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Optimal convergence rates for adaptive FEM for compactly perturbed elliptic problems

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joint work with

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Introduction

Impetus

- adaptivity is well-understood for (BEM for) Poisson problem!
- question: what can be said about (BEM for) Helmholtz problem?
- particular interest: boundary element method for scattering problems
- rumor: adaptivity works for arbitrarily coarse initial mesh?!
- but: Gårding inequality requires meshes to be sufficiently small!
- easier start: consider FEM for interior Helmholtz problem

Model problem

- $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain, $d = 2, 3$

Helmholtz equation

$$\begin{aligned} -\Delta u - \kappa^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- κ^2 is not an eigenvalue of $-\Delta u$ $\rightsquigarrow$ well-posedness
- $\mathcal{S}^p(\mathcal{T}_\star)$ piecewise polynomials on mesh $\mathcal{T}_\star$

Discrete formulation

- find $U_\star \in \mathcal{X}_\star := \mathcal{S}^p(\mathcal{T}_\star) \cap H^1_0(\Omega)$ s.t.

$$(\nabla U_\star, \nabla V_\star)_{L^2} - (\kappa^2 U_\star, V_\star)_{L^2} = (f, V_\star)_{L^2} \qquad \text{for all } V_\star \in \mathcal{X}_\star$$

- well-posedness of discrete problem has to be considered!

Residual error estimator

- $\mathcal{T}_\star$ conforming triangulation of Ω
- $\eta_\star(T)^2 := h_T^2 \, \|f + \Delta U_\star + \kappa^2 U_\star\|_{L^2(T)}^2 + h_T \, \|[\partial_n U_\star]\|_{L^2(\partial T \cap \Omega)}^2$
- where $h_T := |T|^{1/d} \simeq \mathrm{diam}(T)$
- $\eta_\star ::= \Big(\sum_{T \in \mathcal{T}_\star} \eta_\star(T)^2 \Big)^{1/2}$

$\Longrightarrow$ reliable $\|u - U_\star\|_{H^1} \lesssim \eta_\star$

$\Longrightarrow$ efficient $\eta_\star \lesssim \|u - U_\star\|_{H^1} + \mathrm{osc}_\star$

Solve $\longrightarrow$ Estimate $\longrightarrow$ Mark $\longrightarrow$ Refine

- Input: initial mesh $\mathcal{T}_0$, adaptivity parameter $0 < \theta \leq 1$

For all $\ell = 0, 1, 2, 3, \ldots$ iterate

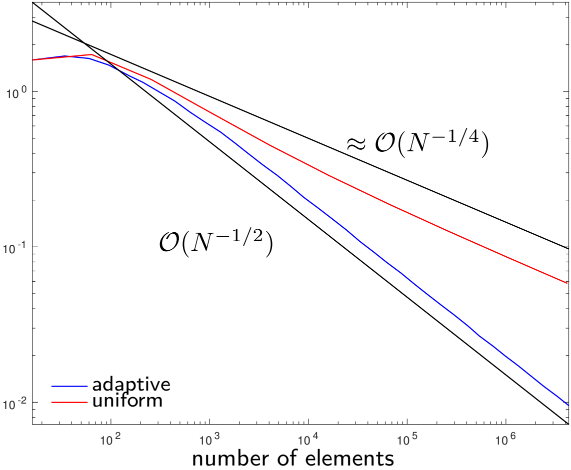
1. if $\nexists\, U_\ell$, refine $\mathcal{T}_\ell$ uniformly and continue with step 1
2. compute discrete solution $U_\ell \in \mathcal{X}_\ell$
3. compute error estimators $\eta_\ell(T)$ for all $T \in \mathcal{T}_\ell$
4. find (almost) minimal set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ s.t.

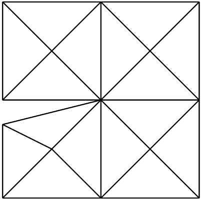
$$\theta\, \eta_\ell^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T)^2$$

5. refine (at least) marked elements $T \in \mathcal{M}_\ell$ to obtain $\mathcal{T}_{\ell+1}$

- Output: approximations U_ℓ and estimators η_ℓ for all $\ell \in \mathbb{N}$

What is all about?





