Non-local Operators

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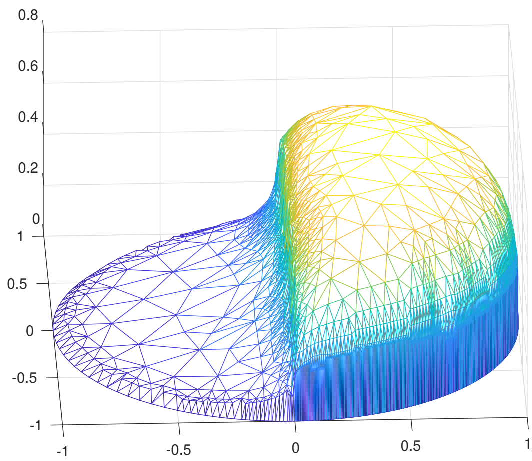

Lecture Notes

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Contents

Part

Introduction

0.1 Non-local Operators

In this lecture, we study the analytical behavior of certain non-local operators and present some numerical methods for these.

We call an operator $\mathcal{A} : X \to Y$ acting between two function spaces X, Y of functions $u : \mathbb{R}^d \to \mathbb{R}$ **local**, if for all $x \in \mathbb{R}^d$ the value $(\mathcal{A}u)(x)$ (if point-evaluation is possible) only depends on the values of $u|_{B_\varepsilon(x)}$ for all $\varepsilon > 0$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \ : \ |x-y| < \varepsilon\}$ denotes the open ball of radius ε around $x \in \mathbb{R}^d$.

Classical examples of local operators are, e.g., differential operators such as the Laplacian

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(x)$$

on the function space $C^2(\mathbb{R}^d)$, since computing derivatives at a point x only needs the function values in a neighborhood of x.

Conversely, if an operator is not local, we call it a **non-local operator**.

Non-local operators appear oftentimes in physics by modeling of non-local effects such as gravity or quantum entanglement. As a simple mathematical example, we consider the integral operator

$$\mathcal{A}u(x) := \int_0^1 (x-y)u(y)\, dy \qquad u \in L^2(0,1).$$

From the definition, one directly sees, that the computation of $\mathcal{A}u(x)$ needs all values $u(y)$ for $y \in (0,1)$. This also leads to the effect, that, even if u is locally supported, e.g., u is the characteristic function of a sub-interval $u = \chi_{(1/4,1/2)}$, we compute that $\operatorname{supp} \mathcal{A}u := \overline{\{x \ : \ \mathcal{A}u(x) \neq 0\}} = [0,1]$, i.e., $\mathcal{A}u$ does have global support.
This observation is particularly important for numerics, since the discretization of local operators (e.g. by finite element methods for the Laplacian) usually leads to sparse linear systems of equations, which can be solved efficiently. In contrast, non-local operators lead to fully-populated matrices, and computations with these can be too expensive. Therefore, an additional challenge for non-local operators is to derive efficient numerical methods that circumvent the problem of fully populated systems. This, however, is out of the scope of this lecture, but the topic of some other special lectures given at TU Wien, such as “matrix-compression and $\mathcal{H}$-matrices”, [FP17].

In this lecture, we are primarily concerned with two classes of non-local operators:

1. Part I: Integral operators of convolution type with singular kernel

   $$\mathcal{A}u(x) := \int k(x,y)u(y)\, dy,$$

   where the kernel function k is the fundamental solution for the Laplacian, see Section 1.3;

2. Part II: Fractional differential operators

   $$(-\Delta)^s \qquad \text{for } s \in (0,1),$$

   where the different formal definitions are given in Chapter 4.

Mainly, we will present the precise mathematical definitions of these operators, prove some mapping properties and boundedness in appropriate function spaces, discuss the solvability of the equations

$$\mathcal{A}u = f$$

for given data f, and briefly introduce numerical approximations to the solutions of these equations.

In the following, we briefly motivate why both classes of operators are of interest.

0.2 Recasting a PDE as Integral Equation

Our model problem is the elliptic PDE

$$-\Delta u = f \qquad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a (bounded) domain, and f is a given right-hand side.
We refer to the PDE lecture, for the fact that, if $k(\cdot,\cdot)$ is a fundamental solution to the PDE, then, the so called **Newton potential**

$$u(x) = \widetilde{N}f(x) := k * f(x) = \int_\Omega k(x,y)f(y)dy \qquad x \in \Omega$$

is a classical solution to the PDE provided $f \in C_0^\infty(\Omega)$ and $\partial\Omega$ is sufficiently smooth. The Newton potential is an example of a non-local operator of convolution type, described in the previous subsection.
One (if not in fact, THE) advantage of the integral equation approach presented in the following is that it works for unbounded domains of the form $\mathbb{R}^d \backslash \overline{\Omega}$, where Ω is bounded, as well.

0.2.1 Direct Integral Method

If we additionally impose boundary conditions (to have a well-posed problem), such as Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u|_{\partial\Omega} &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{0.1}$$

with given boundary data g, additional terms need to be taken into account.
In fact, the representation formula (presented in (1.14)) gives

$$u(x) = \widetilde{N}f(x) + \widetilde{V}\phi(x) - \widetilde{K}g(x) \quad \text{for } x \in \Omega, \tag{0.2}$$

where $\phi := \partial u/\partial n$ is the normal derivative on $\partial\Omega$ (which is unknown), and $\widetilde{V}, \widetilde{K}$ are the so-called **single-layer potential** and **double-layer potential** given by

$$\begin{aligned} \widetilde{V}\phi(x) &:= \int_{\partial\Omega} k(x,y)\phi(y)ds_y \qquad && x \in \Omega, \\ \widetilde{K}g(x) &:= \int_{\partial\Omega} \frac{\partial}{\partial n_y} k(x,y)g(y)ds_y \qquad && x \in \Omega. \end{aligned}$$

The potentials $\widetilde{V}, \widetilde{K}$ again are non-local operators of convolution type. So far, the integral equation (0.2) contains two unknowns: The solution u in Ω and the normal derivative ϕ on $\partial\Omega$. Now, the idea is to consider the limit process $\Omega \ni \widetilde{x} \to x \in \partial\Omega$. The left-hand side then converges to $u(\widetilde{x}) \to u(x) = g(x)$, and we are left with just a single unknown function ϕ.
We now simply state some of the results of the following chapters:

- Because of $f \in C(\overline{\Omega})$, the Newton potential is continuous on $\mathbb{R}^d$ and thus $\widetilde{N}f(\widetilde{x}) \to \widetilde{N}f(x) := N_0 f(x)$. The limit is the integral

$$N_0 f(x) = \int_\Omega G(x-y) f(y)\, dy.$$

- Because of $\phi \in C(\partial\Omega)$, the single-layer potential is continuous on $\mathbb{R}^d$ and thus $\widetilde{V}\phi(\widetilde{x}) \to \widetilde{V}\phi(x) := V\phi(x)$. The limit is the boundary integral

$$V\phi(x) = \int_{\partial\Omega} G(x-y)\phi(y)\, ds_y.$$

- The double-layer potential is more involved: For the limit $\widetilde{x} \to x$, we have convergence $\widetilde{K}g(\widetilde{x}) \to (K - 1/2)g(x)$, where Kg is a Cauchy principal value

$$Kg(x) = \fint_{\partial\Omega} \frac{\partial_y}{\partial n(y)} G(x-y) g(y)\, ds_y.$$

Altogether, the representation formula (1.14) in Ω becomes

$$g(x) = N_0 f(x) + V\phi(x) - (K - 1/2)g(x) \quad \text{for } x \in \partial\Omega.$$

Finally, we thus end up with the following boundary integral equation

$$V\phi = -N_0 f + (K + 1/2)g \quad \text{on } \partial\Omega, \tag{0.3}$$

which is known as **Symm's integral equation**. We have now seen that $\phi = \partial u/\partial n$ is a solution to (0.3). Moreover, it can be shown that V has certain ellipticity properties so that the solution of (0.3) is unique.

The **direct boundary integral method** for the solution of the model problem (0.1) consists of two steps:

- Solve Symm's integral equation to obtain the normal derivative $\phi = \partial u/\partial n$.
- Compute the solution u of the model problem by use of the representation formula.

In praxis it is often not possible to compute the solution ϕ of Symm's integral equation (0.3) analytically. The boundary element method is a numerical scheme to compute an approximate (discrete) solution ϕ_h instead of ϕ.

0.2.2 Indirect Integral Method for the Model Problem

The indirect method avoids the use of the explicit representation formula. Instead one uses the superposition principle for linear equations: According to the representation formula, there holds $-\Delta(\widetilde{N}f) = f$. With the ansatz $u = u_0 + \widetilde{N}f$, the solution of the model problem (0.1) is equivalent to

$$\begin{aligned} -\Delta u_0 &= 0 && \text{in } \Omega, \\ u_0 &= g - N_0 f && \text{on } \partial\Omega. \end{aligned}$$

We will show later that the single-layer potential $\widetilde{V}\phi_0$ is a potential in the sense that it solves

$$-\Delta(\widetilde{V}\phi_0) = 0 \quad \text{in } \Omega.$$

Therefore, for a function ϕ_0 on $\partial\Omega$, we can make the ansatz $u_0 = \widetilde{V}\phi_0$, which leads to the boundary integral equation

$$V\phi_0 = g - N_0 f,$$

since $V\phi_0$ is the trace of $\widetilde{V}\phi_0$ on $\partial\Omega$.

The **indirect boundary integral method** consists of the following steps

- Solve the single-layer potential equation $V\phi_0 = g - N_0 f$.
- Compute the solution $u = \widetilde{V}\phi_0 + \widetilde{N}f$ of the model problem.

This procedure is called *indirect* since the computed function ϕ_0 has no physical meaning.

0.3 Fractional Operators

Recently (since ∼2000), more complex physical and biological models started to take non-local diffusive effects into account. A simple model for such operators is the fractional Laplacian

$$(-\Delta)^s \qquad s \in (0,1),$$

where the choice of s gives an additional model-parameter for more precise modeling.
As such, further applications in peridynamics, finance, image processing and materials science followed.
Mathematically, these operators are also of interest as some crucial differences to the Laplacian appear. As already mentioned, fractional differential operators are non-local, which makes their analysis and numerical approximation challenging. In fact, even the precise mathematical definition of these operators is not straight forward. On the whole space $\mathbb{R}^d$ there are multiple definitions, which turn out to be equivalent. A formally easy way is to use the Fourier transformation to write

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\zeta|^{2s}\mathcal{F}u),$$

which, however, is unpractical for numerical methods. A definition, which is more in line with the singular integral operators of the previous subsection is given by the so called **integral fractional Laplacian** defined pointwise as the Cauchy principal value

$$(-\Delta)^s u(x) := C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy.$$

Restricting oneself from the full-space to a subdomain Ω makes matters more complicated, as different definitions turn out to be not equivalent any more. In this lecture, we discuss two different definitions, the integral fractional Laplacian, which is the formula above applied to functions vanishing outside of the domain, and the so called **spectral fractional Laplacian**, which uses the eigenvalues λ_k and eigenfunctions φ_k of $(-\Delta)$ to define

$$(-\Delta)^s_\sigma u := \sum_{k=1}^{\infty} \lambda_k^s u_k \varphi_k, \qquad u_k := \int_\Omega u\varphi_k dx.$$

There are some other definitions, like the regional fractional Laplacian, which we do not discuss in detail in the following, and given your application in mind you need to choose the appropriate definition accordingly.

0.4 Outlook and Literature

In the following, we briefly sketch the content of the lecture notes and mention the used literature.

0.4.1 Part I

The first part of these lecture notes are an iteration of the course "The Boundary Element Method" given by Dirk Praetorius in 2007. Classical literature for the boundary element method (BEM) and (boundary integral equations) are the books by McLean, [McL00] and Sauter, Schwab, [SS11], on which most of the lecture notes are based. We also mention the book of Steinbach, [Ste08], where both finite and boundary element methods are derived in a compact and readable way.

As you probably know, the Laplace problem (0.1) may lack the existence of classical solutions $u \in C^2(\overline{\Omega})$. Instead one seeks for so-called weak solutions that belong to the Sobolev space $u \in H^1(\Omega)$. A big part of the lecture is thus concerned with the mathematical understanding of the operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, which act between certain Sobolev spaces.

Chapter 1: Function Spaces and Weak Formulations

- We start with the main ingredient of the reformulation of PDEs as (boundary) integral equations, the existence of a fundamental solution. Consequently, we prove the mentioned representation formula for classical solutions.
- We introduce (resp. recall) the Sobolev spaces on domains and boundaries which are needed for the functional analytic framework of the representation formula.
- We recall the main theorems on Sobolev spaces and introduce (different) Sobolev spaces that take homogeneous boundary conditions into account.
- We recall the weak form of the model problem and prove unique existence of solutions for the pure Dirichlet and Neumann problem.

Chapter 2: Integral Operators

- The chapter is concerned with the mathematical framework of the representation formula. We consider the trace as well as the normal derivative of the equation

$$u = \widetilde{N}(-\Delta u) + \widetilde{V}(\partial u/\partial n) - \widetilde{K}(u) \quad \text{in } \Omega,$$

  which leads to the integral equations

$$u = N_0(-\Delta u) + V(\partial u/\partial n) + (1/2 - K)(u) \quad \text{on } \partial\Omega \tag{0.4}$$

  and

$$\frac{\partial u}{\partial n} = N_1(-\Delta u) + (1/2 + K')(\partial u/\partial n) + W(u) \quad \text{on } \partial\Omega, \tag{0.5}$$

  where N_0, N_1, V, W, K are certain integral operators. Writing the last equations as linear system, we see that the Cauchy data $(u, \partial u/\partial n)$ solve the Calderón system

$$\begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} u \\ \frac{\partial u}{\partial n} \end{pmatrix} + \begin{pmatrix} N_0(-\Delta u) \\ N_1(-\Delta u) \end{pmatrix} \tag{0.6}$$

- With the help of the Calderón system, we come up with equivalent integral formulations of our model problem. For instance, if the Dirichlet data $u|_{\partial\Omega}$ are known, the Calderón system resp. (0.4) provides Symm's integral equation

$$V(\partial u/\partial n) = (K + 1/2)(u) - N_0(-\Delta u). \tag{0.7}$$

  Having computed the normal derivative $\partial u/\partial n$, we obtain u from the representation formula.

- An important property of the traces of the potentials $\widetilde{V}, \widetilde{K}$ is that they can be written as integral operators as well, which is essential for implementation of a numerical method.

- Finally, we prove that the operators V and W have certain ellipticity properties, i.e. we are in the context of the Lax-Milgram lemma: In particular, $\partial u/\partial n$ is the unique solution of Symm's integral equation.

Chapter 3: Galerkin Boundary Element Method

- Within the last chapters, we obtained the mathematical framework needed for a Galerkin method. We recall the idea of a Galerkin scheme and directly obtain unique solvability of the Galerkin formulation due to the ellipticity of the operators.

- We present some a-priori estimates to quantify the speed of convergence of the Galerkin discretization in a space of piecewise constant functions on a quasi-uniform mesh (for Symm's integral equations).

0.4.2 Part II

Compared to the BEM, numerical methods for fractional operators are studied quite recently. Therefore, few textbooks suitable for lectures are available, and the course is mainly designed using research articles. As such, this lecture notes can also be seen as a collection of readable literature for an introduction to the topic. Different definitions – and their equivalences – can be found in the rather technical paper of Kwasnicki [Kwa17]. Among them, we mention the ground breaking work of Caffarelli and Silvestre, [CS07], where the fractional Laplacian was identified as a Dirichlet-to-Neumann operator for a degenerated elliptic PDE.
The sections on numerical approximation are based on the overview article by Nochetto et al., [BBN+18], which collects results of Acosta and Borthagaray, [AB17] for the integral fractional Laplacian, as well as Nochetto, Otarola, Salgado, [NOS15] for the spectral fractional Laplacian. For a different approach using the Dunford-Taylor calculus, we refer to the article of Bonito and Pasciak, [BP15].

Chapter 4: Definitions of the Fractional Laplacian

- We start with a probabilistic motivation by looking at a random walk approach with arbitrary long jumps. It turns out that taking the limit of the discrete random walk gives a fractional heat equation $u_t = (-\Delta)^s u$, with the integral fractional Laplacian, whereas a fixed length random-walk gives the classical heat equation.
- We formally introduce different definitions of the fractional Laplacian, the integral definition, the Fourier definition, the definition via a heat-semigroup as well as the famous Caffarelli-Silvestre extension problem, which gives a PDE-approach to fractional diffusion. On the full-space $\mathbb{R}^d$, we prove that all definitions are equivalent.
- Having understood the definitions in $\mathbb{R}^d$, we turn to the case of a bounded subdomain $\Omega \subset \mathbb{R}^d$. Here, we state the integral and spectral definition and show that they are indeed different operators. Moreover, we formulate the Caffarelli-Silvester extension for both operators, which is very useful for analysis and numerical methods.

Chapter 5: Numerical Approximation

- We start by deriving a weak formulation for the equation $(-\Delta)^s u = f$ using the integral fractional Laplacian, which leads to the bilinear form
$$a(u, v) \simeq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} dy\, dx.$$
Hereby, analyzing a Galerkin approximation has two main difficulties, the non-locality of the bilinear form, i.e., plugging in two functions with disjoint support gives in general a non-zero contribution, as well as the non-locality of the energy-norm in a fractional Sobolev-space. The non-locality of the norm does not allow for elementwise a-priori estimates. However, we mention a remedy for this problem by proving a localizable upper bound for the energy norm.
- With a Ceá-type best-approximation estimate, the a-priori analysis of a finite element approximation then comes down to a question of providing a quasi-interpolation operator with

the right approximation properties – here the Scott-Zhang projection – as well as regularity of the solution. Regularity for the integral fractional Laplacian is rather delicate, and it turns out that solutions tend to be not more regular than $H^{1/2+s-\varepsilon}(\Omega)$ even if the geometry and right-hand side are smooth.

- Using the best-approximation property, localization, approximation properties of the Scott-Zhang projection as well as the regularity results, we prove convergence of the FEM approximation with rate $h^{1/2-\varepsilon}$ on quasi-uniform meshes, which can be improved to $h^{1-\varepsilon}$ by using graded meshes.
- Finally, we study a numerical approximation for the spectral fractional Laplacian as well. Hereby, the PDE interpretation using the Caffarelli-Silvestre extension problem is used, and the PDE (in $d+1$-dimensions) is approximated using FEM. Again, questions of approximation properties and regularity are discussed, where especially the regularity in the extended dimension is the limiting factor.
- Using quasi-uniform tensor grids, we prove convergence of order $h^{s-\varepsilon}$, which can be improved using an anisotropic grid (only in the extended variable).

Chapter 6: Dunford-Taylor Approach

- Using Cauchy's integral formula gives yet another definition of the fractional Laplacian with the so-called Dunford-Taylor calculus. In fact, one can deform Cauchy's formula to the real-axis to obtain an operator-valued integral over $\mathbb{R}^+$, the so called Balakrishnan formula.
- With the Balakrishnan formula at hand, one can derive a numerical method using FEM to approximate the shifted Laplacian in the integrand and so-called sinc-Quadrature to approximate the integral, which converges exponentially. Balancing the FEM error and the quadrature error gives convergence rates up to h^{2-s} for highly regular solutions.
- The Balakrishnan formula only holds for the spectral integral Laplacian. However, one can derive a similar formula for the bilinear form corresponding to the integral fractional Laplacian using the Fourier transform.

Part I

Part I: (Boundary-)Integral Equations

Chapter 1

Function Spaces, Weak Formulations

1.1 Model Problem

Throughout the first part of the lecture notes, we consider the Laplace operator

$$\Delta u(x) := \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}(x) \tag{1.1}$$

for a function $u \in C^2(\overline{\Omega})$ on a domain $\Omega \subseteq \mathbb{R}^d$ with $d = 2, 3$. The model problem for a second-order elliptic partial differential equation (PDE) reads: Find u such that

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.2}$$

which satisfies certain boundary conditions, e.g.,

$$u|_\Gamma = g \quad \text{on } \Gamma := \partial\Omega. \tag{1.3}$$

Here, $f \in C(\overline{\Omega})$ is a given volume force and $g \in C(\Gamma)$ are given Dirichlet boundary conditions.

We stress that other second-order elliptic operators lead to the same results. However, it seems to be the right idea to understand the analytical techniques for some precise model example.
One goal of this lecture is to reformulate the PDE (1.2) in the domain Ω into an integral equation only posed on the surface Γ. As the concept of classical solutions, i.e., $u \in C^2(\overline{\Omega})$ is usually too strong, we shall look at weaker formulations both of the PDEs and the integral equations in appropriate function spaces (Sobolev spaces defined below).

We start with some additional assumptions on our domain Ω. For simplicity, we assume that Ω is **bounded** and simply connected. However, most of the results also work for unbounded domains provided $\mathbb{R}^d \backslash \overline{\Omega}$ is bounded.

More important, we impose additional regularity on Γ, we consider so called bounded **Lipschitz domains**. Hereby, Ω is locally only on one side of the boundary Γ, and the boundary Γ can locally be parametrized by Lipschitz continuous functions. This is stated formally in the following definition.

Definition 1.1. *(Lipschitz domain) We start with the definition of the sets* $B_R(0) := \{x \in \mathbb{R}^d \,|\, |x| < R\}$ *and*

$$B_R^+ := \{x \in B_R(0) \,|\, x_d > 0\}, \quad B_R^- := \{x \in B_R(0) \,|\, x_d < 0\}, \quad \text{and} \quad B_R^0 := \{x \in B_R(0) \,|\, x_d = 0\}.$$

Now, a set $\Omega \subset \mathbb{R}^d$ *is called* **Lipschitz domain**, *if* Ω *is an open, bounded, and connected set such that for each* $x \in \Gamma := \partial\Omega$, *there is an open neighborhood* $U_x \subset \mathbb{R}^d$ *of* x *and a bijective function* $\chi_x : B_1(0) \to U_x$ *such that*

- χ_x *as well as* χ_x^{-1} *are Lipschitz continuous,*
- $\chi_x(B_1^0) = \Gamma \cap U_x$, *i.e.* χ_x *provides a local parametrization of* Γ,
- $\chi_x(B_1^-) = \Omega \cap U_x$,
- $\chi_x(B_1^+) = (\mathbb{R}^d \backslash \overline{\Omega}) \cap U_x$,

where the latter assumptions state that Ω *is (locally) only on one side of the boundary.*

We note that the choice of radius $R = 1$ in the definition of the Lipschitz domain is arbitrary.

In the following, we usually consider the case when Ω is a Lipschitz domain. According to the Rademacher theorem, a Lipschitz continuous function is differentiable almost everywhere. Therefore, we may define an outer normal vector $n = n(x)$ for almost every $x \in \Gamma$. The smoothness of the boundary is measured by the smoothness of the local parametrizations χ_x in the definition of a Lipschitz domain.

Definition 1.2. *A function* $f : \Omega \to \mathbb{R}$ *is* **Hölder continuous** *of order* (k, λ), *if* $f \in C^k(\Omega)$ *and all* k*-th derivatives satisfy*

$$\sup_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\lambda} < \infty.$$

The space of all Hölder continuous functions on Ω *is denoted by* $C^{k,\lambda}(\Omega)$. *The domain* Ω *is called a* $\boldsymbol{C^{k,\lambda}}$ **domain** *if it is a Lipschitz domain and if all local parametrizations* χ_x *as well as* χ_x^{-1} *are Hölder continuous of order* (k, λ). □

Remark. Obviously, $C^{0,1}(\Omega)$ is the space of all Lipschitz continuous functions, and Lipschitz domains are just $C^{0,1}$ domains. Moreover, $C^{k,0}$ domains are usually abbreviated as C^k domains. □

Remark. One can show that $C^{k,\lambda}(\Omega)$, associated with the norm

$$\|f\|_{C^{k,\lambda}(\Omega)} := \sup_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq k}} \|\partial^\alpha f\|_{\infty,\Omega} + \sup_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\lambda}$$

is a Banach space. □

1.2 Integration by Parts

From now on, we shall assume that Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma := \partial\Omega$. Here, *smooth* just means that we may use the **integration by parts formula**

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, dx + \int_\Omega u \frac{\partial v}{\partial x_j} \, dx = \int_\Gamma u v n_j \, ds \quad \text{for } u, v \in C^1(\overline{\Omega}), \tag{1.4}$$

where n_j denotes the j-th component of the outer normal vector of Ω. We note some immediate consequences:

- For $f \in C^1(\Omega)^d$, let $\operatorname{div} f := \sum_{j=1}^d \frac{\partial f_j}{\partial x_j}$ denote the divergence operator. Then, there holds the **Gauss divergence theorem**

$$\int_\Omega \operatorname{div} f \, dx = \int_\Gamma f \cdot n \, ds \quad \text{for } f \in C^1(\overline{\Omega})^d. \tag{1.5}$$

- From the identity $-\Delta u = -\operatorname{div}(\nabla u)$, we obtain the **first Green's formula**

$$\int_\Omega (-\Delta u) v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Gamma \frac{\partial u}{\partial n} v \, ds \quad \text{for } u \in C^2(\overline{\Omega}) \text{ and } v \in C^1(\overline{\Omega}). \tag{1.6}$$

- Using the first Green's formula twice, we prove the **second Green's formula**

$$\int_\Omega (-\Delta u) v \, dx + \int_\Gamma \frac{\partial u}{\partial n} v \, ds = \int_\Omega u(-\Delta v) \, dx + \int_\Gamma u \frac{\partial v}{\partial n} \, ds \quad \text{for } u, v \in C^2(\overline{\Omega}). \tag{1.7}$$

In the following sections, we shall see that the requirements $u, v \in C^2(\overline{\Omega})$ can be substantially weakened.

1.3 Fundamental Solutions, Representation Formula

The starting point for the reformulation of the PDE to an integral equation is the so-called *representation formula* (or *third Green's formula*) proven in the next subsection. In order to obtain this formula, we need to have a *fundamental solution* for our PDE.

Definition 1.3. *Let*

$$C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) \ : \ \operatorname{supp} \varphi \subset \Omega \text{ compact}\}$$

be the space of compactly supported, infinitely differentiable functions. Employing the topology defined by the sequential convergence

$$(\varphi_n)_{n\in\mathbb{N}} \to 0 \iff \exists K \subset \Omega \text{ compact} : \operatorname{supp} \varphi_n \subset K, \partial^\alpha \varphi_n \to 0 \text{ uniformly in } K, \forall \alpha \in \mathbb{N}_0^d$$

gives the space of **test functions** $\mathcal{D}(\Omega)$.
Its dual space $\mathcal{D}(\Omega)' := \{\ell : \mathcal{D}(\Omega) \to \mathbb{R}, \text{cont., linear}\}$ *is called the space of* **distributions**.

Example.

- Every function $u \in L^1_{\text{loc}}(\Omega) := \{u \in L^1(K) \quad \forall K \subset \Omega \quad \text{compact}\}$ defines a distribution $\widetilde{u} \in \mathcal{D}(\Omega)'$ by

$$\langle \widetilde{u} \,;\, \varphi \rangle = \widetilde{u}(\varphi) := \int_\Omega u\varphi dx \qquad \varphi \in \mathcal{D}(\Omega).$$

  We write $u = \widetilde{u}$ for such distributions and call them *regular distributions*.

- The *Dirac delta distribution* δ_y defined for fixed $y \in \Omega$ by

$$\langle \delta_y \,;\, \varphi \rangle := \varphi(y) \qquad \varphi \in \mathcal{D}(\Omega)$$

  is an example of a non-regular distribution.

Remark. We stress that distributions are always differentiable, where the derivative of $u \in \mathcal{D}(\Omega)'$ is defined by

$$\langle D^\alpha u \,;\, \varphi \rangle := (-1)^{|\alpha|} \langle u \,;\, D^\alpha \varphi \rangle \qquad \varphi \in \mathcal{D}(\Omega), \alpha \in \mathbb{N}_0^d.$$

For regular distributions $u \in L^1_{\text{loc}}(\Omega)$, we call $D^\alpha u$ the **weak derivative**. If all weak derivatives with $|\alpha| = 1$ satisfy $D^\alpha u \in L^1_{\text{loc}}(\Omega)$, we call u **weakly differentiable**. We stress that if a function is weakly differentiable, its weak gradient is uniquely defined and it coincides with the classical derivative if $u \in C^1(\Omega)$.

Definition 1.4. *Let L be a scalar differential operator. Then, a function $G : \Omega \times \Omega \to \mathbb{R}$ is called a* **fundamental solution** *(or Green's function), if*

$$LG(x, y) = \delta_y$$

in the sense of distributions.

Example.

- For our model problem $L = -\Delta$, the fundamental solution is given by the **Newton kernel**

$$G(x, y) := \begin{cases} \frac{1}{2}(1 - |x - y|), & \text{for } d = 1, \\ -\frac{1}{2\pi} \log |x - y|, & \text{for } d = 2, \\ \frac{1}{4\pi} \frac{1}{|x-y|}, & \text{for } d = 3. \end{cases} \tag{1.8}$$

  Since the fundamental solution does only depend on $|x - y|$, we will write $G(x - y) := G(x, y)$ using only one input argument.

- For the Helmholtz operator $L = -\Delta - k^2 I$ and $d = 3$ the fundamental solution is given by

$$G_k(x, y) := \frac{e^{ik|x-y|}}{4\pi |x - y|}.$$

We refer to the PDE-lecture for the classical result that the convolution of Green's function and right-hand side

$$u(x) = G * f(x) = \int_{\mathbb{R}^d} G(x-y) f(y) dy$$

solves $Lu = f$ in the sense of distributions.

In the following, we only consider the operator $L = -\Delta$ for $d = 2, 3$. We note that there holds $|S_2^2| = 2\pi$ and $|S_2^3| = 4\pi$, where $|S_2^d|$ denotes the measure of the unit sphere in $\mathbb{R}^d$. Our first lemma easily follows from direct calculations and is left to the reader (The last two statements are easily obtained by use of polar coordinates).

Lemma 1.5. (i) *There holds $G \in C^\infty(\mathbb{R}^d \backslash \{0\})$ with first and second derivatives*

$$\partial_j G(z) = -\frac{1}{|S_2^d|} \frac{z_j}{|z|^d} \quad \textit{and} \quad \partial_{jk} G(z) = -\frac{1}{|S_2^d|} \frac{\delta_{jk}|z|^2 - d z_j z_k}{|z|^{d+2}}. \tag{1.9}$$

(ii) *There holds $-\Delta G(z) = 0$ for $z \neq 0$.*
(iii) *$G \in L^p_{\ell oc}(\mathbb{R}^d)$ for $d < 2p/(p-1)$, in particular $G \in L^2_{\ell oc}(\mathbb{R}^d)$.*
(iv) *$\partial_j G \in L^p_{\ell oc}(\mathbb{R}^d)$ for $d < p/(p-1)$, in particular $\partial_j G \in L^1_{\ell oc}(\mathbb{R}^d)$.* ■

The main result of this section is the representation formula. It states that the (smooth) solution of a Laplace problem $-\Delta u = f$ is uniquely determined by its **Cauchy data** $(u, \partial u/\partial n)$ on the boundary Γ, i.e. we know u if we know the Dirichlet and Neumann data on the entire boundary Γ.

Proposition 1.6 (Representation Formula). *Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\Gamma := \partial\Omega$ and $u \in C^2(\overline{\Omega})$. With $f := -\Delta u \in C(\overline{\Omega})$, there holds*

$$u(x) = \int_\Omega G(x-y) f(y)\, dy + \int_\Gamma G(x-y) \frac{\partial u}{\partial n(y)}(y)\, ds_y - \int_\Gamma \frac{\partial_y}{\partial n(y)} G(x-y)\, u(y)\, ds_y$$

for all $x \in \Omega$, where $n(y)$ denotes the outer normal vector at $y \in \Gamma$.

Proof. Fix $x \in \Omega$. We want to apply, for u and $v(y) = G(x-y)$, the second Green's formula which reads in classical terms

$$(-\Delta u\, ;\, v)_\Omega + (\partial u/\partial n\, ;\, v)_\Gamma = (u\, ;\, -\Delta v)_\Omega + (u\, ;\, \partial v/\partial n)_\Gamma \quad \text{for } u, v \in C^2(\overline{\Omega}). \tag{1.10}$$

As $v \notin C^2(\overline{\Omega})$, we cut-off the singularity for $y = x$ and consider (1.10) on $\Omega_\varepsilon := \Omega \backslash B_\varepsilon(x)$. Here, $\varepsilon > 0$ is chosen small enough so that $B_\varepsilon(x) \subset \Omega$. Then, with $\Gamma_\varepsilon := \partial\Omega_\varepsilon$, there holds $\Gamma_\varepsilon = \Gamma \cup \partial B_\varepsilon(x)$ and $\Gamma \cap \partial B_\varepsilon(x) = \emptyset$. The second Green's formula proves

$$(-\Delta u\, ;\, v)_{\Omega_\varepsilon} + (\partial u/\partial n\, ;\, v)_\Gamma - (u\, ;\, \partial v/\partial n)_\Gamma = -\,(\partial u/\partial n\, ;\, v)_{\partial B_\varepsilon(x)} + (u\, ;\, \partial v/\partial n)_{\partial B_\varepsilon(x)}\,.$$

It now remains to consider the convergence of the terms for $\varepsilon \to 0$, where the left-hand side tends to the right-hand side of the representation formula, cf. step 1, and where the right-hand side tends to $u(x)$, cf. step 2 and 3.

1. step. There holds $(-\Delta u\,;\,v)_{\Omega_\varepsilon} \xrightarrow{\varepsilon\to 0} (-\Delta u\,;\,v)_\Omega$ which follows obviously from the Lebesgue dominated convergence theorem as $-\Delta u = f \in L^2(\Omega)$ and $v \in L^2(\Omega)$.

2. step. There holds $(\partial u/\partial n\,;\,v)_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon\to 0} 0$: Note that, for $y \in \partial B_\varepsilon(x)$, there holds

$$v(y) = G(x-y) = \frac{1}{|S_2^d|}\begin{cases} -\log\varepsilon & \text{for } d=2,\\ 1/\varepsilon & \text{for } d=3.\end{cases}$$

Therefore, v is constant on $\partial B_\varepsilon(x)$, and we can estimate

$$|\,(\partial u/\partial n\,;\,v)_{\partial B_\varepsilon(x)}\,| \le \|u\|_{C^1(\overline{\Omega})}|v||\partial B_\varepsilon(x)| \le C\|u\|_{C^1(\overline{\Omega})}\varepsilon^{d-1}\begin{cases} -\log\varepsilon & \text{for } d=2,\\ 1/\varepsilon & \text{for } d=3\end{cases}$$

vanishes with $\varepsilon \to 0$.

