June 24, 2021 – Recent Advances in the Numerical Approximation of PDEs

Goal-oriented adaptive FEMs with optimal computational complexity

Dirk Praetorius

joint work with

Roland Becker, Gregor Gantner, Michael Innerberger

TU Wien
Institute of Analysis and Scientific Computing

Outline

Explaining the title

GOAFEM with exact solver

GOAFEM with iterative solver

Numerical experiment

Conclusion

Explaining the title

Model problem

- symmetric linear elliptic PDE

$$
\begin{aligned}
-\operatorname{div}(\boldsymbol{A}\nabla u^\star) &= f - \operatorname{div}\boldsymbol{f} && \text{in } \Omega \subset \mathbb{R}^d \\
u^\star &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

- linear goal functional $G(v) = \int_\Omega gv + \boldsymbol{g}\cdot\nabla v \,\mathrm{d}x$

- approximate quantity of interest $G(u^\star)$
  - instead of $u^\star$ wrt. energy norm

FEM discretization

- $\mathcal{T}_0$ conforming initial triangulation
- $\mathbb{T} = \mathbb{T}(\mathcal{T}_0)$ set of admissible conforming refinements $\mathcal{T}_\ell \in \mathbb{T}$
- $u_\ell^\star \in \mathbb{V}_\ell := \mathcal{S}_0^p(\mathcal{T}_\ell)$ conforming FEM solution

$$a(u_\ell^\star, v_\ell) = F(v_\ell) \quad \text{for all} \quad v_\ell \in \mathbb{V}_\ell$$

- apply: **SOLVE → ESTIMATE → MARK → REFINE**
- goal: $|G(u^\star) - G(u_\ell^\star)| = \mathcal{O}((\#\mathcal{T}_\ell)^{-\alpha})$ with $\alpha > 0$ as large as possible

Optimal adaptive algorithms?!

- starting point: an adaptive algorithm leads to optimal rates

$\Longleftrightarrow$ error decays with any possible rate along generated sequence $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$

- first question: Which rates are possible?
  - $\|G(u^\star)\|_{\mathbb{A}_\alpha} := \sup\limits_{N\ge\#\mathcal{T}_0}\ \min\limits_{\#\mathcal{T}_{\rm opt}\le N}(N+1)^\alpha|G(u^\star)-G(u^\star_{\rm opt})|$
  - then: $\|G(u^\star)\|_{\mathbb{A}_\alpha} < \infty \quad\Longleftrightarrow\quad$ rate $\alpha > 0$ is possible

- second question: When does algorithm convergence with rate $\alpha > 0$?

$\Longleftrightarrow \quad \sup\limits_{\ell\in\mathbb{N}_0}(\#\mathcal{T}_\ell)^\alpha|G(u^\star)-G(u^\star_\ell)| < \infty$

$\Longrightarrow$ adaptive algorithm guarantees optimal rates if and only if

- $\forall\alpha > 0: \quad \|G(u^\star)\|_{\mathbb{A}_\alpha} \simeq \sup\limits_{\ell\in\mathbb{N}_0}(\#\mathcal{T}_\ell)^\alpha|G(u^\star)-G(u^\star_\ell)|$

Optimal rates vs. optimal complexity

- optimal rates

$$\forall \alpha > 0: \quad \|G(u^\star)\|_{\mathbb{A}_\alpha} \simeq \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^\alpha |G(u^\star) - G(u_\ell^\star)|$$

- idealized optimal computational complexity — because of $u_\ell^\star$

$$\forall \alpha > 0: \quad \|G(u^\star)\|_{\mathbb{A}_\alpha} \simeq \sup_{\ell \in \mathbb{N}_0} \Big(\sum_{\ell'=0}^{\ell} \#\mathcal{T}_{\ell'} \Big)^\alpha |G(u^\star) - G(u_\ell^\star)|$$