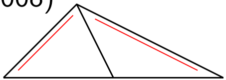
Main results

1. linear convergence $\eta_{\ell+k}^2 \le C\, q^k\, \eta_\ell^2$
   - $\forall\, 0 < \theta \le 1 \quad \exists\, C > 0 \quad \exists\, 0 < q = q(\theta) < 1 \ldots$
2. quasi-optimal convergence behavior $\eta_\ell = \mathcal{O}(N^{-s})$
   - $\exists\, 0 < \theta_\star = \theta_\star(C_{\rm rel}) < 1 \quad \forall\, 0 < \theta < \theta_\star \ldots$
   - for each possible $s > 0$
3. further remarks
   - algorithm works for every conforming initial mesh $\mathcal{T}_0$
   - does not need any a priori information on $\|h_0\|_\infty \le H$
   - analysis covers AFEM for general 2nd order linear elliptic PDEs
     - $\mathcal{L}u := -\mathrm{div}(\boldsymbol{A}\nabla u) + \boldsymbol{b}\cdot\nabla u + cu$
     - $\boldsymbol{A} \in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \quad \boldsymbol{b} \in L^\infty(\Omega;\mathbb{R}^d), \quad c \in L^\infty(\Omega)$

Related work

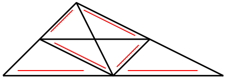
Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

- linear, symmetric, Lax–Milgram



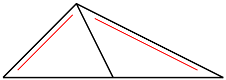
Cascon, Nochetto '12: IMA J. Numer. Anal. 32 (2012)

- linear, non-symmetric, Lax–Milgram
- $\operatorname{div} \boldsymbol{b} = 0, \ c \geq 0$
- initial mesh $\mathcal{T}_0$ sufficiently small



Feischl, Führer, Praetorius '14: SINUM 52 (2014)

- linear, non-symmetric, Lax–Milgram
- no restrictions on $\mathcal{T}_0$



Bespalov, Haberl, Praetorius '17: CMAME 317 (2017)

- now: avoid Lax–Milgram, but only Gårding inequality

Convergence

Abstract setting

- $\mathcal{H} := H^1_0(\Omega)$ separable Hilbert space with dual space $\mathcal{H}^*$

Variational formulation

$$a(u, v) + \langle \mathcal{K} u \, , v \rangle = \langle f \, , v \rangle \qquad \text{for all } v \in \mathcal{H}$$

- $a(\cdot, \cdot)$ symm., elliptic, cont., bilinear, i.e., $\|v\|^2_{\mathcal{H}} \simeq a(v, v) =: |||v|||^2$
- $\mathcal{K} : \mathcal{H} \to \mathcal{H}^*$ compact linear operator
- suppose well-posedness of continuous problem

Discrete formulation

- find $U_\star \in \mathcal{X}_\star \subset \mathcal{H}$ s.t.

$$a(U_\star, V_\star) + \langle \mathcal{K} U_\star \, , V_\star \rangle = \langle f \, , V_\star \rangle \qquad \text{for all } V_\star \in \mathcal{X}_\star$$

- discrete formulation is possibly not well-posed

Céa lemma (based on dual regularity)

Proposition

- suppose H^{1+s} regularity of dual problem, $s > 0$

$\Longrightarrow \quad \exists\, H > 0 \quad \exists\, C > 0 \quad \forall\, \mathcal{T}_\star$ with $\|h_\star\|_{L^\infty} \le H$:

- unique solution $U_\star \in \mathcal{X}_\star$
- $\|u - U_\star\|_{\mathcal{H}} \le C \min_{V_\star \in \mathcal{X}_\star} \|u - V_\star\|_{\mathcal{H}}$

Brenner, Scott: Springer (2008)

Céa lemma (based on compactness argument)

Proposition

- suppose that $\mathcal{H} = \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$

$\Longrightarrow$ $\exists\, \ell_\star \in \mathbb{N}_0 \quad \exists\, C > 0 \quad \forall\, \mathcal{X}_\star \subset \mathcal{H}$ closed with $\mathcal{X}_\star \supseteq \mathcal{X}_{\ell_\star}$:

- unique solution $U_\star \in \mathcal{X}_\star$
- $\|u - U_\star\|_{\mathcal{H}} \le C \min_{V_\star \in \mathcal{X}_\star} \|u - V_\star\|_{\mathcal{H}}$

- uniform refinement $\Longrightarrow$ $\mathcal{H} = \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$

