ (\ell',k') \le (\ell,k)}} \#\mathcal{T}_{\ell'} \Big)^s \|u^\star - u_\ell^k\|$$

- note: later results will be slightly weaker
  - optimal rates / complexity with respect to quasi-error $\Delta_\ell^k \geq \|u^\star - u_\ell^k\|$
  - since error $\|u^\star - u_\ell^k\|$ is not accessible

Stevenson: Found. Comput. Math. 7 (2007)
Carstensen, Gedicke: SIAM J. Numer. Anal. 50 (2012)

AFEM with exact solver

Well-known AFEM algorithm

Input: $\mathcal{T}_0$, $\quad 0 < \theta \le 1$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE:** compute $u_\ell^\star$
- **ESTIMATE:** compute $\eta_\ell(T, u_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^\star)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^\star)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete solutions $u_\ell^\star$ and corresponding estimators $\eta_\ell(u_\ell^\star)$

- in practice: $\mathcal{M}_\ell$ is always chosen with quasi-minimal cardinality

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Dörfler: SIAM J. Numer. Anal. 33 (1996)

Stevenson: Found. Comput. Math. 7 (2007)

Cascon, Kreuzer, Nochetto, Siebert: SIAM J. Numer. Anal. 46 (2008)

Pfeiler, Praetorius: Math. Comp. 89 (2020)

Linear convergence

Theorem (Stevenson '07; ... ; Carstensen, Feischl, Page, P. '14)

- arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q < 1$ such that $\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star) \lesssim q^{\ell-\ell'}\eta_{\ell'}(u_{\ell'}^\star)$

- first proof by Stevenson / Diening & Kreuzer for Poisson problem
  - $\eta_\ell(u_\ell^\star) \simeq \Lambda_\ell^\star := \big[\|u^\star - u_\ell^\star\|^2 + \mathrm{osc}_\ell(F)^2\big]^{1/2}$
- later proof by [CKNS] for symmetric problems
  - $\eta_\ell(u_\ell^\star) \simeq \Lambda_\ell^\star := \big[\|u^\star - u_\ell^\star\|^2 + \gamma\eta_\ell(u_\ell^\star)^2\big]^{1/2}$

$\Longrightarrow$ $\Lambda_{\ell+1}^\star \le q\,\Lambda_\ell^\star$

- required: certain Pythagoras-type quasi-orthogonality

Stevenson: Found. Comput. Math. 7 (2007)

Cascon, Kreuzer, Nochetto, Siebert: SIAM J. Numer. Anal. 46 (2008)

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl. 67 (2014)

Optimal convergence rates

Theorem (Stevenson '07; ... ; Carstensen, Feischl, Page, P. '14)

- arbitrary $s > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup\limits_{N \geq \#\mathcal{T}_0} \min\limits_{\#\mathcal{T}_{\rm opt} \leq N} N^s \, \eta_{\rm opt}(u^\star_{\rm opt}) < \infty$
- sufficiently small $0 < \theta \leq 1$

$\Longrightarrow \quad \sup\limits_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^s \eta_\ell(u^\star_\ell) \simeq \|u^\star\|_{\mathbb{A}_s}$

- unlike intro: definition of $\|u^\star\|_{\mathbb{A}_s}$ in terms of $\eta_{\rm opt}(u^\star_{\rm opt}) \simeq \big[\, |||u^\star - u^\star_{\rm opt}||| + \eta_{\rm opt}(u^\star_{\rm opt}) \,\big]$
- required: linear convergence of $\eta_\ell(u^\star_\ell)$

Stevenson: Found. Comput. Math. 7 (2007)

Cascon, Kreuzer, Nochetto, Siebert: SIAM J. Numer. Anal. 46 (2008)

Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl. 67 (2014)

AFEM with iterative solver

Contractive iterative solver

- let $0 < \kappa < 1$
- consider contractive solver for discrete linear systems

$$|||u_\ell^\star - u_\ell^k||| \le \kappa\, |||u_\ell^\star - u_\ell^{k-1}||| \quad \text{for all} \quad k \in \mathbb{N}$$

- nothing but triangle inequality

$$\Longrightarrow \quad \frac{1-\kappa}{\kappa}\, |||u_\ell^\star - u_\ell^k||| \le |||u_\ell^k - u_\ell^{k-1}||| \le (1+\kappa)\, |||u_\ell^\star - u_\ell^{k-1}|||$$

- example solvers:
  - PCG with multi-level additive Schwarz preconditioner [CNX]
  - geometric multigrid (h-robust [WZ], hp-robust [MPS])

Chen, Nochetto, Xu: Numer. Math. 120 (2012)

Wu, Zheng: Appl. Numer. Math. 113 (2017)

Miraci, Praetorius, Streitberger: in progress (2022+)