3. step. There holds $(u\,;\,\partial v/\partial n)_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon\to 0} u(x)$: The normal vector for $y \in \partial B_\varepsilon(x)$ is given by $n(y) = \frac{1}{\varepsilon}(x-y)$. We plug-in the formula for ∇G to obtain

$$\begin{aligned}\int_{\partial B_\varepsilon(x)} u(y)\frac{\partial_y}{\partial n(y)}G(x-y)\,ds_y &= -\frac{1}{|S_2^d|\varepsilon^d}\int_{\partial B_\varepsilon(x)} u(y)(y-x)\cdot n(y)\,ds_y\\ &= \frac{1}{|S_2^d|\varepsilon^{d-1}}\int_{\partial B_\varepsilon(x)} u(y)\,ds_y.\end{aligned}$$

We write using the Taylor expansion $u(y) = u(x) + (y-x)\cdot\nabla u(\zeta)$ for $\zeta = x + t(y-x)$, $t \in (0,1)$. Then,

$$\frac{1}{|S_2^d|\varepsilon^{d-1}}\int_{\partial B_\varepsilon(x)} u(y)\,ds_y = \frac{1}{|S_2^d|\varepsilon^{d-1}}|\partial B_\varepsilon(x)|u(x) + \frac{1}{|S_2^d|\varepsilon^{d-1}}\int_{\partial B_\varepsilon(x)}(y-x)\cdot\nabla u(\zeta)ds_y.$$

We note that $\frac{1}{|S_2^d|\varepsilon^{d-1}}|\partial B_\varepsilon(x)| = 1$, and the second term on the right-hand side can be estimated by

$$\left|\frac{1}{|S_2^d|\varepsilon^{d-1}}\int_{\partial B_\varepsilon(x)}(y-x)\cdot\nabla u(\zeta)ds_y\right| \le \frac{1}{|S_2^d|\varepsilon^{d-1}}|\partial B_\varepsilon(x)|\;\varepsilon\|u\|_{C^1(\overline{\Omega})} = C\varepsilon \to 0.$$

Therefore, we have proven $(u\,;\,\partial v/\partial n)_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon\to 0} u(x)$ and consequently the representation formula. ∎

The representation formula from Proposition 1.6 allows to represent $u \in C^2(\overline{\Omega})$ in terms of the following three integral operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, namely

- the **Newton potential** of $f : \Omega \to \mathbb{R}$

$$\widetilde{N}f(x) := \int_\Omega G(x-y)f(y)\,dy \quad \text{for } x \in \Omega, \tag{1.11}$$

- the **single layer potential** of $\phi : \Gamma \to \mathbb{R}$

$$\widetilde{V}\phi(x) := \int_\Gamma G(x-y)\phi(y)\,ds_y \quad \text{for } x \in \Omega, \tag{1.12}$$

- the **double layer potential** of $v : \Gamma \to \mathbb{R}$

$$\widetilde{K}v(x) := \int_\Gamma \frac{\partial_y}{\partial n(y)} G(x-y)\, v(y)\, ds_y \quad \text{for } x \in \Omega. \tag{1.13}$$

Obviously, the operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$ are linear operators. Moreover, with this notation, the representation formula can simply be written as follows:

Corollary 1.7 (Representation Formula). *For $u \in C^2(\overline{\Omega})$, there holds*

$$u = \widetilde{N}(-\Delta u) + \widetilde{V}(\partial u/\partial n) - \widetilde{K}(u) \quad \text{in } \Omega, \tag{1.14}$$

which is just the operator statement of Proposition 1.6. ∎

In particular, we see that the Newton kernel G is the fundamental solution of the Laplace operator.

Corollary 1.8. *For $u \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$, there holds $u = \widetilde{N}(-\Delta u)$.*

Proof. The proof follows from the representation formula (1.14) as $u = 0 = \partial u/\partial n$ on Γ. ∎

As already mentioned, the Laplace problem (1.2)–(1.3) may lack the existence of classical solutions $u \in C^2(\overline{\Omega})$. Instead one seeks for so-called weak solutions that belong to the Sobolev space $u \in H^1(\Omega)$. About the first half of the lecture is thus concerned with the mathematical understanding of the operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, which act between certain Sobolev spaces.

1.4 Sobolev Spaces

1.4.1 Sobolev spaces on domains

This section briefly recalls the definition of Sobolev spaces $H^s(\Omega)$, for $s \geq 0$, on domains $\Omega \subseteq \mathbb{R}^d$ and of the corresponding dual spaces $\widetilde{H}^{-s}(\Omega)$. Throughout, Ω is a domain in $\mathbb{R}^d$, i.e. Ω is a connected open subset of $\mathbb{R}^d$.

We start with integer order Sobolev spaces.

Definition 1.9. *The* **Sobolev space** $H^1(\Omega)$ *is defined by*

$$H^1(\Omega) := \{u \in L^2(\Omega) \,|\, u \text{ weakly differentiable}, \nabla u \in L^2(\Omega)\} \tag{1.15}$$

and associated with the graph norm

$$\|u\|_{H^1(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}\big)^{1/2}. \tag{1.16}$$

Higher-order Sobolev spaces of integer order $m \in \mathbb{N}$ *may be defined inductively by*

$$H^m(\Omega) := \{u \in L^2(\Omega) \,|\, u \text{ weakly differentiable}, \nabla u \in H^{m-1}(\Omega)\}, \tag{1.17}$$

with associated norm

$$\|u\|_{H^m(\Omega)} := \big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{H^{m-1}(\Omega)}\big)^{1/2}. \tag{1.18}$$

In this sense, there holds $H^0(\Omega) := L^2(\Omega)$.

We also need Sobolev spaces of non-integer order, which can either be defined by interpolation or by a non-local seminorm as below.

Definition 1.10. *For a fractional order* $0 < \sigma < 1$, *one first defines the* **Sobolev-Slobodeckij seminorm**

$$|u|_{\sigma,\Omega} := \Big(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{d+2\sigma}}\, dy\, dx\Big)^{1/2}. \tag{1.19}$$

As $|\cdot|_{\sigma,\Omega}$ *stems from an inner product, it is clear that it satisfies homogeneity and triangle inequality. In particular, it is a seminorm. Then, for* $m \in \mathbb{N}_0$ *and* $0 < \sigma < 1$, *one defines the* **fractional order Sobolev spaces**

$$H^{m+\sigma}(\Omega) := \{u \in H^m(\Omega) \,|\, |D^m u|_{\sigma,\Omega} < \infty\}, \quad \|u\|_{H^{m+\sigma}(\Omega)} := \big(\|u\|^2_{H^m(\Omega)} + |D^m u|^2_{\sigma,\Omega}\big)^{1/2},$$

where $D^m u$ *denotes the* m*-th (weak) derivative of* u.

By definition, the Sobolev spaces $H^{m+\sigma}(\Omega)$ are subspaces of (product spaces of) $L^2(\Omega)$. Moreover, the norm $\|\cdot\|_{H^{m+\sigma}(\Omega)}$ is obviously induced by an inner product. In fact, $H^{m+\sigma}(\Omega)$ is complete, that is, $H^{m+\sigma}(\Omega)$ is a Hilbert space. For integer order, i.e. $\sigma = 0$, it is rather simple to prove that $H^m(\Omega)$ is a Hilbert space: For $m = 1$, we have to show that $H^1(\Omega)$ is a closed subspace of $L^2(\Omega) \times L^2(\Omega)^d$. One may therefore assume that $(u_n, \nabla u_n)$ converges to (u, g) in $L^2(\Omega) \times L^2(\Omega)^d$. One then has to show, that u is weakly differentiable with $\partial_j u = g_j$. For the fractional order Sobolev spaces $H^{m+\sigma}(\Omega)$ the completeness proof is more involved.

In the following, we shall write $H^s(\Omega)$ for $s \geq 0$ instead of splitting $s = m + \sigma$ into an integer part $m \in \mathbb{N}_0$ and a fractional part $\sigma \in (0, 1)$.

Theorem 1.11. *For $s \geq 0$, $H^s(\Omega)$ is a Hilbert space. Moreover, for $t > s$ there holds the continuous inclusion $H^t(\Omega) \subset H^s(\Omega)$, i.e. the identity $id : H^t(\Omega) \to H^s(\Omega)$ is well-defined and continuous.* ■

Remark. Note that $H^s(\Omega)$ cannot be a closed subspace of $L^2(\Omega)$ with respect to the L^2 norm since there holds $\mathcal{D}(\Omega) \subset H^s(\Omega) \subset L^2(\Omega)$, and $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$. □

Remark. We stress that for Lipschitz domains, an equivalent characterization of Sobolev spaces using the Fourier transformation can be made. Since the Fourier transformation $\mathcal{F}$ turns derivatives into multiplications, we can characterize $H^s(\mathbb{R}^d)$-functions by satisfying

$$|||u|||_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\zeta|^2)^s |\mathcal{F}u(\zeta)|^2 d\zeta < \infty$$

and define $H^s(\Omega) := \{u \in \mathcal{D}(\Omega)' \;:\; u = U|_\Omega, |||U|||_{H^s(\mathbb{R}^d)} < \infty\}$. We refer to [McL00] for the equivalence of the definitions and make use of it in a proof once.

Sobolev spaces are made to provide an existence theory for the solution of elliptic differential equations. For instance, let us consider the Laplace equation with homogeneous boundary conditions,

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.20}$$

The so-called *weak solution* $u \in H^1(\Omega)$ solves a weak formulation, which is obtained by multiplying the differential equation by a test function v and integrating over Ω, i.e., the weak form of the model problem reads

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega (-\Delta u) v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in \mathcal{D}(\Omega), \tag{1.21}$$

where we have used integration by parts to obtain the left-hand side. For $f \in L^2(\Omega)$, the right-hand side defines a continuous linear functional on $H^s(\Omega)$ with operator norm $\leq \|f\|_{L^2(\Omega)}$. This follows from the Cauchy inequality and the definition of $\|\cdot\|_{H^s(\Omega)}$,

$$\Big| \int_\Omega f v \, dx \Big| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{H^s(\Omega)} \quad \text{for all } v \in H^s(\Omega).$$

Therefore, it makes sense to consider the dual space $H^s(\Omega)^*$ of $H^s(\Omega)$ with respect to the extended $L^2(\Omega)$ scalar product, i.e. the duality brackets are defined by

$$\langle f \,;\, v \rangle := \int_\Omega f v \, dx \quad \text{for } v \in H^s(\Omega) \text{ and } f \in H^s(\Omega)^*, \tag{1.22}$$

where the integral is only a symbol if $f \in H^s(\Omega)^* \backslash L^2(\Omega)$ is not L^2 integrable. This can be well-stated mathematically as follows, where $X = H^s(\Omega)$ and $Y = L^2(\Omega)$.

Lemma 1.12. *Let X and Y be real Hilbert spaces with continuous inclusion $X \subseteq Y$. Then, the Riesz mapping $J_Y : Y \to Y^*$, $J_Y y := (y \,;\, \cdot)_Y$ is well-defined as operator $J_Y \in L(Y; X^*)$, and $J_Y(Y)$ is a dense subspace of X^*.*

Proof. According to the assumptions, there holds $\|x\|_Y \leq C\,\|x\|_X$ for all $x \in X$. Thus, the Cauchy inequality proves

$$(y\,;\,x)_Y \leq \|y\|_Y\|x\|_Y \leq C\|y\|_Y\|x\|_X.$$

Thus, $J_Y \in L(Y;X^*)$ is well-defined. Let $J_X : X \to X^*$ denote the Riesz mapping for X. Then, $J_Y(Y)$ is dense in X^* if and only if $V := J_X^{-1}(J_Y(Y))$ is dense in $X = \overline{V} \oplus \overline{V}^\perp$. Therefore, it remains to prove that $\overline{V}^\perp = \{0\}$. Let $x \in \overline{V}^\perp$. Then, for $y \in Y$, there holds

$$0 = \big(x\,;\,J_X^{-1}(J_Y y)\big)_X = (J_Y y)(x) = (y\,;\,x)_Y\,.$$

Choose $y = x \in \overline{V}^\perp \subseteq Y$ to see $x = 0$ in $Y \supseteq X$, which concludes the proof. ■

According to the preceding lemma, equation (1.22) defines the duality brackets on a dense subspace of $H^s(\Omega)^*$. In particular, given $\phi \in H^s(\Omega)^*$ there is a sequence (f_n) in $L^2(\Omega)$ such that

$$\langle \phi\,;\,v\rangle = \lim_{n\to\infty} (f_n\,;\,v)_{L^2(\Omega)} \quad \text{for all } v \in H^s(\Omega).$$

Definition 1.13. *We denote with $\widetilde{H}^{-s}(\Omega)$, for $s \geq 0$, the dual space of $H^s(\Omega)$ with respect to the extended $L^2(\Omega)$ scalar product (1.22). Note that these dual spaces are also called* **Sobolev spaces***, and there holds $\widetilde{H}^0(\Omega) = L^2(\Omega)$.* □

Remark. Note that the definition of $\widetilde{H}^s(\Omega)$ is only a special choice of the representation of the dual space $H^s(\Omega)^*$. As $H^s(\Omega)$ is a Hilbert space, the Riesz theorem states the existence of an element $\widehat{f} \in H^s(\Omega)$ with $\big(\widehat{f}\,;\,v\big)_{H^s(\Omega)} = \int_\Omega f v\,dx$, and this is just another representation of $H^s(\Omega)^*$. However, the representation with the extended $L^2(\Omega)$ scalar product is more convenient for our purposes. □

Remark. A triple (X, Y, X^*) with continuous inclusion $X \subseteq Y$ and continuous and dense inclusion $Y \subseteq X^*$ is usually called **Gelfand triple** in the literature. Here, the inclusions $X \subseteq Y$ and $Y \subseteq X^*$ are usually understood with respect to some injective linear inclusion operator, cf. Lemma 1.12 above. □

1.4.2 Sobolev Spaces on the boundary

In this section we define the Sobolev spaces $H^s(\Gamma)$ and the corresponding dual spaces $H^{-s}(\Gamma)$ for $s \geq 0$ and $\Gamma := \partial\Omega$. However, throughout the first part of the lecture we will never work with the basic definition. Instead, the important space $H^{1/2}(\Gamma)$ will be characterized as the trace space of $H^1(\Omega)$, i.e. u belongs to $H^1(\Omega)$ if and only if the restriction $u|_\Gamma$ belongs to $H^{1/2}(\Gamma)$. The precise mathematical statement follows in Theorem 1.22, Theorem 1.24, and Corollary 1.25.

In the following, we always consider bounded Lipschitz domains, and define Sobolev spaces on Lipschitz boundaries Γ. We present two different equivalent definitions, one by using local charts and one by employing the Sobolev-Slobodeckij seminorm.

We start with the first definition, which employs a few steps:

- For $x \in \Gamma$, let $U_x \subset \mathbb{R}^d$ be the open neighborhood and χ_x be the bi-Lipschitz function according to the definition of a Lipschitz domain.
- Choose $\varepsilon_x > 0$ with $B_{2\varepsilon_x}(x) \subseteq U_x$.
- Since Γ is compact, we may choose finitely many $x_1, \dots x_n$ such that $\Gamma \subseteq \bigcup_{j=1}^n B_{\varepsilon_j}(x_j)$, where $\varepsilon_j := \varepsilon_{x_j}$ etc.
- There are smooth functions $\phi_j \in \mathcal{D}(B_{2\varepsilon_j}(x_j))$ such that $\phi_j \geq 0$ and $\sum_{j=1}^n \phi_j(x) = 1$ for $x \in \Gamma$.
- For a function $v : \Gamma \to \mathbb{R}$, we define $v_j := \phi_j v : \Gamma \to \mathbb{R}$, and we remark that $v = \sum_{j=1}^n v_j$.
- Finally, we may define $\widehat{v}_j := v_j \circ \chi_j : B_1^0 \to \mathbb{R}$.

With the introduced notation, the definition reads as follows:

Definition 1.14. *For $s \geq 0$, the* **Sobolev space** $H^s(\Gamma)$ *is defined as*

$$H^s(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} \,\middle|\, \forall j = 1, \dots, n \quad \widehat{v}_j \in H^s(B_1^0) \right\} \tag{1.23}$$

and associated with the norm

$$\|v\|_{H^s(\Gamma)} := \Big(\sum_{j=1}^n \|\widehat{v}_j\|_{H^s(B_1^0)}^2 \Big)^{1/2}. \tag{1.24}$$

Note that this definition formally depends on the choice of $(\chi_j, \varepsilon_j)_{j=1}^n$ as well as on the corresponding partition of unity $(\phi_j)_{j=1}^n$, and both is non-unique. For the moment, we should therefore write $H^s(\Gamma) = H^s(\Gamma; \pi)$ and $\|\cdot\|_{H^s(\Gamma)} = \|\cdot\|_{H^s(\Gamma;\pi)}$, where π abbreviates the particular choice of the parametrization $(\chi_j, \varepsilon_j, \phi_j)_{j=1}^n$ of Γ. □

The following important theorem states that the Sobolev space $H^s(\Gamma)$ does not depend on the choice of the parametrization $\pi = (\chi_j, \varepsilon_j, \phi_j)_{j=1}^n$. Clearly, there is an upper bound of s involved since the derivatives of χ_j implicitly enter the game according to the chain rule.

Theorem 1.15. *Assume that Ω is a $C^{k-1,1}$ domain and $0 \leq s \leq k$.*
(i) *For π, π' two arbitrary parametrizations of Γ, there holds the set equality $H^s(\Gamma) := H^s(\Gamma; \pi) = H^s(\Gamma; \pi')$, and the corresponding norms are equivalent, $\|\cdot\|_{H^s(\Gamma;\pi)} \sim \|\cdot\|_{H^s(\Gamma;\pi')}$*
(ii) *There holds $L^2(\Gamma) = H^0(\Gamma)$ with equivalent norms.*
(iii) *$H^s(\Gamma)$ is a Hilbert space, and the inclusion $H^t(\Gamma) \subset H^s(\Gamma)$ for $k \geq t > s$ is continuous.* ■

With the continuous inclusion $H^s(\Gamma) \subseteq L^2(\Gamma)$, Lemma 1.12 allows to define the dual space $H^s(\Gamma)^*$ with respect to the extended $L^2(\Gamma)$ scalar product, i.e.

$$\langle f\,;\,v \rangle := \int_\Gamma f v \, ds \quad \text{for } v \in H^s(\Gamma) \text{ and } f \in H^s(\Gamma)^*, \tag{1.25}$$

where the integral is only a symbol and defined by continuous extension if $f \in H^s(\Gamma)^* \backslash L^2(\Gamma)$ is not L^2 integrable.

Definition 1.16. *For Ω a bounded $C^{k-1,1}$ domain and $0 < s \leq k$, we define the* **Sobolev space** *$H^{-s}(\Gamma)$ as dual space $H^s(\Gamma)^*$ with respect to the extended $L^2(\Gamma)$ scalar product.* □

We finish this section with a more practical, equivalent definition of Sobolev spaces on the boundary. For the actual proof of the equivalence, we again refer to [McL00].

Definition 1.17. *Let Ω be a bounded Lipschitz domain and $0 < s < 1$. Then, the* **Sobolev space** *$H^s(\Gamma)$ can also be defined as*

$$H^s(\Gamma) := \{u \in L^2(\Gamma) \ : \ \|u\|_{H^s(\Gamma)} < \infty\},$$

where

$$\|u\|^2_{H^s(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^{d-1+2s}} \, dy \, dx$$

denotes the Sobolev-Slobodeckij norm.

1.4.3 Main Theorems on Sobolev Spaces

Unless otherwise stated, we assume throughout that Ω is (at least) a bounded Lipschitz domain in $\mathbb{R}^d$. The first theorem states that we may always restrict to smooth functions within the proofs.

Theorem 1.18 (Meyers-Serrin). *For each non-negative order $s \geq 0$, $C^\infty(\Omega) \cap H^s(\Omega)$ is a dense subspace of $H^s(\Omega)$. Moreover, for each non-negative order $s \geq 0$, $C^\infty(\overline{\Omega}) := \{u|_\Omega \,|\, u \in C^\infty(\mathbb{R}^d)\}$ is a dense subspace of $H^s(\Omega)$.* ■

If the order $s \geq 0$ is large enough, we are dealing with *classical continuous functions*. Here, large enough means $s > 1/2$ for $d = 1$ (or on the boundary Γ of $\Omega \subset \mathbb{R}^2$), $s > 1$ for $d = 2$ (or on the boundary Γ of $\Omega \subset \mathbb{R}^3$), and $s > 3/2$ for $d = 3$, respectively.

Theorem 1.19 (Sobolev Inequality). *For $d/2 < s$, there holds $H^s(\Omega) \subseteq C(\Omega)$ with continuous embedding, i.e. $\|u\|_\infty \lesssim \|u\|_{H^s(\Omega)}$ for all $u \in H^s(\Omega)$.* ■

Here and in the following, the symbol $\lesssim$ states that there is a multiplicative constant $C > 0$ involved which does neither depend on terms of the right-hand side nor on terms of the left-hand side. For example, the Sobolev inequality reads $\|u\|_\infty \leq C\|u\|_{H^s(\Omega)}$ for all $u \in H^s(\Omega)$, where $C > 0$ depends not on u but (possibly) on Ω and s.

From the classical Arzéla-Ascoli theorem one derives the so-called Rellich theorem which states that the identity operator $id : H^s(\Omega) \hookrightarrow H^t(\Omega)$ for $s > t$ is not only well-defined and continuous, but even is a compact operator, which is a pretty strong result.

Theorem 1.20 (Rellich Compactness Theorem). *For any orders $s > t$, the embedding $H^s(\Omega) \subseteq H^t(\Omega)$ is compact.* ■

A classical tool in the Sobolev space $H^1(\Omega)$ is the Poincaré inequality, which essentially provides estimates for the $L^2(\Omega)$-norm by the $H^1(\Omega)$-seminorm.

Lemma 1.21 (Poincaré Inequality). *For all $u \in H^1(\Omega)$, there holds*

$$\|u\|_{L^2(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)} + \Big| \int_\Omega u\,dx \Big|. \tag{1.26}$$

In particular, we have $\|u\|_{L^2(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)}$ for all $u \in H^1(\Omega)$ with $\int_\Omega u\,dx = 0$.

1.4.4 The Trace Operator

For certain Sobolev functions $u \in H^s(\Omega)$, one can define the trace $\gamma_0^{\mathrm{int}} u$ on the boundary. The induced trace operator is linear and continuous. This is stated in the following theorem.

Theorem 1.22 (Trace Operator). *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{k-1,1}$ domain and $1/2 < s \le k$. For $u \in H^s(\Omega)$ and $x \in \Gamma$, we define formally the trace*

$$\gamma_0^{\mathrm{int}} u(x) := \lim_{\varepsilon \to 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} u(y)\,dy$$

with $\Omega_\varepsilon := \Omega \cap B_\varepsilon(x)$. Then, $\gamma_0^{\mathrm{int}} u$ is defined almost everywhere on Γ and

$$\gamma_0^{\mathrm{int}} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \tag{1.27}$$

is a well-defined bounded linear operator, i.e.

$$\|\gamma_0^{\mathrm{int}} u\|_{H^{s-1/2}(\Gamma)} \lesssim \|u\|_{H^s(\Omega)} \tag{1.28}$$

for all $u \in H^s(\Omega)$. ■

Remark. If $u \in H^s(\Omega)$ is continuous at $x \in \Gamma$, we have $\gamma_0^{\mathrm{int}} u(x) = u(x)$. That is, γ_0^{int} extends the classical trace defined as restriction $u|_\Gamma$ on the boundary for smooth functions $u \in C(\overline{\Omega})$. □

As a first corollary to Theorem 1.22, we can prove that the integration by parts formula also holds for Sobolev functions $u, v \in H^1(\Omega)$.

Corollary 1.23 (Integration by Parts). *For all $u, v \in H^1(\Omega)$, there holds*

$$\int_\Omega u \frac{\partial v}{\partial x_j}\,dx + \int_\Omega \frac{\partial u}{\partial x_j} v\,dx = \int_\Gamma \gamma_0^{\mathrm{int}} u\, \gamma_0^{\mathrm{int}} v\, n_j\,ds. \tag{1.29}$$

Proof. The formula (1.29) holds for $u, v \in C^1(\overline{\Omega})$. All three terms define continuous bilinear forms on $H^1(\Omega) \times H^1(\Omega)$. Therefore (1.29) follows for arbitrary $u, v \in H^1(\Omega)$ from the density of $C^1(\overline{\Omega})$ in $H^1(\Omega)$: Given $u, v \in H^1(\Omega)$, there are sequences (u_n) and (v_n) in $C^1(\overline{\Omega})$ which converge to u resp. v in $H^1(\Omega)$. Therefore, if $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is continuous, there holds $\lim_{n\to\infty} a(u_n, v_n) = a(u, v)$. This concludes the proof. ■

Under the same assumptions as in the Trace Theorem 1.22, one can define a right-inverse of the trace operator. This operator is usually called lifting operator since it maps some boundary values onto some corresponding Sobolev functions in the domain.

Theorem 1.24 (Lifting Operator). *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{k-1,1}$ domain and $1/2 < s \leq k$. Then, there is a continuous linear operator*

$$\mathcal{L} : H^{s-1/2}(\Gamma) \to H^s(\Omega) \tag{1.30}$$

such that $\gamma_0^{\text{int}} \mathcal{L} u = u$ for all $u \in H^{s-1/2}(\Gamma)$. ■

As a remarkable corollary of Theorem 1.22 and Theorem 1.24, we obtain that an equivalent definition of Sobolev spaces on the boundary can be given by use of the trace operator. In particular, the important space $H^{1/2}(\Gamma)$ is just the space of all traces of functions in $H^1(\Omega)$.

Corollary 1.25. *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{k-1,1}$ domain and $1/2 < s \leq k$. Then,*

$$H^{s-1/2}(\Gamma) = \big\{ \gamma_0^{\text{int}} \widehat{u} \,\big|\, \widehat{u} \in H^s(\Omega) \big\}, \tag{1.31}$$

and

$$\|u\| := \inf \big\{ \|\widehat{u}\|_{H^s(\Omega)} \,\big|\, \widehat{u} \in H^s(\Omega) \text{ with } \gamma_0^{\text{int}} \widehat{u} = u \big\} \tag{1.32}$$

defines an equivalent norm on $H^{s-1/2}(\Gamma)$.

Proof. The set inclusion $\supseteq$ follows from the existence of the trace operator. The converse inclusion $\subseteq$ follows from the existence of the lifting operator $\mathcal{L}$ and $\gamma_0^{\text{int}} \mathcal{L} = id$. For $u \in H^{s-1/2}(\Gamma)$, there holds

$$\|u\| \leq \|\mathcal{L}u\|_{H^s(\Omega)} \lesssim \|u\|_{H^{s-1/2}(\Gamma)}$$

with $\mathcal{L}$ the (continuous) lifting operator. To prove the converse estimate, let $\varepsilon > 0$ be arbitrary. According to the definition of an infimum, there is an extension $\widehat{u} \in H^s(\Omega)$ with $\gamma_0^{\text{int}} \widehat{u} = u$ and $\|\widehat{u}\|_{H^s(\Omega)} \leq \|u\| + \varepsilon$. With the (continuous) trace operator, we are led to $\|u\|_{H^{s-1/2}(\Gamma)} = \|\gamma_0^{\text{int}} \widehat{u}\|_{H^{s-1/2}(\Gamma)} \lesssim \|\widehat{u}\|_{H^s(\Omega)}$. Therefore

$$\|u\|_{H^{s-1/2}(\Gamma)} \lesssim \|\widehat{u}\|_{H^s(\Omega)} \leq \|u\| + \varepsilon,$$

for all $\varepsilon > 0$. With $\varepsilon \to 0$, we prove the equivalence of $\|\cdot\|$ and $\|\cdot\|_{H^{s-1/2}(\Gamma)}$. In particular, $\|u\| = 0$ implies $u = 0$, i.e. $\|\cdot\|$ is definite. The homogeneity $\|\lambda u\| = |\lambda| \|u\|$ is clear by definition and the linearity of the trace operator. Finally, for $u, v \in H^{s-1/2}(\Gamma)$, there holds

$$\|u + v\| \leq \|\mathcal{L}(u + v)\|_{H^s(\Omega)} \leq \|\mathcal{L}u\|_{H^s(\Omega)} + \|\mathcal{L}v\|_{H^s(\Omega)}.$$

Taking the infimum over all extensions $\widehat{u}$ and $\widehat{v}$ of u and v, respectively, we prove the triangle inequality. ■

Remark. In fact, if we define $H_0^s(\Omega) := \{u \in H^s(\Omega) \,|\, \gamma_0^{\text{int}} u = 0\}$ for $1/2 < s \leq k$, then $H^{s-1/2}(\Gamma)$ is (isomorphic to) the quotient space $H^s(\Omega)/H_0^s(\Omega)$. Note that the norm $\|\cdot\|$ from (1.32) is nothing but the quotient norm. □

With the trace operator, we can provide a similar result to Poincaré inequality, which will be used for the analytical treatment of the Dirichlet problem, the so-called Friedrichs inequality.

Lemma 1.26 (Friedrichs Inequality). *Assume that the Dirichlet boundary $\Gamma_D \subseteq \Gamma$ has positive surface measure $|\Gamma_D| > 0$. Then, there holds*

$$\|u\|_{L^2(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)} + \|\gamma_0^{\text{int}} u\|_{L^2(\Gamma_D)} \tag{1.33}$$

for all $u \in H^1(\Omega)$. In particular, $\|\nabla u\|_{L^2(\Omega)}$ defines a Hilbert norm on the closed subspace

$$H_0^1(\Omega) := \{u \in H^1(\Omega) \,|\, \gamma_0^{\text{int}} u = 0\} \tag{1.34}$$

of $H^1(\Omega)$, and this norm is equivalent to $\|\cdot\|_{H^1(\Omega)}$ on $H_0^1(\Omega)$.

1.4.5 Sobolev Spaces with Zero Boundary Conditions

In this subsection, we incorporate zero boundary conditions into the function spaces $H^s(\Omega)$. The main observation is that there are multiple ways to do this, and Lemma 1.28 shows that these definitions can be equivalent or different depending on the parameter s.

Definition 1.27. *For $s \in \mathbb{R}_+$ we define* **Sobolev spaces with homogeneous boundary conditions** *as follows:*

- $H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}$, *where the closure is understood with respect to the $\|\cdot\|_{H^s(\Omega)}$-norm.*
- $\widetilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^d) \;:\; \operatorname{supp} u \subset \overline{\Omega}\}$ *with the norm*

$$\|u\|_{\widetilde{H}^s(\Omega)} := \|\widetilde{u}\|_{H^s(\mathbb{R}^d)} \qquad \text{with } \widetilde{u} \text{ denoting the 0-extension of } u$$

- $H^{-s}(\Omega) := \widetilde{H}^s(\Omega)^*$

Remark. For $s > 1/2$ an equivalent definition of the space $H_0^s(\Omega)$ is given as the kernel of the trace operator, i.e.,

$$H_0^s(\Omega) := \{u \in H^s(\Omega) \;:\; \gamma_0^{\text{int}} u = 0\}.$$

Remark. The norm on $\widetilde{H}^s(\Omega)$ defined by the 0-extension is oftentimes hard to work with due to the non-local nature of the norms on H^s for $s \notin \mathbb{N}$. However, for $s \in \mathbb{N}$, we indeed have

$$\|u\|_{\widetilde{H}^s(\Omega)} = \|u\|_{H^s(\Omega)}.$$

For non-integer $s \in (0,1)$ and bounded Lipschitz domains Ω, an equivalent norm on $\widetilde{H}^s(\Omega)$ is given by

$$\|u\|^2_{\widetilde{H}^s(\Omega)} \sim \|u\|^2_{H^s(\Omega)} + \|u/\rho^s\|^2_{L^2(\Omega)},$$

where $\rho(x) := \mathrm{dist}(x, \Gamma)$ for $x \in \Omega$. Replacing u by higher derivatives in the L^2-term gives an equivalent norm for $s > 1$.
The following lemma shows in which cases the space $\widetilde{H}^s(\Omega)$ and $H_0^s(\Omega)$ are equal or not.

Lemma 1.28. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*
(i) *Let $s \notin \frac{1}{2} + n$ for $n \in \mathbb{N}_0$. Then,*

$$\widetilde{H}^s(\Omega) = H_0^s(\Omega)$$

with equivalent norms.
(ii) *Let $s = \frac{1}{2}$. Then,*

$$\widetilde{H}^{1/2}(\Omega) \subsetneq H_0^{1/2}(\Omega) = H^{1/2}(\Omega).$$

(iii) *Let $s < \frac{1}{2}$. Then,*

$$\widetilde{H}^s(\Omega) = H^s(\Omega).$$

Proof. We only provide the main idea for the first statement, for the other two statements, we refer to [Gri11].
For an C^0-domain, we refer to [Neč67] for the density of $C_0^\infty(\Omega)$ in $\widetilde{H}^s(\Omega)$. Therefore, it remains to show the equivalence of the norms onto both spaces.
By definition of the $\widetilde{H}^s$-norm, we obviously have

$$\|u\|_{H^s(\Omega)} \leq \|u\|_{\widetilde{H}^s(\Omega)}.$$

For the converse inequality, we take $v \in C_0^\infty(\Omega)$. Since $v \equiv 0$ on Ω^c, we compute using Fubini's theorem

$$\begin{aligned} |v|^2_{H^s(\mathbb{R}^d)} &= \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} dxdy + 2 \int_\Omega \int_{\Omega^c} \frac{|v(x)|^2}{|x-y|^{d+2s}} dydx \\ &\lesssim |v|^2_{H^s(\Omega)} + \int_\Omega |v(x)|^2 \int_{B_{\rho(x)}(x)^c} \frac{1}{|x-y|^{d+2s}} dydx \\ &\lesssim |v|^2_{H^s(\Omega)} + \int_\Omega \frac{|v(x)|^2}{\rho(x)^{2s}} dx, \end{aligned}$$

where the last inequality follows from using polar coordinates. For a Lipschitz domain Ω and $s - 1/2 \notin \mathbb{N}$ the Hardy-inequality, c.f., [Gri11, Thm. 1.4.4.4],

$$\int_\Omega \frac{|\partial^\alpha v(x)|^2}{\rho(x)^{2(s-|\alpha|)}} dx \leq C \|v\|^2_{H^s(\Omega)} \qquad \forall v \in H_0^s(\Omega), \alpha \in \mathbb{N}^d, |\alpha| \leq s$$

holds, and finally gives

$$\|v\|_{\widetilde{H}^s(\Omega)} = \|v\|_{H^s(\mathbb{R}^d)} \lesssim \|v\|_{H^s(\Omega)},$$

which finishes the proof. ■

1.5 The Dirichlet Problem

Strong Form of Dirichlet Problem. We consider the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{1.35}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary Γ and where the Dirichlet data $g \in H^{1/2}(\Gamma)$ and the volume forces f are given. We recall the first Green's formula

$$\int_\Omega fv\,dx = \int_\Omega (-\Delta u)v\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Gamma \frac{\partial u}{\partial n} v\,ds, \tag{1.36}$$

which holds in a classical sense, provided u and v are smooth enough. The main step in this section is to understand (1.36) in a mathematical sense for Sobolev functions u and v, respectively. For $v \in H_0^1(\Omega) = \{u \in H^1(\Omega) \,|\, \gamma_0^{\text{int}} u = 0\}$, the boundary integral vanishes and we are led to

$$\int_\Omega fv\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx. \tag{1.37}$$

Weak Form of Dirichlet Problem. If $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega) = \ker(\gamma_0^{\text{int}} : H^1(\Omega) \to H^{1/2}(\Gamma))$ with respect to the extended L^2 scalar product, we may write (1.37) in the form

$$\langle f\,;\,v\rangle = (\nabla u\,;\,\nabla v)_\Omega \quad \text{for all } v \in H_0^1(\Omega), \tag{1.38}$$

where $\langle\cdot\,;\,\cdot\rangle$ denote the duality brackets and $(\cdot\,;\,\cdot)_\Omega$ denotes the L^2 scalar product. Therefore, we may state the *weak form* of our model problem (1.35): Given $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{aligned} (\nabla u\,;\,\nabla v)_\Omega &= \langle f\,;\,v\rangle \quad \text{for all } v \in H_0^1(\Omega), \\ \gamma_0^{\text{int}} u &= g. \end{aligned} \tag{1.39}$$

Note that a classical solution u of our model problem (1.35) is also a solution of the weak form since we derived our weak form by nothing but integration by parts. Therefore it is *necessary* to study the (unique) solvability of the weak form (1.39).

