ESI Workshop: Interplay of geometric processing, modelling, and adaptivity in Galerkin methods

July 16–20, 2018 (length: 40 min + 5)

Axioms of adaptivity revisited: Optimal adaptive IGAFEM

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joint work with

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Outline

Introduction

Model problem

general 2nd order linear elliptic PDE

- $\mathcal{L}u := -\nabla \cdot (\boldsymbol{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu = f$ in Ω
- $u = 0$ on $\Gamma := \partial\Omega$

- $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain
- $\boldsymbol{A} \in W^{1,\infty}(\Omega)$, $\boldsymbol{A}(x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with uniformly positive eigenvalues
- $\boldsymbol{b} \in L^\infty(\Omega)$, $\boldsymbol{b}(x) \in \mathbb{R}^d$
- $c \in L^\infty(\Omega)$, $c(x) \in \mathbb{R}$

Galerkin formulation

weak formulation

- find $u \in H^1_0(\Omega)$ s.t. $\quad a(u,v) = \int_\Omega f v \, \mathrm{d}x \quad \forall\, v \in H^1_0(\Omega)$

- $\mathcal{L}u := -\nabla \cdot (\boldsymbol{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu$
- $a(u,v) := \int_\Omega \{\, (\boldsymbol{A}\nabla u) \cdot \nabla v + (\boldsymbol{b} \cdot \nabla u + cu)\, v \,\} \, \mathrm{d}x$
- suppose: $a(\cdot,\cdot)$ elliptic

conforming Galerkin discretization

- find $U_\bullet \in \mathcal{X}_\bullet$ s.t. $\quad a(U_\bullet, V_\bullet) = \int_\Omega f V_\bullet \, \mathrm{d}x \quad \forall\, V_\bullet \in \mathcal{X}_\bullet$

- $\mathcal{X}_\bullet \subset H^1_0(\Omega)$ conforming discrete subspace

Residual a-posteriori error estimator

- $\mathcal{X}_\bullet = \mathcal{X}_\bullet(\mathcal{T}_\bullet)$ related to mesh $\mathcal{T}_\bullet$ of Ω

residual estimator

- $\eta_\bullet = \Big(\sum_{T\in\mathcal{T}_\bullet} \eta_\bullet(T)^2 \Big)^{1/2}$
- $$\eta_\bullet(T)^2 = h_T^2 \, \| \{ -\nabla\cdot(\boldsymbol{A}\nabla U_\bullet) + \boldsymbol{b}\cdot\nabla U_\bullet + cU_\bullet \} - f \|_{L^2(T)}^2 + h_T \, \| [\boldsymbol{A}\nabla U_\bullet \cdot \boldsymbol{n}] \|_{L^2(\partial T\setminus\Gamma)}^2$$

- $h_T = |T|^{1/d}$

Adaptive algorithm

- initial mesh $\mathcal{T}_0$
- adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \dots$ iterate

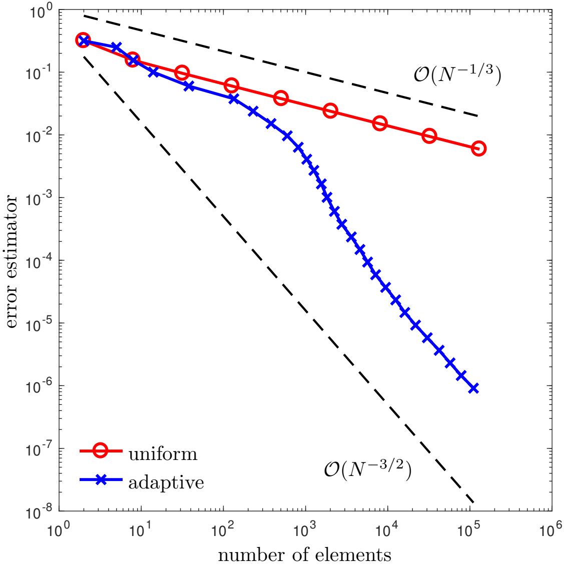
1. SOLVE: compute discrete solution U_ℓ for mesh $\mathcal{T}_\ell$
2. ESTIMATE: compute indicators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
3. MARK: find (minimal) set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

$$\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

4. REFINE: refine (at least) all $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

Dörfler: SINUM 33 (1996)

What is all about?



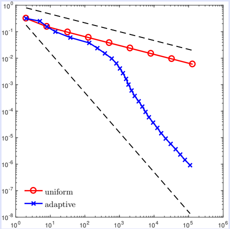
- IGAFEM 2D
- hierar. B-splines
- 3rd order

Main theorem on h-adaptive algorithms

Theorem (Stevenson '07, ..., Carstensen, Feischl, Page, P. '14)

- validity of axioms (A1)–(A4) (+ properties of mesh-refinement)
- $0 < \theta \le 1$

$\Longrightarrow \quad \exists C > 0 \; \exists 0 < q < 1 \; \forall \ell, n \ge 0 \quad \eta_{\ell+n} \le C\, q^n\, \eta_\ell$



- $\mathbb{T}_N := \{\mathcal{T} \in \mathtt{refine}(\mathcal{T}_0) \,:\, \#\mathcal{T} \le N\} \cup \{\mathcal{T}_0\}$
- $s > 0$ arbitrary
- $0 < \theta \ll 1$ sufficiently small
- $\mathcal{M}_\ell$ has (essentially) minimal cardinality

$$\Longrightarrow \quad \sup_{\ell \in \mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell \simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N} \eta_{\mathrm{opt}}\Big)$$

Stevenson: Found. Comput. Math. 7 (2007)

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Axioms of adaptivity (revisited)

First research questions

- Which properties of estimator $\eta_\bullet$ (and mesh-refinement) guarantee:
  - (linear) convergence?
  - optimal algebraic convergence behavior?

$\Longrightarrow$ *axioms of adaptivity* (A1)–(A4)

- Which axioms are sufficient?
- Which axioms are necessary?