$\Longrightarrow$ uniform refinement in step 1 at most finitely many times

$\Longrightarrow$ $\exists\, \ell_0 \in \mathbb{N} \quad \forall\, \ell \ge \ell_0 :$ $\mathcal{T}_\ell$ provides unique solution $U_\ell \in \mathcal{X}_\ell$

Sauter, Schwab: Springer (2011)

Convergence

Proposition (Bespalov, Haberl, P. '17)

- suppose that $\sup_{\ell \geq \ell_0} \|U_\ell\|_{\mathcal{H}} < \infty$

$\Longrightarrow$ $\|u - U_\ell\|_{\mathcal{H}} \lesssim \eta_\ell \to 0$ as $\ell \to \infty$

$\Longrightarrow$ in particular, $u \in \mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$

- for each subsequence (η_{ℓ_i}) of (η_ℓ)
  1. boundedness $\Longrightarrow$ w.l.o.g weak convergence $U_{\ell_i} \rightharpoonup w$ in $\mathcal{H}$
  2. weak formulation $\Longrightarrow$ strong convergence $U_{\ell_i} \to w$ in $\mathcal{H}$
  3. estimator reduction for $\eta_{\ell_{i+1}}^2 \leq \kappa\, \eta_{\ell_i}^2 + C\, \|U_{\ell_{i+1}} - U_{\ell_i}\|_{\mathcal{H}}^2$
  4. implies $\eta_{\ell_i} \to 0$
- overall $\eta_\ell \to 0$

Linear Convergence

Ingredients of symmetric case

- $\mathcal{T}_{\ell+1} \in \mathrm{refine}(\mathcal{T}_\ell) \quad \Longrightarrow \quad \mathcal{S}^p_0(\mathcal{T}_\ell) \subseteq \mathcal{S}^p_0(\mathcal{T}_{\ell+1})$

Estimator reduction

Dörfler marking $\Longrightarrow \quad \eta^2_{\ell+1} \leq \kappa\, \eta^2_\ell + C\, \|U_{\ell+1} - U_\ell\|^2_{\mathcal{H}}$

- $|\!|\!| v |\!|\!|^2 := a(v,v) \simeq \|v\|^2_{\mathcal{H}}$ energy norm of symmetric part

Pythagoras theorem (symmetric case)

$$|\!|\!| u - U_{\ell+1} |\!|\!|^2 + |\!|\!| U_{\ell+1} - U_\ell |\!|\!|^2 = |\!|\!| u - U_\ell |\!|\!|^2$$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Quasi-Pythagoras theorem

- $\|u - U_{\ell+1}\|^2 + \|U_{\ell+1} - U_\ell\|^2 = \|u - U_\ell\|^2$ for symmetric PDE

Proposition (Feischl, Führer, P. '14; Bespalov, Haberl, P. '17)

- suppose that $U_\ell \to u \in \mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$ and well-posedness on $\mathcal{X}_\infty$

$\Longrightarrow$ $\forall\, \varepsilon > 0 \quad \exists\, \ell_1 \geq \ell_0 \quad \forall\, \ell \geq \ell_1$:

$$\|u - U_{\ell+1}\|^2 + \|U_{\ell+1} - U_\ell\|^2 \leq \frac{1}{1-\varepsilon}\, \|u - U_\ell\|^2$$

- mathematical heart of proof:
  - $e_\ell := u - U_\ell \quad \Longrightarrow \quad \dfrac{e_\ell}{\|e_\ell\|} \rightharpoonup 0$ in $\mathcal{H}$
  - $E_\ell := U_{\ell+1} - U_\ell \quad \Longrightarrow \quad \dfrac{E_\ell}{\|E_\ell\|} \rightharpoonup 0$ in $\mathcal{H}$

Feischl, Führer, Praetorius: SINUM 52 (2014)

Quasi-Pythagoras theorem (remainder of proof)