Usually the weak form (1.39) is written in a sloppy way as

$$\begin{aligned} -\Delta u &= f \in H^{-1}(\Omega), \\ \gamma_0^{\text{int}} u &= g \in H^{1/2}(\Gamma). \end{aligned} \tag{1.40}$$

Here, the notation $-\Delta u \in H^{-1}(\Omega)$ is a symbol which is defined by

$$\langle -\Delta u\,;\,v\rangle := (\nabla u\,;\,\nabla v)_\Omega \quad \text{for } v \in H_0^1(\Omega). \tag{1.41}$$

To get familiar with the introduced notation, we show that $-\Delta : H^1(\Omega) \to H^{-1}(\Omega)$ is a continuous linear operator:

$$\| -\Delta u\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{(\nabla u\,;\,\nabla v)_\Omega}{\|v\|_{H^1(\Omega)}} \leq \|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}, \tag{1.42}$$

where the first equality is just the definition and where the first estimate follows from the Cauchy inequality in L^2. Thus, the operator norm satisfies $\| -\Delta : H^1(\Omega) \to H^{-1}(\Omega)\| \le 1$.

Theorem 1.29. *Given $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$, there is a unique weak solution $u \in H^1(\Omega)$ of* (1.39) *resp.* (1.40), *and there holds the stability estimate*

$$\|u\|_{H^1(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}. \tag{1.43}$$

Proof. With the lifting operator $\mathcal{L}$, we define $u_0 := u - \mathcal{L}g \in H^1_0(\Omega)$. Then, (1.39) is equivalent to

$$(\nabla u_0\,;\, \nabla v) = \langle f + \Delta(\mathcal{L}g)\,;\, v\rangle \quad \text{for all } v \in H^1_0(\Omega). \tag{1.44}$$

Note that $f + \Delta(\mathcal{L}g) \in H^{-1}(\Omega)$. According to the Friedrichs inequality, the left-hand side defines an equivalent scalar product on $H^1_0(\Omega)$. Therefore, the Riesz theorem proves the unique existence of a solution $u_0 \in H^1_0(\Omega)$. In particular, $u = u_0 + \mathcal{L}g$ is the unique solution of (1.39). To prove the stability estimate, we first estimate $\|u_0\|_{H^1(\Omega)}$. The Friedrich's inequality proves

$$\begin{aligned}\|u_0\|^2_{H^1(\Omega)} \lesssim \|\nabla u_0\|^2_{L^2(\Omega)} &= (\nabla u\,;\, \nabla u_0)_\Omega - (\nabla\mathcal{L}g\,;\,\nabla u_0)_\Omega\\ &= \langle f\,;\, u_0\rangle - (\nabla\mathcal{L}g\,;\,\nabla u_0)_\Omega\\ &\le \big(\|f\|_{H^{-1}(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)}\big)\|u_0\|_{H^1(\Omega)}.\end{aligned}$$

Finally, we thus obtain

$$\|u\|_{H^1(\Omega)} \le \|u_0\|_{H^1(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)} + \|\mathcal{L}g\|_{H^1(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)},$$

where we have used the continuity of $\mathcal{L}$. ∎

The Conormal Derivative. In the following we assume that $-\Delta u = f \in \widetilde{H}^{-1}(\Omega)$. Note that, by definition, $H^1_0(\Omega) \subset H^1(\Omega)$ and therefore $\widetilde{H}^{-1}(\Omega) \subset H^{-1}(\Omega)$, i.e. the assumption on the volume forces $f \in \widetilde{H}^{-1}(\Omega)$ is now stronger than before. In this setting, we want to explain the first Green's formula mathematically – now for $v \in H^1(\Omega)$ instead of only $v \in H^1_0(\Omega)$. The first Green's formula (1.36) becomes formally, with $\gamma_1^{\rm int}u = \partial u/\partial n$,

$$\langle f\,;\, v\rangle = (\nabla u\,;\,\nabla v)_\Omega - \langle \gamma_1^{\rm int}u\,;\,\gamma_0^{\rm int}v\rangle \tag{1.45}$$

for $u \in H^1(\Omega)$ the weak solution of (1.39) and a test function $v \in H^1(\Omega)$. The first term is well-defined by our assumption on $f \in \widetilde{H}^{-1}(\Omega)$. The second term is well-defined since both gradients are in L^2. However, so far the last term is mathematically undefined since we have not defined the conormal derivative $\gamma_1^{\rm int}u$, yet. Because of $\gamma_0^{\rm int}v \in H^{1/2}(\Gamma)$, we must look for $\gamma_1^{\rm int}u \in H^{-1/2}(\Gamma)$ as an element of the dual space.

Since we would like the first Green's formula (1.45) to hold for any $v \in H^1(\Omega)$, we must define $\gamma_1^{\rm int}u \in H^{-1/2}(\Gamma)$ by

$$\langle \gamma_1^{\rm int}u\,;\, v\rangle := (\nabla u\,;\,\nabla\widehat{v})_\Omega - \langle f\,;\,\widehat{v}\rangle \tag{1.46}$$

for all $v \in H^{1/2}(\Gamma)$ and an arbitrary extension $\widehat{v} \in H^1(\Omega)$ with $\gamma_0^{\rm int}\widehat{v} = v$. However, one has to prove that the right-hand side in (1.46) is independent of the extension $\widehat{v} \in H^1(\Omega)$ of $v \in H^{1/2}(\Gamma)$.

Theorem 1.30. *Let $u \in H^1(\Omega)$ be the unique weak solution of* (1.39) *for given data $f \in \widetilde{H}^{-1}(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Then, the conormal derivative $\gamma_1^{\mathrm{int}} u \in H^{-1/2}(\Gamma)$ from* (1.46) *is well-defined and there holds the first Green's formula* (1.45) *for all $v \in H^1(\Omega)$. Moreover, we have the stability estimate*

$$\|\gamma_1^{\mathrm{int}} u\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}. \tag{1.47}$$

Proof. To prove that $\gamma_1^{\mathrm{int}} u \in H^{-1/2}(\Gamma)$ is well-defined, let $\widehat{v}, \widetilde{v} \in H^1(\Omega)$ satisfy $\gamma_0^{\mathrm{int}} \widehat{v} = v = \gamma_0^{\mathrm{int}} \widetilde{v}$. Then, $\widehat{v} - \widetilde{v} \in H_0^1(\Omega)$ and therefore (1.38) states

$$0 = (\nabla u\,;\,\nabla(\widehat{v} - \widetilde{v}))_\Omega - \langle f\,;\,\widehat{v} - \widetilde{v}\rangle = \big[(\nabla u\,;\,\nabla \widehat{v})_\Omega - \langle f\,;\,\widehat{v}\rangle\big] - \big[(\nabla u\,;\,\nabla \widetilde{v})_\Omega - \langle f\,;\,\widetilde{v}\rangle\big].$$

Thus, the definition (1.46) of $\gamma_1^{\mathrm{int}} u \in H^{-1/2}(\Gamma)$ is mathematically correct, and (1.45) holds for all $v \in H^1(\Omega)$. It only remains to verify the stability estimate. By definition, there holds

$$\|\gamma_1^{\mathrm{int}} u\|_{H^{-1/2}(\Gamma)} = \sup_{\substack{v \in H^{1/2}(\Gamma) \\ v \neq 0}} \frac{\langle \gamma_1^{\mathrm{int}} u\,;\,v\rangle}{\|v\|_{H^{1/2}(\Gamma)}}.$$

For the nominator, we plug-in the first Green's formula to see

$$\langle \gamma_1^{\mathrm{int}} u\,;\,v\rangle = \langle f\,;\,\mathcal{L}v\rangle - (\nabla u\,;\,\nabla(\mathcal{L}v))_\Omega \le \|f\|_{\widetilde{H}^{-1}(\Omega)} \|\mathcal{L}v\|_{H^1(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \|\nabla(\mathcal{L}v)\|_{L^2(\Omega)}.$$

The continuity of the lifting operator $\mathcal{L}$ proves

$$\|\nabla(\mathcal{L}v)\|_{L^2(\Omega)} \le \|\mathcal{L}v\|_{H^1(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}.$$

Altogether, we hence obtain $\|\gamma_1^{\mathrm{int}} u\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$. Finally, the inequality $\|\nabla u\|_{L^2(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}$ follows as in the proof of Theorem 1.29. ■

For the following exercise, recall that $\widetilde{H}^{-1}(\Omega) \subset H^{-1}(\Omega)$. Thus, it is an important question under which circumstances $-\Delta u$, for given $u \in H^1(\Omega)$, does not only belong to $H^{-1}(\Omega)$ but even to $\widetilde{H}^{-1}(\Omega)$. However, this question can very easily been answered.

Exercise 1. Let $u \in H^1(\Omega)$ and $f := -\Delta u \in H^{-1}(\Omega)$. Then, there holds $f \in \widetilde{H}^{-1}(\Omega)$ if and only if there is a functional $\phi \in H^{-1/2}(\Gamma)$ such that

$$\langle f\,;\,v\rangle = (\nabla u\,;\,\nabla v)_\Omega - \langle \phi\,;\,\gamma_0^{\mathrm{int}} v\rangle \quad \text{for all } v \in H^1(\Omega).$$

In this case, there holds $\phi = \gamma_1^{\mathrm{int}} u$. □

Remark. The first Green's formula is also well-defined, if we have $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$, often written as $H^1_\Delta(\Omega) := \{u \in H^1(\Omega) \;:\; \Delta u \in L^2(\Omega)\}$, which allows us to define a conormal derivative by

$$\langle \gamma_1^{\mathrm{int}} u\,;\,\gamma_0^{\mathrm{int}} v\rangle := (\nabla u\,;\,\nabla v)_\Omega + (\Delta u\,;\,v)_\Omega \quad \text{for all } v \in H^1(\Omega).$$

With the same arguments as above, the conormal derivative is well-defined, unique and bounded with

$$\|\gamma_1^{\mathrm{int}} u\|_{H^{-1/2}(\Gamma)} \lesssim \|\Delta u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}, \tag{1.48}$$

and by definition the first Green's formula holds.

1.6 The Neumann Problem

Strong Form of Neumann Problem. We consider the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \partial u/\partial n &= \phi \quad \text{on } \Gamma, \end{aligned} \tag{1.49}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary Γ and where the Neumann data $\phi \in H^{-1/2}(\Gamma)$ and the volume forces $f \in \widetilde{H}^{-1}(\Omega)$ are given. The first Green's formula reads

$$(\nabla u\,;\,\nabla v)_\Omega = \langle f\,;\,v\rangle + \langle \phi\,;\,\gamma_0^{\mathrm{int}} v\rangle \quad \text{for } v \in H^1(\Omega). \tag{1.50}$$

If we plug-in the constant function $v \equiv 1$, we see that the data must satisfy

$$0 = \langle f\,;\,1\rangle + \langle \phi\,;\,1\rangle \tag{1.51}$$

to allow solutions of (1.49). Moreover, additive constants are not fixed in the formulation of the problem, i.e. if u is a solution of (1.49) and $\alpha \in \mathbb{R}$, then $u+\alpha$ is a solution of (1.49). To fix additive constants, we define the Sobolev space

$$H^1_*(\Omega) := \big\{ v \in H^1(\Omega) \,\big|\, \textstyle\int_\Omega v\,dx = 0 \big\}. \tag{1.52}$$

Weak Form of Neumann Problem. The weak form of (1.49) then reads: Find $u \in H^1_*(\Omega)$ such that (1.50) holds.

Theorem 1.31. *Let $f \in \widetilde{H}^{-1}(\Omega)$ and $\phi \in H^{-1/2}(\Gamma)$ satisfy (1.51). Then, there is a unique solution $u \in H^1_*(\Omega)$ of (1.50). There holds the stability estimate*

$$\|u\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\phi\|_{H^{-1/2}(\Gamma)}. \tag{1.53}$$

Proof. For a function $v \in H^1(\Omega)$, we define

$$\widetilde{v} := v - \overline{v} \in H^1_*(\Omega), \quad \text{where} \quad \overline{v} = \frac{1}{|\Omega|}\int_\Omega v\,dx.$$

As the right-hand side of (1.50) reads

$$\langle f\,;\,v\rangle + \langle \phi\,;\,\gamma_0^{\mathrm{int}} v\rangle = \langle f\,;\,\widetilde{v}\rangle + \langle \phi\,;\,\gamma_0^{\mathrm{int}} \widetilde{v}\rangle,$$

(1.50) can be stated equivalently with $H^1_*(\Omega)$ replacing $H^1(\Omega)$. According to the Poincaré inequality, the left-hand side of (1.50) defines an equivalent scalar product on $H^1_*(\Omega)$. Therefore, the Riesz theorem applies and proves the unique existence of a weak solution $u \in H^1_*(\Omega)$. Another application of the Poincaré inequality and $v = u$ in (1.50) proves

$$\begin{aligned}\|u\|^2_{H^1(\Omega)} \lesssim \|\nabla u\|^2_{L^2(\Omega)} &= \langle f\,;\,u\rangle + \langle \phi\,;\,\gamma_0^{\mathrm{int}} u\rangle \\ &\le \|f\|_{\widetilde{H}^{-1}(\Omega)}\|u\|_{H^1(\Omega)} + \|\phi\|_{H^{-1/2}(\Gamma)}\|\gamma_0^{\mathrm{int}} u\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|u\|_{H^1(\Omega)}\,\{\|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\phi\|_{H^{-1/2}(\Gamma)}\}.\end{aligned}$$

This concludes the proof. ∎

Saddle Point Formulation of Neumann Problem. As in the previous section, we may state the weak form (1.50) equivalently as a saddle point problem with solution $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$ in order to eliminate the side constraint $\int_\Omega u\,dx = 0$ contained in the definition of $H^1_*(\Omega)$.

$$\left.\begin{array}{lcl}(\nabla u\,;\,\nabla v)_\Omega + \lambda\int_\Omega v\,dx &=& \langle f\,;\,v\rangle + \langle \phi\,;\,\gamma_0^{\mathrm{int}} v\rangle \\ \mu\int_\Omega u\,dx &=& 0\end{array}\right\}\quad \text{for all } (v,\mu) \in H^1(\Omega)\times\mathbb{R}. \tag{1.54}$$

Proposition 1.32. *The Brezzi theorem (Theorem A.4) applies to the saddle point problem* (1.54)*: Provided $f \in \widetilde{H}^{-1}(\Omega)$ and $\phi \in H^{-1/2}(\Gamma)$ satisfy* (1.51)*,* (1.54) *has a unique solution $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$. There holds $\lambda = 0$ and $u \in H^1_*(\Omega)$, and u is the unique solution of the weak form* (1.50) *of the Neumann problem.*

Proof. With respect to the abstract setting of the Brezzi theorem, we have $X = H^1(\Omega)$, $Y = \mathbb{R}$, $a(u, v) = (\nabla u\,;\,\nabla v)_\Omega$, and $b(u, \lambda) := \lambda\int_\Omega u\,dx$. There holds $X_0 = \{u \in H^1(\Omega)\,|\,b(u,\cdot) = 0\} = H^1_*(\Omega)$, and $a(\cdot,\cdot)$ is an equivalent scalar product on $H^1_*(\Omega)$. Therefore, the Brezzi theorem provides a unique solution $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$. Note that $b(u,\cdot) = 0$ implies $u \in H^1_*(\Omega)$. Plugging-in $v \equiv 1$ into the first equation of (1.54), we obtain $\lambda = 0$. Therefore, the first equation simplifies to the first Green's formula, and u is the unique solution of (1.50). ∎

Remark. It is also possible to consider mixed boundary value problems of the form

$$\begin{aligned} -\Delta u &= f &&\text{in } \Omega,\\ u &= g_D &&\text{on } \Gamma_D,\\ \partial u/\partial n &= g_N &&\text{on } \Gamma_N,\end{aligned} \tag{1.55}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with boundary Γ. Γ_D and Γ_N are open subsets of Γ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \Gamma$, where we assume $|\Gamma_D| > 0$. To come up with a functional analytic setting, we would have to specify the Sobolev spaces to which g_D and g_N belong, which leads to Sobolev spaces $H^s(\gamma)$ on screens $\gamma \subset \Gamma$. The essential ideas are hereby the same as in the previous sections, but the discussion is a bit more technical. For details, we refer, e.g., to [Ste08].

Chapter 2

Integral Operators

In Chapter 1, we have proven the **representation formula**

$$u = \widetilde{N}(-\Delta u) + \widetilde{V}(\partial u/\partial n) - \widetilde{K}(u) \quad \text{in } \Omega \tag{2.1}$$

only for $u \in C^2(\overline{\Omega})$, cf. Proposition 1.6. Here, we abbreviate notation by use of three linear integral operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$, namely

- the **Newton potential** of $f : \Omega \to \mathbb{R}$

$$\widetilde{N}f(\widetilde{x}) := \int_\Omega G(\widetilde{x} - y)f(y)\, dy \quad \text{for } \widetilde{x} \in \Omega, \tag{2.2}$$

- the **single layer potential** of $\phi : \Gamma \to \mathbb{R}$

$$\widetilde{V}\phi(\widetilde{x}) := \int_\Gamma G(\widetilde{x} - y)\phi(y)\, ds_y \quad \text{for } \widetilde{x} \in \Omega, \tag{2.3}$$

- the **double layer potential** of $v : \Gamma \to \mathbb{R}$

$$\widetilde{K}v(\widetilde{x}) := \int_\Gamma \frac{\partial_y}{\partial n(y)} G(\widetilde{x} - y)\, v(y)\, ds_y \quad \text{for } \widetilde{x} \in \Omega. \tag{2.4}$$

The goal of this chapter is threefold: First, we want to study the mapping properties of the three operators with respect to our functional analytic setting. This leads to a general statement of the representation formula (2.1) as well as to the introduction of the Calderón projector. Second, we derive integral equations which provide — at a fist glance — necessary conditions for a function $u \in H^1(\Omega)$ to solve the model problem $-\Delta u = f$ for Dirichlet and Neumann boundary conditions, respectively. In the next chapter, we shall show that these integral equations are in fact equivalent formulations of our model problems. Finally, we want to derive integral representations of the operators for the case $\widetilde{x} \to x \in \Gamma := \partial\Omega$. Throughout, we assume that Ω is a Lipschitz domain in $\mathbb{R}^d$.

Besides the function spaces of the last chapter, the following easy result from functional analysis states the most important mathematical tool for the entire section.

Lemma 2.1. *Let X and Y be Banach spaces, D be a dense subspace of X, and $T \in L(D;Y)$. Then, there is a unique extension $\widehat{T} \in L(X;Y)$, i.e., $\widehat{T}x = Tx$ for all $x \in D$. Moreover, there holds $\|\widehat{T} : X \to Y\| = \|T : D \to Y\|$ for the corresponding operator norms.*

2.1 Newton, Single-Layer, and Double-Layer Potential

In the following, we study the mapping properties of the three potential operators $\widetilde{N}$, $\widetilde{V}$, and $\widetilde{K}$. We start with the Newton potential $\widetilde{N}f$ of a function $f : \Omega \to \mathbb{R}$ as well as with its trace and its conormal derivative

$$N_0 := \gamma_0^{\rm int}\widetilde{N} \quad \text{and} \quad N_1 := \gamma_1^{\rm int}\widetilde{N}, \tag{2.5}$$

respectively. We start with an elementary observation, which allows to verify the mapping properties for smooth functions only.

Lemma 2.2. *For each domain Ω and $s \geq 0$, $\mathcal{D}(\Omega)$ is a dense subspace of both, $H^{-s}(\Omega)$ as well as $\widetilde{H}^{-s}(\Omega)$.*

Proof. We consider the case of $\widetilde{H}^{-s}(\Omega) = H^s(\Omega)^*$. Recall that $\|\cdot\|_{L^2(\Omega)} \leq \|\cdot\|_{H^s(\Omega)}$, whence $\|\cdot\|_{\widetilde{H}^{-s}(\Omega)} \leq \|\cdot\|_{L^2(\Omega)}$. To conclude the proof, we recall that $\mathcal{D}(\Omega)$ is a dense subspace of $L^2(\Omega)$ with respect to the L^2-norm and that $L^2(\Omega)$ is a dense subspace of $\widetilde{H}^{-s}(\Omega)$ with respect to the $\widetilde{H}^{-s}$-norm. Altogether, we thus have density of $\mathcal{D}(\Omega)$ in $\widetilde{H}^{-s}(\Omega)$. The same arguments work for $H^{-s}(\Omega) = H^1_0(\Omega)^*$. ■

The following theorem gathers the most important mapping properties of $\widetilde{N}$ together.

Theorem 2.3. (i) *There holds $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$ for $f \in \mathcal{D}(\Omega)$.*
(ii) *$\widetilde{N}$ allows for a unique extension $\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^1(\Omega)\big)$ from $\mathcal{D}(\Omega)$ to $\widetilde{H}^{-1}(\Omega)$.*
(iii) *$-\Delta(\widetilde{N}f) = f$ for all $f \in \widetilde{H}^{-1}(\Omega)$.*
(iv) *$N_0 := \gamma_0^{\rm int}\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{1/2}(\Gamma)\big)$.*
(v) *$N_1 := \gamma_1^{\rm int}\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big)$.*

Lemma 2.4. *For $f \in \mathcal{D}(\Omega)$, there holds $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$ as well as $f = \widetilde{N}(-\Delta f) = -\Delta(\widetilde{N}f)$. Moreover, the partial derivatives satisfy $\partial^\alpha(G * f) = G * \partial^\alpha f$.*

Proof. **1. step.** To prove $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$, recall that $\widetilde{N}f = G * f$ and $G \in L^1_{\ell oc}(\mathbb{R}^d)$. We now prove that

$$\forall R > 0 \exists G_R \in L^1(\mathbb{R}^d) \quad (G * f)|_{B_R(0)} = (G_R * f)|_{B_R(0)},$$

so that usual results on convolution apply and yield $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$: Given $R > 0$, we assume without loss of generality that $\Omega \subseteq B_R(0)$. Choose $\chi_R \in \mathcal{D}(\mathbb{R}^d)$ with $\chi_R|_{B_{2R}(0)} \equiv 1$ and define $G_R := G\chi_R$. Provided $|y| \geq 2R$ and $|x| \leq R$, there holds $x - y \notin \Omega$ since $|x-y| \geq |y| - |x| \geq R$. In particular, we have $f(x-y) = 0$ and consequently

$$G * f(x) = \int f(x-y)G(y)\,dy = \int_{|y|\leq 2R} f(x-y)G(y)\,dy = G_R * f(x) \quad \text{for all } x \in B_R(0)$$

since $G(y) = G_R(y)$ for $|y| \leq 2R$.
2. step. The equality $f = \widetilde{N}(-\Delta f)$ follows from the representation formula (1.14) as $f = 0 = \partial f/\partial n$ on Γ.

3. step. To prove the equality $f = -\Delta(\widetilde{N}f)$, we apply the fundamental theorem of calculus: For $g \in \mathcal{D}(\Omega)$, integration by parts and the application of the Fubini theorem prove

$$\begin{aligned}\Big(-\Delta(\widetilde{N}f)\,;\,g\Big)_\Omega = \Big(\widetilde{N}f\,;\,-\Delta g\Big)_\Omega &= \int_\Omega(-\Delta g)(x)\int_\Omega G(x-y)f(y)\,dy\,dx\\ &= \int_\Omega f(y)\int_\Omega G(x-y)(-\Delta g)(x)\,dx\,dy\\ &= (f\,;\,g)_\Omega\,.\end{aligned}$$

Here, we used that $G(x-y) = G(y-x)$ so that the inner integral is just $\widetilde{N}(-\Delta g)(y) = g(y)$ as $g \in \mathcal{D}(\Omega)$. This implies $-\Delta(\widetilde{N}f) = f$ almost everywhere in Ω and thus everywhere in Ω according to continuity. ■

Proof of Theorem 2.3. The statement (i) follows directly from the previous lemma.
Proof of (ii): The proof is rather technical and makes use of the equivalent definition of the Sobolev spaces $H^s(\Omega)$ in terms of the Fourier transform and the so called Bessel potential. Here, we sketch the arguments and refer to [Ste08] for the complete details. We have to show that

$$\|\widetilde{N}f\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)}.$$

It suffices - by using a density argument - to show this result for $f \in \mathcal{D}(\Omega)$.
1. step. The property $\operatorname{supp} f \subset \Omega$ gives

$$\|f\|_{H^{-1}(\mathbb{R}^d)} = \sup_{v\in H^1(\mathbb{R}^d)}\frac{(f,v)_{H^1(\mathbb{R}^d)}}{\|v\|_{H^1(\mathbb{R}^d)}} \le \sup_{v\in H^1(\Omega)}\frac{(f,v)_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}} = \|f\|_{\widetilde{H}^{-1}(\Omega)},$$

which allows us to work in the full-space $\mathbb{R}^d$. Applying the definition of Sobolev spaces via Fourier transformation gives for $s \in \mathbb{R}$

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(\mathcal{F}f(\zeta))^2(1+|\zeta|^2)^s d\zeta. \tag{2.6}$$

2. step. Let $u = \widetilde{N}f$ and $u_\chi(x) := \int_\Omega \chi(|x-y|)G(x-y)f(y)dy$ for $x \in \mathbb{R}^d$, where $\chi \in C_0^\infty([0,\infty))$ is a cut-off function satisfying $0 \le \chi \le 1$, $\chi(r) \equiv 1$ for $r \in [0, 2R]$, and $R > 0$ is sufficiently large such that $\Omega \subset B_R(0)$.
Then, we have $u_\chi = u$ on Ω as well as $u_\chi \in H^1(\mathbb{R}^d)$. This gives together with (2.6) that

$$\|u\|^2_{H^1(\Omega)} = \|u_\chi\|^2_{H^1(\Omega)} \le \|u_\chi\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(\mathcal{F}u_\chi(\zeta))^2(1+|\zeta|^2)d\zeta. \tag{2.7}$$

Therefore, it remains to estimate the Fourier transform of u_χ. We write using that the Fourier transform turns convolutions into multiplications

$$\mathcal{F}u_\chi(\zeta) = \mathcal{F}f(\zeta)I(|\zeta|) \qquad \text{with} \quad I(|\zeta|) := \int_{\mathbb{R}^d} e^{-iz\cdot\zeta}\chi(|z|)G(z)\,dz.$$

We note, that indeed $I(\zeta) = I(|\zeta|)$ is rotational symmetric, since the Fourier transform preserves this property and the integrand is rotational symmetric. Using some properties of Bessel potentials, some tedious calculations (see [Ste08]) provide estimates for I

$$|I(|\zeta|)| \lesssim \begin{cases} |\zeta|^{-2} & |\zeta| \geq 1, \\ 1 & |\zeta| < 1. \end{cases}$$

Inserting everything into (2.7), we may estimate

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{F}u_\chi(\zeta))^2(1+|\zeta|^2)d\zeta &= \int_{|\zeta|\geq 1}(1+|\zeta|^2)|I(|\zeta|)|^2|\mathcal{F}f(\zeta)|^2 d\zeta + \int_{|\zeta|<1}(1+|\zeta|^2)|I(|\zeta|)|^2|\mathcal{F}f(\zeta)|^2 d\zeta \\ &\lesssim 4\int_{\mathbb{R}^d} \frac{1}{1+|\zeta|^2}|\mathcal{F}f(\zeta)|^2 d\zeta = \|f\|^2_{H^{-1}(\mathbb{R}^d)}, \end{aligned}$$

where we used the bound on $I(|\zeta|)$ and elementary estimates to show $I(|\zeta|)^2(1+|\zeta|^2) \leq 4\frac{1}{1+|\zeta|^2}$ for both cases for $|\zeta|$ as well as (2.6) for $s = -1$. With step 1, this finishes the proof of (ii).

Proof of (iv): (iv) follows immediately from (ii) and the mapping properties of the trace operator.

Proof of (iii) *and* (v): Both statements are proven simultaneously by use of density arguments:

1. step. There holds $-\Delta(\widetilde{N}f) = f$ for all test functions $f \in \mathcal{D}(\Omega)$ as proven in Lemma 2.4.

2. step. There is a unique operator $N_1 \in L(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma))$ with $N_1 = \gamma_1^{\mathrm{int}}\widetilde{N}$ on $\mathcal{D}(\Omega)$: For $f \in \mathcal{D}(\Omega)$, there holds $\widetilde{N}f \in C^\infty(\overline{\Omega})$ with $-\Delta(\widetilde{N}f) = f \in \mathcal{D}(\Omega) \subset \widetilde{H}^{-1}(\Omega)$. We may therefore apply Theorem 1.30 and derive that $N_1 f := \gamma_1^{\mathrm{int}}\widetilde{N}f \in H^{-1/2}(\Gamma)$ is well-defined with

$$\|N_1 f\|_{H^{-1/2}(\Gamma)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)} + \|\widetilde{N}f\|_{H^1(\Omega)} \lesssim \|f\|_{\widetilde{H}^{-1}(\Omega)},$$

where we have used the continuity of $\widetilde{N}$ in the final estimate. As $\mathcal{D}(\Omega)$ is a dense subspace of $\widetilde{H}^{-1}(\Omega)$, there is a unique extension of N_1 from $\mathcal{D}(\Omega)$ to an operator $N_1 \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big)$.

3. step. There holds $-\Delta(\widetilde{N}f) = f$ for all $f \in \widetilde{H}^{-1}(\Omega)$: By definition, we have to show that there is an element $\phi \in H^{-1/2}(\Gamma)$ such that

$$\langle f\,;\, v\rangle = \Big(\nabla(\widetilde{N}f)\,;\,\nabla v\Big)_\Omega - \langle \phi\,;\,\gamma_0^{\mathrm{int}}v\rangle \quad \text{for all } v \in H^1(\Omega). \tag{2.8}$$

We choose $\phi := N_1 f$ with N_1 the extended operator from step 2. Now, let (f_n) be a sequence in $\mathcal{D}(\Omega)$ with $\lim\limits_{n\to\infty} f_n = f \in \widetilde{H}^{-1}(\Omega)$. For each $n \in \mathbb{N}$, there holds

$$\langle f_n\,;\, v\rangle = \Big(\nabla(\widetilde{N}f_n)\,;\,\nabla v\Big)_\Omega - \langle N_1 f_n\,;\,\gamma_0^{\mathrm{int}}v\rangle \quad \text{for all } v \in H^1(\Omega) \tag{2.9}$$

as $-\Delta(\widetilde{N}f_n) = f_n$ and $N_1 f_n = \gamma_1^{\mathrm{int}}\widetilde{N}f_n$. Note that $\lim_{n\to\infty}\widetilde{N}f_n = \widetilde{N}f \in H^1(\Omega)$ according to (ii). Moreover, $\lim_{n\to\infty} N_1 f_n = N_1 f \in H^{-1/2}(\Gamma)$. Thus, the equality (2.9) implies the equality (2.8) in the continuous limit $n \to \infty$.

4. step. The operator N_1 from step 2 satisfies $N_1 f = \gamma_1^{\mathrm{int}}\widetilde{N}f$ for all $f \in \widetilde{H}^{-1}(\Omega)$: From step 3, we derive that $\gamma_1^{\mathrm{int}}\widetilde{N}f \in H^{-1/2}(\Gamma)$ is well-defined. Moreover, from the definition of $\gamma_1^{\mathrm{int}}\widetilde{N}f$ in Theorem 1.30, we obtain that $\phi = \gamma_1^{\mathrm{int}}\widetilde{N}f$ in (2.8). As we have just proven $\phi = N_1 f$, we conclude $N_1 f = \gamma_1^{\mathrm{int}}\widetilde{N}f$. ∎

We next consider the single-layer potential $\widetilde{V}\phi$ of a function $\phi : \Gamma \to \mathbb{R}$ defined as

$$\widetilde{V}\phi = \int_\Gamma G(x-y)\phi(y)ds_y,$$

which is well-defined for $x \in \mathbb{R}^d \backslash \Gamma$ and $\phi \in L^1(\Gamma)$.
The following lemma generalizes some known facts from basic analysis for the convolution, which will be applied for the Newton kernel G and its derivatives $\partial^\alpha G$.

Lemma 2.5. *For any $\phi \in L^1(\Gamma)$ and $g \in C^k(\mathbb{R}^d \backslash \{0\})$, the function $u(x) := \int_\Gamma g(x-y)\phi(y)\, ds_y$ belongs to $C^k(\mathbb{R}^d \backslash \Gamma)$. Moreover, there holds*

$$\partial^\alpha u(x) = \int_\Gamma \partial_x^\alpha g(x-y)\phi(y)\, ds_y \quad \text{for all } x \in \mathbb{R}^d \backslash \Gamma \text{ and all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \le k.$$

Proof. Clearly, the lemma follows from the special case $k = 1$ by induction. We thus only have to consider the cases $k = 0$ and $k = 1$ to derive that $u \in C^1(\mathbb{R}^d \backslash \Gamma)$.

1. step. We first show that u is continuous in $\mathbb{R}^d \backslash \Gamma$: To that end, fix $x_0 \in \mathbb{R}^d \backslash \Gamma$ and $\eta > 0$ such that $B_\eta(x_0) \subset \mathbb{R}^d \backslash \Gamma$. Note that

$$|u(x) - u(x_0)| = \Big| \int_\Gamma \big(g(x-y) - g(x_0-y)\big)\phi(y)\, ds_y \Big| \le \|\phi\|_{L^1(\Gamma)} \sup_{y \in \Gamma} \big|g(x-y) - g(x_0-y)\big|$$

for all $x \in \mathbb{R}^d$. It thus remains to show that for all $\varepsilon > 0$ exists a $\delta > 0$ such that

$$\sup_{y \in \Gamma} \big|g(x-y) - g(x_0-y)\big| \le \varepsilon \quad \forall x \in B_\delta(x_0).$$

We define $K := \big\{x - y \,\big|\, x \in B_\eta(x_0), y \in \Gamma\big\}$ and note that K is compact. Letting $\varepsilon > 0$, we choose $\delta > 0$ with respect to the uniform continuity of g on K. For $x \in B_\delta(x_0)$ and any $y \in \Gamma$ holds $|(x-y) - (x_0-y)| < \delta$. This yields $\big|g(x-y) - g(x_0-y)\big| \le \varepsilon$ for any $y \in \Gamma$.