Stevenson: Found. Comput. Math. 7 (2007)

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Feischl, Führer, Praetorius: SINUM 52 (2014)

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Axioms of adaptivity

$\forall \mathcal{T}_H \quad \forall \mathcal{T}_h \in \mathtt{refine}(\mathcal{T}_H)$

(A1) $$\Big| \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_h(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_H(T)^2 \Big)^{1/2} \Big| \le C_{\mathrm{stab}} \, \|U_h - U_H\|$$

(A2) $$\sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_H} \eta_h(T)^2 \le q_{\mathrm{red}} \sum_{T \in \mathcal{T}_H \setminus \mathcal{T}_h} \eta_H(T)^2 + C_{\mathrm{red}} \, \|U_h - U_H\|^2$$

(A3) $$\|U_h - U_H\|^2 \le C_{\mathrm{rel}}^2 \sum_{T \in \mathcal{R}_{Hh}} \eta_H(T)^2$$

where $\mathcal{T}_H \setminus \mathcal{T}_h \subseteq \mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ and $\#\mathcal{R}_{Hh} \le C_{\mathrm{rel}} \, \#(\mathcal{T}_H \setminus \mathcal{T}_h)$

$\forall \ell, N \ge 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(\|U_{k+1} - U_k\|^2 - \varepsilon \eta_k^2 \big) \le C_{\mathrm{orth}}(\varepsilon) \, \eta_\ell^2$$

Second research questions

- Let us restrict to residual error estimator!
    - and conforming FEM
- What properties of the FEM spaces (and mesh-refinement) guarantee
    - *axioms of adaptivity* (A1)–(A4)
- Are these properties satisfied for
    - adaptive IGAFEM with hierarchical splines?
    - adaptive IGAFEM with ...?

Nochetto, Veeser: Primer on AFEMs (2012)

Gantner, Haberlik, Praetorius: M3AS 27 (2017)

(A1) Stability and (A2) Reduction

$\forall \mathcal{T}_H \quad \forall \mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$

(A1) $$\Big| \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_h(T)^2 \Big)^{1/2} - \Big(\sum_{T \in \mathcal{T}_H \cap \mathcal{T}_h} \eta_H(T)^2 \Big)^{1/2} \Big| \le C_{\rm stab} \, |\!|\!| U_h - U_H |\!|\!|$$

(A2) $$\sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_H} \eta_h(T)^2 \le q_{\rm red} \sum_{T \in \mathcal{T}_H \setminus \mathcal{T}_h} \eta_H(T)^2 + C_{\rm red} \, |\!|\!| U_h - U_H |\!|\!|^2$$

- $|\!|\!| \cdot |\!|\!|$ is H^1-norm
- $$\eta_\bullet(T)^2 = h_T^2 \, \| \{ -\nabla \cdot (\boldsymbol{A} \nabla U_\bullet) + \boldsymbol{b} \cdot \nabla U_\bullet + c U_\bullet \} - f \|_{L^2(T)}^2 + h_T \, \| [\boldsymbol{A} \nabla U_\bullet \cdot \boldsymbol{n}] \|_{L^2(\partial T \setminus \Gamma)}^2$$
- verification: triangle inequality plus scaling arguments

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Elementary requirements

Meshes

1. neighboring elements have comparable size
2. number of neighbors is uniformly bounded
3. validity of trace inequality

Refinement

1. father is union of its sons
2. refinement leads to contraction of local mesh-size

Space

1. refinement leads to nested spaces
2. local inverse estimate

(A3) Discrete reliability

$\forall \mathcal{T}_H \quad \forall \mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$

(A3) $\|U_h - U_H\|^2 \le C_{\rm rel}^2 \sum_{T \in \mathcal{R}_{Hh}} \eta_H(T)^2$

where $\mathcal{T}_H \backslash \mathcal{T}_h \subseteq \mathcal{R}_{Hh} \subseteq \mathcal{T}_H$ and $\#\mathcal{R}_{Hh} \le C_{\rm rel}\, \#(\mathcal{T}_H \backslash \mathcal{T}_h)$

$\Longrightarrow$ "classical" reliability $\|u - U_H\| \le C_{\rm rel}\, \eta_H$

- verification: "classical" reliability proof
  - but Galerkin orthogonality exploited with Scott-Zhang projector

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Stevenson: Found. Comput. Math. 7 (2007)

Advanced requirements on space

Space+: discrete functions are defined locally

3. $\mathcal{T}_h \in \texttt{refine}(\mathcal{T}_H)$, $\quad T \in \mathcal{T}_H \cap \mathcal{T}_h$ with $\mathrm{patch}(T) \subseteq \mathcal{T}_H \cap \mathcal{T}_h$

$\Longrightarrow \quad \mathcal{X}_h|_{\mathrm{patch}(T)} = \mathcal{X}_H|_{\mathrm{patch}(T)}$

Space+: existence of Scott–Zhang projector $J_H : H^1(\Omega) \to \mathcal{X}_H$

4. $v|_{\mathrm{patch}(T)} \in \mathcal{X}_H|_{\mathrm{patch}(T)} \quad \Longrightarrow \quad (J_H v)|_T = v|_T$ — local projection
5. $\|(1 - J_H)v\|_{L^2(T)} \lesssim h_T \, \|v\|_{H^1(\mathrm{patch}(T))}$ — approximation
6. $\|\nabla(1 - J_H)v\|_{L^2(T)} \lesssim \|v\|_{H^1(\mathrm{patch}(T))}$ — H^1-stability

$\Longrightarrow$ discrete reliability with $\mathcal{R}_{Hh} := \mathrm{patch}(\mathcal{T}_H \backslash \mathcal{T}_h)$

- $\mathrm{patch}(\cdot)$ can be of arbitrary (but fixed) order

Plain AFEM convergence by (A1)–(A3)

- (A1) + (A2) + Dörfler marking

$$\Longrightarrow \quad \eta_{\ell+1}^2 \le q\,\eta_\ell^2 + C\,|||U_{\ell+1} - U_\ell|||^2$$

- nestedness + Céa lemma

$$\Longrightarrow \quad U_\ell \to U_\infty \quad \text{as } \ell \to \infty$$

$$\Longrightarrow \quad \eta_\ell \to 0 \quad \text{as } \ell \to \infty$$

- (A3) reliability

$$\Longrightarrow \quad |||u - U_\ell||| \lesssim \eta_\ell \to 0 \quad \text{as } \ell \to \infty$$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Morin, Siebert, Veeser: M3AS 18 (2008)