Stopping criterion for algebraic solver

- reliability & stability of residual error estimator

$$\|u^\star - u_\ell^k\| \le \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| \overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$$

$\Longrightarrow$ contractive solver allows for a-posteriori error control

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

$\Longrightarrow$ stop algebraic solver for $K = k$ as soon as $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ a-posteriori error control for all u_ℓ^k but u_0^0

AFEM with contractive solver

Input: $\mathcal{T}_0$, u_0^0, $0 < \theta \le 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \dots, K$, **repeat**
  - compute u_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

  until $\| u_\ell^K - u_\ell^{K-1} \| \le \lambda \, \eta_\ell(u_\ell^K)$

- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$,

Output: Discrete solutions u_ℓ^k and corresponding estimator $\eta_\ell(u_\ell^k) + \| u_\ell^k - u_\ell^{k-1} \|$

- note: number of solver steps $K = K(\ell)$ might vary with ℓ
- weaker criterion $\| u_\ell^K - u_\ell^{K-1} \| \le \lambda \, \eta_\ell(u_\ell^0)$ yields convergence (but: without rates!)

Main result 1: Full linear convergence

Theorem (Gantner, Haberl, P., Schimanko '21)

- $\mathcal{Q} := \big\{(\ell,k) \in \mathbb{N}_0^2 \,:\, u_\ell^k$ computed by algorithm$\big\}$
- $|\ell,k| \;:= \#\big\{(\ell',k') \in \mathcal{Q} \,:\, u_{\ell'}^{k'}$ computed earlier than $u_\ell^k\big\}$
- quasi-error $\quad \Delta_\ell^k := [\![\, |\!|\!| u^\star - u_\ell^k |\!|\!| + \eta_\ell(u_\ell^k)]\!]$
- arbitrary $\quad u_0^0, \quad$ arbitrary $0 < \theta \le 1, \quad$ sufficiently small $\lambda > 0$

$\Longrightarrow\quad$ exists $0 < q < 1$ such that $\quad |\!|\!| u^\star - u_\ell^k |\!|\!| \lesssim \Delta_\ell^k \lesssim q^{|\ell,k| - |\ell',k'|} \Delta_{\ell'}^{k'}$

- note: linear convergence $\iff$ $$\sum_{\substack{(\ell,k) \in \mathcal{Q} \\ |\ell,k| > |\ell',k'|}} \Delta_\ell^k \lesssim \Delta_{\ell'}^{k'}$$
- for exact solvers: $\Delta_\ell^\star \simeq \eta_\ell(u_\ell^\star)$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Perturbation argument = Why sufficiently small λ?

- $$\eta_\ell(u_\ell^K) \le \eta_\ell(u_\ell^\star) + C_{\rm stab}\, \|u_\ell^\star - u_\ell^K\| \le \eta_\ell(u_\ell^\star) + C_{\rm stab}\frac{\kappa}{1-\kappa}\, \|u_\ell^K - u_\ell^{K-1}\| \le \eta_\ell(u_\ell^\star) + C_{\rm stab}\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$$
- similarly: $\eta_\ell(u_\ell^\star) \le \eta_\ell(u_\ell^K) + C_{\rm stab}\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$

$\Longrightarrow \quad (1-C\lambda)\,\eta_\ell(u_\ell^K) \le \eta_\ell(u_\ell^\star) \le (1+C\lambda)\,\eta_\ell(u_\ell^K)$

- moreover: Dörfler marking for (θ, u_ℓ^K) $\Longleftrightarrow$ Dörfler marking for $(\widetilde\theta, u_\ell^\star)$

$\Longrightarrow$ linear convergence of $\eta_\ell(u_\ell^K) \simeq \eta_\ell(u_\ell^\star)$

- $\Delta_\ell^k = \|u^\star - u_\ell^k\| + \eta_\ell(u_\ell^k) \simeq \|u_\ell^\star - u_\ell^k\| + \eta_\ell(u_\ell^k)$
  - therefore: summability by geometric series + contraction

Main result 2: Optimal complexity

Theorem (Gantner, Haberl, P., Schimanko '21)

- arbitrary $s > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \ge \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \le N} N^s \Delta^\star_{\rm opt} < \infty$
- sufficiently small $0 < \theta \le 1$ and $\lambda > 0$

$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_s} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \lesssim \max\big\{\|u^\star\|_{\mathbb{A}_s}, \Delta_0^0\big\}$$

- recall: $\|u^\star - u_\ell^k\| \le \Delta_\ell^k := \|u^\star - u_\ell^k\| + \eta_\ell(u_\ell^k)$
- and: $\Delta_\ell^\star \simeq \eta_\ell(u_\ell^\star)$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

The wonder of full linear convergence!