2. step. Next, we sketch the proof of the differentiability of u: Fix $x \in \mathbb{R}^d \backslash \Gamma$. For $h > 0$, we define the j-th difference quotient Δ_j^h by

$$\Delta_j^h u(x) := \frac{u(x + he_j) - u(x)}{h},$$

which is well-defined for $h < \varepsilon$ with $B_\varepsilon(x) \subset \mathbb{R}^d \backslash \Gamma$. With $v_j := \int_\Gamma \partial_{j,x} g(x-y)\phi(y)\, ds_y$, there holds

$$\begin{aligned} |\Delta_j^h u(x) - v_j(x)| &= \Big| \int_\Gamma \big(\Delta_{j,x}^h g(x-y) - \partial_{j,x} g(x-y)\big)\phi(y)\, ds_y \Big| \\ &\le \|\phi\|_{L^1(\Gamma)} \|\Delta_{j,x}^h g(x-\cdot) - \partial_{j,x} g(x-\cdot)\|_{L^\infty(\Gamma)}. \end{aligned}$$

Note that Γ is compact. Thus, the right-hand side converges to 0 with $h \to 0$ since $\Delta_{j,x}^h g(x-\cdot)$ and $\partial_j g(x-\cdot)$ are uniformly continuous for fixed $x \notin \Gamma$ and variable $y \in \Gamma$. The details follow as in the proof of Theorem A.8 and are left to the reader. This proves $u \in C^1(\mathbb{R}^d \backslash \Gamma)$ with $\partial_j u = v_j$. ■

With this result in hand, we directly obtain that on $\mathbb{R}^d\backslash\Gamma$ the single-layer potential is infinitely differentiable. In fact, the main theorem for $\widetilde{V}$ reads as follows:

Theorem 2.6. (i) *There holds* $\widetilde{V}\phi \in C^\infty(\mathbb{R}^d\backslash\Gamma)$ *at least for* $\phi \in L^2(\Gamma)$.
(ii) $\widetilde{V}$ *allows for a unique extension* $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega)\big)$ *from* $L^2(\Gamma)$ *to* $H^{-1/2}(\Gamma)$.
(iii) $-\Delta\widetilde{V}\phi = 0 \in \widetilde{H}^{-1}(\Omega)$ *for all* $\phi \in H^{-1/2}(\Gamma)$.
(iv) $V := \gamma_0^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$.
(v) $\gamma_1^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$.

Remark. We shall see later that for $\phi \in L^\infty(\Gamma)$, there does not only hold $\widetilde{V}\phi \in C^\infty(\mathbb{R}^d\backslash\Gamma)$ but also $\widetilde{V}\phi \in C(\mathbb{R}^d)$. □

Proof of Theorem 2.6. The proof is split into several steps, from which only the first step may be innovative. The remaining steps just follow the proof of Theorem 2.3. However, we first stress that, for $\phi \in L^2(\Gamma)$, the preceding lemma implies $\widetilde{V}\phi \in C^\infty(\mathbb{R}^d\backslash\Gamma)$. Moreover $-\Delta G = 0$ in $\mathbb{R}^d\backslash\{0\}$ proves $-\Delta(\widetilde{V}\phi) = 0$ in $\mathbb{R}^d\backslash\Gamma$ in this case.

1. step. For $\phi \in L^2(\Gamma)$, there holds $\widetilde{V}\phi \in H^1(\Omega)$ with $\|\widetilde{V}\phi\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{-1/2}(\Gamma)}$: As $\widetilde{V}\phi \in C^\infty(\Omega)$, it only remains to estimate the H^1-norm. With the density of $\mathcal{D}(\Omega)$ in $\widetilde{H}^{-1}(\Omega)$, the Hahn-Banach theorem yields

$$\|\widetilde{V}\phi\|_{H^1(\Omega)} = \sup_{f\in\widetilde{H}^{-1}(\Omega)\backslash\{0\}} \frac{\langle\widetilde{V}\phi\,;\,f\rangle}{\|f\|_{\widetilde{H}^{-1}(\Omega)}} = \sup_{f\in\mathcal{D}(\Omega)\backslash\{0\}} \frac{\langle\widetilde{V}\phi\,;\,f\rangle}{\|f\|_{\widetilde{H}^{-1}(\Omega)}}.$$

Let $f \in \mathcal{D}(\Omega)$ and recall that $\widetilde{N}f \in C^\infty(\mathbb{R}^d)$. The Fubini theorem and the symmetry $G(x-y) = G(y-x)$ prove

$$\begin{aligned}\langle\widetilde{V}\phi\,;\,f\rangle = \int_\Omega f(x)\int_\Gamma G(x-y)\phi(y)\,ds_y\,dx &= \int_\Gamma \phi(y)\int_\Omega G(x-y)f(x)\,dx\,ds_y \\ &= \langle\phi\,;\,N_0 f\rangle \\ &\le \|\phi\|_{H^{-1/2}(\Gamma)}\|N_0 f\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|\phi\|_{H^{-1/2}(\Gamma)}\|f\|_{\widetilde{H}^{-1}(\Omega)},\end{aligned}$$

i.e. $\|\widetilde{V}\phi\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{-1/2}(\Gamma)}$.

2. step. As $L^2(\Gamma)$ is a dense subspace of $H^{-1/2}(\Gamma)$, $\widetilde{V}$ may be uniquely extended from $L^2(\Gamma)$ to an operator $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega)\big)$.

3. step. The operator $V := \gamma_0^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is well-defined.

4. step. There holds $-\Delta(\widetilde{V}\phi) = 0 \in \widetilde{H}^{-1}(\Omega)$: For $\phi \in L^2(\Gamma)$, there holds $-\Delta(\widetilde{V}\phi) = 0$ almost everywhere, whence in $L^2(\Omega) \subset \widetilde{H}^{-1}(\Omega)$. Thus, $V_1\phi := \gamma_1^{\rm int}\widetilde{V}\phi \in H^{-1/2}(\Gamma)$ is well-defined with $\|V_1\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|\widetilde{V}\phi\|_{H^1(\Omega)} \lesssim \|\phi\|_{H^{-1/2}(\Gamma)}$ according to Theorem 1.30. Therefore, we may extend V_1 from $L^2(\Gamma)$ to an operator $V_1 \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$. With continuity arguments, we prove that

$$0 = \Big(\nabla(\widetilde{V}\phi)\,;\,\nabla v\Big)_\Omega - \langle V_1\phi\,;\,\gamma_0^{\rm int}v\rangle \quad \text{for all } \phi \in \widetilde{H}^{-1/2}(\Gamma) \text{ and } v \in H^1(\Omega),$$

since the equality holds for all $\phi \in L^2(\Gamma)$ and $v \in H^1(\Omega)$. By definition, this proves $-\Delta(\widetilde{V}\phi) = 0$ in $\widetilde{H}^{-1}(\Omega)$.

5. step. The operator $\gamma_1^{\mathrm{int}}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$ is well-defined. Moreover, there holds $\gamma_1^{\mathrm{int}}\widetilde{V} = V_1$ with the operator V_1 from step 4. ∎

Finally, we consider the double layer potential $\widetilde{K}v$ of a function $v : \Gamma \to \mathbb{R}$ given by

$$\widetilde{K}v = \int_\Gamma \gamma_{1,y}^{\mathrm{int}} G(x-y)v(y)ds_y,$$

which again is well-defined for $x \in \mathbb{R}^d \backslash \Gamma$ and $v \in L^1(\Gamma)$.
The main result for $\widetilde{K}$ reads as follows:

Theorem 2.7. (i) *There holds $\widetilde{K}v \in C^\infty(\mathbb{R}^d \backslash \Gamma)$ for all $v \in H^{1/2}(\Gamma)$.*
(ii) *There holds $\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^1(\Omega)\big)$.*
(iii) *$-\Delta\widetilde{K}v = 0 \in \widetilde{H}^{-1}(\Omega)$ for all $v \in H^{1/2}(\Gamma)$.*
(iv) *$\gamma_0^{\mathrm{int}}\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$.*
(v) *$W := -\gamma_1^{\mathrm{int}}\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$.*

Proof of Theorem 2.7. **1. step.** For $w \in L^1(\Gamma)$, we show

$$\partial^\alpha(\widetilde{K}w)(x) = \int_\Gamma \gamma_{1,y}^{\mathrm{int}} \partial_x^\alpha G(x-y)w(y)\,ds_y \quad \text{for } x \in \mathbb{R}^d \backslash \Gamma \text{ and } \alpha \in \mathbb{N}_0^d,$$

which implies $\widetilde{K}w \in C^\infty(\mathbb{R}^d \backslash \Gamma)$ as well as $-\Delta(\widetilde{K}w) = 0$.
To prove that, we note that, because of $x \in \Omega$, $\gamma_{1,y}^{\mathrm{int}} G(x-y) = \frac{\partial_y}{\partial n_y} G(x-y) = n(y) \cdot \nabla_y G(x-y)$. In particular, the Schwartz theorem applies and proves that one may interchange the order of derivatives to obtain $\partial_x^\alpha \gamma_{1,y}^{\mathrm{int}} G(x-y) = \gamma_{1,y}^{\mathrm{int}} \partial_x^\alpha G(x-y)$. Moreover, the trivial observation

$$\begin{aligned}
|\Delta_{j,x}^h \widetilde{K}v(x) - \partial_j \widetilde{K}v(x)| &\le \|v\|_{L^1(\Gamma)} \sup_{y\in\Gamma} |\gamma_{1,y}^{\mathrm{int}} \Delta_{j,x}^h G(x-y) - \gamma_{1,y}^{\mathrm{int}} \partial_{j,x} G(x-y)| \\
&\le \|v\|_{L^1(\Gamma)} \sup_{y\in\Gamma} |\Delta_{j,x}^h \nabla_y G(x-y) - \partial_{j,x} \nabla_y G(x-y)|
\end{aligned}$$

allows to apply the arguments of Lemma 2.5 for the kernel $g(z) = \nabla G(z)$.

2. step. For $f \in \mathcal{D}(\Omega)$, there holds $\widetilde{N}f \in C^\infty(\Omega)$ with $\partial_j(\widetilde{N}f)(x) = \int_\Omega \partial_{j,x} G(x-y) f(y)\,dy$: Recall that Lemma 2.4 implies $\widetilde{N}f \in C^\infty(\Omega)$ and

$$\partial_j(\widetilde{N}f)(x) = \int_{\mathbb{R}^d} G(y)\partial_{j,x} f(x-y)\,dy.$$

With the Lebesgue theorem and integration by parts, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} G(y)\partial_{j,x} f(x-y)\,dy &= -\int_{\mathbb{R}^d} G(y)\partial_{j,y} f(x-y)\,dy \\
&= -\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d \backslash B_\varepsilon(0)} G(y)\partial_{j,y} f(x-y)\,dy \\
&= -\lim_{\varepsilon\to 0} \Big(-\int_{\mathbb{R}^d \backslash B_\varepsilon(0)} \partial_{j,y} G(y)\, f(x-y)\,dy + \int_{|y|=\varepsilon} G(y) f(x-y) n_j(y)\,ds_y \Big).
\end{aligned}$$

Note that the boundary integral vanishes for $\varepsilon \to 0$, whereas the volume integral exists due to $\partial_j G \in L^1_{\ell oc}(\mathbb{R}^d)$. This proves

$$\partial_j(\widetilde{N}f)(x) = \int_{\mathbb{R}^d} \partial_{j,y}G(y)\,f(x-y)\,dy = (\partial_j G) * f = \int_\Omega \partial_{j,x}G(x-y)\,f(y)\,dy.$$

3. step. There holds $\|\widetilde{K}v\|_{H^1(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}$: We proceed along the lines of the proof for the single-layer potential: Let $f \in \mathcal{D}(\Omega)$. Then, the Fubini theorem proves

$$\begin{aligned}\left(\widetilde{K}v\,;\,f\right)_{L^2} = \int_\Omega f(x)\int_\Gamma \gamma^{\rm int}_{1,y}G(x-y)v(y)\,ds_y\,dx &= \int_\Gamma v(y)\,n(y)\cdot\int_\Omega \nabla_y G(x-y)\,f(x)\,dx\,ds_y\\ &= \int_\Gamma v(y)\,n(y)\cdot\nabla\int_\Omega G(x-y)\,f(x)\,dx\,ds_y\\ &= \langle v\,;\,N_1 f\rangle\\ &\lesssim \|v\|_{H^{1/2}(\Gamma)}\|f\|_{\widetilde{H}^{-1}(\Omega)}.\end{aligned}$$

From the Hahn-Banach theorem, we now derive $\|\widetilde{K}v\|_{H^1(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)}$, which proves (ii), from which (iv) immediately follows.

4. step. Note that $-\Delta(\widetilde{K}v) = 0$ in $\mathbb{R}^d\backslash\Gamma$ and hence in $\widetilde{H}^{-1}(\Omega)$. In particular $\gamma^{\rm int}_1\widetilde{K}v \in H^{-1/2}(\Gamma)$ is well-defined.

5. step. There holds $\gamma^{\rm int}_1\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$: For $v \in H^{1/2}(\Gamma)$ holds

$$\|\gamma^{\rm int}_1\widetilde{K}v\|_{H^{-1/2}(\Gamma)} \lesssim \|\widetilde{K}v\|_{H^1(\Omega)} \lesssim \|v\|_{H^{1/2}(\Gamma)},$$

since $-\Delta(\widetilde{K}v) = 0 \in \widetilde{H}^{-1}(\Omega)$. This concludes the proof. ■

2.2 Representation Formula and Calderón Projector

So far, we have introduced the boundary integral operators arising from the Newton potential

$$N_0 := \gamma^{\rm int}_0\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{1/2}(\Gamma)\big) \quad \text{and} \quad N_1 := \gamma^{\rm int}_1\widetilde{N} \in L\big(\widetilde{H}^{-1}(\Omega); H^{-1/2}(\Gamma)\big). \tag{2.10}$$

Furthermore, we now define the **single-layer operator**

$$V := \gamma^{\rm int}_0\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big) \tag{2.11}$$

and the **hypersingular operator**

$$W := -\gamma^{\rm int}_1\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{-1/2}(\Gamma)\big) \tag{2.12}$$

as well as the **double-layer operator**

$$K := \frac{1}{2} + \gamma^{\rm int}_0\widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big) \tag{2.13}$$

and the **adjoint double-layer operator**

$$K' := -\frac{1}{2} + \gamma_1^{\mathrm{int}} \widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big). \tag{2.14}$$

The notation for the operators K and K' stems from the fact that there holds

$$\langle Kv\,;\,\phi\rangle = \langle v\,;\,K'\phi\rangle \quad \text{for all } v \in H^{1/2}(\Gamma) \text{ and } \phi \in H^{-1/2}(\Gamma), \tag{2.15}$$

i.e. K' is the adjoint operator for K in the functional analytic sense. However, we postpone the proof of (2.15) to a later section and take a second glance on the representation formula:

Theorem 2.8 (Representation Formula). *For $u \in H^1(\Omega)$ with $-\Delta u = f \in \widetilde{H}^{-1}(\Omega)$, there holds*

$$u = \widetilde{N} f + \widetilde{V}(\gamma_1^{\mathrm{int}} u) - \widetilde{K}(\gamma_0^{\mathrm{int}} u). \tag{2.16}$$

Proof. **1. step.** We first prove (2.16) for $-\Delta u = f \in L^2(\Omega)$: So far, we have proven (2.16) only pointwise in Ω for $u \in C^2(\overline{\Omega})$,

$$u(x) = -\int_\Omega G(x-y)\Delta u(y)\,dy + \int_\Gamma G(x-y)\frac{\partial u}{\partial n_y}(y)\,ds_y - \int_\Gamma \frac{\partial_y}{\partial n_y} G(x-y)\,u(y)\,ds_y,$$

which is the pointwise statement of (2.16), cf. Proposition 1.6.
The main ideas of the proof are similar to the proof of Proposition 1.6, but the estimates using the C^1-norm and Taylor expansion are replaced by application of the Lebesgue differentiation theorem, Theorem A.6.
For a fix $x \in \Omega$, we cut-off the singularity for $y = x$ and consider the second Green's formula on $\Omega_\varepsilon := \Omega \backslash B_\varepsilon(x)$ to obtain

$$(-\Delta u\,;\,v)_{\Omega_\varepsilon} + (\partial u/\partial n\,;\,v)_\Gamma - (u\,;\,\partial v/\partial n)_\Gamma = -\,(\partial u/\partial n\,;\,v)_{\partial B_\varepsilon(x)} + (u\,;\,\partial v/\partial n)_{\partial B_\varepsilon(x)}$$

as in the proof of Proposition 1.6. It remains to analyze the convergence of the terms for $\varepsilon \to 0$.

- There holds $(-\Delta u\,;\,v)_{\Omega_\varepsilon} \xrightarrow{\varepsilon\to 0} (-\Delta u\,;\,v)_\Omega$ which follows obviously from the Lebesgue dominated convergence theorem as $-\Delta u = f \in L^2(\Omega)$ and $v \in L^2(\Omega)$.
- There holds $(\partial u/\partial n\,;\,v)_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon\to 0} 0$: The Gauss divergence theorem gives

  $$\int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial n}\,ds = -\int_{B_\varepsilon(x)} \Delta u\,dy = -|B_\varepsilon(x)| \fint_{B_\varepsilon(x)} \Delta u\,dy,$$

  where here and in the following $\fint_B \cdot\,dy := |B|^{-1}\int_B \cdot\,dy$ denotes the integral mean. There holds $|B_\varepsilon(x)| = |B_2^d|\varepsilon^d$, and the integral mean $\fint_{B_\varepsilon(x)} \Delta u\,dy$ converges to $\Delta u(x)$ as $\varepsilon \to 0$ almost everywhere due to the Lebesgue differentiation theorem. Moreover, for $y \in \partial B_\varepsilon(x)$, we have $v(y) = G(x-y) = \frac{1}{|S_2^d|}\begin{cases} -\log\varepsilon & \text{for } d=2,\\ 1/\varepsilon & \text{for } d=3,\end{cases}$ which leads to

  $$(\partial u/\partial n\,;\,v)_{\partial B_\varepsilon(x)} = -\frac{|B_2^d|}{|S_2^d|}\fint_{B_\varepsilon(x)} \Delta u\,dy \cdot \begin{cases} \varepsilon^2|\log\varepsilon| & \text{for } d=2,\\ \varepsilon^2 & \text{for } d=3,\end{cases}$$

  vanishing for $\varepsilon \to 0$.

- There holds $(u\,;\partial v/\partial n)_{\partial B_\varepsilon(x)} \xrightarrow{\varepsilon\to 0} u(x)$: We plug-in the formula for ∇G and use the Gauss divergence theorem to obtain

$$\begin{aligned}\int_{\partial B_\varepsilon(x)} u(y)\frac{\partial_y}{\partial n(y)}G(x-y)\,ds_y &= -\frac{1}{|S_2^d|\varepsilon^d}\int_{\partial B_\varepsilon(x)} u(y)(y-x)\cdot n(y)\,ds_y \\ &= \frac{1}{|S_2^d|\varepsilon^d}\int_{B_\varepsilon(x)} \nabla u(y)\cdot(y-x)+du(y)\,dy.\end{aligned}$$

With $|B_\varepsilon(x)| = |B_2^d|\varepsilon^d$, there holds

$$\frac{1}{|S_2^d|\varepsilon^d}\Big|\int_{B_\varepsilon(x)} \nabla u(y)\cdot(y-x)\,dy\Big| \le \frac{|B_2^d|\varepsilon}{|S_2^d|}\fint_{B_\varepsilon(x)} |\nabla u|\,dy \xrightarrow{\varepsilon\to 0} 0,$$

since the integral mean converges to $|\nabla u(x)|$ by the Lebesgue differentiation theorem. For the remaining term, we obtain

$$\frac{d}{|S_2^d|\varepsilon^d}\int_{B_\varepsilon(x)} u\,dy = \underbrace{\frac{d|B_2^d|}{|S_2^d|}}_{=1}\fint_{B_\varepsilon(x)} u\,dy \xrightarrow{\varepsilon\to 0} u(x),$$

which finishes the proof for $f\in L^2(\Omega)$.

2. step. We prove (2.16) for $-\Delta u = f \in \widetilde{H}^{-1}(\Omega)$: As $L^2(\Omega)$ is a dense subspace of $\widetilde{H}^{-1}(\Omega)$, we choose a sequence (f_n) in $L^2(\Omega)$ which converges to f in $\widetilde{H}^{-1}(\Omega)$. Then, let $u_n\in H^1(\Omega)$ be the unique weak solution of the Dirichlet problem

$$-\Delta u_n = f_n \text{ in } \Omega \quad\text{and}\quad \gamma_0^{\text{int}}u_n = \gamma_0^{\text{int}}u \text{ on } \Gamma.$$

According to Theorem 1.29, there holds $\|u-u_n\|_{H^1(\Omega)} \lesssim \|f-f_n\|_{\widetilde{H}^{-1}(\Omega)}$, whence (u_n) converges to u in $H^1(\Omega)$. Moreover, Theorem 1.30 states $\|\gamma_1^{\text{int}}u-\gamma_1^{\text{int}}u_n\|_{H^{-1/2}(\Gamma)} \lesssim \|f-f_n\|_{\widetilde{H}^{-1}(\Omega)}+\|u-u_n\|_{H^1(\Omega)}$, whence $(\gamma_1^{\text{int}}u_n)$ converges to $\gamma_1^{\text{int}}u$ in $H^{-1/2}(\Gamma)$. As (2.16) is already proven for u_n, the continuity of the involved operators concludes the proof for the limit case $n\to\infty$. ∎

Corollary 2.9. *We define the* **Calderón projector** *as operator matrix*

$$C := \begin{pmatrix} \frac12 - K & V \\ W & \frac12 + K' \end{pmatrix} : H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma). \tag{2.17}$$

If $u\in H^1(\Omega)$ satisfies $-\Delta u = f\in\widetilde{H}^{-1}(\Omega)$, then the **Cauchy data** $(\gamma_0^{\text{int}}u, \gamma_1^{\text{int}}u)$ *satisfy*

$$\begin{pmatrix}\gamma_0^{\text{int}}u\\ \gamma_1^{\text{int}}u\end{pmatrix} = C\begin{pmatrix}\gamma_0^{\text{int}}u\\ \gamma_1^{\text{int}}u\end{pmatrix} + \begin{pmatrix}N_0 f\\ N_1 f\end{pmatrix}, \tag{2.18}$$

i.e. the Cauchy data solve the so-called **Calderón system**. *Moreover, the Calderón projector has the projector property $C^2 = C$.*

Proof. We start from the representation formula (2.16) and consider the trace and the conormal derivative. From $V = \gamma_0^{\rm int}\widetilde{V}$ and $-\frac{1}{2} + K = \gamma_0^{\rm int}\widetilde{K}$, we obtain

$$\gamma_0^{\rm int}u = \gamma_0^{\rm int}\widetilde{N}f + \gamma_0^{\rm int}\widetilde{V}\gamma_1^{\rm int}u - \gamma_0^{\rm int}\widetilde{K}\gamma_0^{\rm int}u = N_0 f + V\gamma_1^{\rm int}u + \Big(\frac{1}{2} - K\Big)\gamma_0^{\rm int}u.$$

From $\frac{1}{2} + K' = \gamma_1^{\rm int}\widetilde{V}$ and $W = -\gamma_1^{\rm int}\widetilde{K}$, we obtain

$$\gamma_1^{\rm int}u = \gamma_1^{\rm int}\widetilde{N}f + \gamma_1^{\rm int}\widetilde{V}\gamma_1^{\rm int}u - \gamma_1^{\rm int}\widetilde{K}\gamma_0^{\rm int}u = N_1 f + \Big(\frac{1}{2} + K'\Big)\gamma_1^{\rm int}u + W\gamma_0^{\rm int}u.$$

Writing the latter equations in a 2×2-system, we prove (2.18). To prove $C^2 = C$, let $(v, \phi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and define $u := \widetilde{V}\phi - \widetilde{K}v \in H^1(\Omega)$. Note that $-\Delta u = 0$ so that the Calderón system simplifies to

$$\begin{pmatrix} \gamma_0^{\rm int}u \\ \gamma_1^{\rm int}u \end{pmatrix} = C \begin{pmatrix} \gamma_0^{\rm int}u \\ \gamma_1^{\rm int}u \end{pmatrix}.$$

From $\gamma_0^{\rm int}u = V\phi + (\frac{1}{2} - K)v$ and $\gamma_1^{\rm int}u = (\frac{1}{2} + K')\phi + Wv$ we derive

$$\begin{pmatrix} \gamma_0^{\rm int}u \\ \gamma_1^{\rm int}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} = C \begin{pmatrix} v \\ \phi \end{pmatrix}.$$

Plugging this into the Calderón system, we obtain

$$C \begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} \gamma_0^{\rm int}u \\ \gamma_1^{\rm int}u \end{pmatrix} = C \begin{pmatrix} \gamma_0^{\rm int}u \\ \gamma_1^{\rm int}u \end{pmatrix} = C^2 \begin{pmatrix} v \\ \phi \end{pmatrix}.$$

As (v, ϕ) is arbitrary, this proves $C^2 = C$. ■

Using the projector property $C^2 = C$, elementary calculations prove the following additional relations of the boundary integral operators V, W, K and K':

Corollary 2.10. *There hold the following equations:*

$$\begin{aligned} VW &= \big(\tfrac{1}{2} + K\big)\big(\tfrac{1}{2} - K\big), & VK' &= KV, \\ K'W &= WK, & WV &= \big(\tfrac{1}{2} + K'\big)\big(\tfrac{1}{2} - K'\big). \end{aligned} \tag{2.19}$$

Proof. From the projector property $C = C^2$, we obtain

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \\ &= \begin{pmatrix} (\frac{1}{2} - K)^2 + VW & (\frac{1}{2} - K)V + V(\frac{1}{2} + K') \\ W(\frac{1}{2} - K) + (\frac{1}{2} + K')W & WV + (\frac{1}{2} + K')^2 \end{pmatrix}. \end{aligned}$$

This yields (2.19) according to some elementary calculations. ■

2.3 Integral Formulation of Dirichlet Problem

We consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= f \in \widetilde{H}^{-1}(\Omega), \\ \gamma_0^{\text{int}} u &= v \in H^{1/2}(\Gamma). \end{aligned} \tag{2.20}$$

According to Section 1.5, there is a unique weak solution $u \in H^1(\Omega)$. With the (unknown) conormal derivative $\phi = \gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$, the Calderón system states

$$\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}$$

according to Corollary 2.9. The first equation can be read in the form

$$V\phi = \Big(\frac{1}{2} + K\Big) v - N_0 f, \tag{2.21}$$

which is known as **Symm's integral equation**. The following theorem states that the Dirichlet problem (2.20) and Symm's integral equation (2.21) are equivalent formulations of the same problem.

Theorem 2.11. (i) *If $u \in H^1(\Omega)$ solves* (2.20)*, the conormal derivative $\phi := \gamma_1^{\text{int}} u \in H^{-1/2}(\Gamma)$ solves Symm's integral equation* (2.21)*.*
(ii) *Conversely, if $\phi \in H^{-1/2}(\Gamma)$ solves* (2.21)*, the function $u := \widetilde{N} f + \widetilde{V}\phi - \widetilde{K} v \in H^1(\Omega)$ solves the Dirichlet problem* (2.20)*.*

Proof. (i) has just been proven to derive Symm's integral equation. (ii) According to the mapping properties of the three potential operators, there holds $-\Delta u = f$ as well as

$$\gamma_0^{\text{int}} u = N_0 f + V\phi + \Big(\frac{1}{2} - K\Big) v = \Big(\frac{1}{2} + K\Big) v + \Big(\frac{1}{2} - K\Big) v = v,$$

where we have plugged-in $V\phi$ from (2.21). ■

Note that the theorem does not state the unique solvability of Symm's integral equation. We still have to show that $\phi = \gamma_1^{\text{int}} u$ is in fact the unique solution, which will be postponed to a later subsection.

2.4 Integral Formulation of Neumann Problem

We consider the Neumann problem

$$\begin{aligned} -\Delta u &= f \in \widetilde{H}^{-1}(\Omega), \\ \gamma_1^{\text{int}} u &= \phi \in H^{-1/2}(\Gamma). \end{aligned} \tag{2.22}$$

The weak form, i.e. the first Green's formula, then reads

$$\langle f \,;\, v \rangle = (\nabla u \,;\, \nabla v)_\Omega - \langle \phi \,;\, \gamma_0^{\text{int}} v \rangle \quad \text{for all } v \in H^1(\Omega). \tag{2.23}$$

The choice $v \equiv 1$ therefore shows that

$$\langle f\,;1\rangle + \langle \phi\,;1\rangle = 0 \tag{2.24}$$

is a necessary condition to allow weak solutions $u \in H^1(\Omega)$ of (2.22). Moreover, solutions can only be unique up to additive constants: According to Section 1.6, there is a unique solution $u \in H^1_*(\Omega) = \{u \in H^1(\Omega) \,|\, (u\,;1)_\Omega = 0\}$. With the (unknown) trace $v := \gamma_0^{\rm int} u \in H^{1/2}(\Gamma)$, the Calderón system states

$$\begin{pmatrix} v \\ \phi \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - K & V \\ W & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}.$$

The second equation can be read in the form

$$Wv = \Big(\frac{1}{2} - K'\Big)\phi - N_1 f \tag{2.25}$$

and is then called **hypersingular integral equation**. The following theorem states that the Neumann problem (2.22) and the hypersingular integral equation (2.25) are equivalent.

Theorem 2.12. (i) *If $u \in H^1(\Omega)$ solves* (2.22)*, the trace $v := \gamma_0^{\rm int} u \in H^{1/2}(\Gamma)$ solves the hypersingular integral equation* (2.25)*.*
(ii) *Conversely, if $v \in H^{1/2}(\Gamma)$ solves* (2.25)*, the function $u := \widetilde{N} f + \widetilde{V}\phi - \widetilde{K} v \in H^1(\Omega)$ solves the Neumann problem* (2.22)*.*

Proof. (i) has just been proven to derive the hypersingular integral equation. (ii) According to the mapping properties of the three potential operators, there holds $-\Delta u = f$ as well as

$$\gamma_1^{\rm int} u = N_1 f + \Big(\frac{1}{2} + K'\Big)\phi + Wv = \Big(\frac{1}{2} + K'\Big)\phi + \Big(\frac{1}{2} - K'\Big)\phi = \phi,$$

where we have plugged-in Wv from (2.25). ∎

Note that the theorem again does not state the unique solvability of the hypersingular integral equation. Moreover, neither the necessary assumption (2.24) nor the assumption $\int_\Omega u\,dx = 0$ have been used in the proof of Theorem 2.12. In particular, we may expect that $u \in H^1(\Omega)$ from (ii) won't satisfy $\int_\Omega u\,dx = 0$ in general.

The subsequent proposition states some elementary mapping properties of the hypersingular integral operator. For the statement, we recall the definition of the spaces

$$H^{1/2}_*(\Gamma) = \{v \in H^{1/2}(\Gamma) \,|\, \langle 1\,;v\rangle = 0\} \quad \text{and} \quad H^{-1/2}_*(\Gamma) = \{\psi \in H^{-1/2}(\Gamma) \,|\, \langle \psi\,;1\rangle = 0\}. \tag{2.26}$$

We first note that $H^{-1/2}_*(\Gamma)$ is the dual space of $H^{1/2}_*(\Gamma)$.

Lemma 2.13. *$H^{-1/2}_*(\Gamma)$ is the dual space of $H^{1/2}_*(\Gamma)$ with respect to the extended L^2-scalar product.*

Proof. Clearly, each $\psi \in H_*^{-1/2}(\Gamma)$ belongs to the dual space of $H_*^{1/2}(\Gamma)$. Conversely, one has to show that, given $\Psi \in H^{-1/2}(\Gamma)$, there is a $\psi \in H_*^{-1/2}(\Gamma)$ such that $\langle \psi \,;\, v\rangle = \langle \Psi \,;\, v\rangle$ for all $v \in H_*^{1/2}(\Gamma)$. Obviously,

$$\psi := \Psi - \tfrac{\langle \Psi \,;\, 1\rangle}{\langle 1 \,;\, 1\rangle}$$

does this job. ∎

Proposition 2.14. (i) *For any $v \in H^{1/2}(\Gamma)$ holds $Wv \in H_*^{1/2}(\Gamma)$.*
(ii) *For a constant function $c \in H^{1/2}(\Gamma)$ holds $Wc = 0$.*
(iii) *Provided $f \in \widetilde{H}^{-1}(\Omega)$, $\phi \in H^{-1/2}(\Gamma)$ satisfy (2.24), there holds $\big(\frac{1}{2} - K'\big)\phi - N_1 f \in H_*^{-1/2}(\Gamma)$.*

Proof. (i) The first Green's formula reads $(-\nabla u \,;\, w) = (\nabla u \,;\, \nabla w) - \langle \gamma_1^{\rm int} u \,;\, \gamma_0^{\rm int} w\rangle$. If we plug-in $u = \widetilde{K}v$ and $w \equiv 1$, we obtain $0 = -\langle \gamma_1^{\rm int}\widetilde{K}v \,;\, 1\rangle = \langle Wv \,;\, 1\rangle$.

(ii) We consider the constant function c on Ω. Then, the representation formula shows $c = -\widetilde{K}c$ almost everywhere in Ω, whence $Wc = -\gamma_1^{\rm int}\widetilde{K}c = 0$.

(iii) Using $u = \widetilde{V}\phi$ and $w \equiv 1$ in the first Green's formula, we see that $\langle (\frac{1}{2} + K')\phi \,;\, 1\rangle = \langle \gamma_1^{\rm int}\widetilde{V}\phi \,;\, 1\rangle = 0$. Moreover, there holds $\langle N_1 f \,;\, 1\rangle = \langle \gamma_1^{\rm int}\widetilde{N}f \,;\, 1\rangle = -\langle f \,;\, 1\rangle$ as $-\Delta\widetilde{N}f = f$. Therefore,

$$\langle \big(\tfrac{1}{2} - K'\big)\phi - N_1 f \,;\, 1\rangle = \langle \phi \,;\, 1\rangle - \langle \big(\tfrac{1}{2} + K'\big)\phi \,;\, 1\rangle + \langle f \,;\, 1\rangle = 0$$

which concludes the proof. ∎

From (ii), we see that W cannot be elliptic on the entire space $H^{1/2}(\Gamma)$. However, factoring the constant functions out, we obtain unique solvability on $H_*^{1/2}(\Gamma)$, which is also the subject of a later subsection.

2.5 Boundary Integral Operators

The main goal in this section is to derive integral representations of the introduced boundary integral operators (2.10)–(2.14). Of course, this is a fundamental point for a numerical method since one needs to know what has to be implemented.

The discretization is usually based on piecewise polynomials. Therefore, our discrete functions always belong to $L^\infty(\Omega)$ and $L^\infty(\Gamma)$, respectively. The first theorem shows that, for $f \in L^\infty(\Omega)$, the Newton potential belongs to $C^1(\mathbb{R}^d)$ with

$$N_0 f(x) := \gamma_0^{\rm int}\widetilde{N}f(x) = \int_\Omega G(x-y)f(y)\,dy$$
$$N_1 f(x) := \gamma_1^{\rm int}\widetilde{N}f(x) = \int_\Omega \frac{\partial_x}{\partial n_x}G(x-y)f(y)\,dy$$

for $x \in \Gamma$.