- optimal computational complexity — with $u_\ell^k \approx u_\ell^\star$ and $G_\ell^k \approx G(u_\ell^\star)$

$$\forall \alpha > 0: \quad \|G(u^\star)\|_{\mathbb{A}_\alpha} \simeq \sup_{(\ell,k) \in \mathcal{Q}} \Big(\sum_{\substack{(\ell',k') \in \mathcal{Q} \\ (\ell',k') \le (\ell,k)}} \#\mathcal{T}_{\ell'} \Big)^\alpha |G(u^\star) - G_\ell^k|$$

- note: later results will be slightly weaker
  - optimal rates / complexity wrt. quasi-error quantity $\Delta_\ell^k \gtrsim |G(u^\star) - G_\ell^k|$
  - since error $|G(u^\star) - G_\ell^k|$ is not accessible

GOAFEM with exact solver

Dual problem & goal error estimate

- $u^\star \in \mathbb{V} = H^1_0(\Omega)$ primal solution

$$a(u^\star, v) = F(v) \quad \text{for all} \quad v \in \mathbb{V}$$

- with FEM approximation $u^\star \approx u^\star_\ell \in \mathbb{V}_\ell$

- $z^\star \in \mathbb{V}$ dual solution

$$a(v, z^\star) = G(v) \quad \text{for all} \quad v \in \mathbb{V}$$

- with FEM approximation $z^\star \approx z^\star_\ell \in \mathbb{V}_\ell$

- residual a-posteriori error estimators $\eta_\ell(u^\star_\ell)$ and $\zeta_\ell(z^\star_\ell)$

$$\Longrightarrow \quad G(u^\star) - G(u^\star_\ell) = G(u^\star - u^\star_\ell) = a(u^\star - u^\star_\ell, z^\star) = a(u^\star - u^\star_\ell, z^\star - z^\star_\ell)$$

$$\Longrightarrow \quad \boxed{|G(u^\star) - G(u^\star_\ell)| \le |||u^\star - u^\star_\ell||| \, |||z^\star - z^\star_\ell||| \lesssim \eta_\ell(u^\star_\ell)\zeta_\ell(z^\star_\ell)}$$

Giles, Süli: Acta Numer. 11 (2002)

GOAFEM with exact solver

Input: $\mathcal{T}_0$, $0 < \theta \le 1$

For each $\ell = 0, 1, 2, \dots$ do

- **SOLVE:** compute $u_\ell^\star$ and $z_\ell^\star$
- **ESTIMATE:** compute $\eta_\ell(T, u_\ell^\star)$ and $\zeta_\ell(T, z_\ell^\star)$ for all $T \in \mathcal{T}_\ell$
- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \varrho_\ell(T)^2$

  where $\varrho_\ell(T)^2 = \eta_\ell(u_\ell^\star)^2 \zeta_\ell(T, z_\ell^\star)^2 + \eta_\ell(T, u_\ell^\star)^2 \zeta_\ell(z_\ell^\star)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$

Output: Discrete goal values $G(u_\ell^\star)$ and estimator products $\eta_\ell(u_\ell^\star)\,\zeta_\ell(z_\ell^\star)$

- same structure as AFEM loop, but modified marking step
- $\mathcal{M}_\ell$ is always chosen with quasi-minimal cardinality

Mommer, Stevenson: SIAM J. Numer. Anal., 47 (2009)

Becker, Estecahandy, Trujillo: SIAM J. Numer. Anal., 49 (2011)

Mommer & Stevenson '09

- choose $\mathcal{M}_\ell^u \subseteq \mathcal{T}_\ell$ such that $\theta\,\eta_\ell(u_\ell^\star)^2 \le \sum_{T\in\mathcal{M}_\ell^u} \eta_\ell(T,u_\ell^\star)^2$
- choose $\mathcal{M}_\ell^z \subseteq \mathcal{T}_\ell$ such that $\theta\,\zeta_\ell(z_\ell^\star)^2 \le \sum_{T\in\mathcal{M}_\ell^z} \zeta_\ell(T,z_\ell^\star)^2$
- choose $\mathcal{M}_\ell \in \{\mathcal{M}_\ell^u, \mathcal{M}_\ell^z\}$ with $\#\mathcal{M}_\ell = \min\{\#\mathcal{M}_\ell^u, \#\mathcal{M}_\ell^z\}$