$$M := \sup_{n\in\mathbb{N}_0} (\#\mathcal{T}_n)^s \,\Delta_n \le \sup_{n\in\mathbb{N}_0} \Big(\sum_{m=0}^{n} \#\mathcal{T}_m \Big)^s \Delta_n \overset{?!}{\le} M$$

- definition of M $\Longrightarrow$ $\#\mathcal{T}_m \le M^{1/s}\, \Delta_m^{-1/s}$
- linear convergence $\Longrightarrow$ $\displaystyle\sum_{m=0}^{n} \#\mathcal{T}_m \le M^{1/s} \sum_{m=0}^{n} \Delta_m^{-1/s} \lesssim M^{1/s}\, \Delta_n^{-1/s}$

$$\Longrightarrow \quad \sup_{n\in\mathbb{N}_0} \Big(\sum_{m=0}^{n} \#\mathcal{T}_m \Big)^s \Delta_n \lesssim M$$

Feischl: PhD thesis, TU Wien (2015)

New argument = full linear convergence

NumPDEs
ASC ▸ TUWIEN

- full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k$$

and $\quad \sup_{(0,k)\in\mathcal{Q}} (\#\mathcal{T}_0)^s \Delta_0^k \lesssim \Delta_0^0$

- sufficiently small $\lambda > 0$

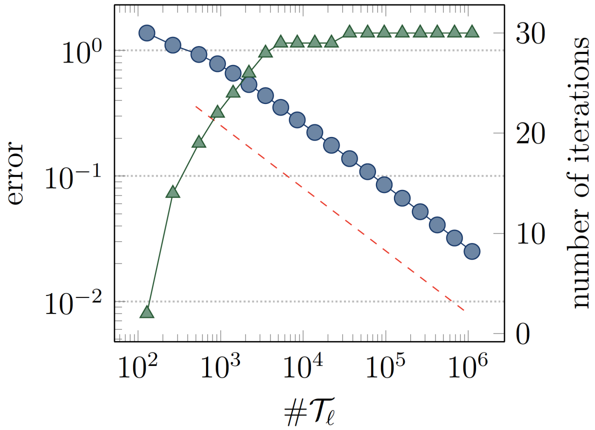
$$\Longrightarrow \quad \sup_{\substack{(\ell,k)\in\mathcal{Q} \\ \ell>0}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k \lesssim \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^K \simeq \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^\star \simeq \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star)$$

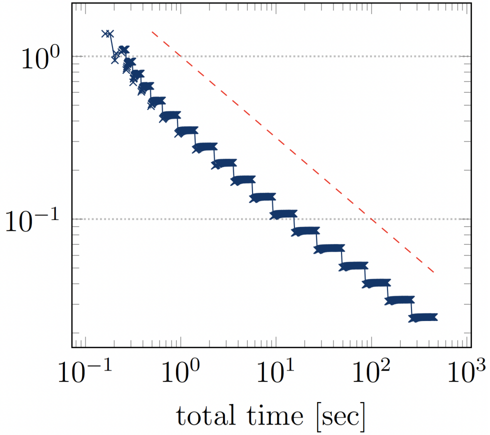
- sufficiently small $0 < \theta \le 1$

$$\Longrightarrow \quad \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_s}$$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Visualizing the theory?!





Miraci, Praetorius, Streitberger: in progress (2022+)

State of the art

Symmetric linear elliptic PDEs

- Standard model problem

$$\begin{aligned} -\mathrm{div}\, \boldsymbol{A}\nabla u^\star + c u^\star &= f - \mathrm{div}\, \vec{\boldsymbol{f}} \quad && \text{in } \Omega \subset \mathbb{R}^d \\ u^\star &= 0 && \text{on } \partial\Omega \end{aligned}$$

$\Longrightarrow$ full linear convergence holds for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda > 0$

  - optimal complexity for sufficiently small θ, λ

- proof extends the original argument of CKNS to $\Lambda_\ell^k := \|\!|u^\star - u_\ell^k|\!\|^2 + \gamma\, \eta_\ell(u_\ell^k)^2$
  - $\Lambda_\ell^{k+1} \le q\, \Lambda_\ell^k$
  - $\Lambda_{\ell+1}^0 \le q\, \Lambda_\ell^{K-1}$

- open: analysis applies to general linear elliptic PDEs, but contractive solver is unclear!