The proof of Theorem 2.15 uses the elementary properties of the convolution that $g * f \in C(\mathbb{R}^d)$ provided $f \in L^\infty(\Omega)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, which is stated in Lemma A.9.

Theorem 2.15. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and $f \in L^\infty(\Omega)$. Then, $\widetilde{N} f := G * f \in C^1(\mathbb{R}^d)$ with $\partial_j(\widetilde{N} f) = \partial_j G * f$.*

Proof. We define $w := \widetilde{N} f = G * f$ and $v_j := \partial_j G * f$. According to Lemma A.9, there holds $w, v_j \in C(\mathbb{R}^d)$. Let $\eta \in C^1(\mathbb{R})$ be a cut-off function which satisfies

$$0 \leq \eta \leq 1, \quad \eta|_{\{|x| \leq 1\}} = 0, \quad \text{and} \quad \eta|_{\{|x| \geq 2\}} = 1.$$

For $\varepsilon > 0$, we define $G_\varepsilon \in C^1(\mathbb{R}^d)$ by $G_\varepsilon(x) := G(x)\eta(|x|/\varepsilon)$ for $x \in \mathbb{R}^d$. According to Lemma A.9, there holds $w_\varepsilon := G_\varepsilon * f \in C^1(\mathbb{R}^d)$. Note that $|G_\varepsilon| \leq |G|$ and $G(y) - G_\varepsilon(y) = G(y)\big(1 - \eta(|y|/\varepsilon)\big) = 0$ for $|y| \geq 2\varepsilon$. Moreover, elementary calculations with polar coordinates prove

$$\int_{B_\varepsilon(0)} |G(z)|\, dz = \mathcal{O}(\varepsilon^2 \log \varepsilon) \quad \text{and} \quad \int_{B_\varepsilon(0)} |\partial_j G(z)|\, dz = \mathcal{O}(\varepsilon).$$

Therefore,

$$\begin{aligned} |w(x) - w_\varepsilon(x)| = |(G - G_\varepsilon) * f(x)| &\leq \|f\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d} |G(x-y) - G_\varepsilon(x-y)|\, dy \\ &= \|f\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d} |G(z) - G_\varepsilon(z)|\, dz \\ &= \|f\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d} \big(1 - \eta(z/|\varepsilon|)\big)\, |G(z)|\, dz \\ &\leq \|f\|_{L^\infty(\Omega)} \int_{B_{2\varepsilon}(0)} |G(z)|\, dz \xrightarrow{\varepsilon \to 0} 0, \end{aligned}$$

i.e. we have uniform convergence $w_\varepsilon \to w \in C(\mathbb{R}^d)$ as $\varepsilon \to 0$. We now show uniform convergence $\partial_j w_\varepsilon \to v_j$ as $\varepsilon \to 0$. With the same techniques as before, we are led to

$$\begin{aligned} |v_j(x) - \partial_j w_\varepsilon| &\leq \|f\|_{L^\infty(\Omega)} \int_{B_{2\varepsilon}(0)} |\partial_j G(z) - \partial_j G_\varepsilon(z)|\, dz \\ &\leq \|f\|_{L^\infty(\Omega)} \int_{B_{2\varepsilon}(0)} \big|\partial_j \big\{\big((1 - \eta(|z|/\varepsilon)\big)\, G(z)\big\}\big|\, dz \\ &\leq \|f\|_{L^\infty(\Omega)} \Big(\int_{B_{2\varepsilon}(0)} \underbrace{|\,\partial_j \eta(|z|/\varepsilon)|}_{\leq \|\eta'\|_{L^\infty}/\varepsilon} |G(z)|\, dz + \int_{B_{2\varepsilon}(0)} \underbrace{|1 - \eta(|y|/\varepsilon)|}_{\leq 1} |\partial_j G(z)|\, dz \Big) \\ &\xrightarrow{\varepsilon \to 0} 0. \end{aligned}$$

Altogether, we have uniform convergence $w_\varepsilon \to w$ and $\partial_j w_\varepsilon \to v_j$, and it only remains to prove $\partial_j w = v$. However, this is a 1D problem as we are considering only the derivative in the x_j-direction. In 1D, the fundamental theorem of calculus states

$$w_\varepsilon(x) - w_\varepsilon(y) = \int_y^x w_\varepsilon'\, dt.$$

According to the uniform convergence, the left-hand side converges to $w(x) - w(y)$ as $\varepsilon \to 0$, whereas the right-hand side converges to $\int_y^x v_j \, dt$. Thus, the fundamental theorem of calculus states $w \in C^1(\mathbb{R})$ with derivative $w' = v_j$. ■

Next, we consider the single-layer potential. The following theorem proves that

$$V\phi(x) := \gamma_0^{\mathrm{int}} \widetilde{V}\phi(x) = \int_\Gamma G(x-y)\phi(y)\, ds_y \quad \text{for } x \in \Gamma.$$

Theorem 2.16. *For $\phi \in L^\infty(\Gamma)$, there holds $\widetilde{V}\phi \in C(\mathbb{R}^d)$. Moreover, $\widetilde{V} \in L\big(L^\infty(\Gamma); L^\infty(\Gamma)\big)$.*

The technical difficulties of the proof of Theorem 2.16 are stated in the following lemma.

Lemma 2.17. *Given $\varepsilon > 0$, let $g_\varepsilon(z) := |z|^{-(d-2+\varepsilon)}$. Then, for any $\phi \in L^\infty(\Gamma)$, the function $\Phi(x) := \int_\Gamma g_\varepsilon(x-y)\phi(y)\, ds_y$ is globally continuous on $\mathbb{R}^d$. In particular, there hold $M := \sup_{x\in\Gamma} \|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma)} < \infty$ and $\|\Phi\|_{L^\infty(\Gamma)} \le M\, \|\phi\|_{L^\infty(\Gamma)}$.*

Proof. **1. step.** Let us assume that we had already proven continuity of Φ for any $\phi \in L^\infty(\Gamma)$. Choose the constant function $\phi \equiv 1$ and observe that $\|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma)} = \Phi(x)$. Since Φ attains its maximum on the compact set Γ, we infer $M < \infty$. Now, for arbitrary $\phi \in L^\infty(\Gamma)$, the Hölder inequality proves $|\Phi(x)| \le M\, \|\phi\|_{L^\infty(\Gamma)}$ for any $x \in \Gamma$.

2. step. We now go two steps back and prove that, for fixed $x \in \Gamma$, there holds $g_\varepsilon(x-\cdot) \in L^1(\Gamma)$: Let U_x be an open neighborhood of x and let $\chi : B_1(0) \to U_x$ be the bi-Lipschitz mapping according to the definition of a Lipschitz boundary. Without loss of generality, we may assume $\widehat{x} := \chi^{-1}(x) = 0$. [Otherwise, we choose $\delta > 0$ with $B_\delta(\widehat{x}) \subset B_1(0)$ and consider the restriction $\chi : B_\delta(\widehat{x}) \to \chi(B_\delta(\widehat{x})) =: U_x$. Note that $B_\delta(\widehat{x})$ can be mapped onto $B_1(0)$ with a (bi-Lipschitz) affine transformation.] Then,

$$\|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma)} = \|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma\setminus U_x)} + \|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma\cap U_x)},$$

where the first contribution is finite since $g_\varepsilon(x-\cdot)$ is smooth on $\Gamma\setminus U_x$. According to the bi-Lipschitz property of χ, there holds

$$|\widehat{y} - \widehat{z}| \lesssim |\chi(\widehat{y}) - \chi(\widehat{z})| \lesssim |\widehat{y} - \widehat{z}| \quad \text{for all } \widehat{y}, \widehat{z} \in B_1(0).$$

Therefore, with $\lambda \in L^\infty(B_1(0))$ the surface element[1], we obtain

$$\begin{aligned}\|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma\cap U_x)} = \int_{B_1^0} \big|g_\varepsilon\big(x - \chi(\widehat{y},0)\big)\big| \lambda(\widehat{y})\, d\widehat{y} &= \int_{B_1^0} \frac{1}{|\chi(0) - \chi(\widehat{y},0)|^{d-2+\varepsilon}} \lambda(\widehat{y})\, d\widehat{y} \\ &\lesssim \int_{B_1^0} \frac{1}{|(\widehat{y},0)|^{d-2+\varepsilon}}\, d\widehat{y} < \infty\end{aligned}$$

by use of polar coordinates.

[1]The surface element reads $\lambda(\widehat{y}) = \big[\det\big(J_\chi(\widehat{y},0)^T J_\chi(\widehat{y},0)\big)\big]^{1/2}$ with the Jacobian $J_\chi(\widehat{y},0) \in \mathbb{R}^{3\times 2}$.

3. step. It remains to prove the continuity of Φ on $\mathbb{R}^d$. With the general observations from Lemma 2.5, Φ is continuous in $\mathbb{R}^d\backslash\Gamma$, and it remains to prove continuity on Γ: Fix $x \in \Gamma$ and adopt the notation of step 2. Note that there holds

$$\lim_{\widetilde{x}\to x} \int_{\Gamma\backslash U_x} g_\varepsilon(\widetilde{x}-y)\phi(y)\,ds_y = \int_{\Gamma\backslash U_x} g_\varepsilon(x-y)\phi(y)\,ds_y,$$

since both sides have smooth integrands. It therefore remains to consider the integral over $\Gamma\cap U_x$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ with $x_n \to x$. We define $\widehat{x}_n := \chi^{-1}(x_n)$ and assume, without loss of generality, $\widehat{x}_n \in B_1(0)$. We write $\widehat{x}_n = (\widehat{y}_n, \alpha_n) \in B_1^0 \times \mathbb{R}$. Note that $\widehat{x}_n \to \widehat{x} = 0$ and thus $\widehat{y}_n \to 0$. There holds

$$\begin{aligned}
\int_{\Gamma\cap U_x} g_\varepsilon(x_n-y)\phi(y)\,ds_y &= \int_{B_1^0} g_\varepsilon\big(x_n-\chi(\widehat{y},0)\big)\phi(\widehat{y},0)\lambda(\widehat{y})\,d\widehat{y}\\
&= \int_{\mathbb{R}^{d-1}} \chi_{B_1^0}(\widehat{y}) g_\varepsilon\big(x_n-\chi(\widehat{y},0)\big)\phi(\widehat{y},0)\lambda(\widehat{y})\,d\widehat{y}\\
&= \int_{\mathbb{R}^{d-1}} \underbrace{\chi_{B_1^0}(\widehat{y}+\widehat{y}_n) g_\varepsilon\big(x_n-\chi(\widehat{y}+\widehat{y}_n,0)\big)\phi(\widehat{y}+\widehat{y}_n,0)\lambda(\widehat{y}+\widehat{y}_n)}_{=:f_n(\widehat{y})}\,d\widehat{y}.
\end{aligned}$$

Note that $f_n \in L^1(\mathbb{R}^{d-1})$ and that f_n converges pointwise to $f(\widehat{y}) := \chi_{B_1^0}(\widehat{y})g_\varepsilon(x-\chi(\widehat{y},0))\phi(\widehat{y},0)\lambda(\widehat{y})$. To establish $f_n \to f \in L^1(\mathbb{R}^{d-1})$, it remains to prove that $|f_n|$ is pointwise bounded. With $\widehat{x}_n = (\widehat{y}_n,\alpha_n) \in B_1^0$ and $x_n = \chi(\widehat{x}_n) = \chi(\widehat{y}_n,\alpha_n)$ follows

$$|f_n(\widehat{y})| \lesssim \frac{\chi_{B_1^0}(\widehat{y}+\widehat{y}_n)}{|\widehat{x}_n-(\widehat{y}+\widehat{y}_n,0)|^{d-2+\varepsilon}} = \frac{\chi_{B_1^0}(\widehat{y}+\widehat{y}_n)}{|(\widehat{y},\alpha_n)|^{d-2+\varepsilon}} \le \frac{\chi_{B_2^0}(\widehat{y})}{|\widehat{y}|^{d-2+\varepsilon}} =: g(\widehat{y}),$$

where we have used $|\widehat{y}_n| < 1$ as well as $\phi, \lambda \in L^\infty(\Gamma)$. As $g \in L^1(\mathbb{R}^{d-1})$, the Lebesgue dominated convergence theorem yields

$$\int_{\Gamma\cap U_x} g_\varepsilon(x_n-y)\phi(y)\,ds_y = \int_{\mathbb{R}^{d-1}} f_n(\widehat{y})\,d\widehat{y} \xrightarrow{n\to\infty} \int_{\mathbb{R}^{d-1}} f(\widehat{y})\,d\widehat{y} = \int_{\Gamma\cap U_x} g_\varepsilon(x-y)\phi(y)\,ds_y.$$

Altogether, we have proven

$$\lim_{\widetilde{x}\to x} \int_\Gamma g_\varepsilon(\widetilde{x}-y)\phi(y)\,ds_y = \int_\Gamma g_\varepsilon(x-y)\phi(y)\,ds_y,$$

which is just the continuity of Φ. ■

Proof of Theorem 2.16. **1. step.** There holds $G(x-\cdot) \in L^1(\Gamma)$ for all $x \in \Gamma$ and $M := \sup_{x\in\Gamma} \|G(x-\cdot)\|_{L^1(\Gamma)} < \infty$: For $d = 3$, there holds $|G(z)| \lesssim g_0(z)$ and we may choose $\varepsilon = 0$. For $d = 2$, we fix an arbitrary $\varepsilon \in (0,1)$. As the function $t^\varepsilon \log t$ is globally continuous on $\mathbb{R}_{\ge 0}$, there holds $|G(x-y)| \lesssim g_\varepsilon(x-y)$ for all $x, y \in \Gamma$. For both cases, we hence obtain $\|G(x-\cdot)\|_{L^1(\Gamma)} \lesssim \|g_\varepsilon(x-\cdot)\|_{L^1(\Gamma)}$, and thus an upper bound of M by use of Lemma 2.17.

2. step. There holds $\widetilde{V} \in L\big(L^\infty(\Gamma); L^\infty(\Gamma)\big)$: For $x \in \Gamma$, a Hölder inequality proves

$$|\widetilde{V}\phi(x)| = \Big|\int_\Gamma G(x-y)\phi(y)\,ds_y\Big| \le \|\phi\|_{L^\infty(\Gamma)}\|G(x-\cdot)\|_{L^1(\Gamma)} \le M\,\|\phi\|_{L^\infty(\Gamma)}.$$

Thus, $\|\widetilde{V}\phi\|_{L^\infty(\Gamma)} \le M\,\|\phi\|_{L^\infty(\Gamma)}$ which is our claim.

3. step. For $\phi \in L^\infty(\Gamma)$, there holds $\widetilde{V}\phi \in C(\mathbb{R}^d)$: For $d = 3$, the proof follows from Lemma 2.17 and $G(z) = \frac{1}{4\pi}\,g_0(z)$. For $d = 2$, the reader may imitate the proof of the preceding lemma — now for a logarithmic instead of an algebraic singularity of the kernel function. This is left as an exercise. ■

Next, we stress some immediate consequence of Theorem 2.16, namely the symmetry of V on $H^{-1/2}(\Gamma)$. In Section 2.7 we shall see that V even induces an equivalent scalar product on $H^{-1/2}(\Gamma)$.

Corollary 2.18. *The single-layer potential $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is a symmetric operator, i.e. $\langle V\phi\,;\,\psi\rangle = \langle \phi\,;\,V\psi\rangle$ for all $\phi, \psi \in H^{-1/2}(\Gamma)$.*

Proof. Note that $L^\infty(\Gamma)$ is a dense subspace of $H^{-1/2}(\Gamma)$. We have to show that the bilinear form

$$\langle\!\langle \phi\,;\,\psi \rangle\!\rangle := \langle V\phi\,;\,\psi\rangle,$$

is symmetric. Due to continuity, it thus suffices to consider $\phi, \psi \in L^\infty(\Gamma)$. We apply the Fubini theorem to verify

$$\langle\!\langle \phi\,;\,\psi \rangle\!\rangle = \int_\Gamma \psi(x) \int_\Gamma G(x-y)\phi(y)\,ds_y\,ds_x = \int_\Gamma \phi(y) \int_\Gamma G(y-x)\psi(x)\,ds_x\,ds_y = \langle\!\langle \psi\,;\,\phi \rangle\!\rangle$$

as the kernel is symmetric, i.e. $G(y-x) = G(x-y)$. ■

Next, we investigate the operators K and K'. To this end, we restrict to the case that Γ is not only a Lipschitz boundary but even piecewise C^2. The results can also be obtained in a more general setting but this would lead to even more technical difficulties. However, the assumption seems not to be too restrictive: In numerical simulations Γ is almost always a piecewise polygonal boundary and thus even piecewise C^∞.

Definition 2.19. *We first define the so-called* **reference element**

$$T_{\mathrm{ref}} = \begin{cases} (0,1)\times\{0\} & \text{for } d = 2,\\ \big\{x \in \mathbb{R}^2 \,\big|\, x_1, x_2 > 0, x_1 + x_2 < 1\big\} \times \{0\} & \text{for } d = 3, \end{cases} \tag{2.27}$$

which correspond to the open unit simplices in $\mathbb{R}^{d-1}$. □

We now extend the idea of Lipschitz boundaries to the definition of piecewise smooth boundaries, where the reader might want to recall the definition of Lipschitz domains from Section 1.4.2. We stress that the following definition is also used to introduce the boundary element method later-on.

Definition 2.20. *The boundary $\Gamma = \partial\Omega$ of a Lipschitz domain Ω in $\mathbb{R}^d$ is* **piecewise C^2** *provided there hold the following: There are finitely many relatively open boundary pieces $\Gamma_1, \dots, \Gamma_N \subseteq \Gamma$ with $\Gamma_j \cap \Gamma_k = \emptyset$ for $j \neq k$. To each Γ_j belong open sets $U_j, V_j \subset \mathbb{R}^d$ and a C^2-diffeomorphism $\chi_j : U_j \to V_j$ such that*

- *there hold the set inclusions $T_{\mathrm{ref}} \subset U_j$ and $\Gamma_j \subset V_j$,*
- *and $\chi_j^0 := \chi_j|_{T_{\mathrm{ref}}}$ is a parametrization of Γ_j, i.e. $\Gamma_j = \chi_j(T_{\mathrm{ref}})$.*

The subsets Γ_j are called **smooth screens** *in the following.* □

We first consider the (abstract) operators K and K', which are defined by

$$K := \frac{1}{2} + \gamma_0^{\mathrm{int}} \widetilde{K} \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$$

and

$$K' := -\frac{1}{2} + \gamma_1^{\mathrm{int}} \widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big),$$

respectively. To derive integral representations, we further introduce integral operators K_0 and K_0' on $L^\infty(\Gamma)$. In fact, it will essentially turn out that $K = K_0$ and $K' = K_0'$, respectively.

Lemma 2.21. *Assume that Γ is piecewise C^2. Then, for any smooth screen Γ_ℓ and $x \in \Gamma_\ell$, there holds $\gamma_{1,x}^{\mathrm{int}} G(x - \cdot), \gamma_{1,y}^{\mathrm{int}} G(x - \cdot) \in L^1(\Gamma)$. In particular, the* **double layer potential**

$$K_0 v(x) := \int_\Gamma \gamma_{1,y}^{\mathrm{int}} G(x - y) v(y)\, ds_y \tag{2.28}$$

and the **adjoint double layer potential**

$$K_0' \phi(x) := \int_\Gamma \gamma_{1,x}^{\mathrm{int}} G(x - y) \phi(y)\, ds_y \tag{2.29}$$

are well-defined for $v, \phi \in L^\infty(\Gamma)$ and almost all $x \in \Gamma$.

Proof. **1. step.** We first prove that $\gamma_{1,x}^{\mathrm{int}} G(x - \cdot), \gamma_{1,y}^{\mathrm{int}} G(x - \cdot) \in L^1(\Gamma)$: Let $\chi : U \to V$ be the C^2-diffeomorphism from the definition of a C^2-piecewise boundary, where U and V are the open neighborhoods of T_{ref} and Γ_ℓ, respectively. Consider the restriction χ^0 to T_{ref} and the corresponding Jacobian $J(\widehat{y}) \in \mathbb{R}^{d \times (d-1)}$ of χ^0 at $\widehat{y} \in T_{\mathrm{ref}}$. Note that $J(\widehat{y})$ gives the tangential plane of Γ at $y = \chi^0(\widehat{y})$, whence $n(y) \cdot [J(\widehat{y})(\widehat{y} - \widehat{x})] = 0$ for any $\widehat{x}, \widehat{y} \in T_{\mathrm{ref}}$. Given $x = \chi^0(\widehat{x}), y = \chi^0(\widehat{y})$, the mean value theorem provides $\widehat{\zeta} \in \mathrm{conv}\{\widehat{x}, \widehat{y}\}$ such that

$$\begin{aligned} n(y) \cdot (y - x) = n(y) \cdot \big[\chi^0(\widehat{y}) - \chi^0(\widehat{x})\big] &= n(y) \cdot \big[J(\widehat{\zeta})(\widehat{y} - \widehat{x})\big] \\ &= n(y) \cdot \big[\big(J(\widehat{\zeta}) - J(\widehat{y})\big)(\widehat{y} - \widehat{x})\big]. \end{aligned}$$

Therefore, the bi-Lipschitz property proves

$$|n(y) \cdot (y - x)| \lesssim |\widehat{y} - \widehat{x}|^2 \lesssim |y - x|^2,$$

where the constants only depends on the smooth screen Γ_ℓ. Consequently, we have

$$|\gamma_{1,x}^{\text{int}} G(x-y)| + |\gamma_{1,y}^{\text{int}} G(x-y)| \lesssim \frac{1}{|x-y|^{d-2}} \quad \text{for all } x, y \in \Gamma_\ell.$$

According to Lemma 2.17, we finally obtain

$$\int_\Gamma |\gamma_{1,x}^{\text{int}} G(x-y)|\, ds_y = \int_{\Gamma\backslash\Gamma_\ell} |\gamma_{1,x}^{\text{int}} G(x-y)|\, ds_y + \int_{\Gamma_\ell} |\gamma_{1,x}^{\text{int}} G(x-y)|\, ds_y < \infty$$

as well as

$$\int_\Gamma |\gamma_{1,y}^{\text{int}} G(x-y)|\, ds_y = \int_{\Gamma\backslash\Gamma_\ell} |\gamma_{1,y}^{\text{int}} G(x-y)|\, ds_y + \int_{\Gamma_\ell} |\gamma_{1,y}^{\text{int}} G(x-y)|\, ds_y < \infty,$$

where we remark that the integrals over $\Gamma\backslash\Gamma_\ell$ contain the smooth part of the kernels and are thus obviously finite.

2. step. In particular, the Hölder inequality now proves that $K_0 v(x)$ and $K_0'\phi(x)$ are well-defined. This concludes the proof. ■

Theorem 2.22. *Assume that Γ is piecewise C^2. Then, there holds the following:*
(i) *The double layer potential operator K_0 from (2.28) allows a unique extension from $\gamma_0^{\text{int}}(C^\infty(\overline{\Omega}))$ to an operator $K_0 \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$, and there holds $K_0 = K := \frac{1}{2} + \gamma_0^{\text{int}}\widetilde{K}$.*
(ii) *The adjoint double layer potential operator K_0' from (2.29) allows a unique extension from $L^\infty(\Gamma)$ to an operator $K_0' \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$, and there holds $K_0' = K' := -\frac{1}{2} + \gamma_1^{\text{int}}\widetilde{V}$.*
(iii) *The adjoint double layer potential $K' \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$ is the adjoint operator of $K \in L\big(H^{1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ in the functional analytic sense, i.e.*

$$\langle Kv\,;\,\phi\rangle = \langle v\,;\,K'\phi\rangle \quad \textit{for all } v \in H^{1/2}(\Gamma) \textit{ and } \phi \in H^{-1/2}(\Gamma). \tag{2.30}$$

Before we prove Theorem 2.22, we need a small technical lemma which explains that one has to expect jump terms in the limit case for $\gamma_1^{\text{int}}\widetilde{V}$ and $\gamma_0^{\text{int}}\widetilde{K}$.

Lemma 2.23. *Assume that Γ is piecewise C^2. For $x \in \mathbb{R}^d$, we consider the function*

$$\kappa : \mathbb{R}^d \to \mathbb{R}, \quad \kappa(x) = \int_\Gamma \gamma_{1,y}^{\text{int}} G(x-y)\, ds_y. \tag{2.31}$$

Then, the range of κ is discrete, and there holds

$$\kappa(x) = \begin{cases} -1 & \text{for } x \in \Omega, \\ -1/2 & \text{for } x \in \Gamma \text{ and } \Gamma \text{ smooth around } x, \\ 0 & \text{for } x \in \Omega^{\text{ext}} := \mathbb{R}^d\backslash\overline{\Omega}. \end{cases} \tag{2.32}$$

Moreover, for any $x \in \Gamma$, holds $\kappa(x) = -\lim\limits_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon(x)\cap\Omega} \gamma_{1,y}^{\text{int}} G(x-y)\, ds_y$, and the limit exists.

Proof. **1. step.** For $x \in \Omega$ holds $\kappa(x) = \widetilde{K}(1) = -1$ according to the representation formula for the constant function 1.

2. step. For $x \in \Omega^{\mathrm{ext}}$, we choose $\varepsilon > 0$ with $B_\varepsilon(x) \subset \Omega^{\mathrm{ext}}$ and consider $\widetilde{\Omega} := \Omega \cup B_\varepsilon(x)$. Applying step 1 for the domains $\widetilde{\Omega}$ and $B_\varepsilon(x)$, respectively, we see

$$-1 = \int_{\partial\widetilde{\Omega}} \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y = \kappa(x) + \int_{\partial B_\varepsilon(x)} \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y = \kappa(x) - 1,$$

which proves $\kappa(x) = 0$.

3. step. Finally, we consider the case $x \in \Gamma$ and assume that, for some $\varepsilon > 0$, $\Gamma \cap B_\varepsilon(x)$ is smooth. We have already proven, that $\gamma_{1,y}^{\mathrm{int}} G(x - \cdot) \in L^1(\Gamma)$. Hence, the Lebesgue dominated convergence theorem proves

$$\begin{aligned}\kappa(x) = \int_\Gamma \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y &= \lim_{\varepsilon\to 0^+} \int_{\Gamma\backslash B_\varepsilon(x)} \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y \\ &= -\lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon(x)\cap\Omega} \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y,\end{aligned}$$

where we have applied integration by parts on $\Omega_\varepsilon := \Omega\backslash B_\varepsilon(x)$ in the last step, namely $0 = \int_{\Omega_\varepsilon} \Delta G(x-y) = \int_{\partial\Omega_\varepsilon} \gamma_{1,y}^{\mathrm{int}} G(x-y)\, ds_y$. Note that the integrand over $\partial B_\varepsilon(x) \cap \Omega$ reads

$$\gamma_{1,y}^{\mathrm{int}} G(x-y) = -\frac{1}{|S_2^d|} \frac{y-x}{|x-y|^d} \cdot n(y) = -\frac{1}{|S_2^d|} \frac{y-x}{|x-y|^d} \cdot \frac{x-y}{|x-y|} = +\frac{1}{|S_2^d|\varepsilon^{d-1}} = \frac{1}{|\partial B_\varepsilon(x)|}.$$

Altogether, we thus obtain

$$\kappa(x) = -\lim_{\varepsilon\to 0^+} \frac{|\partial B_\varepsilon(x) \cap \Omega|}{|\partial B_\varepsilon(x)|}.$$

As Γ is smooth around x, it is asymptotically flat and therefore $\partial B_\varepsilon(x) \cap \Omega$ asymptotically is a half sphere, i.e. we obtain $\kappa(x) = -1/2$. ∎

Proof of Theorem 2.22. Throughout, let Γ_ℓ be some smooth screen and $x \in \Gamma$ and $(x_n)_{n\in\mathbb{N}}$ in Ω a sequence that converges to x.

1. step. First, we show that for any $v \in C^\infty(\overline{\Omega})$ holds

$$\lim_{n\to\infty} \int_\Gamma \gamma_{1,y}^{\mathrm{int}} G(x_n - y)\big(v(y) - v(x_n)\big)\, ds_y = \int_\Gamma \gamma_{1,y}^{\mathrm{int}} G(x-y)\big(v(y) - v(x)\big)\, ds_y, \tag{2.33}$$

where the existence of the right-hand side follows from Lemma 2.17. Obviously, convergence holds at least pointwise almost everywhere in Γ. Choose an open neighborhood U_x of Γ and a bi-Lipschitz function $\chi_x : B_1(0) \to U_x$ according to the definition of a Lipschitz boundary. We stress that the convergence (2.33) is clear for the integration domain $\Gamma\backslash U_x$. It remains to prove convergence for the domain $\Gamma \cap U_x$. Without loss of generality, we assume $\widehat{x} = \chi^{-1}(x) = 0$. Let $\widehat{x}_n := \chi^{-1}(x_n)$. We may then assume that $\widehat{x}_n = (\widehat{y}_n, \alpha_n)$ with $\widehat{y}_n \in B_1^0$ and $\alpha_n \in \mathbb{R}$. We now proceed as in the proof of

Lemma 2.17 to apply the Lebesgue differentiation theorem. There holds

$$\begin{aligned}
\int_{\Gamma\cap U_x} \gamma_{1,y}^{\rm int} G(x_n - y)\big(v(y) - v(x_n)\big)\, ds_y &= -\frac{1}{|S_2^d|}\int_{\Gamma\cap U_x} \frac{(y-x_n)\cdot n(y)}{|y-x_n|^d}\big(v(y)-v(x_n)\big)\, ds_y \\
&= -\frac{1}{|S_2^d|}\int_{B_1^0} \frac{(\chi(\widehat{y},0)-x_n)\cdot n(\chi(\widehat{y},0))}{|(\widehat{y},0)-x_n|^d}\big(v(\chi(\widehat{y},0)) - v(x_n)\big)\,\lambda(\widehat{y})\, d\widehat{y} \\
&= -\frac{1}{|S_2^d|}\int_{\mathbb{R}^{d-1}} \chi_{B_1^0}(\widehat{y})\,\frac{(\chi(\widehat{y},0)-x_n)\cdot n(\chi(\widehat{y},0))}{|(\widehat{y},0)-x_n|^d}\big(v(\chi(\widehat{y},0)) - v(x_n)\big)\,\lambda(\widehat{y})\, d\widehat{y}.
\end{aligned}$$

Again, we use the translation $\widehat{y}\mapsto \widehat{y}+\widehat{y}_n$ in the integrand, which yields the integrand

$$f_n(\widehat{y}) := \chi_{B_1^0}(\widehat{y}+\widehat{y}_n)\,\frac{(\chi(\widehat{y}+\widehat{y}_n,0)-x_n)\cdot n(\chi(\widehat{y}+\widehat{y}_n,0))}{|(\widehat{y}+\widehat{y}_n,0)-x_n|^d}\big(v(\chi(\widehat{y}+\widehat{y}_n,0)) - v(x_n)\big)\,\lambda(\widehat{y}+\widehat{y}_n).$$

We now plug-in $x = \chi(\widehat{y}_n,\alpha_n)$ and use $\lambda\in L^\infty(\mathbb{R}^d)$, the bi-Lipschitz property of χ, and the Lipschitz property of v to see

$$\begin{aligned}
|f_n(\widehat{y})| \lesssim \chi_{B_1^0}(\widehat{y}+\widehat{y}_n)\,\frac{1}{|(\widehat{y}+\widehat{y}_n,0)-(\widehat{y}_n,\alpha)|^{d-2}} &\le \chi_{B_1^0}(\widehat{y}+\widehat{y}_n)\,\frac{1}{|(\widehat{y},\alpha)|^{d-2}} \\
&\le \chi_{B_2^0}(\widehat{y})\,\frac{1}{|\widehat{y}|^{d-2}}.
\end{aligned}$$

Note that the upper bound belongs to $L^1(\mathbb{R}^{d-1})$. Therefore, (2.33) follows from the Lebesgue dominated convergence theorem.

2. step. For $v\in C^\infty(\overline{\Omega})$ holds $\gamma_0^{\rm int}\widetilde{K}v = \big(-\frac{1}{2}+K_0\big)v$ almost everywhere on Γ: We apply Lemma 2.23 for $x_n\in\Omega$ and $x\in\Gamma$, which yields

$$\begin{aligned}
\widetilde{K}v(x_n) &= v(x_n)\int_\Gamma \gamma_{1,y}^{\rm int}G(x_n-y)\, ds_y + \int_\Gamma \gamma_{1,y}^{\rm int}G(x_n-y)\,\big(v(y)-v(x_n)\big)\, ds_y \\
&\xrightarrow{n\to\infty} -v(x) + \int_\Gamma \gamma_{1,y}^{\rm int}G(x-y)\,\big(v(y)-v(x)\big)\, ds_y = -\frac{1}{2}\,v(x) + K_0 v(x).
\end{aligned}$$

3. step. By definition of K, the last step implies $Kv = \big(\frac{1}{2}+\gamma_0^{\rm int}\widetilde{K}\big)v = K_0 v$ for all $v\in \gamma_0^{\rm int}\big(C^\infty(\overline{\Omega})\big) =: V$. Since $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$, the space V is dense in $H^{1/2}(\Gamma)$. In particular, K is the unique extension of K_0 to the entire space $H^{1/2}(\Gamma)$.