Split differential operator

- $\mathcal{L} := \mathcal{A} + \mathcal{K} \in L(\mathcal{H}, \mathcal{H}^*)$
- $\langle \mathcal{A}u\,, v\rangle = (\boldsymbol{A}\nabla u, \nabla v)_\Omega$ symmetric, elliptic, linear, continuous
- $\langle \mathcal{K}u\,, v\rangle = (\boldsymbol{b}\cdot\nabla u + cu, v)_\Omega$ non-symm., linear, cont., compact

$$
\begin{aligned}
\|e_{\ell+1}\|^2 + \|E_\ell\|^2 &= \|e_\ell\|^2 + \langle \mathcal{K}E_\ell\,, e_{\ell+1}\rangle - \langle \mathcal{K}e_{\ell+1}\,, E_\ell\rangle + \dots \\
&\le \|e_\ell\|^2 + \|\mathcal{K}E_\ell\|_*\|e_{\ell+1}\| + \|\mathcal{K}e_{\ell+1}\|_*\|E_\ell\| + \dots \\
&\le \|e_\ell\|^2 + \underbrace{\Big(\Big\|\mathcal{K}\frac{E_\ell}{\|E_\ell\|}\Big\|_* + \Big\|\mathcal{K}\frac{e_{\ell+1}}{\|e_{\ell+1}\|}\Big\|_*\Big)}_{\longrightarrow 0}\|e_{\ell+1}\|\|E_\ell\| + \dots \\
&\le \|e_\ell\|^2 + 2\varepsilon\,\|e_{\ell+1}\|\|E_\ell\| + \dots
\end{aligned}
$$

$$
\Longrightarrow \quad \|e_{\ell+1}\|^2 + \|E_\ell\|^2 \le \frac{1}{1-\varepsilon}\,\|e_\ell\|^2 \quad \forall \ell \ge \ell_1(\varepsilon)
$$

Linear convergence

Theorem (Bespalov, Haberl, P. 17)

- suppose that $U_\ell \to u \in \mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$ and well-posedness on $\mathcal{X}_\infty$

$\Longrightarrow$ $\forall\, 0 < \theta \le 1 \quad \exists\, 0 < q < 1 \quad \exists\, C > 0 \quad \forall\, \ell \ge 0 \quad \forall\, k \in \mathbb{N}:$

$$\eta_{\ell+k}^2 \le C q^k \eta_\ell^2$$

- estimator reduction & quasi-Pythagoras theorem & reliability

$\Longrightarrow$ $\exists\, \gamma\ \forall\, \ell \ge \ell_1\ \forall\, k: \quad \big(|||u - U_{\ell+k}|||^2 + \gamma\, \eta_{\ell+k}^2\big) \le q^k \big(|||u - U_\ell|||^2 + \gamma\, \eta_\ell^2\big)$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Optimal Céa lemma (in energy norm)

Theorem (Bespalov, Haberl, P. '17)

- suppose that $U_\ell \to u \in \mathcal{X}_\infty := \overline{\bigcup_{\ell=0}^\infty \mathcal{X}_\ell}$ and well-posedness on $\mathcal{X}_\infty$
- $\exists\, C_\ell \geq 1$ with $C_\ell \to 1 \quad \forall\, \ell \geq \ell_0$:

$$\|u - U_\ell\| \leq C_\ell \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\|$$

- Galerkin orth. $\implies$ $\|u - U_\ell\| \leq \underbrace{\frac{1 + C\,\|\mathcal{K}\frac{e_\ell}{\|e_\ell\|}\|_*}{1 - C\,\|\mathcal{K}\frac{e_\ell}{\|e_\ell\|}\|_*}}_{\longrightarrow 1} \min_{V_\ell \in \mathcal{X}_\ell} \|u - V_\ell\|$

How to ensure well-posedness on $\mathcal{X}_\infty$?

- required: $\gamma_\ell := \inf_{V_\ell \in \mathcal{X}_\ell \setminus \{0\}} \sup_{W_\ell \in \mathcal{X}_\ell \setminus \{0\}} \dfrac{|a(W_\ell, V_\ell) + \langle \mathcal{K} W_\ell \,, V_\ell \rangle|}{\|V_\ell\|_{\mathcal{H}} \|W_\ell\|_{\mathcal{H}}} > 0$!
- question: $\gamma_\ell > 0$ for all $\ell \geq \ell_0$, but $\gamma_\infty = 0$ for $\mathcal{X}_\infty = \overline{\bigcup_{\ell=0}^{\infty} \mathcal{X}_\ell}$?