(A4) Quasi-orthogonality ✓

$\forall \ell, N \geq 0 \quad \forall \varepsilon > 0$

(A4) $$\sum_{k=\ell}^{N} \big(|||U_{k+1} - U_k|||^2 - \varepsilon \eta_k^2 \big) \leq C_{\rm orth}(\varepsilon)\, \eta_\ell^2$$

- suppose: $\mathcal{L}$ symmetric, $|||\cdot|||$ induced energy norm

$$\Longrightarrow \quad \sum_{k=\ell}^{N} \big(|||U_{k+1} - U_k|||^2 \big) = \sum_{k=\ell}^{N} \big(|||u - U_k|||^2 - |||u - U_{k+1}|||^2 \big) \leq |||u - U_\ell|||^2$$

$\Longrightarrow \quad C_{\rm orth}(\varepsilon) = C_{\rm rel}^2$ independent of u

- general: $\mathcal{L}$ is symmetric + compact perturbation

$\Longrightarrow$ perturbation argument with $(u - U_\ell)/|||u - U_\ell||| \rightharpoonup 0$ in $H^1(\Omega)$

$\Longrightarrow \quad C_{\rm orth}(\varepsilon)$ depends also on u

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Feischl, Führer, Praetorius: SINUM 52 (2014)

Bespalov, Haberl, Praetorius: CMAME 317 (2017)

Finally: Advanced requirements on mesh-refinement

Refinement+

3. number of sons is uniformly finite

$$\#\mathtt{refine}(\mathcal{T}_H, \mathcal{T}_H) \lesssim \#\mathcal{T}_H$$

4. overlay estimate

$$\exists\, \mathcal{T}_h \in \mathtt{refine}(\mathcal{T}_H') \cap \mathtt{refine}(\mathcal{T}_H'') : \quad \#\mathcal{T}_h \le \#\mathcal{T}_H' + \#\mathcal{T}_H'' - \#\mathcal{T}_0$$

5. closure estimate

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \lesssim \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \quad \text{for all } \mathcal{T}_\ell$$

Binev, Dahmen, DeVore: Numer. Math. 97 (2004)

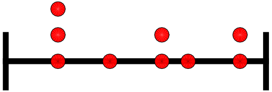
Stevenson: Found. Comput. Math. 7 (2007)

Stevenson: Math. Comp. 77 (2007)

Adaptive IGAFEM

Splines in 1D

- polynomial degree p
- knots $\mathcal{K}$ on $[0,1]$ with multiplicity $\leq p+1$

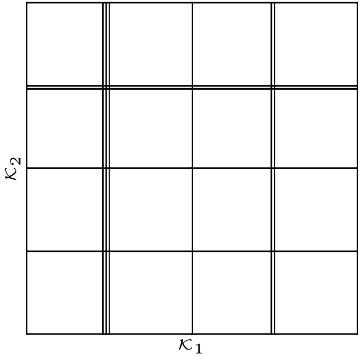


- $\mathcal{S}^p(\mathcal{K}) := \{s \text{ pw. polynomial} : s \text{ is } p - \mathrm{mult}(z) \text{ cont. diff. at } z \in \mathcal{K}\}$

Tensor splines

- polynomial degrees p_1 and p_2

- 1D knots $\mathcal{K}_1$ and $\mathcal{K}_2$ $\quad\Longrightarrow\quad$ tensor-mesh $\mathcal{T}$



- $\mathcal{S}^{(p_1,p_2)}(\mathcal{K}_1,\mathcal{K}_2) := \big\{ s_1(x)\cdot s_2(y) \,:\, s_1 \in \mathcal{S}^{p_1}(\mathcal{K}_1) \text{ and } s_2 \in \mathcal{S}^{p_2}(\mathcal{K}_2) \big\}$

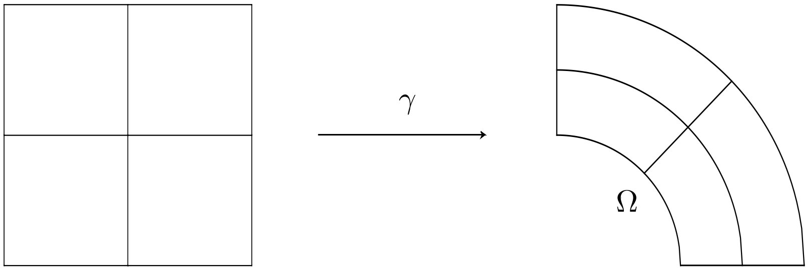
Rational tensor splines

- for geometry representation in CAD, splines are not enough

$\Longrightarrow$ fix $w \in \mathcal{S}^{(p_1,p_2)}(\mathcal{K}_1,\mathcal{K}_2)$ with $w > 0$ and define

$$\mathcal{S}_w^{(p_1,p_2)}(\mathcal{K}_1,\mathcal{K}_2) := \Big\{ \frac{s}{w} \,:\, s \in \mathcal{S}^{(p_1,p_2)}(\mathcal{K}_1,\mathcal{K}_2) \Big\}$$

- $\gamma : [0,1]^2 \to \Omega$ with components in $\mathcal{S}_w^{(p_1,p_2)}(\mathcal{K}_1,\mathcal{K}_2)$

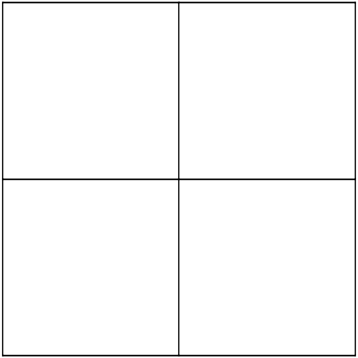


Uniform mesh-refinement

- with $p_1, p_2, \mathcal{K}_1, \mathcal{K}_2, w$ from γ, choose initial ansatz space

$$\mathcal{X}_0 := \{r \circ \gamma^{-1} : r \in \mathcal{S}_w^{(p_1,p_2)}(\mathcal{K}_1, \mathcal{K}_2)\}$$

- refine by inserting new knots in $\mathcal{K}_1, \mathcal{K}_2$, while p_1, p_2, w are fixed