4. step. There holds $(K_0 v\,;\,\phi)_\Omega = (v\,;\,K_0'\phi)$ for all $v\in\gamma_0^{\rm int}\big(C^\infty(\overline{\Omega})\big)$ and $\phi\in L^\infty(\Gamma)$: From step 3, we infer that $K_0 v\in H^{1/2}(\Gamma)\subseteq L^1(\Gamma)$. Therefore, the product on the left-hand side is well-defined. Moreover, Fubini's theorem proves

$$\begin{aligned}
\big(v\,;\,K_0'\phi\big) = \int_\Gamma v(x)\int_\Gamma \gamma_{1,x}^{\rm int}G(x-y)\phi(y)\, ds_y\, ds_x &= \int_\Gamma \phi(y)\int_\Gamma \gamma_{1,x}^{\rm int}G(x-y)v(x)\, ds_x\, ds_y \\
&= \int_\Gamma \phi(y)\int_\Gamma \gamma_{1,x}^{\rm int}G(y-x)v(x)\, ds_x\, ds_y \\
&= (K_0 v\,;\,\phi)\,.
\end{aligned}$$

5. step. For $\phi\in L^\infty(\Gamma)$ and $v\in C^\infty(\overline{\Omega})$ holds

$$\langle \gamma_1^{\rm int}\widetilde{V}\phi\,;\,\gamma_0^{\rm int}v\rangle = \langle\big(\tfrac{1}{2}+K_0'\big)\phi\,;\,\gamma_0^{\rm int}v\rangle,$$

where the right-hand side exists as L^2-scalar product: Starting from the first Green's formula, we have

$$\langle\gamma_1^{\rm int}\widetilde{V}\phi\,;\,\gamma_0^{\rm int}v\rangle=\big(\nabla\widetilde{V}\phi\,;\,\nabla v\big)_\Omega=\int_\Omega\nabla v(x)\cdot\Big(\int_\Gamma\nabla_x G(x-y)\phi(y)\,ds_y\Big)\,dx,$$

where we have used, for $x\in\Omega$ fixed, that one may interchange integral and differential for the single-layer potential, cf. Lemma 2.5. We now apply the Fubini theorem and obtain

$$\langle\gamma_1^{\rm int}\widetilde{V}\phi\,;\,\gamma_0^{\rm int}v\rangle=\int_\Gamma\phi(y)\int_\Omega\nabla v(x)\cdot\nabla_x G(x-y)dx\,ds_y. \tag{2.34}$$

Fix $y\in\Gamma$. Since $\nabla_x G(x-y)\in L^1_{\ell oc}(\mathbb{R}^d)$ and $\nabla v\in L^\infty(\mathbb{R}^d)$, we may use the Lebesgue theorem to obtain

$$\int_\Omega\nabla v(x)\cdot\nabla_x G(x-y)dx=\lim_{\varepsilon\to0}\int_{\Omega\backslash B_\varepsilon(y)}\nabla v(x)\cdot\nabla_x G(x-y)dx.$$

For $\varepsilon>0$, integration by parts yields

$$\int_{\Omega\backslash B_\varepsilon(y)}\nabla v(x)\cdot\nabla_x G(x-y)\,dx=\int_{\Gamma\backslash B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)v(x)\,ds_x+\int_{\Omega\cap\partial B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)v(x)\,ds_x.$$

In the limit $\varepsilon\to0$, the first boundary integral converges to $K_0v(y)$ for almost all $y\in\Gamma$. The second boundary integral is split into

$$\begin{aligned}&\int_{\Omega\cap\partial B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)v(x)\,ds_x\\&\qquad=v(y)\int_{\Omega\cap\partial B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)\,ds_x+\int_{\Omega\cap\partial B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)\big[v(x)-v(y)\big]\,ds_x.\end{aligned}$$

According to Lemma 2.23, the first term converges to $\frac12\,v(y)$ for $\varepsilon\to0$. The second integral vanishes because of

$$\Big|\int_{\Omega\cap\partial B_\varepsilon(y)}\gamma_{1,x}^{\rm int}G(x-y)\big[v(x)-v(y)\big]\,ds_x\Big|\le\sup_{x\in B_\varepsilon(y)}|v(x)-v(y)|\,\underbrace{\int_{\Omega\cap\partial B_\varepsilon(y)}|\gamma_{1,x}^{\rm int}G(x-y)|\,ds_x}_{\le1}.$$

Altogether, we thus obtain

$$\int_\Omega\nabla v(x)\cdot\nabla_x G(x-y)dx=K_0v(y)+\frac12\,v(y)\quad\text{for almost all }y\in\Gamma.$$

Plugging this into (2.34) and using the adjointness from step 4, we finally obtain

$$\langle\gamma_1^{\rm int}\widetilde{V}\phi\,;\,\gamma_0^{\rm int}v\rangle=(\phi\,;\,K_0v)_\Gamma+\frac12\,(\phi\,;\,v)_\Gamma=\big(K_0'\phi\,;\,v\big)_\Gamma+\frac12\,(\phi\,;\,v)_\Gamma=\big(\big(\tfrac12+K_0'\big)\phi\,;\,v\big)_\Gamma.$$

6. step. Since $\gamma_0^{\rm int}\big(C^\infty(\overline{\Omega})\big)$ is dense in $H^{1/2}(\Gamma)$, step 5 and the Hahn-Banach theorem imply $K'\phi=\big(-\frac12+\gamma_1^{\rm int}\widetilde{V}\big)\phi=K_0'\phi$ for all $\phi\in L^\infty(\Gamma)$. Since $L^\infty(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$, we infer that K' is the unique extension of K_0' to the entire space $H^{-1/2}(\Gamma)$.

7. step. There holds $\langle Kv\,;\,\phi\rangle = \langle v\,;\,K'\phi\rangle$ for all $v \in H^{1/2}(\Gamma)$ and all $\phi \in H^{-1/2}(\Gamma)$: According to step 4, this equality holds on dense subspaces. Therefore, continuity arguments prove the equality for the entire duality pairing. ∎

The integral representation of the hypersingular integral operator W is more involved, and we only state the corresponding results. For the proof, we refer to [SS11].

Theorem 2.24. *Assume that Γ is piecewise C^2 and let $v \in H^{1/2}(\Gamma) \cap C(\Gamma)$. Then, for $x \in \Gamma$, the operator W has the representation*

$$Wv(x) = -\int_\Gamma \frac{\partial}{\partial n_x}\frac{\partial}{\partial n_y} G(x-y)(v(y)-v(x))ds_y,$$

where the integral is understood as a Cauchy principal value, i.e., as the limit $\lim_{\varepsilon\to 0}\int_{\Gamma\setminus B\varepsilon(x)}$.

For implementation of the hypersingular integral operator usually a different representation using the single-layer operator is used. For that, we define the surface curl

$$\mathrm{curl}_\Gamma\, u := \begin{pmatrix} \partial_{x_2} u \\ -\partial_{x_1} u \end{pmatrix} \cdot n$$

for $d = 2$ and

$$\mathrm{curl}_\Gamma\, u := n \times \nabla u$$

for $d = 3$.

Lemma 2.25. *Assume that Γ is piecewise C^2 and let $u, v \in H^{1/2}(\Gamma) \cap C(\Gamma) \cap C^1_{\mathrm{pw}}(\Gamma)$. Then, we have*

$$(Wu\,;\,v)_\Gamma = \int_\Gamma\int_\Gamma \mathrm{curl}_\Gamma\, v(x)\cdot \mathrm{curl}_\Gamma\, u(y) G(x-y)\; ds_y\; ds_x = (V\,\mathrm{curl}_\Gamma\, u\,;\,\mathrm{curl}_\Gamma\, v)_\Gamma$$

as a weakly singular integral.

2.6 Exterior Trace and Conormal Derivative

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with boundary $\Gamma := \partial\Omega$. We define the **exterior domain**

$$\Omega^{\mathrm{ext}} := \mathbb{R}^d \backslash \overline{\Omega}.$$

For a sufficiently large radius $R > 0$ with $\overline{\Omega} \subset B_R(0)$, we use the abbreviate notation

$$\Omega^{\mathrm{ext}}_R := \Omega^{\mathrm{ext}} \cap B_R(0) \quad \text{with boundary} \quad \Gamma_R := \partial\Omega^{\mathrm{ext}}_R.$$

Note that Ω_R^{ext} is again a Lipschitz domain and that Γ_R is the disjoint union of Γ and the sphere $\partial B_R(0)$. In this section, we introduce the exterior trace $\gamma_0^{\text{ext}}u$ and the exterior conormal derivative $\gamma_1^{\text{ext}}u$ for a function $u \in H^1_{\ell oc}(\Omega^{\text{ext}})$. To that end, $H^1_{\ell oc}(\widehat{\Omega})$ denotes the set of all local H^1-functions

$$H^1_{\ell oc}(\widehat{\Omega}) := \big\{ v \in L^2_{\ell oc}(\widehat{\Omega}) \,\big|\, \forall R > 0 \quad v \in H^1(\widehat{\Omega} \cap B_R(0)) \big\}. \tag{2.35}$$

Next, we give the definition of the exterior trace $\gamma_0^{\text{ext}}u$, which shall satisfy at least $\gamma_0^{\text{ext}}u|_\Gamma = u|_\Gamma$ for all $u \in C(\overline{\Omega^{\text{ext}}})$.

Proposition 2.26. (i) *Let ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $\overline{\Omega} \subset \omega$, i.e. $\Gamma \cap \partial\omega = \emptyset$. Then, $\widehat{\omega} := \omega\backslash\Omega$ is a bounded Lipschitz domain with boundary $\widehat{\Gamma} := \partial\widehat{\omega} = \Gamma \cup \partial\omega$. Therefore, the trace operator $\gamma_{0,\omega}^{\text{int}} \in L\big(H^1(\widehat{\omega}); H^{1/2}(\widehat{\Gamma})\big)$ with respect to $\widehat{\omega}$ induces an operator*

$$\gamma_0^{\text{ext}} \in L\big(H^1(\widehat{\omega}); H^{1/2}(\Gamma)\big), \quad \gamma_0^{\text{ext}}u := (\gamma_{0,\omega}^{\text{int}}u)|_\Gamma. \tag{2.36}$$

(ii) *For $u \in H^1_{\ell oc}(\Omega^{\text{ext}})$, the* **exterior trace** *$\gamma_0^{\text{ext}}u \in H^{1/2}(\Gamma)$ is independent of ω, i.e.*

$$\gamma_0^{\text{ext}}u := (\gamma_{0,\omega}^{\text{int}}u)|_\Gamma = (\gamma_{0,\widetilde{\omega}}^{\text{int}}u)|_\Gamma \tag{2.37}$$

for all bounded Lipschitz domains $\omega, \widetilde{\omega} \subset \mathbb{R}^d$ with $\overline{\Omega} \subset \omega \cap \widetilde{\omega}$.

Proof. To prove (i), one only has to realize that the restriction of $v \mapsto v|_\Gamma$ is a continuous linear operator from $H^{1/2}(\widehat{\Gamma})$ to $H^{1/2}(\Gamma)$.
(ii) Without loss of generality, we assume $\widetilde{\omega} \subseteq \omega$ since we may otherwise consider $\omega \cap \widetilde{\omega}$ instead of $\widetilde{\omega}$. For $u \in H^1(\widehat{\omega})$, we find a sequence $u_n \in C^\infty(\widehat{\omega})$ which converges to u in $H^1(\widehat{\omega})$ and whence in $H^1(\widehat{\widetilde{\omega}})$. Recall that, e.g., $\gamma_{0,\omega}^{\text{int}}u_n = u_n|_{\widehat{\Gamma}}$. Thus, there holds $\gamma_0^{\text{ext}}u_n = u_n|_\Gamma$ independently of whether we consider $\widehat{\omega}$ or $\widehat{\widetilde{\omega}}$. In particular, the convergence $u_n|_\Gamma = \gamma_0^{\text{ext}}u_n \to \gamma_0^{\text{ext}}u \in H^{1/2}(\Gamma)$ concludes the proof. ∎

Second, we want to define the exterior conormal derivative which shall satisfy $\gamma_1^{\text{ext}}u = (\partial u/\partial n)|_\Gamma$ for $u \in C^1(\overline{\Omega^{\text{ext}}})$, where n is the outer normal vector on Γ with respect to Ω and thus the inner normal vector with respect to Ω^{ext}. In particular, we want to formalize the first Green's formula on Ω^{ext}, which reads, for $u \in C^2(\overline{\Omega^{\text{ext}}})$, in classical terms

$$(-\Delta u \,;\, v)_{\Omega^{\text{ext}}} = (\nabla u \,;\, \nabla v)_{\Omega^{\text{ext}}} + \big(\gamma_1^{\text{ext}}u \,;\, \gamma_0^{\text{ext}}v\big)_\Gamma \quad \text{for all } v \in \mathcal{D}(\mathbb{R}^d).$$

Note the plus sign on the right-hand side which stems from the fixed orientation of the normal vector n on Γ, i.e. n is the outer normal vector of Ω so that $-n$ is the outer normal vector of Ω^{ext} on Γ.

The following proposition generalizes the exterior conormal derivative for Sobolev spaces and provides a stability estimate, similar to Theorem 1.30. As the proofs do not provide any new insight, we leave them to the reader as an exercise.

Proposition 2.27. *Let ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with $\overline{\Omega} \subset \omega$ and define the bounded Lipschitz domain $\widehat{\omega} := \omega \backslash \Omega$, i.e. $\widehat{\Gamma} := \partial\widehat{\omega} = \Gamma \cup \partial\omega$ and $\Gamma \cap \partial\omega = \emptyset$. Then, there hold:*
(i) *The conormal derivative $\gamma_{1,\omega}^{\text{int}} \in L\big(H^1_\Delta(\widehat{\omega}); H^{-1/2}(\widehat{\Gamma})\big)$ induces an operator*

$$\gamma_1^{\text{ext}} \in L\big(H^1_\Delta(\widehat{\omega}); H^{-1/2}(\Gamma)\big), \quad \gamma_1^{\text{ext}} u := -\widehat{\phi} \text{ for } \phi := \gamma_{1,\omega}^{\text{int}} u \in H^{-1/2}(\widehat{\Gamma}), \tag{2.38}$$

where $\widehat{\phi} \in H^{-1/2}(\Gamma)$ denotes the functional defined by $\langle \widehat{\phi} \,;\, v\rangle := \langle \phi \,;\, \widehat{v}\rangle$ for $\phi \in H^{-1/2}(\widehat{\Gamma})$, $v \in H^{1/2}(\Gamma)$ and its zero-extension $\widehat{v} \in H^{1/2}(\widehat{\Gamma})$.
(ii) *For $u \in H^1_{\ell oc}(\Omega^{\text{ext}})$ with $-\Delta u \in L^2_{\ell oc}(\Omega^{\text{ext}})$, the* **exterior conormal derivative** *$\gamma_1^{\text{ext}} u \in H^{-1/2}(\Gamma)$ is independent of ω, and there holds*

$$(-\Delta u \,;\, v)_{\Omega^{\text{ext}}} = (\nabla u \,;\, \nabla v)_{\Omega^{\text{ext}}} + \langle \gamma_1^{\text{ext}} u \,;\, \gamma_0^{\text{ext}} v\rangle \quad \text{for all } v \in \mathcal{D}(\mathbb{R}^d). \tag{2.39}$$

(iii) *Let $u \in H^1_{\ell oc}(\Omega^{\text{ext}})$ with $-\Delta u \in L^2(\Omega^{\text{ext}})$. Moreover, we assume that u has finite energy, i.e. $\nabla u \in L^2(\Omega^{\text{ext}})$. Then, there holds the stability estimate*

$$\|\gamma_1^{\text{ext}} u\|_{H^{-1/2}(\Gamma)} \lesssim \| -\Delta u\|_{L^2(\Omega^{\text{ext}})} + \|\nabla u\|_{L^2(\Omega^{\text{ext}})}. \tag{2.40}$$

2.7 Ellipticity of Single-Layer Operator

We have already seen that the single-layer operator $V = \gamma_0^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is a symmetric operator, cf. Corollary 2.18. In this section, we prove that V is also elliptic. To that end, we need some more facts about $\widetilde{V}\phi$. The first proposition provides the jump relations of $\widetilde{V}\phi$. Throughout, we use the notation $\Omega^{\rm ext} := \mathbb{R}^d\backslash\overline{\Omega}$ and $\Omega_R^{\rm ext} := \Omega^{\rm ext} \cap B_R(0)$ as introduced in the previous section.

Proposition 2.28. *For $\phi \in H^{-1/2}(\Gamma)$ holds $\widetilde{V}\phi \in H^1_{\ell oc}(\mathbb{R}^d)$ with $-\Delta(\widetilde{V}\phi) = 0$ weakly in $\mathbb{R}^d\backslash\Gamma$. In particular, the following jump relations are well-defined:*
(i) $[\gamma_0\widetilde{V}\phi] := \gamma_0^{\rm ext}\widetilde{V}\phi - \gamma_0^{\rm int}\widetilde{V}\phi = 0 \in H^{1/2}(\Gamma)$.
(ii) $[\gamma_1\widetilde{V}\phi] := \gamma_1^{\rm ext}\widetilde{V}\phi - \gamma_1^{\rm int}\widetilde{V}\phi = -\phi \in H^{-1/2}(\Gamma)$.

Proof. **1. step.** Exterior trace $\gamma_0^{\rm ext}\widetilde{V}$ and conormal derivative $\gamma_1^{\rm ext}\widetilde{V}$ are well-defined: For $\phi \in H^{-1/2}(\Gamma)$, we have already proven that $\widetilde{V}\phi \in H^1(\Omega)$ with $-\Delta\widetilde{V}\phi = 0 \in \widetilde{H}^{-1}(\Omega)$. The crucial step in the proof was to provide the identity

$$\left(\widetilde{V}\phi\,;\,f\right)_\Omega = (\phi\,;\,N_0 f)_\Gamma \quad \text{for } \phi \in L^2(\Gamma) \text{ and } f \in \mathcal{D}(\Omega).$$

The remaining steps were just based on abstract functional analysis, namely density arguments and mapping properties of $N_0 = \gamma_0^{\rm int}\widetilde{N}$. We then obtained $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega)\big)$ with $-\Delta\widetilde{V}\phi = 0$ weakly in Ω for all $\phi \in H^{-1/2}(\Gamma)$. From this, we derived that the operators

$$\gamma_0^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big) \quad \text{and} \quad \gamma_1^{\rm int}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$$

are well-defined. It is important to notice that even the same arguments apply for $\Omega_R^{\rm ext}$ and thus prove $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(\Omega_R^{\rm ext})\big)$ with $-\Delta\widetilde{V}\phi = 0$ weakly in $\Omega_R^{\rm ext}$ for all $\phi \in H^{-1/2}(\Gamma)$. In particular, the operators

$$\gamma_0^{\rm ext}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big) \quad \text{and} \quad \gamma_1^{\rm ext}\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big)$$

are well-defined. Moreover, we have seen in Section 2.6 that neither $\gamma_0^{\rm ext}\widetilde{V}$ nor $\gamma_1^{\rm ext}\widetilde{V}$ depend on the radius $R > 0$. Altogether, the jump operators also satisfy

$$[\gamma_0\widetilde{V}] \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big) \quad \text{and} \quad [\gamma_1\widetilde{V}] \in L\big(H^{-1/2}(\Gamma); H^{-1/2}(\Gamma)\big).$$

2. step. There holds $[\gamma_0\widetilde{V}] = 0$: For $\phi \in L^\infty(\Gamma)$, holds $\widetilde{V}\phi \in C(\mathbb{R}^d)$ and therefore $[\gamma_0\widetilde{V}\phi] = 0$. From density $L^\infty(\Gamma)$ in $H^{-1/2}(\Gamma)$, we thus obtain $[\gamma_0\widetilde{V}\phi] = 0$ for any $\phi \in H^{-1/2}(\Gamma)$.

3. step. There holds $\widetilde{V}\phi \in H^1(B_R(0))$ for all $\phi \in H^{-1/2}(\Gamma)$ and $R > 0$: We know that $\widetilde{V}\phi \in H^1(\Omega) \cap H^1(\Omega_R^{\rm ext})$ for all $R > 0$. Moreover, for $\phi \in L^\infty(\Gamma)$, there holds $\widetilde{V}\phi \in C(\mathbb{R}^d)$. This and $B_R(0) = \Omega \cup \Omega_R^{\rm ext}$ imply $\widetilde{V}\phi \in H^1(B_R(0))$ together with the continuity estimate

$$\|\widetilde{V}\phi\|^2_{H^1(B_R(0))} = \|\widetilde{V}\phi\|^2_{H^1(\Omega)} + \|\widetilde{V}\phi\|^2_{H^1(\Omega_R^{\rm ext})} \lesssim \|\phi\|^2_{H^{-1/2}(\Gamma)}.$$

By density and continuity, this proves that $\widetilde{V} \in L\big(H^{-1/2}(\Gamma); H^1(B_R(0))\big)$, where the operator norm — of course — depends on the radius $R > 0$.

4. step. There holds $\widetilde{V}\phi \in H^1_{\ell oc}(\mathbb{R}^d)$ for all $\phi \in H^{-1/2}(\Gamma)$: Formally, $\widetilde{V}\phi \in H^1(B_R(0))$ is obtained by continuous extension of the operator $\widetilde{V}$ from $L^\infty(\Gamma)$ to $H^{-1/2}(\Gamma)$ and thus $\widetilde{V}\phi = \widetilde{V}_R\phi$ depends on the radius R. However, for $\phi \in L^\infty(\Gamma)$, there holds $\widetilde{V}\phi \in C(\mathbb{R}^d)$ and thus $\widetilde{V}_r\phi = (\widetilde{V}\phi)|_{B_r(0)} = (\widetilde{V}_R\phi)|_{B_r(0)}$ for any $0 < r < R$. Therefore, the function $\widetilde{V}\phi$ is independent of the chosen radius and $\widetilde{V}\phi \in H^1_{\ell oc}(\mathbb{R}^d)$ in the limit $R \to \infty$.

5. step. There holds $[\gamma_1\widetilde{V}] = -\phi$: For $\phi \in L^2(\Gamma)$ and $f \in \mathcal{D}(\mathbb{R}^d)$, the first Green's formula on $\Omega^{\rm ext}$ resp. Ω yields

$$\begin{aligned}\langle[\gamma_1\widetilde{V}\phi] \,;\, \gamma_0 f\rangle &= \langle\gamma_1^{\rm ext}\widetilde{V}\phi \,;\, \gamma_0^{\rm ext} f\rangle - \langle\gamma_1^{\rm int}\widetilde{V}\phi \,;\, \gamma_0^{\rm int} f\rangle\\ &= -\Big(\nabla\widetilde{V}\phi \,;\, \nabla g\Big)_{\Omega^{\rm ext}} - \Big(\nabla\widetilde{V}\phi \,;\, \nabla g\Big)_{\Omega}\\ &= -\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla_x \int_\Gamma G(x-y)\phi(y)\,ds_y\,dx.\end{aligned}$$

For $x \in \mathbb{R}^d\backslash\Gamma$, we may interchange the gradient and the integration over the boundary Γ. Moreover, the Fubini theorem gives

$$= -\int_\Gamma \phi(y)\int_{\mathbb{R}^d} \nabla_x G(x-y)\cdot\nabla f(x)\,dx\,ds_y.$$

For fix $y \in \Gamma$, we use integration by parts, which yields

$$\int_{\mathbb{R}^d} \nabla_x G(x-y)\cdot\nabla f(x)\,dx = \int_{\mathbb{R}^d} G(x-y)\big(-\Delta f(x)\big)\,dx = \widetilde{N}(-\Delta f)(y) = f(y)$$

since $f \in \mathcal{D}(\mathbb{R}^d)$. We therefore end up with

$$\langle[\gamma_1\widetilde{V}\phi] \,;\, \gamma_0 f\rangle = -\langle\phi \,;\, \gamma_0 f\rangle \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^d).$$

Since $\gamma_0\big(\mathcal{D}(\mathbb{R}^d)\big) = \gamma_0^{\rm int}\big(C^\infty(\overline{\Omega})\big)$ is dense in $H^{1/2}(\Gamma)$, the Hahn-Banach theorem yields the conormal jump $[\gamma_1\widetilde{V}\phi] = -\phi$. ∎

The following exercise has been used in step 3 of the proof of Proposition 2.28.

Exercise 2. Let Ω_1, Ω_2 be two open sets in $\mathbb{R}^d$ and $\Omega := \text{interior}(\overline{\Omega_1 \cup \Omega_2})$. Let $u \in C(\Omega)$ such that $u \in H^1(\Omega_1)$ and $u \in H^1(\Omega_2)$. Prove that u is weakly differentiable in Ω and conclude that therefore $u \in H^1(\Omega)$. □

Theorem 2.22 proves $\gamma_1^{\rm int}\widetilde{V} = \frac{1}{2} + K'$. In particular, we thus obtain an explicit formula for the exterior conormal derivative $\gamma_1^{\rm ext}\widetilde{V}$ as well.

Corollary 2.29. *If the boundary Γ is piecewise C^2, there holds $\gamma_1^{\rm ext}\widetilde{V} = -\frac{1}{2} + K'$.*

Proof. There holds $-\phi = [\gamma_1\widetilde{V}\phi] = \gamma_1^{\rm ext}\widetilde{V}\phi - \gamma_1^{\rm int}\widetilde{V} = \gamma_1^{\rm ext}\widetilde{V}\phi - \big(\frac{1}{2} + K'\big)\phi$. ∎

With Proposition 2.28, we have provided the necessary tool to prove the ellipticity of V. However, since the kernel function $G(x-y)$ has a logarithmic singularity in 2D and may be of different signs, we have to treat the 2D and the 3D case separately. As the analysis is simpler, we shall start with the 3D case. To be more precise, Theorem 2.30 holds for $d = 3$ only, whereas Theorem 2.33 holds for both $d = 2, 3$.

Theorem 2.30. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Then, the single-layer potential $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is an elliptic and symmetric isomorphism. In particular, given $v \in H^{1/2}(\Gamma)$, there is a unique $\phi \in H^{-1/2}(\Gamma)$ such that $V\phi = v$. Moreover, $\langle\!\langle \phi\,;\,\psi \rangle\!\rangle := \langle V\phi\,;\,\psi\rangle$ defines a scalar product on $H^{-1/2}(\Gamma)$, and the induced norm $|\!|\!|\phi|\!|\!| := \langle\!\langle \phi\,;\,\phi\rangle\!\rangle^{1/2}$ is an equivalent norm on $H^{-1/2}(\Gamma)$.*

Before we prove Theorem 2.30, we provide some elementary observations on the decay of $\widetilde{V}\phi$ at infinity.

Lemma 2.31. *Let $d = 3$, i.e. Ω is a bounded Lipschitz domain in $\mathbb{R}^3$. Let $R > 0$ with $\Omega \subseteq B_R(0)$ and $y_0 \in \Omega$. Then, there holds, for any $\phi \in L^1(\Omega)$ and $x \in \mathbb{R}^3$ with $|x - y_0| \geq 3R$,*

$$|\widetilde{V}\phi(x)| \lesssim \frac{1}{|x-y_0|} \leq \frac{2}{|x|} = \mathcal{O}\big(1/|x|\big) \quad \textit{and} \quad |\nabla\widetilde{V}\phi(x)| \lesssim \frac{1}{|x-y_0|^2} \leq \frac{4}{|x|^2} = \mathcal{O}\big(1/|x|^2\big)$$

as $|x| \to \infty$. The constant only depends on ϕ. In particular, there holds $\nabla\widetilde{V}\phi \in L^2(\mathbb{R}^3)^3$.

Proof. **1. step.** For any $y \in B_R(0) \supset \Gamma$ holds $1/|x-y| \leq 3/|x-y_0|$, which follows from

$$|x-y_0| \leq |x-y| + |y-y_0| \leq |x-y| + 2R \leq |x-y| + \frac{2}{3}\,|x-y_0|$$

and whence $|x-y_0| \leq 3\,|x-y|$.

2. step. There holds $1/|x-y_0| \leq 2/|x|$: From $y_0 \in \Omega$, we obtain $|y_0| \leq R$ and whence $|x| \geq |x-y_0| - |y_0| \geq 2R$. Consequently, we have

$$\frac{1}{|x-y_0|} \leq \frac{1}{|x|-|y_0|} = \frac{1}{|x|}\,\frac{1}{1-|y_0|/|x|} \leq \frac{1}{|x|}\,\frac{1}{1-1/2} = \frac{2}{|x|}.$$

3. step. Estimate $|\widetilde{V}\phi(x)|$ and $|\nabla\widetilde{V}\phi(x)|$: The preceding observations and $|G(x-y)| \sim 1/|x-y|$ lead to

$$|\widetilde{V}\phi(x)| \leq \int_\Gamma |G(x-y)||\phi(y)|\,ds_y \lesssim \frac{3}{|x-y_0|}\,\|\phi\|_{L^1(\Gamma)} \leq \frac{6}{|x|}\,\|\phi\|_{L^1(\Gamma)}.$$

Recall that $|\nabla G(x-y)| \sim 1/|x-y|^2$. Therefore, the same arguments prove

$$|\nabla\widetilde{V}\phi(x)| \lesssim \frac{9}{|x-y_0|^2}\,\|\phi\|_{L^1(\Gamma)} \leq \frac{36}{|x|^2}\,\|\phi\|_{L^1(\Gamma)}.$$

4. step. It now only remains to prove $\nabla\widetilde{V}\phi \in L^2(\mathbb{R}^3)^3$: By use of polar coordinates, we see

$$\int_{|x|\geq R} |\nabla\widetilde{V}\phi|^2\,dx \lesssim \int_{|x|\geq R} \frac{1}{|x|^4}\,dx = 4\pi\int_{r\geq R}\frac{1}{r^4}\,r^2\,dr < \infty.$$

Since $\widetilde{V}\phi \in H^1_{\ell oc}(\mathbb{R}^3)$, this yields $\nabla\widetilde{V}\phi \in L^2(\mathbb{R}^3)^3$. ■

Proof of Theorem 2.30. **1. step.** For $\phi \in L^2(\Gamma)$ and $u := \widetilde{V}\phi$, there holds $\|\nabla u\|^2_{L^2(\Omega^{\mathrm{ext}})} = -\langle\gamma_1^{\mathrm{ext}}u\,;\,\gamma_0^{\mathrm{ext}}u\rangle$: For sufficiently large $R > 0$, we consider the bounded Lipschitz domain $\Omega_R^{\mathrm{ext}} := \Omega^{\mathrm{ext}} \cap U_R(0)$ and apply the first Green formula to see

$$\|\nabla u\|^2_{L^2(\Omega_R^{\mathrm{ext}})} = -\langle\gamma_1^{\mathrm{ext}}u\,;\,\gamma_0^{\mathrm{ext}}u\rangle_\Gamma + \langle\gamma_1^{\mathrm{int}}u\,;\,\gamma_0^{\mathrm{int}}u\rangle_{\partial B_R(0)}. \tag{2.41}$$

As $\nabla u \in L^2(\Omega^{\mathrm{ext}})$, the left-hand side in (2.41) converges to $\|\nabla u\|_{L^2(\Omega^{\mathrm{ext}})}$ for $R \to \infty$ according to the Lebesgue dominated convergence theorem. The boundary integral over the sphere $\partial B_R(0)$ satisfies

$$\langle\gamma_1^{\mathrm{int}}u\,;\,\gamma_0^{\mathrm{int}}u\rangle_{\partial B_R(0)} = \int_{\partial B_R(0)} u(x)\nabla u(x)\cdot n(x)\,ds_x = \mathcal{O}(|\partial B_R(0)|/R^3) = \mathcal{O}(R^{-1})$$

and thus vanishes in the limit $R \to \infty$. Altogether, this proves $\|\nabla u\|^2_{L^2(\Omega^{\mathrm{ext}})} = -\langle\gamma_1^{\mathrm{ext}}u\,;\,\gamma_0^{\mathrm{ext}}u\rangle$.

2. step. For $\phi \in L^2(\Gamma)$ and $u := \widetilde{V}\phi$, holds ellipticity $\|\phi\|^2_{H^{-1/2}(\Gamma)} \lesssim \langle V\phi\,;\,\phi\rangle$: First, the jump condition $[\gamma_1 u] = -\phi$ proves that

$$\|\phi\|_{H^{-1/2}(\Gamma)} = \|[\gamma_1 u]\|_{H^{-1/2}(\Gamma)} \le \|\gamma_1^{\mathrm{int}}u\|_{H^{-1/2}(\Gamma)} + \|\gamma_1^{\mathrm{ext}}u\|_{H^{-1/2}(\Gamma)} \lesssim \|\nabla u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega^{\mathrm{ext}})}$$

according to the stability estimates (1.48)–(2.40) for the conormal derivative. Second, the exterior and interior Green formula prove

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega^{\mathrm{ext}})} &= \langle\gamma_1^{\mathrm{int}}u\,;\,\gamma_0^{\mathrm{int}}u\rangle - \langle\gamma_1^{\mathrm{ext}}u\,;\,\gamma_0^{\mathrm{ext}}u\rangle\\ &= \langle\gamma_1^{\mathrm{int}}u - \gamma_1^{\mathrm{ext}}u\,;\,V\phi\rangle\\ &= -\langle[\gamma_1 u]\,;\,V\phi\rangle\\ &= \langle\phi\,;\,V\phi\rangle,\end{aligned}$$

where we have used the jump condition $[\gamma_1 u] = -\phi$ as well as $[\gamma_0 u] = 0$, whence $\gamma_0^{\mathrm{ext}}u = \gamma_0^{\mathrm{int}}u = V\phi$. Combining both estimates, we obtain the ellipticity estimate for $\phi \in L^2(\Gamma)$.

3. step. The single-layer potential operator $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is elliptic: The proof follows from the ellipticity of V on the dense subspace $L^2(\Gamma)$ by continuity arguments.

4. step. All remaining claims now follow from the Lax-Milgram lemma (Appendix, Theorem A.1). ■

A closer look on the proof of Theorem 2.30 shows that only Lemma 2.31 is crucial. For the 2D kernel, the analogous result is slightly more involved since we have to deal with the logarithmic singularity of the single-layer potential kernel.

Lemma 2.32. *Let $d = 2$, i.e. Ω is a bounded Lipschitz domain in $\mathbb{R}^2$. Let $R > 0$ with $\Omega \subseteq B_R(0)$ and $y_0 \in \Omega$. Let $\phi \in L^1_*(\Gamma) := \big\{\psi \in L^1(\Gamma)\,\big|\,(\psi\,;\,1)_\Gamma = 0\big\}$. Then, there holds, for any $x \in \mathbb{R}^3$ with $|x - y_0| \ge 3R$,*

$$|\widetilde{V}\phi(x)| \lesssim \frac{1}{|x-y_0|} \le \frac{2}{|x|} = \mathcal{O}\big(1/|x|\big) \quad \text{and} \quad |\nabla\widetilde{V}\phi(x)| \lesssim \frac{1}{|x-y_0|^2} \le \frac{4}{|x|^2} = \mathcal{O}\big(1/|x|^2\big)$$

as $|x| \to \infty$. The constant only depends on ϕ and R. In particular, there holds $\nabla\widetilde{V}\phi \in L^2(\mathbb{R}^2)^2$.

Proof. **1. step.** As in the proof of Lemma 2.31 for the 3D case, there holds

$$\frac{1}{|x-y|} \le \frac{3}{|x-y_0|} \quad \text{as well as} \quad \frac{1}{|x-y_0|} \le \frac{2}{|x|}$$

2. step. Decay of $\widetilde{V}\phi(x)$ at infinity: For $y \in B_R(0)$ holds $\operatorname{conv}\{y, y_0\} \subset \mathbb{R}^2 \backslash \{x\}$ since otherwise $x = y_0 + \lambda(y - y_0)$ with some $\lambda \in [0,1]$ would imply $3R \le |x - y_0| = \lambda\,|y - y_0| \le 2R$. Therefore, we may use the Taylor expansion for the function $f(t) := \log|y_0 + t(y - y_0) - x|$ defined for $t \in [0,1]$ to prove

$$\begin{aligned}\log|y-x| = f(1) &= f(0) + \int_0^1 f'(t)\,dt \\ &= \log|y_0 - x| + \int_0^1 \frac{y_0 + t(y-y_0) - x}{|y_0 + t(y-y_0) - x|^2} \cdot (y - y_0)\,dt.\end{aligned}$$

Note that $y_0 + t(y - y_0) \in B_R(0)$ according to convexity and $y_0, y \in B_R(0)$. Therefore, step 1 proves

$$\Big|\frac{y_0 + t(y-y_0) - x}{|y_0 + t(y-y_0) - x|^2} \cdot (y - y_0)\Big| \le \frac{|y - y_0|}{|y_0 + t(y-y_0) - x|} \le \frac{6R}{|x - y_0|}.$$

Finally, $\phi \in L^1_*(\Gamma)$ and thus $\int_\Gamma \log|y_0 - x|\phi(y)\,ds_y = 0$ imply

$$|\widetilde{V}\phi(x)| \sim \Big|\int_\Gamma \phi(y) \int_0^1 \frac{y_0 + t(y-y_0) - x}{|y_0 + t(y-y_0) - x|^2} \cdot (y - y_0)\,dt\,ds_y\Big| \le \frac{6R}{|x - y_0|}\,\|\phi\|_{L^1(\Gamma)}.$$

3. step. Decay of $\nabla\widetilde{V}\phi(x)$ at infinity: As $\partial_j \widetilde{V}\phi(x) \sim \int_\Gamma \partial_{j,x} \log|y - x|\phi(y)\,ds_y$, we my apply the same technique for $f(t) = \partial_{j,x}\log|y_0 + t(y - y_0) - x|$. With $|\partial_j \log|z|| \lesssim 1/|z|$, we then obtain $|\nabla\widetilde{V}\phi(x)| \lesssim 1/|x - y_0|^2 \lesssim 1/|x|^2$.