Remedy: Ensure $\mathcal{X}_\infty = \mathcal{H}$ by marking strategy

- add one largest element $T \in \mathcal{T}_\ell$ to $\mathcal{M}_\ell$
- $\mathcal{M}_\ell$ remains (almost) minimal

$\Longrightarrow$ $\mathcal{X}_\infty = \mathcal{H}$ and $\gamma_\infty > 0$

$\Longrightarrow$ $\sup_{\ell \geq \ell_0} \|U_\ell\|_{\mathcal{H}} < \infty$

$\Longrightarrow$ linear convergence and optimal Céa lemma

Optimal Convergence Rates

Approximation class

- $\mathbb{T}$ set of possible meshes obtained from NVB
- $\mathbb{T}_N = \{\mathcal{T}_\star \in \mathbb{T} : \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N \text{ and discrete well-posedness}\}$
- $\widetilde{\mathbb{T}}_N = \{\mathcal{T}_\star \in \mathbb{T} : \#\mathcal{T}_\star - \#\mathcal{T}_0 \le N\}$

- $\|u\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_\star\in\mathbb{T}_N} \eta_\star\Big) < \infty$

$\Longleftrightarrow$

- $\sup_{N\in\mathbb{N}_0} \Big((N+1)^s \min_{\mathcal{T}_\star\in\widetilde{\mathbb{T}}_N} \min_{V_\star\in\mathcal{X}_\star} \big(\|u - V_\star\|_{H^1} + \mathrm{osc}_\star(V_\star)\big)\Big) < \infty$

Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Carstensen, Feischl, Page, Praetorius: Comput. Math. Appl. 67 (2014)

Optimal convergence rates

Theorem (Bespalov, Haberl, P. '17)

- $\exists\, \ell_\star \in \mathbb{N}_0 \quad \exists\, 0 < \theta_\star < 1 \quad \forall\, 0 < \theta < \theta_\star \quad \forall\, s > 0$

$$\|u\|_{\mathbb{A}_s} < \infty \quad \Longleftrightarrow \quad \forall \ell \geq \ell_\star : \quad \eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^{-s}$$

- proof follows well-known ideas of standard AFEM
- but: take care with discrete solvability for overlay

Stevenson: Found. Comput. Math. 7 (2007)

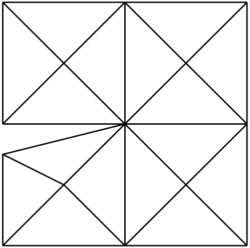
Cascon, Kreuzer, Nochetto, Siebert: SINUM 46 (2008)

Carstensen, Feischl, Page, Praetorius: Comput. Math. Appl. 67 (2014)

Numerical Experiments

2D Helmholtz equation

$$\begin{aligned} -\Delta u - \kappa^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_D \\ u &= g \quad \text{on } \Gamma_N \end{aligned}$$