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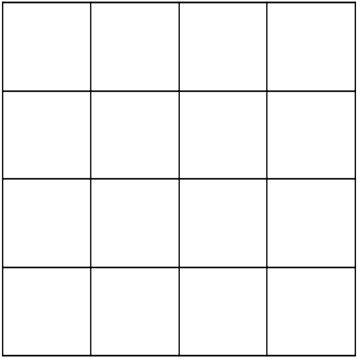

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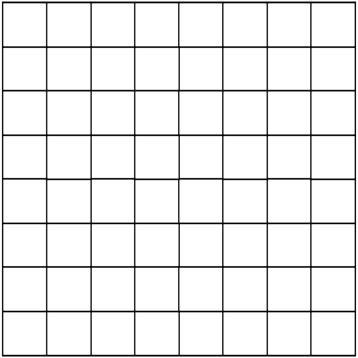
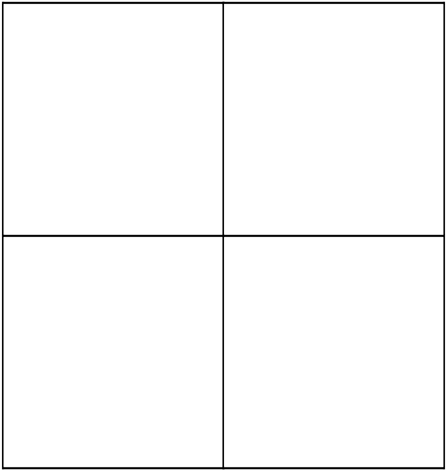
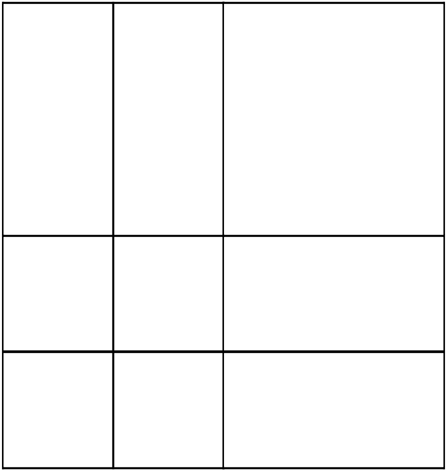


Adaptive mesh-refinement without hanging nodes



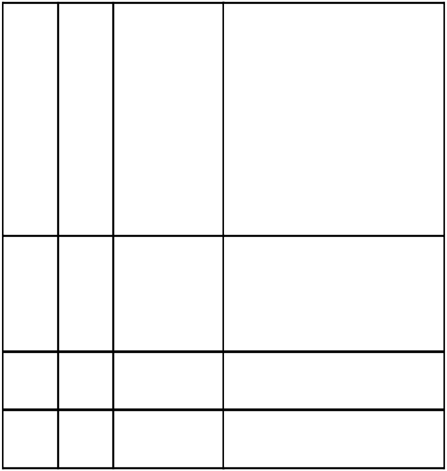
- initial mesh

Adaptive mesh-refinement without hanging nodes



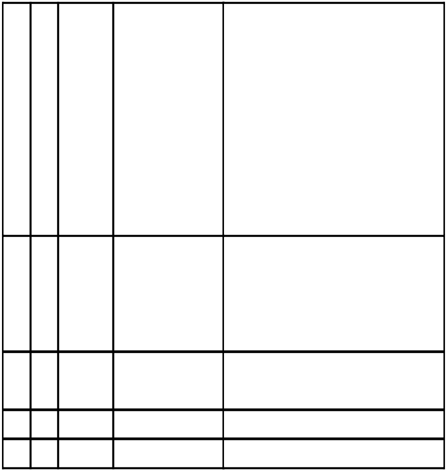
- Step 1: 1 marked $\Longrightarrow$ 3 refined

Adaptive mesh-refinement without hanging nodes



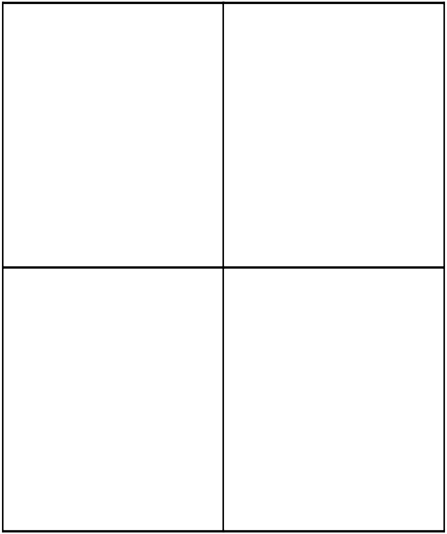
- Step 2: 1 marked $\Longrightarrow$ 5 refined

Adaptive mesh-refinement without hanging nodes



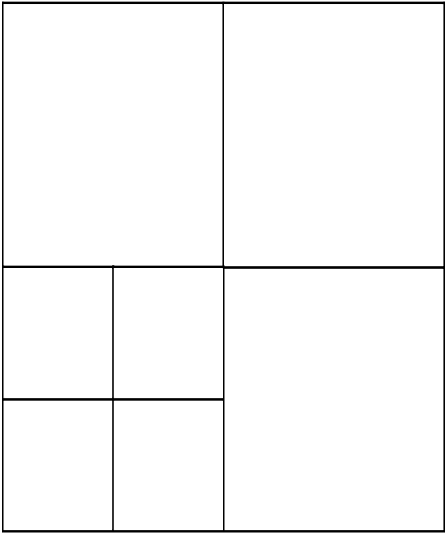
- Step 3: 1 marked $\Longrightarrow$ 7 refined
- Step k: 1 marked $\Longrightarrow$ $2k+1$ refined