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

Strongly monotone nonlinearities

- energy minimization problems $\mathcal{E}(u^\star) = \min_{v \in \mathbb{V}} \mathcal{E}(v)$
- where Euler–Lagrange operator $\mathcal{A} := \mathrm{d}\mathcal{E}$ is strongly monotone & Lipschitz continuous
- replace: $\| u^\star - u_\ell^k \|^2 \quad \rightsquigarrow \quad \mathrm{d}^2(u^\star, u_\ell^k) := \mathcal{E}(u_\ell^k) - \mathcal{E}(u^\star)$
- now: contractive solver $\quad \rightsquigarrow \quad$ contractive linearization $\mathrm{d}^2(u_\ell^\star, u_\ell^{k+1}) \le \kappa \, \mathrm{d}^2(u_\ell^\star, u_\ell^k)$
  - examples: Zarantonello iteration, Kačanov iteration, damped Newton method

- note: analysis is abstract, but assumptions only verified in [GMZ] for
  - scalar nonlinearities $\mathcal{A}(u) = -\mathrm{div}\,\big(\mu(|\nabla u|)\nabla u\big)$
  - lowest-order FEM

$\Longrightarrow$ full linear convergence holds for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda > 0$
  - "optimal complexity" for sufficiently small θ, λ

Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl. 5 (2012)
Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)
Heid, Praetorius, Wihler: Comput. Meth. Appl. Math., 21 (2021)

Full adaptivity (with nested solvers)

- energy minimization problem with strongly monotone & Lipschitz continuous $\mathcal{A} := \mathrm{d}\mathcal{E}$
- Zarantonello iteration leads to sequence of (linear) Poisson model problems

$\Longrightarrow$ nested contractions (linearization & algebraic solver)

- [HPSV] requires sufficiently small $\lambda_{\rm lin}$, $\lambda_{\rm alg}$ for full linear convergence
  - optimal complexity for sufficiently small θ, $\lambda_{\rm lin}$, $\lambda_{\rm alg}$
- [MPV] requires sufficiently small $\lambda_{\rm alg}$ but allows arbitrary $\lambda_{\rm lin}$ for full linear convergence
  - and: new stopping criterion avoids $\lambda_{\rm alg}$
  - but: optimality analysis remains still open for new stopping criterion

Haberl, Praetorius, Schimanko, Vohralik: Numer. Math. 147 (2021)
Miraci, Praetorius, Vohralik: in progress (2022+)

Semi-linear PDEs

- monotone semilinear PDEs

$$\begin{aligned} -\mathrm{div}\, \boldsymbol{A}\nabla u^\star + b(u^\star) &= f \quad \text{in } \Omega \subset \mathbb{R}^d \\ u^\star &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- with $b' \geq 0$ and algebraic growth condition on $b(\xi)$

$\Longrightarrow$ energy minimization with strongly monotone & locally Lipschitz continuous $\mathcal{A} := \mathrm{d}\mathcal{E}$

- analysis allows FEM with arbitrary polynomials, but requires sufficiently small λ

$\Longrightarrow$ full linear convergence for arbitrary $0 < \theta \leq 1$ and sufficiently small $\lambda > 0$
  - optimal complexity for sufficiently small θ, λ

Becker, Brunner, Innerberger, Melenk, Praetorius: arXiv:2112.06687 (2021)

Brunner, Innerberger, Praetorius: in progress (2022+)

Goal-oriented adaptivity

- Standard model problem

$$-\mathrm{div}\,(\boldsymbol{A}\nabla u^\star) + c u^\star = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u^\star = 0 \quad \text{on } \partial\Omega$$

- linear goal functional $G(v) = \int_\Omega g v \, \mathrm{d}x$

- requires: discrete goal $G(u_\ell^k, z_\ell^k) = G(u_\ell^k) + \big[F(z_\ell^k) - a(u_\ell^k, z_\ell^k)\big]$

- estimate: $|G(u^\star) - G(u_\ell^k, z_\ell^k)| \lesssim \Delta_\ell^k := \big[\eta_\ell(u_\ell^k) + |||u_\ell^k - u_\ell^{k-1}|||\big]\big[\zeta_\ell(z_\ell^k) + |||z_\ell^k - z_\ell^{k-1}|||\big]$

$\Longrightarrow$ full linear convergence for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda > 0$

- optimal complexity for sufficiently small θ, λ

Becker, Gantner, Innerberger, Praetorius: arXiv:2101.11407 (2021)

Conclusion & open questions

Conclusion

- analysis of optimal complexity (= cost) of AFEM
  - main concept = full linear convergence
- requires contractive solver / linearization
- nested linearization / solver for nonlinear problems is possible
- adaptivity is steered via natural equidistribution of estimated error components
  - $|||u_\ell^k - u_\ell^{k-1}||| \le \lambda \eta_\ell(u_\ell^k)$
  - λ is arbitrary for full linear convergence for many problems

Open questions

- available analysis only for conforming (primal) methods!
  - e.g., nested iteration
- contractive solvers for non-symmetric linear PDEs?
- contractive solvers for mixed formulations?
- only simple quasi-linear energy minimization problems!
- what about stronger non-linearities like the p-Laplace problem?
- adaptive steering of parameters (e.g., θ, λ_{lin}, λ_{alg}, solver damping)?

Thank you for your attention!

Habilitation
01/2005

Happy birthday, Carsten,

... and thank you for putting me on the academic track!