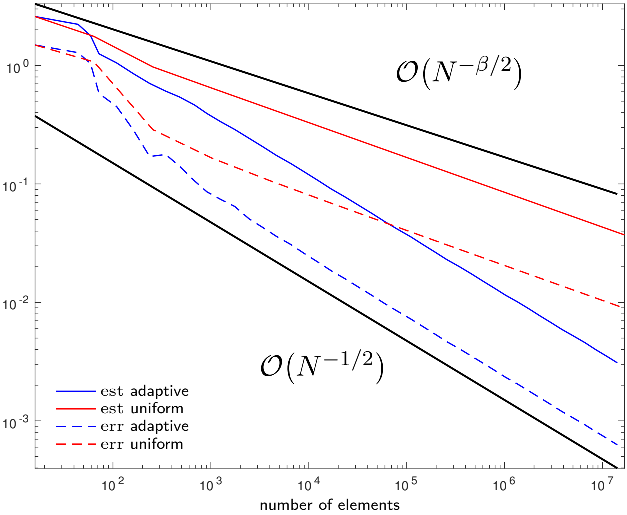


- exact PDE solution $u(x,y)$ in polar coordinates

$$u(x,y) = r^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right)$$

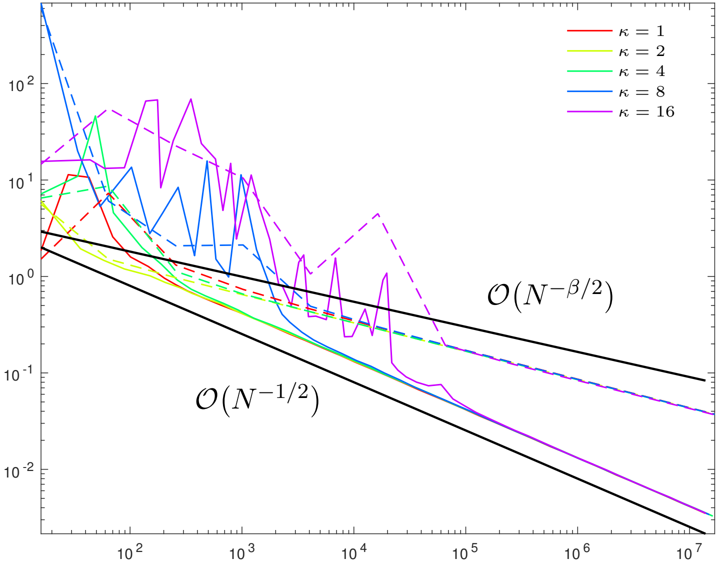
- P^1-FEM $\mathcal{X}_\star := \mathcal{S}^1(\mathcal{T}_\star) \cap H^1_D(\Omega)$
- goal: aim for convergence rate $\|u - U_\ell\|_{H^1} \lesssim \eta_\ell \lesssim N^{-1/2}$