Adaptive mesh-refinement with hanging nodes



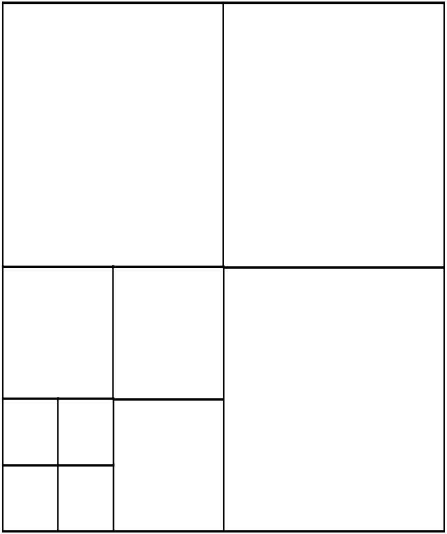
- initial mesh

Adaptive mesh-refinement with hanging nodes



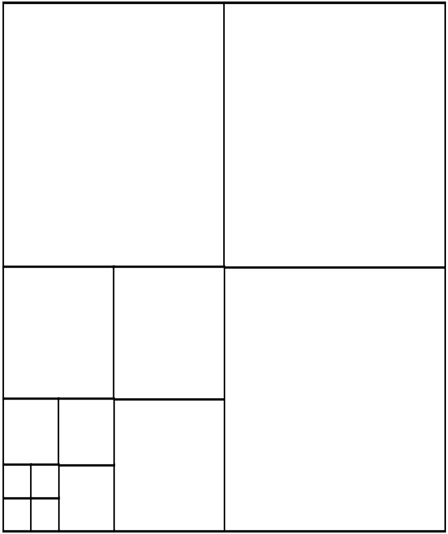
- Step 1

Adaptive mesh-refinement with hanging nodes



- Step 2

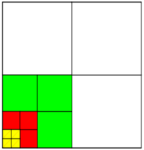
Adaptive mesh-refinement with hanging nodes



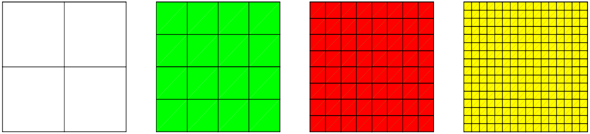
- Step 3, etc.

Hierarchical B-splines 1/2

- $\mathcal{T}_\bullet$ adaptive mesh, $p_1, p_2, \mathcal{K}_1, \mathcal{K}_2, w$ from γ



- uniform refinements $\mathcal{K}_1^\ell$ and $\mathcal{K}_2^\ell$

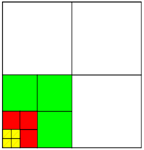


- $\Omega^\ell := \bigcup\{T \in \mathcal{T}_\bullet : \mathrm{level}(T) \geq \ell\}$

Giannelli, Jütler, Simeon, Vuong: CMAME 200 (2011)

Hierarchical B-splines 2/2

- $\mathcal{T}_\bullet$ adaptive mesh, $p_1, p_2, \mathcal{K}_1, \mathcal{K}_2, w$ from γ



- $\Omega^\ell := \bigcup\{T \in \mathcal{T}_\bullet : \mathrm{level}(T) \geq \ell\}$
- tensor-product B-spline basis $\mathcal{B}^\ell$ of $\mathcal{S}_w^{(p_1,p_2)}(\mathcal{K}_1^\ell, \mathcal{K}_2^\ell)$
- $\mathcal{H}_\bullet := \bigcup_{\ell \geq 0} \{\beta \in \mathcal{B}^\ell : \mathrm{supp}(\beta) \subseteq \Omega^\ell, \quad \mathrm{supp}(\beta) \not\subseteq \Omega^{\ell+1}\}$

$\Longrightarrow$ $\mathcal{H}_\bullet$ is linearly independent

- hierarchical splines $\mathcal{S}_w^{(p_1,p_2)}(\mathcal{T}_\bullet) := \mathrm{span}(\mathcal{H}_\bullet)$

Giannelli, Jütler, Simeon, Vuong: CMAME 200 (2011)

Admissible meshes

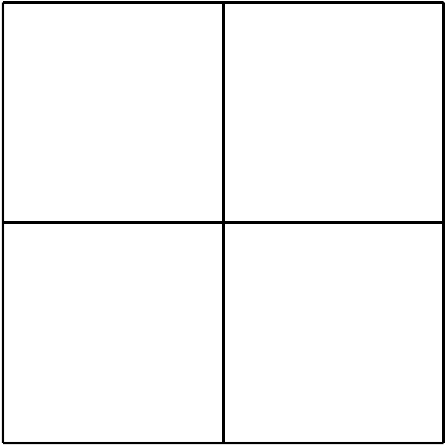
Admissibility condition

- $|\mathrm{level}(T) - \mathrm{level}(T')| \le 1$ for all $T \in \mathcal{T}_\bullet$, $T' \in \mathcal{N}_\bullet(T)$
- with neighbors $\mathcal{N}_\bullet(T) := \{T' \in \mathcal{T}_\bullet \,:\, \exists \beta \in \mathcal{H}_\bullet \quad T, T' \subseteq \mathrm{supp}(\beta)\}$

- will be ensured through refinement algorithm
  - idea: if T is marked for refinement
    $\Longrightarrow$ additionally mark neighbors with $\mathrm{level}(T') = \mathrm{level}(T) - 1$
- consequences:
  - good: number of basis functions supported on $T \in \mathcal{T}_\bullet$ uniformly bounded
  - good: number of elements in support of $\beta \in \mathcal{H}_\bullet$ uniformly bounded
  - bad: requires certain over-refinement

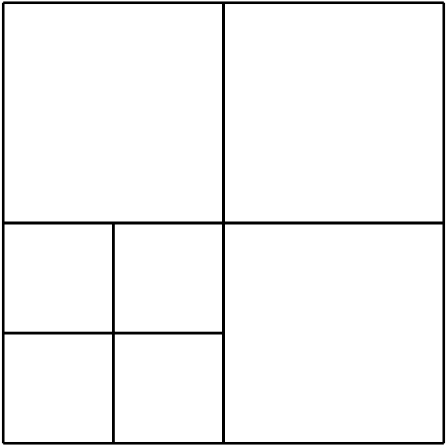
Gantner, Haberlik, Praetorius: M3AS 27 (2017)