Becker, Estecahandy, Trujillo '11

- choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that

$$2\theta\,\eta_\ell(u_\ell^\star)^2\zeta_\ell(z_\ell^\star)^2 \le \eta_\ell(u_\ell^\star)^2 \sum_{T\in\mathcal{M}_\ell} \zeta_\ell(T,z_\ell^\star)^2 + \zeta_\ell(u_\ell^\star)^2 \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T,u_\ell^\star)^2$$

- MS$(\theta,\mathcal{M}_\ell)$ $\Longrightarrow$ BET$(\theta/2,\mathcal{M}_\ell)$
- BET$(\theta,\mathcal{M}_\ell)$ $\Longrightarrow$ Dörfler$(\eta_\ell,\theta,\mathcal{M}_\ell)$ or Dörfler$(\zeta_\ell,\theta,\mathcal{M}_\ell)$

Mommer, Stevenson: SIAM J. Numer. Anal., 47 (2009)

Becker, Estecahandy, Trujillo: SIAM J. Numer. Anal., 49 (2011)

Linear convergence

Theorem (Mommer, Stevenson '09; Feischl, P., van der Zee '16)

- arbitrary $0 < \theta \le 1$
- BET marking (or MS marking)

$\Longrightarrow$ exists $0 < q < 1$ such that

$$|G(u^\star) - G(u_\ell^\star)| \lesssim \eta_\ell(u_\ell^\star)\zeta_\ell(z_\ell^\star) \lesssim q^{\ell-\ell'}\eta_{\ell'}(u_{\ell'}^\star)\zeta_{\ell'}(z_{\ell'}^\star)$$

- first proof by MS for Poisson problem
  - $\eta_\ell(u_\ell^\star)\zeta_\ell(z_\ell^\star) \simeq \Lambda_\ell^\star := \big[|||u^\star - u_\ell^\star|||^2 + \mathrm{osc}_\ell(F)^2\big]^{1/2}\big[|||z^\star - z_\ell^\star|||^2 + \mathrm{osc}_\ell(G)^2\big]^{1/2}$
  - $\Lambda_{\ell+1}^\star \le q\,\Lambda_\ell^\star$ by monotonicity of oscillations
  - does not generalize to symmetric problems
- proof by FPZ for general 2nd order linear elliptic PDEs

Mommer, Stevenson: SIAM J. Numer. Anal., 47 (2009)

Feischl, Praetorius, van der Zee: SIAM J. Numer. Anal., 54 (2016)

Optimal convergence rates

Theorem (Mommer, Stevenson '09; Feischl, P., van der Zee '16)

- arbitrary $s, t > 0$
- BET marking (or MS marking)
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} (N+1)^s \eta_{\rm opt}(u^\star_{\rm opt}) < \infty$
- $\|z^\star\|_{\mathbb{A}_t} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} (N+1)^t \zeta_{\rm opt}(z^\star_{\rm opt}) < \infty$
- sufficiently small $0 < \theta \leq 1$

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^{s+t} \eta_\ell(u^\star_\ell) \zeta_\ell(z^\star_\ell) \lesssim \|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t}$$

- note: involves separate approximation classes for $u^\star$ and $z^\star$ (unlike intro!)
- proof closely related to optimality of standard AFEM

Mommer, Stevenson: SIAM J. Numer. Anal., 47 (2009)

Feischl, Praetorius, van der Zee: SIAM J. Numer. Anal., 54 (2016)

GOAFEM with iterative solver

Discrete goal & goal error estimate

- exact FEM solutions $u_\ell^\star, z_\ell^\star \in \mathbb{V}_\ell$

$$G(u^\star) - G(u_\ell^\star) = a(u^\star - u_\ell^\star, z^\star) \overset{\text{orth}}{=} a(u^\star - u_\ell^\star, z^\star - z_\ell^\star)$$