Error vs. estimator

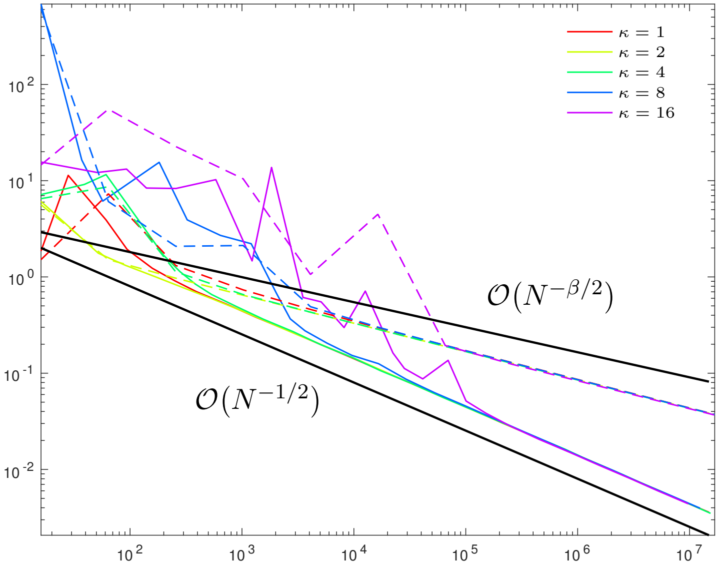


- reduced uniform rate of $\beta/2 \approx 0.26$ with $\beta = \pi/\alpha$

Influence of κ for $\theta = 0.2$

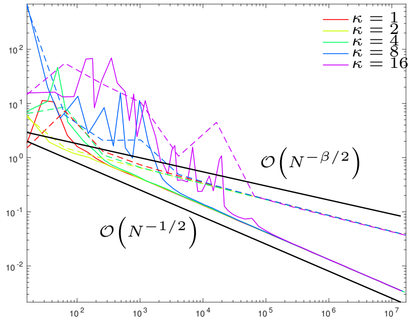


Influence of κ for $\theta = 0.5$

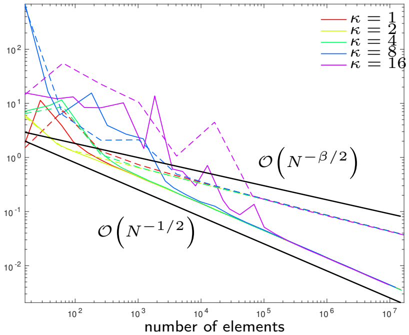


Influence of κ

$\theta = 0.2$

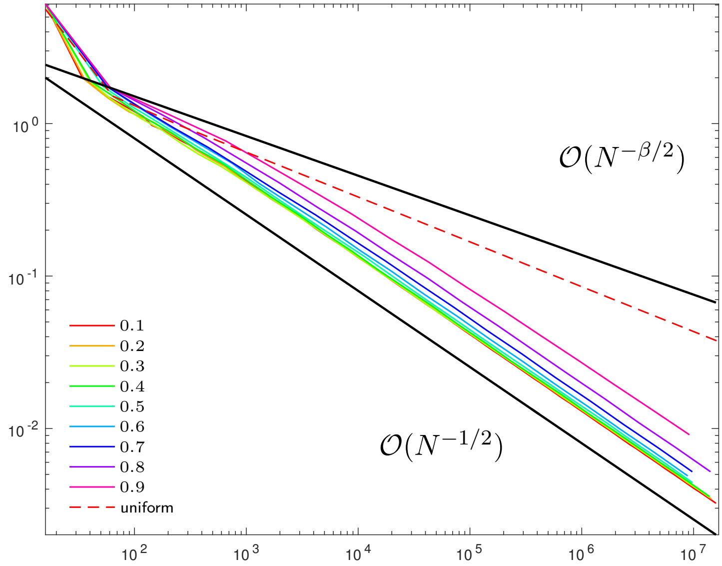


$\theta = 0.5$

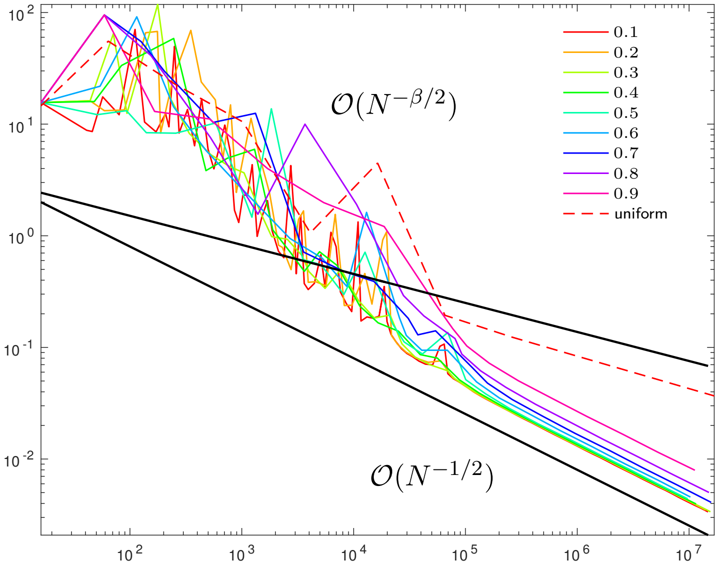


- increase of κ just effects the pre-asymptotic phase
- convergence rate independent of κ

Influence of θ for $\kappa = 2$

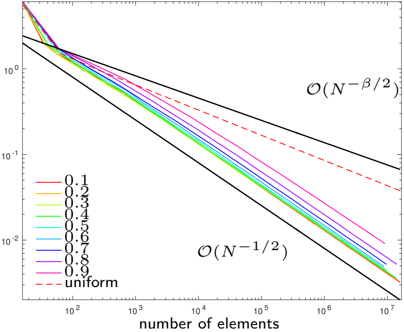


Influence of θ for $\kappa = 16$

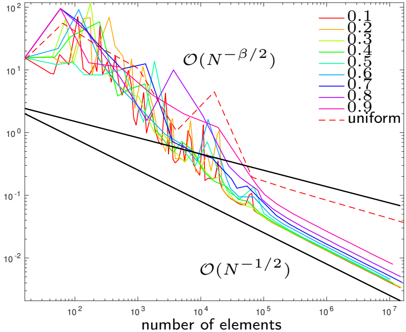


Adaptivity parameter $0 < \theta \leq 1$

$\kappa = 2$



$\kappa = 16$



- rate stable in θ, while $0 < \theta \ll 1$ in analysis
- κ affects the pre-asymptotics

Conclusion

Conclusion

- optimal convergence of AFEM
  - for general $\mathcal{L}u := -\mathrm{div}(\boldsymbol{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu$
  - with coarse initial mesh $\mathcal{T}_0$
  - covers, e.g., Helmholtz
- minimal modification (solve, mark) of adaptive algorithm
  - uniform refinement, if discrete system admits no solution
  - enlarge marked elements by *one* largest element
- analysis is done in abstract framework for the estimator
  - stability on non-refined elements
  - reduction on refined elements
  - discrete reliability
  - applicable to FEM, BEM, IGAFEM
- can also prove estimator properties for Helmholtz BEM

Thanks for listening!

- Alex Bespalov, Alexander Haberl, Dirk Praetorius:
  Adaptive FEM with coarse initial mesh guarantees optimal convergence rates for compactly perturbed elliptic problems
  CMAME 317 (2017), 31–340

- Alex Bespalov, Timo Betcke, Alexander Haberl, Dirk Praetorius:
  Adaptive BEM for accoustic scattering
  in progress (2017)

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