Example for $p = 1$



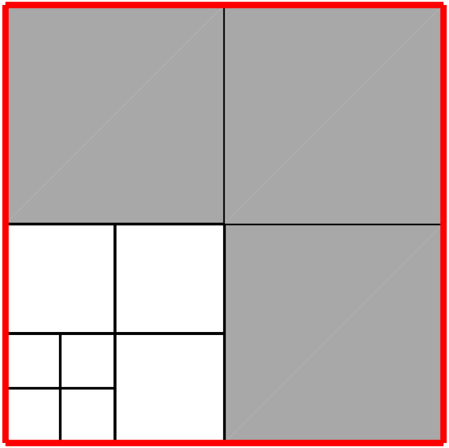
- initial mesh

Example for $p = 1$



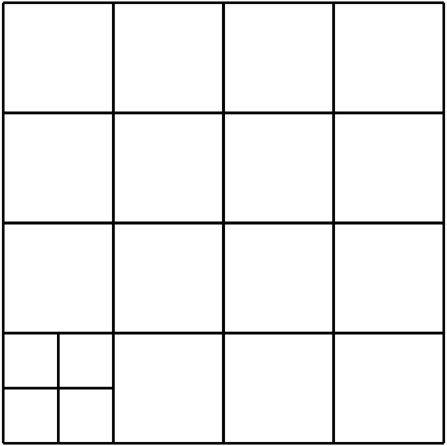
- admissible

Example for $p = 1$



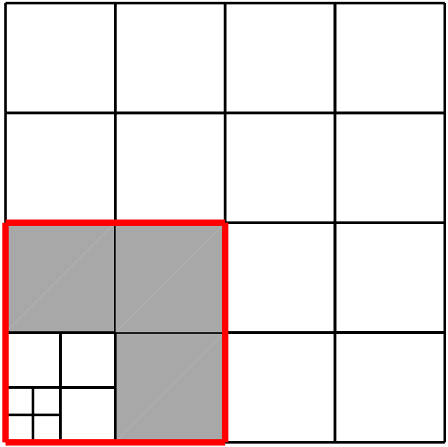
- inadmissible

Example for $p = 1$



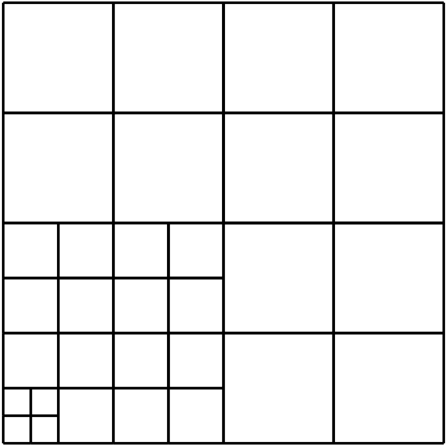
- admissible

Example for $p = 1$



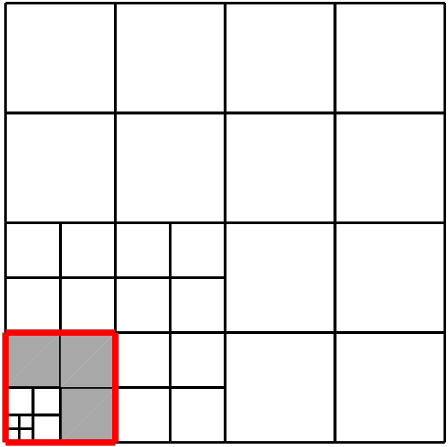
- inadmissible

Example for $p = 1$



- admissible

Example for $p = 1$



- inadmissible, etc.

Main theorem on adaptive IGAFEM

- admissible refinement satisfies crucial refinement properties

Theorem (Gantner, Haberlik, P. '17)

- IGAFEM with hierarchical splines
- proposed mesh-refinement strategy (= admissibility)

$\Longrightarrow$ residual a-posteriori error estimator satisfies (A1)–(A4)

$\Longrightarrow$ adaptive algorithm leads to linear convergence with optimal rates

- remember: optimality is contrained by estimator + refinement strategy

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Morgenstern, Peterseim: CAGD 34 (2015)

Buffa, Giannelli: M3AS 26 (2016)

Buffa, Giannelli: M3AS 27 (2017)

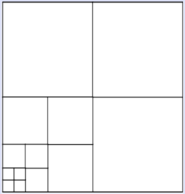
Gantner, Haberlik, Praetorius: M3AS 27 (2017)

Admissibility $\Longrightarrow$ worse rates? NO!

- admissibility $\Longrightarrow$ $\eta_\bullet \simeq \min\limits_{V_\bullet \in \mathcal{X}_\bullet} \big(\|u - V_\bullet\| + \mathrm{osc}_\bullet(V_\bullet)\big)$
- conformity $\Longrightarrow$ RHS is monotone under refinement
- refinement with simple hanging nodes satisfies mesh-closure estimate

Proposition (Gantner, P. '18+)

- $\mathcal{T} \in \mathbb{T}_N$ admissible refinement of $\mathcal{T}_0$, $\#\mathcal{T} \le N$
- $\mathcal{T} \in \widehat{\mathbb{T}}_N$ hierarchical refinement of $\mathcal{T}_0$, $\#\mathcal{T} \le N$
- $s > 0$

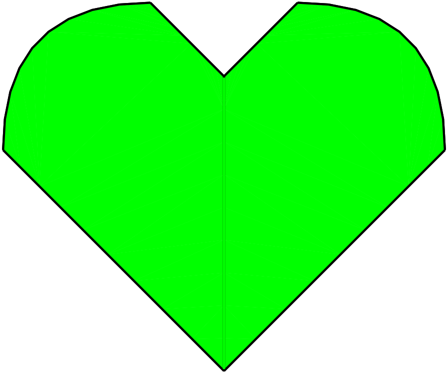


$$\Longrightarrow \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \eta_{\rm opt}\Big)$$
$$\simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt} \in \mathbb{T}_N} \min_{V_{\rm opt} \in \mathcal{X}_{\rm opt}} \big(\|u - V_{\rm opt}\| + \mathrm{osc}_{\rm opt}(V_{\rm opt})\big)\Big)$$
$$\simeq \sup_{N>0} \Big(N^s \min_{\mathcal{T}_{\rm opt} \in \widehat{\mathbb{T}}_N} \min_{V_{\rm opt} \in \mathcal{X}_{\rm opt}} \big(\|u - V_{\rm opt}\| + \mathrm{osc}_{\rm opt}(V_{\rm opt})\big)\Big)$$

Bonito, Nochetto: SINUM 48 (2010)