4. step. By use of polar coordinates, we obtain $\nabla\widetilde{V}\phi \in L^2(\mathbb{R}^2)^2$. This concludes the proof. ∎

Theorem 2.33. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. Then, there is a unique* **equilibrium density** $\phi_{\rm eq} \in H^{-1/2}(\Gamma)$ *and a unique* **capacity** $\lambda_{\rm eq} \in \mathbb{R}$ *such that*

$$V\phi_{\rm eq} \equiv \lambda_{\rm eq} \quad \text{and} \quad \langle \phi_{\rm eq}\,;\,1\rangle = 1. \tag{2.42}$$

With the space

$$H^{1/2}_{\rm eq} := \big\{v \in H^{1/2}(\Gamma)\,\big|\,\langle\phi_{\rm eq}\,;\,v\rangle = 0\big\}, \tag{2.43}$$

there hold the following assertions on the single-layer potential operator:
(i) $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ *is a symmetric operator.*
(ii) $V \in L\big(H^{-1/2}_*(\Gamma); H^{1/2}_{\rm eq}(\Gamma)\big)$ *is a well-defined and elliptic isomorphism.*
(iii) $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ *is elliptic if and only if $\lambda_{\rm eq} > 0$.*
(iv) *For $d = 2$ and $\operatorname{diam}(\Omega) < 1$, there holds $\lambda_{\rm eq} > 0$ and thus ellipticity of V on $H^{-1/2}(\Gamma)$.*

Proof. **1. step.** For $\phi \in H^{-1/2}_*(\Gamma)$ and $u := \widetilde{V}\phi$ holds ellipticity $\|\phi\|^2_{H^{-1/2}(\Gamma)} \lesssim \langle\!\langle \phi\,;\,\phi \rangle\!\rangle$: For $d = 3$, this claim holds even for general $\phi \in H^{-1/2}(\Gamma)$. For $d = 2$, the proof of Theorem 2.30 works for $H^{-1/2}_*(\Gamma)$ since $L^2_*(\Gamma)$ is dense in $H^{-1/2}_*(\Gamma)$ and since functions $\phi \in L^2_*(\Gamma)$ lead to the proper decay of $\widetilde{V}\phi$ at infinity: One only has to replace Lemma 2.31 by Lemma 2.32. The reader may want to check this in detail.

2. step. Unique existence of $(\phi_{\rm eq}, \lambda_{\rm eq})$: We consider the saddle-point problem

$$\begin{array}{lcrcl} a(\phi_{\rm eq}, \psi) & + & b(\psi, \lambda_{\rm eq}) & = & 0, \\ b(\phi_{\rm eq}, \mu) & & & = & -\mu, \end{array} \tag{2.44}$$

for all $(\psi, \mu) \in H^{-1/2}(\Gamma) \times \mathbb{R}$, where $a(\phi, \psi) = \langle V\phi\,;\,\psi\rangle$ and $b(\psi, \mu) = -\mu\,\langle \psi\,;\,1\rangle$. With the notation of the Brezzi Theorem A.4, there holds $\ker(B_1) := \big\{\phi \in H^{-1/2}(\Gamma) \,\big|\, \forall\lambda \in \mathbb{R} \quad b(\phi, \lambda) = 0\big\} = H^{-1/2}_*(\Gamma)$. Since $a(\cdot,\cdot)$ is elliptic on $\ker(B_1)$ and since B_1 is surjective, (2.44) has a unique solution $(\phi_{\rm eq}, \lambda_{\rm eq}) \in H^{-1/2}(\Gamma) \times \mathbb{R}$. From the first equation and the Hahn-Banach theorem, we derive $V\phi_{\rm eq} \equiv \lambda_{\rm eq}$. The second equation yields $\langle \phi_{\rm eq}\,;\,1\rangle = 1$.

3. step. $V \in L\big(H^{-1/2}_*(\Gamma); H^{1/2}_{\rm eq}(\Gamma)\big)$ is a well-defined and elliptic isomorphism: From step 1, we know that V is elliptic on $H^{-1/2}_*(\Gamma)$. For $\phi \in H^{-1/2}_*(\Gamma)$ holds

$$\langle V\phi\,;\,\phi_{\rm eq}\rangle = \langle \phi\,;\,V\phi_{\rm eq}\rangle = \lambda_{\rm eq}\langle \phi\,;\,1\rangle = 0,$$

whence $V\phi \in H^{1/2}_{\rm eq}(\Gamma)$. Therefore, $V \in L\big(H^{-1/2}_*(\Gamma); H^{1/2}_{\rm eq}(\Gamma)\big)$ is well-defined. We now show that $H^{1/2}_{\rm eq}(\Gamma)$ is the dual space of $H^{-1/2}_*(\Gamma)$. To that end, recall that we have already proven that $H^{1/2}_*(\Gamma)$ is the dual space of $H^{-1/2}_*(\Gamma)$. We thus only need to prove the following:

- Given $v \in H^{1/2}_{\rm eq}(\Gamma)$, there is an element $w \in H^{1/2}_*(\Gamma)$ such that $\langle \phi\,;\,v\rangle = \langle \phi\,;\,w\rangle$ for all $\phi \in H^{-1/2}_*(\Gamma)$: The obvious definition of $w := v - \langle v\,;\,1\rangle/\langle 1\,;\,1\rangle \in H^{1/2}_*(\Gamma)$ does the job.
- Given $w \in H^{1/2}_*(\Gamma)$, there is an element $v \in H^{1/2}_{\rm eq}(\Gamma)$ such that $\langle \phi\,;\,v\rangle = \langle \phi\,;\,w\rangle$ for all $\phi \in H^{-1/2}_*(\Gamma)$: Since $\langle \phi_{\rm eq}\,;\,1\rangle = 1$, the definition of $v := w - \langle \phi_{\rm eq}\,;\,w\rangle \in H^{1/2}_{\rm eq}(\Gamma)$ works.

Thus, $H^{-1/2}(\Gamma)^* = H^{1/2}_{\rm eq}(\Gamma)$ in the sense of the extended L^2-scalar product. Therefore, the Lax-Milgram lemma proves that V is an isomorphism between $H^{-1/2}_*(\Gamma)$ and $H^{1/2}_{\rm eq}(\Gamma)$.

4. step. If V is $H^{-1/2}(\Gamma)$-elliptic, there holds $\lambda_{\rm eq} > 0$ since $0 < \|\phi_{\rm eq}\|^2_{H^{-1/2}(\Gamma)} \lesssim \langle V\phi_{\rm eq}\,;\,\phi_{\rm eq}\rangle = \lambda_{\rm eq}\langle \phi_{\rm eq}\,;\,1\rangle = \lambda_{\rm eq}$.

5. step. Provided $\lambda_{\rm eq} > 0$, the operator V is elliptic on $H^{-1/2}(\Gamma)$: Let $\phi \in H^{-1/2}(\Gamma)$ and define $\widetilde{\phi} := \phi - \langle \phi\,;\,1\rangle\,\phi_{\rm eq}$. Note that $\widetilde{\phi} \in H^{-1/2}_*(\Gamma)$. Moreover, the definition of $\phi_{\rm eq}$ proves

$$\langle V\phi\,;\,\phi\rangle = \langle V\widetilde{\phi}\,;\,\widetilde{\phi}\rangle + 2\langle \phi\,;\,1\rangle \underbrace{\langle V\phi_{\rm eq}\,;\,\widetilde{\phi}\rangle}_{=0} + |\langle \phi\,;\,1\rangle|^2 \underbrace{\langle V\phi_{\rm eq}\,;\,\phi_{\rm eq}\rangle}_{=\lambda_{\rm eq}}.$$

Recall that we have already proven that V is at least elliptic on $H^{-1/2}_*(\Gamma)$. As $\|\phi_{\rm eq}\|_{H^{-1/2}(\Gamma)}$ and $\lambda_{\rm eq}$ are just positive constants, the triangle inequality $\|\phi\|_{H^{-1/2}(\Gamma)} \le \|\widetilde{\phi}\|_{H^{-1/2}(\Gamma)} + |\langle \phi\,;\,1\rangle|\,\|\phi_{\rm eq}\|_{H^{-1/2}(\Gamma)}$ yields

$$\|\phi\|^2_{H^{-1/2}(\Gamma)} \lesssim \|\widetilde{\phi}\|^2_{H^{-1/2}(\Gamma)} + |\langle \phi\,;\,1\rangle|^2 \lesssim \langle V\widetilde{\phi}\,;\,\widetilde{\phi}\rangle + \lambda_{\rm eq}|\langle \phi\,;\,1\rangle|^2 = \langle V\phi\,;\,\phi\rangle.$$

6. step. It only remains to prove that $\mathrm{diam}(\Omega) < 1$ implies $\lambda_{\mathrm{eq}} > 0$. This is, however, a rather deep result from complex analysis. The reader is therefore referred to the literature. ■

The following corollary is an immediate consequence of Theorem 2.33 and the Lax-Milgram lemma. However, we state it explicitly to stress the ellipticity of the single-layer potential in 2D.

Corollary 2.34. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^2$ with $\mathrm{diam}(\Omega) < 1$. Then, the single-layer potential $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is an elliptic and symmetric isomorphism. In particular, given $v \in H^{1/2}(\Gamma)$, there is a unique $\phi \in H^{-1/2}(\Gamma)$ such that $V\phi = v$. Moreover, $\langle\!\langle \phi\,;\,\psi \rangle\!\rangle := \langle V\phi\,;\,\psi\rangle$ defines a scalar product on $H^{-1/2}(\Gamma)$, and the induced norm $|\!|\!|\phi|\!|\!| := \langle\!\langle \phi\,;\,\phi \rangle\!\rangle^{1/2}$ is an equivalent norm on $H^{-1/2}(\Gamma)$.* ■

Remark. (i) One can show that the single-layer potential operator $V \in L\big(H^{-1/2}(\Gamma); H^{1/2}(\Gamma)\big)$ is an isomorphism if and only if $\lambda_{\mathrm{eq}} \neq 0$.
(ii) Moreover, one can check (numerically) for ellipticity as follows:

- Solve $V\phi \equiv 1$. — If the corresponding linear system has no solutions, V is not elliptic.
- Compute $\langle \phi\,;\,1\rangle$. — If the value is less or equal zero, V is not elliptic.
- Otherwise, define $\lambda := 1/\langle \phi\,;\,1\rangle$ and $\widetilde{\phi} := \lambda\phi$. Then, $V\widetilde{\phi} \equiv \lambda$ and $\langle \widetilde{\phi}\,;\,1\rangle = 1$, whence $(\widetilde{\phi}, \lambda)$ solves (2.44). Thus, $(\widetilde{\phi}, \lambda) = (\phi_{\mathrm{eq}}, \lambda_{\mathrm{eq}})$.

In particular, this algorithm computes the capacity λ_{eq}. □

2.8 Ellipticity of Hypersingular Integral Operator

The following elementary lemma is left to the reader. The proof follows along the lines of the proof of Lemma 2.31 and 2.32.

Lemma 2.35. *Let $R > 0$ with $\Omega \subset B_R(0)$ and $y_0 \in \Omega$. Then, there holds, for any $v \in H^{1/2}(\Omega)$ and $x \in \mathbb{R}^d$ with $|x - y_0| \geq 3R$*

$$|\widetilde{K}v(x)| \lesssim \frac{1}{|x-y_0|^{d-1}} = \mathcal{O}(1/|x|^{d-1}) \quad \textit{and} \quad |\nabla\widetilde{K}v(x)| \lesssim \frac{1}{|x-y_0|^{d}} = \mathcal{O}(1/|x|^{d})$$

as $|x| \to \infty$. The constant depends only on ϕ. In particular, there holds $\nabla\widetilde{K}v \in L^2(\Omega^{\mathrm{ext}})$. ■

For the following results, we assume that Γ is piecewise C^2 so that we have integral representations of the double-layer potential and the hypersingular integral operator.

Proposition 2.36. *For $v \in H^{1/2}(\Gamma)$ holds $\widetilde{K}v \in C^\infty(\mathbb{R}^d\backslash\Gamma) \cap H^1(\Omega) \cap H^1_{\ell oc}(\Omega^{\mathrm{ext}})$ with $-\Delta(\widetilde{K}v) = 0$ weakly in $\mathbb{R}^d\backslash\Gamma$. Moreover, the following jump relations are well-defined:*
(i) $[\gamma_0\widetilde{K}v] := \gamma_0^{\mathrm{ext}}\widetilde{K}v - \gamma_0^{\mathrm{int}}\widetilde{K}v = v \in H^{1/2}(\Gamma)$.
(ii) $[\gamma_1\widetilde{K}v] := \gamma_1^{\mathrm{ext}}\widetilde{K}v - \gamma_1^{\mathrm{int}}\widetilde{K}v = 0 \in H^{-1/2}(\Gamma)$.

Proof. **1. step.** We know that $\widetilde{K}v \in C^\infty(\mathbb{R}^d\backslash\Gamma)$ with $-\Delta(\widetilde{K}v) = 0$ pointwise in $\mathbb{R}^d\backslash\Gamma$. The fundamental step in the proof of Theorem 2.7 was the identity

$$\langle \widetilde{K}v\,;\,f\rangle = \langle v\,;\,N_1 f\rangle \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^d),$$

which led to $\widetilde{K}v \in H^1(\Omega)$. The same techniques prove $\widetilde{K}v \in H^1(\Omega_R^{\mathrm{ext}})$ for any $R > 0$, whence $\widetilde{K}v \in H^1_{\ell oc}(\Omega^{\mathrm{ext}})$.

2. step. There holds $[\gamma_0\widetilde{K}v] = v$: Theorem 2.22 provides $\gamma_0^{\mathrm{int}}\widetilde{K} = -\frac{1}{2} + K$. The reader may check the proof to see that $\gamma_0^{\mathrm{ext}}\widetilde{K} = \frac{1}{2} + K$. In particular, the jump satisfies $[\gamma_0\widetilde{K}v] = \gamma_0^{\mathrm{ext}}\widetilde{K}v - \gamma_0^{\mathrm{int}}\widetilde{K}v = v$.

3. step. There holds $[\gamma_1\widetilde{K}v] = 0$: For $f \in \mathcal{D}(\mathbb{R}^d)$, integration by parts proves

$$\langle \gamma_1^{\mathrm{int}}\widetilde{K}v\,;\,\gamma_0^{\mathrm{int}}f\rangle = \Big(\nabla\widetilde{K}v\,;\,\nabla f\Big)_\Omega = \Big(\widetilde{K}v\,;\,-\Delta f\Big)_\Omega + \langle \gamma_0^{\mathrm{int}}\widetilde{K}v\,;\,\gamma_1^{\mathrm{int}}f\rangle$$

as well as

$$\langle \gamma_1^{\mathrm{ext}}\widetilde{K}v\,;\,\gamma_0^{\mathrm{ext}}f\rangle = \Big(\nabla\widetilde{K}v\,;\,\nabla f\Big)_{\Omega^{\mathrm{ext}}} = \Big(\widetilde{K}v\,;\,-\Delta f\Big)_{\Omega^{\mathrm{ext}}} - \langle \gamma_0^{\mathrm{ext}}\widetilde{K}v\,;\,\gamma_1^{\mathrm{ext}}f\rangle.$$

Therefore, $\gamma_0^{\mathrm{int}}f = \gamma_0^{\mathrm{int}}f = \gamma_0 f$ and $\gamma_1^{\mathrm{int}}f = \gamma_1^{\mathrm{ext}}f = \gamma_1 f$ prove

$$\langle[\gamma_1\widetilde{K}v]\,;\,\gamma_0 f\rangle = \Big(\widetilde{K}v\,;\,-\Delta f\Big)_{\mathbb{R}^d} - \langle[\gamma_0\widetilde{K}v]\,;\,\gamma_1 f\rangle = \Big(\widetilde{K}v\,;\,-\Delta f\Big)_{\mathbb{R}^d} - \langle v\,;\,\gamma_1 f\rangle.$$

We consider the scalar product over $\mathbb{R}^d$, which reads

$$\begin{aligned}
\Big(\widetilde{K}v\,;\,-\Delta f\Big)_{\mathbb{R}^d} &= -\int_{\mathbb{R}^d}\Delta f(x)\int_\Gamma \gamma_{1,y}^{\mathrm{int}}G(x-y)v(y)\,ds_y\,dx \\
&= -\int_\Gamma v(y)\gamma_{1,y}^{\mathrm{int}}\int_{\mathbb{R}^d}G(x-y)\Delta f(x)\,dx\,ds_y \\
&= \langle v\,;\,\gamma_1^{\mathrm{int}}\widetilde{N}(-\Delta f)\rangle \\
&= \langle v\,;\,\gamma_1^{\mathrm{int}}f\rangle.
\end{aligned}$$

Here, we have used the Fubini theorem as well as the fact that the Newton potential $\widetilde{N}(-\Delta f)$ belongs to C^∞. The combination of the latter two equalities proves

$$\langle[\gamma_1\widetilde{K}v]\,;\,\gamma_0^{\mathrm{int}}f\rangle = 0 \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^d).$$

Since $\gamma_0^{\mathrm{int}}\big(\mathcal{D}(\mathbb{R}^d)\big)$ is dense in $H^{1/2}(\Gamma)$, the Hahn-Banach theorem proves $[\gamma_1\widetilde{K}v] = 0$. ∎

Corollary 2.37. *For a piecewise C^2 boundary Γ holds $\gamma_0^{\mathrm{ext}}\widetilde{K} = \frac{1}{2} + K$.*

Proof. The proof follows from $v = [\gamma_0\widetilde{K}v] = \gamma_0^{\mathrm{ext}}\widetilde{K}v - \gamma_0^{\mathrm{int}}\widetilde{K}v = \gamma_0^{\mathrm{ext}}\widetilde{K}v - (-\frac{1}{2} + K)v$. ∎

The following theorem states ellipticity of W in $H_*^{-1/2}(\Gamma)$. For its proof, we refer to [SS11].

Theorem 2.38. *The hypersingular integral operator $W \in L\big(H_*^{1/2}(\Gamma); H_*^{-1/2}(\Gamma)\big)$ is an elliptic and symmetric isomorphism. In particular, given $\phi \in H_*^{-1/2}(\Gamma)$, there is a unique $v \in H_*^{1/2}(\Gamma)$ such that $Wv = \phi$. Moreover, $\langle\!\langle v\,;\,w\rangle\!\rangle := \langle Wv\,;\,w\rangle$ defines a scalar product on $H_*^{1/2}(\Gamma)$, and the induced norm $|\!|\!|v|\!|\!| := \langle\!\langle v\,;\,v\rangle\!\rangle^{1/2}$ is an equivalent norm on $H_*^{1/2}(\Gamma)$.*

Chapter 3

Galerkin Boundary Element Method

3.1 Abstract Galerkin Methods

Throughout this section, H is a Hilbert space and $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ is a continuous and elliptic bilinear form on H. For given $F \in H^*$, the Lax-Milgram lemma proves the unique existence of a solution $u \in H$ of

$$\langle\!\langle u\,;\,v \rangle\!\rangle = F(v) \quad \text{for all } v \in H, \tag{3.1}$$

for what we use the short-hand notation

$$\langle\!\langle u\,;\,\cdot \rangle\!\rangle = F \in H^* \tag{3.2}$$

to indicate the set of test functions. Now, the Galerkin projection simply consists in replacing the continuous space H by some finite dimensional subspace: Let X_h be a finite-dimensional (and hence closed) subspace of H. Since the Lax-Milgram lemma applies to the Hilbert space X_h as well, there is a unique **Galerkin solution** $u_h := \mathbb{G}_h u \in X_h$ such that

$$\langle\!\langle \mathbb{G}_h u\,;\,\cdot \rangle\!\rangle = F \in X_h^*. \tag{3.3}$$

For $u \in H$ and the corresponding functional $\langle\!\langle u\,;\,\cdot \rangle\!\rangle \in H^*$, this defines the **Galerkin projection**

$$\mathbb{G}_h : H \to X_h \quad \text{where } \mathbb{G}_h u \in X_h \text{ solves} \quad \langle\!\langle \mathbb{G}_h u\,;\,\cdot \rangle\!\rangle = \langle\!\langle u\,;\,\cdot \rangle\!\rangle \in X_h^*. \tag{3.4}$$

Note that $\mathbb{G}_h u \in X_h$ is characterized by the **Galerkin orthogonality**

$$\langle\!\langle u - \mathbb{G}_h u\,;\,v_h \rangle\!\rangle = 0 \quad \text{for all } v_h \in X_h. \tag{3.5}$$

If $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ is additionally symmetric, it is a scalar product, and the induced norm $|\!|\!| u |\!|\!| := \langle\!\langle u\,;\,u \rangle\!\rangle^{1/2}$ is an equivalent norm on H. In this case $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ and $|\!|\!| \cdot |\!|\!|$ are called **energy scalar product** and **energy norm**, respectively.

Before we proceed with the theoretical analysis of Galerkin schemes, we treat an implementational issue. The following theorem is the fundamental observation: Usually, only the bilinear form $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ and the right-hand side $F \in H^*$ are known, whereas the exact solution $u \in H^*$ of (3.1) is unknown. Then, the Galerkin solution $\mathbb{G}_h u \in X_h$ can be computed without knowing u by solving a linear system of equations.

Theorem 3.1. *Let $\{\phi_1, \dots, \phi_N\}$ be a basis of X_h. We define the Galerkin matrix $A \in \mathbb{R}^{N \times N}$ and the vector $b \in \mathbb{R}^N$ by*

$$A_{jk} := \langle\!\langle \phi_k \,;\, \phi_j \rangle\!\rangle \quad \text{and} \quad b_j := F(\phi_j). \tag{3.6}$$

Then, A is a regular matrix, and $\mathbb{G}_h u = \sum_{j=1}^N x_j \phi_j$, where the vector $x \in \mathbb{R}^N$ solves $Ax = b$. Moreover, if $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$ is symmetric, the Galerkin matrix A is symmetric and positive definite.

Proof. **1. step.** To show that A is regular, we only need to show that A is injective: For any $x \in \mathbb{R}^N$ and $v_h := \sum_{j=1}^N x_j \phi_j$ holds

$$\|v_h\|_H^2 \lesssim \langle\!\langle v_h \,;\, v_h \rangle\!\rangle = \sum_{j,k=1}^N x_j x_k \langle\!\langle \phi_j \,;\, \phi_k \rangle\!\rangle = x \cdot Ax.$$

Therefore, $Ax = 0$ implies $\|v_j\|_H = 0$ and finally $x = 0$.
2. step. Determine Galerkin solution: Let $x \in \mathbb{R}^n$ be the unique solution of the linear Galerkin system $Ax = b$. We use the basis representation $\mathbb{G}_h u = \sum_{j=1}^N y_j \phi_j$ of the Galerkin solution with some coefficient vector $y \in \mathbb{R}^n$. By use of the linearity of $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$, equation (3.3) becomes

$$b_k = F(\phi_k) = \langle\!\langle \mathbb{G}_h u \,;\, \phi_k \rangle\!\rangle = \sum_{j=1}^N y_j \langle\!\langle \phi_j \,;\, \phi_k \rangle\!\rangle = (Ay)_k \quad \text{for all } k = 1, \dots, N.$$

Therefore, the coefficient vector $y \in \mathbb{R}^N$ satisfies $Ay = b$. This proves $x = y$, i.e., we obtain $\mathbb{G}_h u$ by solving $Ax = b$.
3. step. If $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$ is symmetric, the matrix A is symmetric as well. Moreover, step 1 proves even positive definiteness of A. ∎

Remark. We just remark that Theorem 3.1 can be applied for *any* orthogonal-type projection, e.g., the L^2-orthogonal projection onto a discrete space. □

We now proceed with the abstract analysis of Galerkin schemes. The following two lemmata provide elementary properties of the Galerkin projection. The first lemma, proves stability of the method with respect to changes of the right-hand side F.

Lemma 3.2. *The Galerkin projection $\mathbb{G}_h$ is a linear and continuous projection onto X_h. If $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$ is symmetric, $\mathbb{G}_h$ is the orthogonal projection onto X_h with respect to the energy scalar product $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$.*

Proof. For $u_h \in X_h$, the Galerkin orthogonality (3.5) implies $\mathbb{G}_h u_h = u_h$. Therefore $\mathbb{G}_h$ is a projection onto X_h. Also the linearity of $\mathbb{G}_h$ follows from the Galerkin orthogonality (3.5). To see the continuity of $\mathbb{G}_h$ it remains to estimate the operator norm: For $u \in H$ holds

$$\|\mathbb{G}_h u\|_H^2 \lesssim \langle\!\langle \mathbb{G}_h u \,;\, \mathbb{G}_h u \rangle\!\rangle = \langle\!\langle u \,;\, \mathbb{G}_h u \rangle\!\rangle \lesssim \|u\|_H \|\mathbb{G}_h u\|_H,$$

where we have used ellipticity and continuity of $\langle\!\langle \cdot \,;\, \cdot \rangle\!\rangle$. This proves $\|\mathbb{G}_h u\|_H \lesssim \|u\|_H$ and thus continuity of $\mathbb{G}_h$, where the operator norm is bounded by the quotient of continuity and ellipticity constant. Finally, if $\langle\!\langle \cdot\,;\cdot \rangle\!\rangle$ is a scalar product, the *unique* orthogonal projection with respect to this scalar product, is characterized by the orthogonality relation (3.5). ∎

The following Céa lemma states that the **Galerkin error** $\|u - \mathbb{G}_h u\|_H$ is quasi-optimal, i.e. it behaves like the best approximation error up to multiplicative constants, which depend only on the continuous setting but not on X_h.

Lemma 3.3 (Céa). *The Galerkin error is quasi-optimal, i.e.*

$$\|u - \mathbb{G}_h u\|_H \lesssim \min_{v_h \in X_h} \|u - v_h\|_H \quad \textit{for all } u \in H, \tag{3.7}$$

where the constant depends only on the ellipticity and the continuity of $\langle\!\langle \cdot\,;\cdot \rangle\!\rangle$. If $\langle\!\langle \cdot\,;\cdot \rangle\!\rangle$ is symmetric, there holds

$$|\!|\!| u - \mathbb{G}_h u |\!|\!| = \min_{v_h \in X_h} |\!|\!| u - v_h |\!|\!| \quad \textit{for all } u \in H, \tag{3.8}$$

i.e. the Galerkin solution $\mathbb{G}_h u$ is the best approximation of u with respect to the energy norm.

Proof. For arbitrary $v_h \in X_h$, the Galerkin orthogonality (3.5) proves

$$\|u - \mathbb{G}_h u\|_H^2 \lesssim \langle\!\langle u - \mathbb{G}_h u\,;u - \mathbb{G}_h u \rangle\!\rangle = \langle\!\langle u - \mathbb{G}_h u\,;u - v_h \rangle\!\rangle \lesssim \|u - \mathbb{G}_h u\|_H \|u - v_h\|_H$$

with the same arguments as in the proof of the last lemma. This leads to (3.7) with an infimum on the right-hand side. If we replace $\mathbb{G}_h$ by the orthogonal projection Π_h onto X_h with respect to $\|\cdot\|_H$, we see that all inequalities of our estimate hold with constant 1. This proves that the minimum in (3.7) is attained for $v_h = \Pi_h u$. Even the same argument proves (3.8). ∎

A major advantage of Galerkin methods is, that one can prove convergence to the exact solution $u \in H$. In the following, think of the subscript $h > 0$ as a mesh-size parameter with corresponding finite dimensional spaces X_h:

Proposition 3.4. *We assume that there is a dense subspace D of H with approximation property, namely*

$$\lim_{h \to 0} \min_{v_h \in X_h} \|v - v_h\|_H = 0 \quad \textit{for all } v \in D. \tag{3.9}$$

Then, for any $u \in H$, there holds

$$\lim_{h \to 0} \|u - \mathbb{G}_h u\|_H = 0, \tag{3.10}$$

i.e. the sequence of Galerkin solutions converges to the exact solution u.

Proof. For $v \in D$ holds the estimate

$$\begin{aligned}\|u - \mathbb{G}_h u\|_H &\leq \|u - v\|_H + \|v - \mathbb{G}_h v\|_H + \|\mathbb{G}_h u - \mathbb{G}_h v\|_H \\ &\lesssim \|u - v\|_H + \min_{v_h \in X_h} \|v - v_h\|_H \\ &=: C\big(\|u - v\|_H + \min_{v_h \in X_h} \|v - v_h\|_H\big)\end{aligned}$$

by use of continuity of $\mathbb{G}_h$ and the quasi-optimality estimate (3.7). We have to show that

$$\exists C > 0 \forall \varepsilon > 0 \exists h_0 > 0 \forall h \in (0, h_0) \quad \|u - \mathbb{G}_h u\|_H \leq \varepsilon.$$

For $\varepsilon > 0$, let $v \in D$ with $\|u - v\|_H \leq \varepsilon$. Choose $h_0 > 0$ according to the approximation assumption (3.10) so that $\|v - \mathbb{G}_h v\|_H \leq \varepsilon$ for all $h \in (0, h_0)$. We thus finally obtain $\|u - \mathbb{G}_h u\|_H \leq 2C\varepsilon$, which concludes the proof. ■

Although the result of the preceding lemma seems to be very attractive, we stress, however, that the convergence of a Galerkin scheme can be arbitrarily slow. We argue in the abstract setting: If H is a separable Hilbert space, e.g., H is one of the introduced Sobolev spaces, there is a countable orthonormal basis $\{\phi_j \,|\, j \in \mathbb{N}\}$. Any $u \in H$ can be written as $u = \sum_{j=1}^{\infty} x_j \phi_j$ with coefficients $(x_n) \in \ell_2$. If we define $X_j := \operatorname{span}\{\phi_1, \dots, \phi_j\}$, there holds

$$\min_{v_h \in X_h} \|u - v_h\|_H^2 = \sum_{j=k+1}^{\infty} x_j^2.$$

Finally, the decrease of the right-hand side can be very slow. One may think of, e.g., $x_j^2 = j^{-(1+\varepsilon)}$ for any $\varepsilon > 0$, so that the series converges but is — in the beginning — almost the divergent harmonic series.

Another important fact is that the Galerkin scheme is stable with respect to certain perturbations of the bilinear form $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ or the right-hand side F due to the so called Strang lemma.

3.2 A-Priori Error Analysis

We now turn our attention to Symm's integral equation of finding $\phi \in H^{-1/2}(\Gamma)$ such that

$$V\phi = f \tag{3.11}$$

for given right-hand side $f \in H^{1/2}(\Gamma)$. Our goal is to provide an *a-priori* estimate for the Galerkin error to quantify the speed of convergence.
For the Galerkin boundary element method, we choose piecewise polynomial spaces X_h: Let $\mathcal{T}_h$ be a triangulation of Γ, i.e.

- $\mathcal{T}_h = \{T_1, \dots, T_N\}$ is a finite set of subsets $T_j \subseteq \Gamma$,
- each $T_j \in \mathcal{T}_h$ is (relatively) open and connected with positive surface measure $|T_j| > 0$,
- for $T_j, T_k \in \mathcal{T}_h$ with $j \neq k$ holds $T_j \cap T_k = \emptyset$,

- $\Gamma = \bigcup \{ \overline{T} \,|\, T \in \mathcal{T}_h \}$, i.e. $\mathcal{T}_h$ is a covering of Γ.

For the ease of presentation, we additionally assume that the elements $T \in \mathcal{T}_h$ are flat, i.e. there is an open and connected set $V_T \subset \mathbb{R}^{d-1}$ and an affine bijection $\Phi_T : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi_T(V_T) = T$. Moreover, we assume that the elements are convex — this clearly holds in 2D and is the common case in 3D, where the elements usually are flat triangles or rectangles. With χ_T the characteristic function of a set T, we consider the space

$$\mathcal{P}^0(\mathcal{T}_h) := \operatorname{span}\{ \chi_T \,|\, T \in \mathcal{T}_h \} \tag{3.12}$$

of all $\mathcal{T}_h$-piecewise constant functions. We define the **local mesh-width**

$$h \in \mathcal{P}^0(\mathcal{T}_h), \quad h|_{T_j} = h_{T_j} := \operatorname{diam}(T_j) := \sup_{x,y \in T_j} |x - y| \tag{3.13}$$

as well as the **maximal mesh-width**

$$h_{\max} := \|h\|_{L^\infty(\Gamma)} = \max_{T \in \mathcal{T}_h} h_T. \tag{3.14}$$

Moreover, we define the **shape regularity constant**

$$\sigma(\mathcal{T}_h) := \max_{T \in \mathcal{T}_h} \frac{h_T^{d-1}}{|T|} \tag{3.15}$$

to measure the degeneracy of the elements in $\mathcal{T}_h$.
Remark. In the finite element analysis, the shape regularity constant from (3.15) involves h_T^d instead of h_T^{d-1}. We stress that in the context of boundary elements h_T^{d-1} coincides to the fact that we are dealing with $(d-1)$-dimensional manifolds.

Theorem 3.5 (Approximation Theorem). *Let $\Pi_h : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_h)$ denote the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$. For $\psi \in L^2(\Gamma) \cap H^1(\mathcal{T}_h)$, holds*

$$\|\psi - \Pi_h \psi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{3/2} \nabla_{\mathcal{T}} \psi\|_{L^2(\Gamma)}, \tag{3.16}$$

where the constant only depends on the shape regularity constant $\sigma(\mathcal{T}_h)$. Here, $\psi \in H^1(\mathcal{T}_h)$ means that $\psi \in H^1(T)$ for all $T \in \mathcal{T}_H$, and $\nabla_{\mathcal{T}}$ thus denotes the $\mathcal{T}_h$-elementwise gradient.

Proof. The elementary proof of (3.16) is split into four steps.
1. step. The L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$ can explicitly be written as

$$(\Pi_h v)|_T = \frac{1}{|T|} \int_T v \, ds \quad \text{for all } v \in L^2(\Gamma) \text{ and all } T \in \mathcal{T}_h. \tag{3.17}$$

This follows from the orthogonality property

$$0 = (v - \Pi_h v \,;\, \chi_T)_{L^2(\Gamma)} = \int_T v \, ds - \int_T (\Pi_h v) \, ds = \int_T v \, ds - |T| \, (\Pi_h v)|_T \quad \text{for all } T \in \mathcal{T}_h.$$

2. step. According to the Poincaré inequality, there holds, for any $T \in \mathcal{T}_h$,

$$\|\psi - \Pi_h \psi\|_{L^2(T)} \le \frac{1}{\pi} h_T \|\nabla \psi\|_{L^2(T)},$$

where convexity of $T \in \mathcal{T}_h$ provides the Poincaré constant $1/\pi$.