- for general $u_\ell^k, z_\ell^k \in \mathbb{V}_\ell$

$$G(u^\star) - G(u_\ell^k) = a(u^\star - u_\ell^k, z^\star) = a(u^\star - u_\ell^k, z^\star - z_\ell^k) + a(u^\star - u_\ell^k, z_\ell^k)$$

$\Longrightarrow$ discrete goal $G(u_\ell^k, z_\ell^k) = G(u_\ell^k) + \big[F(z_\ell^k) - a(u_\ell^k, z_\ell^k)\big]$ satisfies

$$\boxed{|G(u^\star) - G(u_\ell^k, z_\ell^k)| = a(u^\star - u_\ell^k, z^\star - z_\ell^k) \leq |||u^\star - u_\ell^k|||\,|||z^\star - z_\ell^k|||}$$

- note: discrete goal must be adapted, since Galerkin orthogonality fails

Giles, Süli: Acta Numer. 11 (2002)

Contractive iterative solver

- let $0 < \kappa < 1$

- consider contractive solver for discrete linear systems

$$\|u_\ell^\star - u_\ell^k\| \le \kappa \, \|u_\ell^\star - u_\ell^{k-1}\| \quad \text{for all} \quad k \in \mathbb{N}$$

- nothing but triangle inequality

$$\Longrightarrow \quad \frac{1-\kappa}{\kappa} \, \|u_\ell^\star - u_\ell^k\| \le \|u_\ell^k - u_\ell^{k-1}\| \le (1+\kappa) \, \|u_\ell^\star - u_\ell^{k-1}\|$$

- example solvers:
  - PCG with multi-level additive Schwarz preconditioner
  - geometric multigrid

Chen, Nochetto, Xu: Numer. Math. 120 (2012)

Wu, Zheng: Appl. Numer. Math. 113 (2017)

Stopping criterion for algebraic solver

- reliability & stability of residual error estimator

$$\|u^\star - u_\ell^k\| \le \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| \overset{\text{rel}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{stab}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{solve}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$$

- note: contractive solver allows for a-posteriori error control

$$|G(u^\star) - G(u_\ell^k, z_\ell^k)| \lesssim \big[\eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|\big]\big[\zeta_\ell(z_\ell^k) + \|z_\ell^k - z_\ell^{k-1}\|\big]$$

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

$\Longrightarrow$ stop algebraic solver for $K = k$ as soon as $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ a-posteriori error control for all u_ℓ^k but u_0^0

GOAFEM with contractive solver

Input: $\mathcal{T}_0$, u_0^0, z_0^0, $0 < \theta \le 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \ldots, K$, **repeat**
  - compute u_ℓ^k and z_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ and $\zeta_\ell(T, z_\ell^k)$ for all $T \in \mathcal{T}_\ell$
- **until** $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$ and $\|z_\ell^K - z_\ell^{K-1}\| \le \lambda\, \zeta_\ell(z_\ell^K)$
- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \varrho_\ell(T)^2 \le \sum_{T \in \mathcal{M}_\ell} \varrho_\ell(T)^2$

  where $\varrho_\ell(T)^2 = \eta_\ell(u_\ell^K)^2 \zeta_\ell(T, z_\ell^K)^2 + \eta_\ell(T, u_\ell^K)^2 \zeta_\ell(z_\ell^K)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$, $z_{\ell+1}^0 := z_\ell^K$

Output: Discrete goal values $G(u_\ell^K, z_\ell^K)$ and goal error control

- note: number of solver steps $K = K(\ell)$ might vary with ℓ

Main result 1: Full linear convergence

Theorem (Becker, Gantner, Innerberger, P. '21)

- $\mathcal{Q} = \big\{(\ell,k) \in \mathbb{N}_0^2 \,:\, u_\ell^k$ computed by algorithm$\big\}$
- $|\ell,k| \;:= \#\big\{(\ell',k') \in \mathcal{Q} \,:\, u_{\ell'}^{k'}$ computed earlier than $u_\ell^k\big\}$
- $\Delta_\ell^k := \big[\| u^\star - u_\ell^k \| + \eta_\ell(u_\ell^k)\big]\big[\| z^\star - z_\ell^k \| + \zeta_\ell(z_\ell^k)\big]$
- arbitrary u_0^0, z_0^0, $0 < \theta \le 1$, $\lambda > 0$