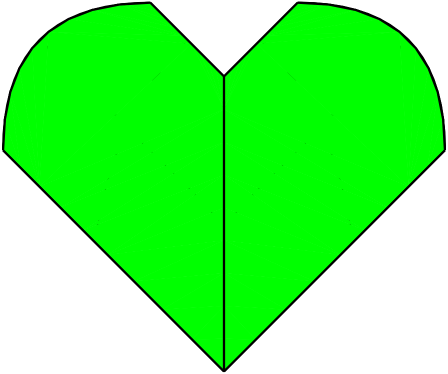
Numerical experiment

Example: Heart-shape



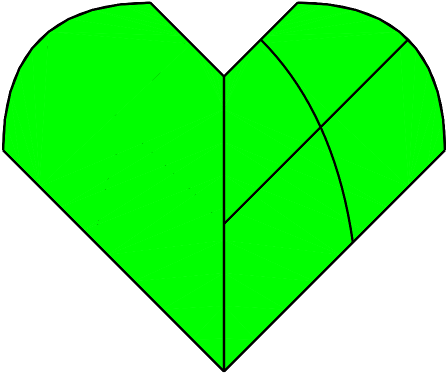
- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape



- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape



- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
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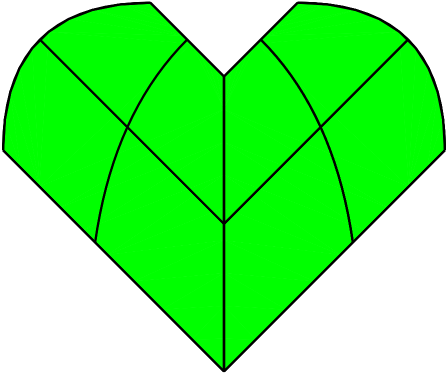
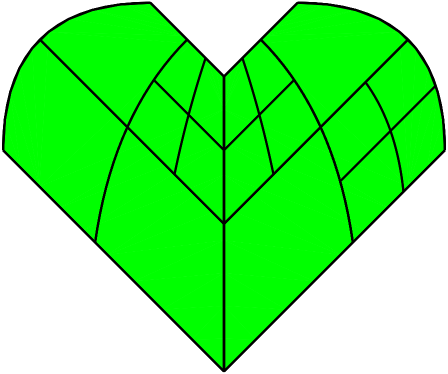
Example: Heart-shape



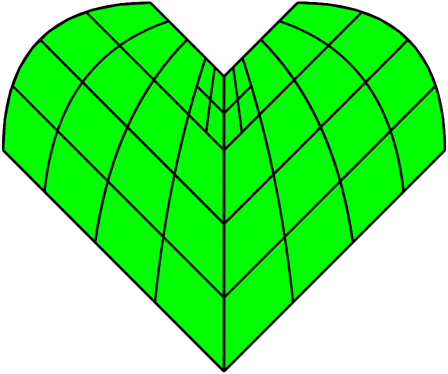
- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape



- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape



- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
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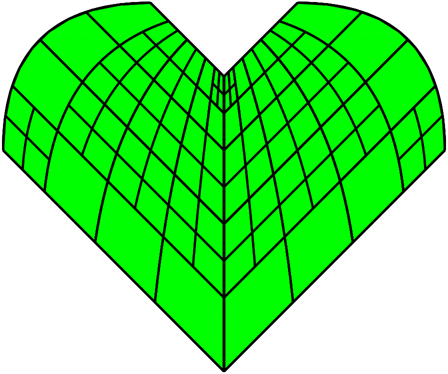
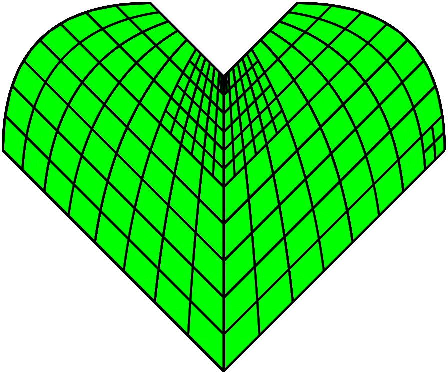
Example: Heart-shape



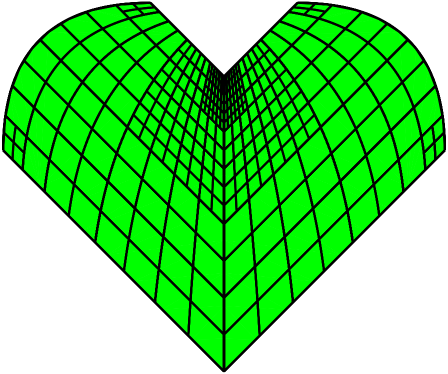
- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape



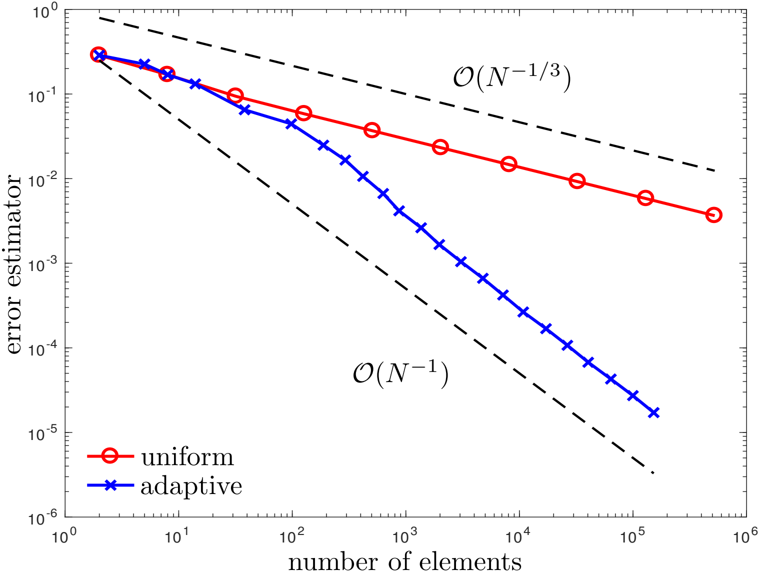
- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