$\Longrightarrow$ exists $0 < q < 1$ such that

$$|G(u^\star) - G(u_\ell^k, z_\ell^k)| \lesssim \Delta_\ell^k \lesssim q^{|\ell,k| - |\ell',k'|} \Delta_{\ell'}^{k'}$$

- note: linear convergence $\iff \displaystyle\sum_{\substack{(\ell,k)\in\mathcal{Q} \\ |\ell,k|>|\ell',k'|}} \Delta_\ell^k \lesssim \Delta_{\ell'}^{k'}$
- for exact solvers: $\Delta_\ell^\star \simeq \eta_\ell(u_\ell^\star)\zeta_\ell(u_\ell^\star)$

Becker, Gantner, Innerberger, Praetorius: preprint arXiv:2101.11407

Main result 2: Optimal complexity

Theorem (Becker, Gantner, Innerberger, P. '21)

- arbitrary $s, t > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} (N+1)^s \eta_{\rm opt}(u^\star_{\rm opt}) < \infty$
- $\|z^\star\|_{\mathbb{A}_t} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} (N+1)^t \zeta_{\rm opt}(z^\star_{\rm opt}) < \infty$
- sufficiently small $0 < \theta \leq 1$ and $\lambda > 0$

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\leq|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^{s+t} \Delta_\ell^k \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t} \, , \, \Delta_0^0 \big\}$$

- recall: $|G(u^\star) - G(u_\ell^k, z_\ell^k)| \lesssim \Delta_\ell^k$

Becker, Gantner, Innerberger, Praetorius: preprint arXiv:2101.11407

New argument = full linear convergence

- full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^{s+t} \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^{s+t} \Delta_\ell^k$$

$$\text{and} \quad \sup_{(0,k)\in\mathcal{Q}} (\#\mathcal{T}_0)^{s+t} \Delta_0^k \lesssim \Delta_0^0$$

- sufficiently small $\lambda > 0$

$$\Longrightarrow \quad \sup_{\substack{(\ell,k)\in\mathcal{Q} \\ \ell>0}} (\#\mathcal{T}_\ell)^{s+t} \Delta_\ell^k \simeq \sup_{(\ell,0)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^{s+t} \Delta_\ell^\star \simeq \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^{s+t} \eta_\ell(u_\ell^\star)\zeta_\ell(u_\ell^\star)$$

- sufficiently small $0 < \theta \le 1$

$$\Longrightarrow \quad \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^{s+t} \eta_\ell(u_\ell^\star)\zeta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_s} \|z^\star\|_{\mathbb{A}_t}$$

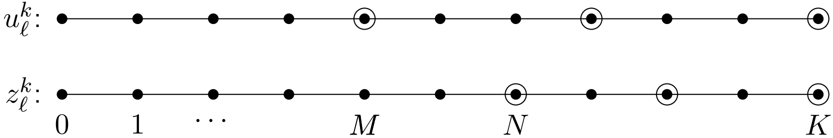
Gantner, Haberl, Praetorius, Schimanko: Math. Comp., in print 2021

Feischl, Praetorius, van der Zee: SIAM J. Numer. Anal., 54 (2016)

Algorithmic variants

- presented algorithm used BET marking strategy
- same results hold for MS (and FPZ) marking strategy

- presented algorithm used simultaneous stopping of algebraic solver
- same results for decoupled stopping criteria



- recall: $\| u_\ell^M - u_\ell^{M-1} \| \le \lambda \, \eta_\ell(u_\ell^M)$ and $\| z_\ell^N - z_\ell^{N-1} \| \le \lambda \, \zeta_\ell(z_\ell^N)$

Mommer, Stevenson: SIAM J. Numer. Anal., 47 (2009)

Becker, Estecahandy, Trujillo: SIAM J. Numer. Anal., 49 (2011)

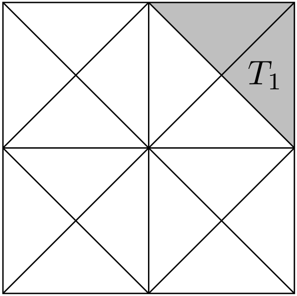
Feischl, Praetorius, van der Zee: SIAM J. Numer. Anal., 54 (2016)

Numerical experiment

Smooth solution, but singular goal

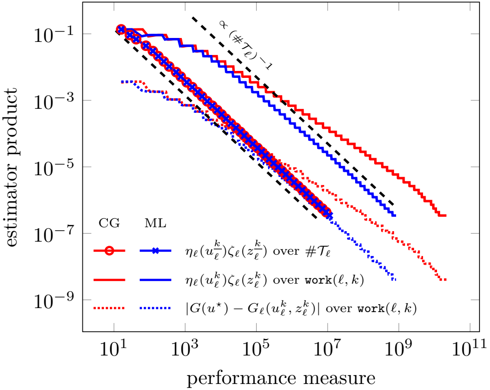
- $\Omega = (0,1)^2 \subset \mathbb{R}^2$
- Prescribe exact solution

  $u^\star(x) := x_1 x_2 (1-x_1)(1-x_2)$
- $G(u^\star) = \int_{T_1} \frac{\partial u^\star}{\partial x_1} \, \mathrm{d}x = 11/960$



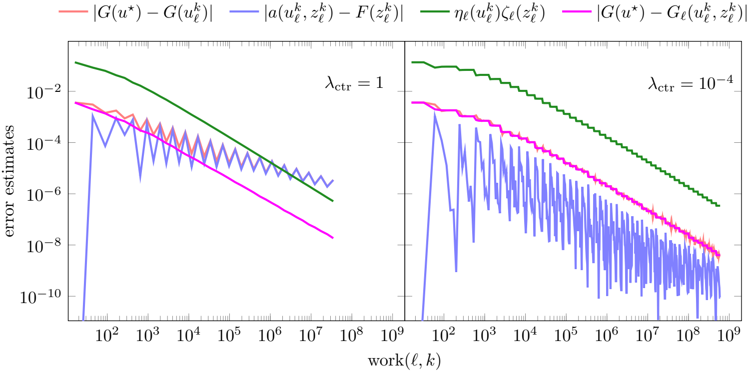
Comparison of iterative solvers

- CG = CG without preconditioner
- ML = PCG with optimal preconditioner
- usually: rates wrt. $\#\mathcal{T}_\ell$
- new: rates wrt. $\texttt{work}(\ell,k) = \sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'}$



Necessity of corrector term

- Recall corrected goal $G(u_\ell^k, z_\ell^k) = G(u_\ell^k) + F(z_\ell^k) - a(u_\ell^k, z_\ell^k)$
- Order of convergence of $G(u_\ell^k)$ depends on solver parameter λ



Conclusion

Summary

- algorithm for goal-oriented adaptive FEM with iterative solvers
- requires corrected goal quantity $G(u_\ell^k, z_\ell^k)$
- stopping criteria for contractive iterative solver
- goal error controlled by quasi-error product $|G(u^\star) - G(u_\ell^k, z_\ell^k)| \lesssim \Delta_\ell^k$
- full linear convergence of quasi-error product Δ_ℓ^k
- optimal complexity = rates wrt. total computational cost

Thank you for your attention!

Roland Becker, Gregor Gantner, Michael Innerberger, Dirk Praetorius
Goal-oriented adaptive finite element methods with optimal computational complexity
preprint at `arXiv:2101.11407`

Dirk Praetorius

TU Wien

Institute of Analysis
and Scientific Computing

NumPDEs group

`dirk.praetorius@asc.tuwien.ac.at`

`https://www.asc.tuwien.ac.at/~praetorius`

NumPDEs
Workgroup on Numerics of PDEs