Example: Heart-shape

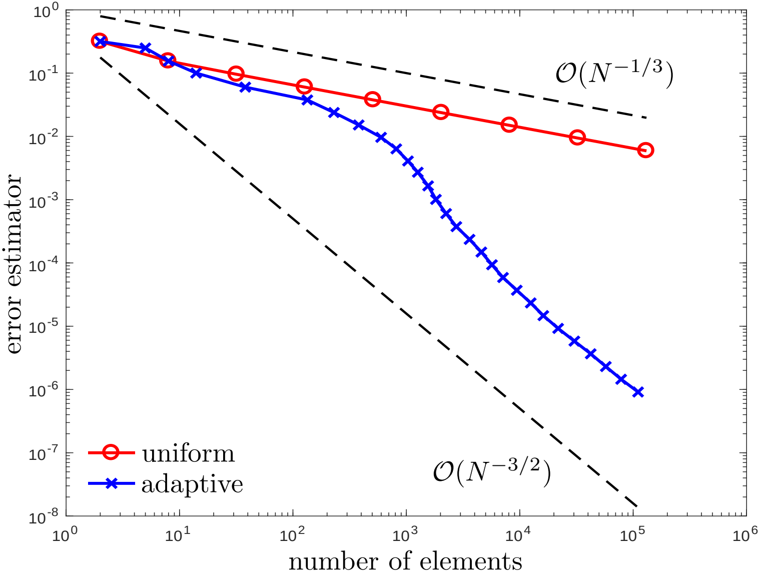


- solve $-\Delta u = 1$ in Ω subject to $u = 0$ on Γ
- splines with $p := p_1 = p_2 = 2$

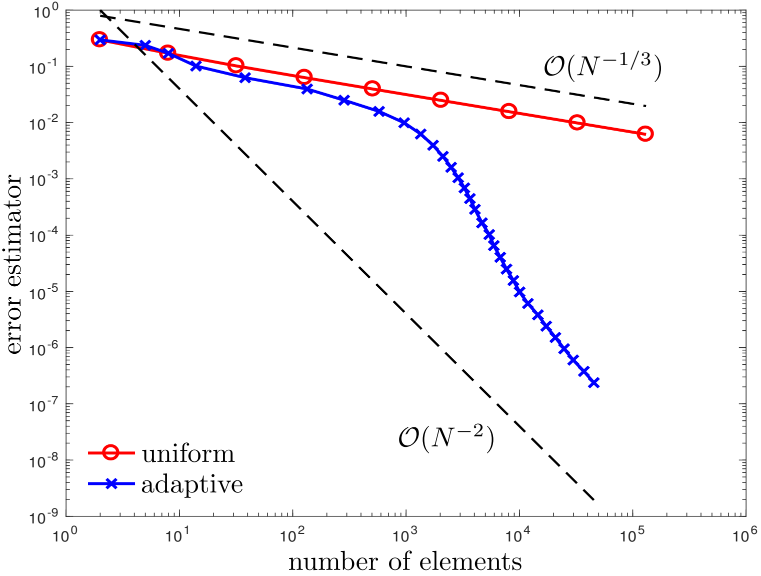
Convergence for $p = 2$



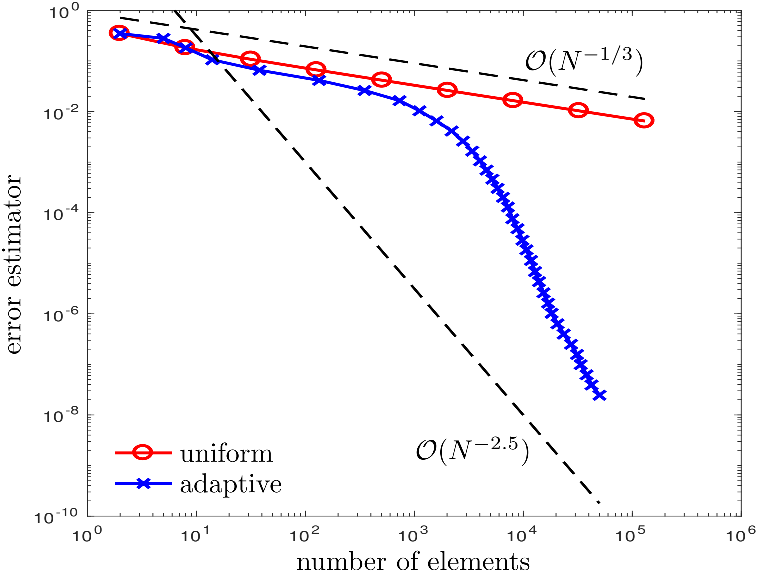
Convergence for $p = 3$



Convergence for $p = 4$



Convergence for $p = 5$



Conclusions

Conclusions

- optimal convergence of adaptive IGAFEM with hierarchical splines
- fits into frame of *axioms of adaptivity*
- over-refinement from admissibility constraint does not matter
- analogous results hold for adaptive IGABEM
- unpublished: adaptive IGAFEM with T-splines is also optimal!

Gantner, Haberlik, Praetorius: M3AS 27 (2017)

Gantner: PhD thesis, TU Wien (2017)

Buffa, Giannelli: M3AS 27 (2017)

Morin, Nochetto, Pauletti: Work in progress (2018+)

The new role of efficiency

- prior: analysis relied on reliability + efficiency of η_ℓ
- then: $\eta_\ell \simeq |||u - U_\ell||| + \mathrm{osc}_\ell(U_\ell) \simeq \min_{V_\ell \in \mathcal{X}_\ell} \big(|||u - V_\ell||| + \mathrm{osc}_\ell(V_\ell)\big)$
- often: main theorem formulated for this *(best) total error*
- now: two step procedure:
  1. prove optimal rates for estimator (without using efficiency)
  2. derive optimal rates for (best) total error (using efficiency)
- advantages of new approach:
  - proof of main theorem is simplified (verify only 4 axioms)
  - less dependencies of constants
  - some problems lack efficiency of estimator (e.g., nonlinear, BEM)

Stevenson: Found. Comput. Math. 7 (2007)

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Carstensen, Feischl, Page, Praetorius: CAMWA 67 (2014)

Thanks for listening!

Gregor Gantner, Daniel Haberlik, Dirk Praetorius:
Adaptive IGAFEM with optimal convergence rates: Hierarchical B-splines,
M3AS 27 (2017)

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