3. step. For any $v \in H^{1/2}(\Gamma)$ and $T \in \mathcal{T}_h$ holds

$$\|v - \Pi_h v\|_{L^2(T)} \leq \sigma(\mathcal{T}_h)^{1/2}\, h_T^{1/2}\, |v|_{H^{1/2}(T)}.$$

We recall that the $H^{1/2}$-Sobolev-Slobodeckij seminorm is defined by

$$|v|_{H^{1/2}(T)} = \Big(\int_T \int_T \frac{|v(x) - v(y)|^2}{|x-y|^d}\, ds_y\, ds_x \Big)^{1/2}.$$

For fixed $x \in T$, the closed form of $\Pi_h v$ from step 1 and the Cauchy inequality prove

$$\begin{aligned}
|v(x) - \Pi_h v(x)|^2 &= \frac{1}{|T|^2} \Big(\int_T v(x) - v(y)\, ds_y \Big)^2 \\
&\leq \frac{1}{|T|^2} \Big(\int_T \frac{|v(x) - v(y)|^2}{|x-y|^d}\, ds_y \Big) \Big(\int_T |x-y|^d\, ds_y \Big) \\
&\leq \frac{h_T^{d-1}}{|T|}\, h_T \int_T \frac{|v(x) - v(y)|^2}{|x-y|^d}\, ds_y.
\end{aligned}$$

Integration over T now yields

$$\|v - \Pi_h v\|_{L^2(T)}^2 \leq \sigma(\mathcal{T}_h)\, h_T\, |v|_{H^{1/2}(T)}^2.$$

4. step. Finally, we estimate the dual norm

$$\|\psi - \Pi_h \psi\|_{H^{-1/2}(\Gamma)} = \sup_{v \in H^{1/2}(\Gamma) \setminus \{0\}} \frac{\langle \psi - \Pi_h \psi\,;\, v \rangle}{\|v\|_{H^{1/2}(\Gamma)}}.$$

Let $v \in H^{1/2}(\Gamma)$. We stress that the duality brackets are just the L^2-scalar product since both $\psi - \Pi_h \psi, v \in L^2(\Gamma)$. Orthogonality of Π_h provides

$$\langle \psi - \Pi_h \psi\,;\, v \rangle = (\psi - \Pi_h \psi\,;\, v - \Pi_h v)_{L^2(\Gamma)} = \sum_{T \in \mathcal{T}_h} (\psi - \Pi_h \psi\,;\, v - \Pi_h v)_{L^2(T)}.$$

For fixed $T \in \mathcal{T}_h$ holds

$$\begin{aligned}
(\psi - \Pi_h \psi\,;\, v - \Pi_h v)_{L^2(T)} &\leq \|\psi - \Pi_h \psi\|_{L^2(T)} \|v - \Pi_h v\|_{L^2(T)} \\
&\leq \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi}\, h_T^{3/2}\, \|\nabla \psi\|_{L^2(T)} |v|_{H^{1/2}(T)}.
\end{aligned}$$

Therefore, the Cauchy inequality proves

$$\begin{aligned}
\langle \psi - \Pi_h \psi\,;\, v \rangle &\leq \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi} \Big(\sum_{T \in \mathcal{T}} h_T^3 \|\nabla \psi\|_{L^2(T)}^2 \Big)^{1/2} \Big(\sum_{T \in \mathcal{T}} |v|_{H^{1/2}(T)}^2 \Big)^{1/2} \\
&\leq \frac{\sigma(\mathcal{T}_h)^{1/2}}{\pi}\, \|h^{3/2} \nabla_{\mathcal{T}} \psi\|_{L^2(\Gamma)}\, |v|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

This concludes the proof. ∎

Remark. We stress that the same techniques as in the preceding proof yield

$$\|\psi - \Pi_h\psi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{1/2}(\psi - \Pi_h\psi)\|_{L^2(\Gamma)} \le \|h^{1/2}\psi\|_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma). \tag{3.18}$$

Here, the second inequality follows from the Pythagoras theorem (i.e. the L^2-orthogonality)

$$\|\psi - \Pi_h\psi\|^2_{L^2(T)} = \|\psi\|^2_{L^2(T)} - \|\Pi_h\psi\|^2_{L^2(T)} \le \|\psi\|^2_{L^2(T)}.$$

This $\mathcal{T}_h$-elementwise estimate is simply weighted by h_T and then added over all $T \in \mathcal{T}_h$.
The combination of Céa-Lemma and approximation theorem provides an a-priori error estimate.

Corollary 3.6 (A-Priori Estimate for Galerkin Error). *Provided that the exact solution $\phi \in H^{-1/2}(\Gamma)$ of (3.11) satisfies $\phi \in L^2(\Gamma) \cap H^1(\mathcal{T}_h)$, there holds*

$$|||\phi - \phi_h||| \lesssim \|h^{3/2}\nabla\phi\|_{L^2(\Gamma)}, \tag{3.19}$$

where the constant only depends on Γ and the shape regularity constant $\sigma(\mathcal{T}_h)$.

Proof. First recall that the energy norm $|||\cdot|||$ is an equivalent norm on $H^{-1/2}(\Gamma)$, i.e.

$$C_{\text{lower}}\|\psi\|_{H^{-1/2}(\Gamma)} \le |||\psi||| \le C_{\text{upper}}\|\psi\|_{H^{-1/2}(\Gamma)} \quad \text{for all } \psi \in H^{-1/2}(\Gamma).$$

The lower constant $C_{\text{lower}} > 0$ is just the square-root of the ellipticity constant of V, whereas $C_{\text{upper}} > 0$ is the square-root of the operator norm of V, i.e. both constants depend on Γ only. With the L^2-orthogonal projection onto $\mathcal{P}^0(\mathcal{T}_h)$, the Céa-Lemma proves

$$|||\phi - \phi_h||| \le |||\phi - \Pi_h\phi||| \approx \|\phi - \Pi_h\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|h^{3/2}\nabla\phi\|_{L^2(\Gamma)},$$

where we have used that the energy norm $|||\cdot|||$ is an equivalent norm on $H^{-1/2}(\Gamma)$. ∎

The preceding corollary proves that

$$|||\phi - \phi_h||| = \mathcal{O}(h^{3/2}_{\max})$$

in the case that ϕ is sufficiently regular and that the shape regularity constant remains bounded. Finally, we prove that — even without any further regularity assumptions on the exact solution $\phi \in H^{-1/2}(\Gamma)$ — the sequence of Galerkin solutions ϕ_h converges to ϕ. To that end, we consider a sequence $\mathcal{T}_h^{(n)}$ of triangulations with

$$\mathcal{P}^0(\mathcal{T}_h^{(n)}) \subseteq \mathcal{P}^0(\mathcal{T}_h^{(n+1)}),$$

i.e. $\mathcal{T}_h^{(n+1)}$ is obtained from certain refinements of $\mathcal{T}_h^{(n)}$. Let $\phi_h^{(n)} \in \mathcal{P}^0(\mathcal{T}_h^{(n)})$ the sequence of corresponding Galerkin solutions.

Corollary 3.7 (Convergence of Galerkin Method). *Provided that*

$$\sigma := \sup_{n\in\mathbb{N}} \sigma(\mathcal{T}_h^{(n)}) < \infty \quad \textit{and} \quad \lim_{n\to\infty} h^{(n)}_{\max} = 0,$$

there holds convergence $\lim_{n\to\infty} |||\phi - \phi_h^{(n)}||| = 0.$

Proof. Note that $H^1(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$. Given $\varepsilon > 0$, we therefore find $\psi \in H^1(\Gamma)$ such that $|||\phi - \psi||| \leq \varepsilon$. According to the a priori error estimate, there holds $|||\psi - \mathbb{G}_h^{(n)}\psi||| = \mathcal{O}((h_{\max}^{(n)})^{3/2})$. Therefore, there is $n_0 \in \mathbb{N}$ such that

$$|||\psi - \mathbb{G}_h^{(n)}\psi||| \leq \varepsilon$$

for any $n \geq n_0$. The triangle inequality now proves

$$|||\phi - \phi_h^{(n)}||| \leq |||\phi - \psi||| + |||\psi - \mathbb{G}_h^{(n)}\psi||| + |||\mathbb{G}_h^{(n)}\psi - \phi_h^{(n)}||| \leq 3\varepsilon \quad \text{for all } n \geq n_0,$$

where we have used $\mathbb{G}_h^{(n)}\psi - \phi_h^{(n)} = \mathbb{G}_h^{(n)}(\psi - \phi)$ as well as $|||\mathbb{G}_h^{(n)}(\psi - \phi)||| \leq |||\psi - \phi|||$. This proves convergence. ■

Remark. Since the step functions are dense in $L^2(\Gamma)$, one can prove that

$$\mathcal{P}^0(\mathcal{T}_h^{(n)}) \subseteq \mathcal{P}^0(\mathcal{T}_h^{(n+1)}) \quad \text{and} \quad \lim_{n\to\infty} h_{\max}^{(n)} = 0$$

implies that $X := \bigcup_{n\in\mathbb{N}} \mathcal{P}^0(\mathcal{T}_h^{(n)})$ is dense in $L^2(\Gamma)$ as well. Recalling that $L^2(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$, we derive that X is dense in $H^{-1/2}(\Gamma)$ as well. In particular, this proves convergence of the Galerkin boundary element method without the additional assumption of

$$\sigma := \sup_{n\in\mathbb{N}} \sigma(\mathcal{T}_h^{(n)}) < \infty.$$

We stress, however, that this is a special observation for piecewise constant ansatz functions and negative-order Sobolev spaces. The proof of Corollary 3.7 even applies for the finite element method and positive-order Sobolev spaces, e.g., $H^1(\Omega)$.

Remark. We finish the part about the boundary element method by stressing some advantages and disadvantages of the Galerkin BEM for the boundary integral equation compared to the finite element method (FEM) that is often used to compute numerical solutions to the PDE.
The advantages are:

+ BEM is suitable for exterior (unbounded) problems as well as transmission problems that often appear in applications such as wave scattering.
+ The BEM converges with order $h^{3/2}$ in the energy norm for sufficiently smooth solutions, whereas the FEM converges with order h in the energy norm.
+ The boundary integral formulation gives rise to a formulation in $\mathbb{R}^{d-1}$, i.e., the discretization has to be done in one less dimension.
+ The condition number of the Galerkin matrix (for a uniform mesh) is $\kappa(A) \sim \mathcal{O}(h^{-1})$.

However, there are significant disadvantages as well, such as:

- For basis functions ϕ_i, ϕ_j with $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j \neq \emptyset$, the integrals $\int_{\operatorname{supp}\phi_i}\int_{\operatorname{supp}\phi_j} G(x-y)\phi_j(y)\phi_i(x)ds_y ds_x$ have a singular integrand and can therefore not be treated with classical Gaussian quadrature. However, these integrals can be computed (semi-)analytical or be treated with so-called Sauter-Schwab quadrature (see [SS11]). A main idea hereby, is to use a Duffy transformation to get rid of the point singularity.

- Therefore, obtaining a stable implementation is way harder!
- Boundary integral operators are non-local. Therefore, the Galerkin discretization leads to fully populated matrices $A \in \mathbb{R}^{N\times N}$. Storing these or doing matrix-vector multiplications leads to complexity $\mathcal{O}(N^2)$, which is not feasibly for large N. This gave rise to a whole lot of research on fast boundary element methods, such as the fast multipole method (FMM) or hierarchical matrices ($\mathcal{H}$-matrices). A main idea hereby is to approximate the kernel function of the integral operators on suitably separated subsets, since the kernel function is smooth provided $x \neq y$. Doing this in a right way gives computable approximations with storage/matrix-vector-multiplication complexity of $\mathcal{O}(N\log(N))$ and error that decays exponentially.

Part II

Part II: Fractional Differential Operators

Chapter 4

Definitions of the Fractional Laplacian

4.1 The Integral Fractional Laplacian

4.1.1 Probabilistic Motivation

As a motivation for the fractional Laplacian, we start with a random walk approach with arbitrary long jumps.
Let P be a probability measure on $\mathbb{N}$ given by

$$P(I) := c(s) \sum_{j \in I} \frac{1}{j^{1+2s}} \qquad \text{for } I \subset \mathbb{N},$$

where $c(s) := \left(\sum_{j=1}^{\infty} \frac{1}{j^{1+2s}} \right)^{-1}$ is a normalization constant (depending on s) such that $P(\mathbb{N}) = 1$.

We study the motion of a particle described as follows: We assume that the particle does discrete jumps and denote by h the minimal possible jump-width in space and by τ the step size in time and link them by $\tau = h^{2s}$.
By $u(x,t)$ we denote the probability of a particle being at time t at the place x. The particle moves in the way that for each timestep τ, it chooses a random direction $\nu \in \partial B_1(0) \subset \mathbb{R}^d$ according to a uniform distribution as well as a $j \in \mathbb{N}$ according to the probability distribution P and it makes a step in space in the direction $jh\nu$.

Example. In one space dimension $d = 1$ and allowing only forward/backward jumps of length h both with probability $1/2$ gives a classical random walk. Then, we have

$$u(x, t+\tau) = \frac{1}{2} u(x+h, t) + \frac{1}{2} u(x-h, t)$$

and assuming $2\tau = h^2$

$$\frac{u(x, t+\tau) - u(x,t)}{\tau} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}.$$

Taking the limit $h, \tau \to 0$ gives the heat equation

$$u_t = \Delta u. \qquad \blacksquare$$

In contrast to the previous example, we allow jumps of arbitrary length (although long jumps have small probability).
The value $u(x, t+\tau)$ is given by the sum of all probabilities of the particle being at time t at location $x + jh\nu$ multiplied with the probability of jumping from there to x, i.e.,

$$u(x, t+\tau) = \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \sum_{j \in \mathbb{N}} \frac{u(x + jh\nu, t)}{j^{1+2s}} ds_\nu,$$

where the constant in front of the integral is the right normalization of both probability distributions. This gives

$$u(x, t+\tau) - u(x,t) = \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \sum_{j \in \mathbb{N}} \frac{u(x + jh\nu, t) - u(x,t)}{j^{1+2s}} ds_\nu.$$

For fixed $\nu \in \partial B_1(0)$, we define $\psi_\nu(z, x, t) := \frac{u(x+z\nu,t)-u(x,t)}{|z|^{1+2s}}$ and using $\tau = h^{2s}$, we obtain

$$\frac{u(x, t+\tau) - u(x,t)}{\tau} = \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \sum_{j \in \mathbb{N}} h\psi_\nu(jh, x, t) ds_\nu.$$

Now, the integrand is just a Riemann-sum for the integral $\int_0^\infty \psi_\nu(z, x, t) dz$. Taking (formally) the limit $h \to 0$ and using polar coordinates $y = ze^{i\varphi}$, we arrive at

$$\begin{aligned} u_t(x,t) &= \frac{c(s)}{|\partial B_1(0)|} \int_{\partial B_1(0)} \int_0^\infty \psi_\nu(z,x,t) dz ds_\nu \\ &= \int_{\mathbb{R}^d} \frac{u(x+y,t) - u(x,t)}{|y|^{d+2s}} dy \\ &=: C(d,s)(-\Delta)^s u(x,t). \end{aligned}$$

Therefore, the limit of a random walk with arbitrary long jumps leads to the so-called fractional heat equation.

4.1.2 The Integral Fractional Laplacian

The previous example motivates the definition of the fractional Laplacian as a non-local singular integral operator.
We, at first, formally define it on Schwarz-functions in the space

$$\mathcal{S} := \Big\{ u \in C^\infty(\mathbb{R}^d) \; : \; \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta u(x)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^d \Big\}.$$

Definition 4.1. *Let $s \in (0,1)$ and $u \in \mathcal{S}$. Then, the pointwise defined operator*

$$(-\Delta)^s u(x) := C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \tag{4.1}$$
$$= C(d,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy$$

is called **integral fractional Laplacian**. *Here, P.V. denotes the Cauchy principal value and the appearing normalization constant is given by $C(d,s) := \frac{2^{2s} s\Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}$ with the gamma-function $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$.*

Remark. The operator $(-\Delta)^s$ is well-defined pointwise for functions in $\mathcal{S}$. Let $\varepsilon > 0$ be fixed and choose an arbitrary $R > \varepsilon$. Then,

$$\int_{\mathbb{R}^d \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy = \int_{B_R(x) \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy + \int_{B_R(x)^c} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy. \tag{4.2}$$

For the first integral, we use Taylor expansion $u(x)-u(y) = (x-y)\cdot\nabla u(x) + \frac{1}{2}(x-y)^T D^2 u(\zeta)(x-y)$, where $\zeta = x + t(y-x)$ for some $t \in (0,1)$, and Polar coordinates to obtain

$$\int_{B_R(x)\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy = \int_{B_R(x)\setminus B_\varepsilon(x)} \frac{(x-y)\cdot\nabla u(x)}{|x-y|^{d+2s}} dy + \int_{B_R(x)\setminus B_\varepsilon(x)} \frac{(x-y)^T D^2 u(\zeta)(x-y)}{2|x-y|^{d+2s}} dy$$
$$= \int_{r=\varepsilon}^{R} \int_{\nu \in S^{d-1}} \frac{r\nu \cdot \nabla u(x)}{r^{d+2s}} r^{d-1}\, d\nu dr + \int_{r=\varepsilon}^{R} \int_{\nu\in S^{d-1}} \frac{r\nu^T D^2 u(\zeta) r\nu}{2r^{d+2s}} r^{d-1}\, d\nu dr.$$

Here, the first integral vanishes since the integral in r is bounded and the inner integral vanishes:

$$\int_{r=\varepsilon}^{R} \int_{\nu \in S^{d-1}} \frac{r\nu \cdot \nabla u(x)}{r^{d+2s}} r^{d-1}\, d\nu dr = \nabla u(x) \cdot \int_{r=\varepsilon}^{R} r^{-2s} \int_{\nu\in S^{d-1}} \nu d\nu dr = 0.$$

For the second integral, we estimate

$$\int_{r=\varepsilon}^{R} \int_{\nu\in S^{d-1}} \frac{r\nu^T D^2 u(\zeta) r\nu}{2r^{d+2s}} r^{d-1}\, d\nu dr \lesssim \|u\|_{C^2(\mathbb{R}^d)} \int_{r=\varepsilon}^{R} r^{1-2s} = C(R^{2-2s} - \varepsilon^{2-2s}) \lesssim C \quad \forall \varepsilon > 0,$$

i.e., the limit $\varepsilon \to 0$ exists. Finally, the last term in (4.2) can be simply bounded using Polar coordinates by

$$\int_{B_R(x)^c} \frac{u(x)-u(y)}{|x-y|^{d+2s}} dy \lesssim 2\|u\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(x)^c} \frac{1}{|x-y|^{d+2s}} dy \lesssim \int_{r=R}^{\infty} \frac{1}{r^{1+2s}} dr \simeq R^{-2s} < \infty.$$

Therefore, we have shown that the pointwise definition is well-defined. ■

Remark. For $s \in (0, 1/2)$ the Cauchy principal value in the integral in the definition of the fractional Laplacian is not necessary, since it exists as an improper integral. To see this, we again employ the splitting of (4.2) (with $\varepsilon = 0$) and only investigate the first term (the second term follows directly as in the previous remark). Using the Taylor expansion $u(x) - u(y) = (x - y) \cdot \nabla u(\zeta)$ and Polar coordinates, we obtain

$$\int_{B_R(x)} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy \leq \|u\|_{C^1(\mathbb{R}^d)} \int_{B_R(x)} \frac{1}{|x - y|^{d+2s-1}} dy \simeq \int_{r=0}^{R} \frac{1}{r^{-2s}} dr < \infty,$$

since $s < 1/2$ implies $1 - 2s > 0$ and, consequently, the last integral exists. ∎

The definition employing the Cauchy principal value is oftentimes not handy for computations. The following lemma gives an equivalent representation of the integral fractional Laplacian without the need of the principal value (even for $s \geq 1/2$) by using a weighted difference quotient of second order.

Lemma 4.2. *Let $s \in (0, 1)$, $u \in \mathcal{S}$ and $(-\Delta)^s u$ given by Definition 4.1. Then, for $x \in \mathbb{R}^d$, we have*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(d, s) \int_{\mathbb{R}^d} \frac{u(x + y) - 2u(x) + u(x - y)}{|y|^{d+2s}} dy.$$

Proof. Using the transformations $z = y - x$ as well as $\widehat{z} = -z$, we obtain

$$\begin{aligned}(-\Delta)^s u(x) &= -C(d, s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x + z) - u(x)}{|z|^{d+2s}} dz \\ &= -C(d, s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x - \widehat{z}) - u(x)}{|\widehat{z}|^{d+2s}} dz.\end{aligned}$$

Relabeling $\widehat{z}$ to z and adding both equations leads to

$$2(-\Delta)^s u(x) = -C(d, s)\ P.V. \int_{\mathbb{R}^d} \frac{u(x + z) - 2u(x) + u(x - z)}{|z|^{d+2s}} dz.$$

It remains to show that the integral indeed exists as an improper integral. We use the same arguments as in the previous remarks, i.e., we split $\mathbb{R}^d = \mathbb{B}_R(x) \cup \mathbb{B}_R(x)^c$ and use Taylor expansion. Here, the second order difference quotient has the Taylor expansion

$$\begin{aligned}u(x + z) - 2u(x) + u(x - z) &= u(x) + z \cdot \nabla u(x) + \frac{1}{2} z^T D^2 u(\zeta_1) z - 2u(x) \\ &\quad + u(x) - z \cdot \nabla u(x) + \frac{1}{2} z^T D^2 u(\zeta_2) z \\ &= \mathcal{O}(z^2).\end{aligned}$$

Therefore, we obtain with Polar coordinates

$$\int_{B_R(x)} \frac{u(x + z) - 2u(x) + u(x - z)}{|z|^{d+2s}} dz \lesssim \|u\|_{C^2(\mathbb{R}^d)} \int_{B_R(x)} \frac{1}{|z|^{d+2s-2}} dz \simeq \int_{r=0}^{R} r^{1-2s} dr < \infty.$$

Since

$$\int_{B_R(x)^c} \frac{u(x + z) - 2u(x) + u(x - z)}{|z|^{d+2s}} dz \lesssim 3\|u\|_{L^\infty(\mathbb{R}^d)} \int_{B_R(x)^c} \frac{1}{|z|^{d+2s}} dz \simeq \int_{r=R}^{\infty} r^{-1-2s} dr < \infty,$$

this gives the existence of the integral. ∎

4.2 The Fourier Definition

We have already mentioned that the Sobolev spaces $H^t(\mathbb{R}^d)$ for $t \in \mathbb{R}$ can be characterized using the Fourier transformation $\mathcal{F}$ as

$$H^t(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} (1 + |\zeta|^2)^t |\mathcal{F}u(\zeta)|^2 \, d\zeta < \infty\}$$

due to the fact that the Fourier transformation turns derivatives into multiplications in the Fourier image.
Using this observation, a possible way of defining fractional powers of differential operators would be to do the multiplication with $|\zeta|^{2s}$ in the Fourier space (which is well-defined for $u \in \mathcal{S}$) and then transform back, i.e., define the fractional Laplacian as

$$\mathcal{F}^{-1}(|\zeta|^{2s}\mathcal{F}u(\zeta)).$$

For an operator $\mathcal{A}$, in literature, a function $S_{\mathcal{F}}(\zeta) : \mathbb{R}^d \to \mathbb{R}$ is called Fourier-symbol of $\mathcal{A}$ if

$$\mathcal{F}(\mathcal{A}u)(\zeta) = S_{\mathcal{F}}(\zeta)\mathcal{F}u(\zeta).$$

The following theorem shows, that the Fourier-symbol of the integral fractional Laplacian $(-\Delta)^s$ is indeed $|\zeta|^{2s}$, i.e., the integral and the Fourier definition are equivalent.

Theorem 4.3. *Let $s \in (0,1)$ and $(-\Delta)^s$ be the integral fractional Laplacian. Then, for $u \in \mathcal{S}$, we have*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\zeta|^{2s}\mathcal{F}u).$$

The proof of this theorem needs the following technical Lemma, which provides a different representation for the constant $C(d,s)$.

Lemma 4.4. *The constant $C(d,s)$ from Definition 4.1 satisfies*

$$C(d,s) = \left(\int_{\mathbb{R}^d} \frac{1-\cos(z_1)}{|z|^{d+2s}} dz\right)^{-1}.$$

Proof. See [BV16, Lem. 3.1.3.]. ∎

Proof of Theorem 4.3. We use Lemma 4.2 to write for $x \in \mathbb{R}^d$

$$(-\Delta)^s u(x) = -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}} dy.$$

1. step. We show that the integrand is in $L^1(\mathbb{R}^d)$: With the same arguments as in the proof of Lemma 4.2 (Taylor-expansion), we estimate

$$\begin{aligned}\frac{|u(x+y) - 2u(x) + u(x-y)|}{|y|^{d+2s}} &\leq \chi_{B_1(0)}(y)|y|^{2-d-2s}\|D^2 u\|_{L^\infty(B_1(x))} \\ &\quad + \chi_{B_1(0)^c}(y)|y|^{-d-2s}|u(x+y) - 2u(x) + u(x-y)| \\ &\lesssim (1+|x|^{d+1})^{-1}\Big(\chi_{B_1(0)}(y)|y|^{2-d-2s} + \chi_{B_1(0)^c}(y)|y|^{-d-2s}\Big) \in L^1(\mathbb{R}^{2d})\end{aligned}$$

using that $u \in \mathcal{S}$. Therefore, we may use Fubini's theorem.

2. step. We use $\mathcal{F}(u(x+y))(\zeta) = e^{i\zeta\cdot y}\mathcal{F}(u(x))$ and interchange the integral and the Fourier transform to obtain

$$\begin{aligned}\mathcal{F}((-\Delta)^s u)(\zeta) &= -\frac{1}{2}C(d,s)\int_{\mathbb{R}^d}\frac{\mathcal{F}(u(x+y)-2u(x)+u(x-y))}{|y|^{d+2s}}dy\\ &= -\frac{1}{2}C(d,s)\int_{\mathbb{R}^d}\frac{e^{i\zeta\cdot y}+e^{-i\zeta\cdot y}-2}{|y|^{d+2s}}dy\;(\mathcal{F}u)(\zeta)\\ &= C(d,s)\int_{\mathbb{R}^d}\frac{1-\cos(\zeta\cdot y)}{|y|^{d+2s}}dy\;(\mathcal{F}u)(\zeta).\end{aligned}$$

3. step. We show that

$$C(d,s)I(\zeta) := C(d,s)\int_{\mathbb{R}^d}\frac{1-\cos(\zeta\cdot y)}{|y|^{d+2s}}dy = |\zeta|^{2s}, \tag{4.3}$$

which directly gives the Fourier symbol of $(-\Delta)^s$ and proves the theorem.
For $\zeta = (\zeta_1,\dots,\zeta_d)$ with $|\zeta|$ small, we estimate

$$\frac{1-\cos(\zeta_1)}{|\zeta|^{d+2s}} \le \frac{|\zeta_1|^2}{|\zeta|^{d+2s}} \le \frac{1}{|\zeta|^{d+2s-2}}.$$

This shows that the integral

$$\int_{\mathbb{R}^d}\frac{1-\cos(\zeta_1)}{|\zeta|^{d+2s}}d\zeta$$

is indeed finite. We show that $I(\zeta) = I(|\zeta|e_1)$, where $e_1 = (1,0,\dots,0) \in \mathbb{R}^d$.
For $d = 1$ this is obvious since $I(\zeta) = I(-\zeta)$ due to the same property for the cosine.
For $d \ge 2$, we take a rotation R such that $R(|\zeta|e_1) = \zeta$. Noting that $\det R = 1$, we use the transformation $y \mapsto R^T y$ to write

$$\begin{aligned}I(\zeta) &= \int_{\mathbb{R}^d}\frac{1-\cos(R(|\zeta|e_1)\cdot y)}{|y|^{d+2s}}dy = \int_{\mathbb{R}^d}\frac{1-\cos(|\zeta|e_1\cdot R^T y)}{|R^T y|^{d+2s}}dy\\ &= \int_{\mathbb{R}^d}\frac{1-\cos(|\zeta|e_1\cdot y)}{|y|^{d+2s}}dy = I(|\zeta|e_1).\end{aligned}$$

With this equality, we may use the transformation $z = |\zeta|y$ and compute

$$\begin{aligned}I(\zeta) = I(|\zeta|e_1) &= \int_{\mathbb{R}^d}\frac{1-\cos(|\zeta|y_1)}{|y|^{d+2s}}dy = \frac{1}{|\zeta|^d}\int_{\mathbb{R}^d}\frac{1-\cos(z_1)}{|z/|\zeta||^{d+2s}}dz\\ &= C(d,s)^{-1}|\zeta|^{2s},\end{aligned}$$

where the last step follows from the preceding Lemma. This finishes the proof of the theorem. ∎

Remark. From the previous theorem we can immediately deduce for $u \in \mathcal{S}$ and $x \in \mathbb{R}^d$

$$\begin{aligned}\lim_{s\to 0^+}(-\Delta)^s u(x) &= u(x),\\ \lim_{s\to 1^-}(-\Delta)^s u(x) &= -\Delta u(x).\end{aligned}$$

We finally give a representation of the $H^s(\mathbb{R}^d)$-seminorm using the fractional Laplacian.

Theorem 4.5. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$. Then,*

$$|u|_{H^s(\mathbb{R}^d)} = \frac{2}{C(d,s)} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^d)}.$$

Proof. Using the transformation $z = x - y$, we obtain

$$\begin{aligned} |u|_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dydx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| \frac{u(z+y)-u(y)}{|z|^{d/2+s}} \right|^2 dydz \\ &= \int_{\mathbb{R}^d} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{d/2+s}} \right\|^2_{L^2(\mathbb{R}^d)} dz. \end{aligned}$$

Now, Plancherel's formula gives

$$\begin{aligned} \int_{\mathbb{R}^d} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{d/2+s}} \right\|^2_{L^2(\mathbb{R}^d)} dz &= \int_{\mathbb{R}^d} \left\| \mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{d/2+s}} \right) \right\|^2_{L^2(\mathbb{R}^d)} dz \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|e^{i\zeta\cdot z}-1|^2}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 d\zeta dz \\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1-\cos(\zeta\cdot z)}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 dzd\zeta. \end{aligned}$$

With (4.3) this is further equal to

$$\begin{aligned} |u|_{H^s(\mathbb{R}^d)} &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1-\cos(\zeta\cdot z)}{|z|^{d+2s}} |\mathcal{F}u(\zeta)|^2 dzd\zeta = 2C(d,s)^{-1}\int_{\mathbb{R}^d} |\zeta|^{2s}|\mathcal{F}u(\zeta)|^2 d\zeta \\ &= 2C(d,s)^{-1}\||\zeta|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

We stress that this argument shows the equivalence of the Sobolev-norms defined via Slobodeckij-seminorm and via Fourier transformation. Now for $u \in \mathcal{S}$, Theorem 4.3 gives with Plancherel's formula

$$|u|_{H^s(\mathbb{R}^d)} = 2C(d,s)^{-1}\||\zeta|^s\mathcal{F}u\|^2_{L^2(\mathbb{R}^d)} = \|\mathcal{F}((-\Delta)^{s/2}u)\|^2_{L^2(\mathbb{R}^d)} = \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^d)}.$$

Since $\mathcal{S}$ is dense in $H^s(\mathbb{R}^d)$, we can extend this to the case of $u \in H^s(\mathbb{R}^d)$. ■

As a corollary of the previous theorems, we obtain that the fractional Laplacian is an operator of order $2s$.

Corollary 4.6.

(i) $(-\Delta)^s$ *is an operator of order* $2s$, *i.e.,* $(-\Delta)^s : H^\ell(\mathbb{R}^d) \to H^{\ell-2s}(\mathbb{R}^d)$ *is bounded for arbitrary* $\ell \in \mathbb{R}$.

(ii) If $u \in L^2(\mathbb{R}^d)$ *solves* $(-\Delta)^s u = f$ *in* $\mathbb{R}^d$ *with* $f \in H^\ell(\mathbb{R}^d)$, *we have* $u \in H^{\ell+2s}(\mathbb{R}^d)$.

We will see later on, that the second statement is not true for bounded domains $\Omega \subset \mathbb{R}^d$ due to the effect of boundary singularities.

4.3 Definition via Heat Semigroup

In this subsection, we give yet another equivalent definition of the fractional Laplacian, this time by using the semigroup of the heat equation.

As a motivation, we look at the Gamma-function, use integration by parts and the scaling $\tau = \lambda t$ for arbitrary $\lambda > 0$. Then,

$$\begin{aligned} -s\Gamma(-s) = \Gamma(1-s) &= \int_0^\infty \tau^{-s} e^{-\tau}\, d\tau = -\int_0^\infty \tau^{-s} \frac{d}{d\tau}(e^{-\tau} - 1)\, d\tau \\ &= -s \int_0^\infty \tau^{-s-1}(e^{-\tau} - 1)\, d\tau \\ &= -s\lambda^{-s} \int_0^\infty t^{-s-1}(e^{-\lambda t} - 1)\, dt. \end{aligned}$$

Solving for λ^s gives the expression

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(e^{-\lambda t} - 1)\, dt. \tag{4.4}$$

Using this approach for the fractional power, we formally replace $\lambda = -\Delta$ to obtain a possible definition of $(-\Delta)^s$

$$(-\Delta)^s = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(e^{t\Delta} - I)\, dt.$$

Here, $e^{t\Delta}$ is the heat semigroup since the function $U(x,t) := e^{t\Delta}u(x)$ solves the PDE

$$\begin{aligned} \partial_t U &= \frac{\partial}{\partial t}(e^{t\Delta}u(x)) = \Delta(e^{t\Delta}u(x)) = \Delta U, \\ U(x,0) &= u(x), \end{aligned}$$

i.e., the heat equation on $\mathbb{R}^d \times \mathbb{R}^+$ with initial data u.

The following theorem shows that this formal approach can indeed be justified.

Theorem 4.7. *Let $s \in (0,1)$, $u \in \mathcal{S}$ and $U(x,t)$ solve*

$$\begin{aligned} \partial_t U &= \Delta U \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ U(x,0) &= u(x) \qquad x \in \mathbb{R}^d. \end{aligned}$$

Then, the integral fractional Laplacian $(-\Delta)^s$ satisfies

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt.$$

Proof. We refer to the PDE-lecture for the fact that

$$U(x,t) = \int_{\mathbb{R}^d} G(x-y,t)u(y)dy = \int_{\mathbb{R}^d} G(y,t)u(x-y)dy$$

with the heat kernel (Green's function) $G(x,t) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}$ satisfying $\int_{\mathbb{R}^d} G(y,t)dy = 1$. Since G is normalized, we may write

$$\int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt = \int_0^\infty \int_{\mathbb{R}^d} t^{-s-1} G(y,t)(u(x-y) - u(x))\, dy\, dt$$

Now, we employ the substitution $\tau := \frac{|y|^2}{4t}$ noting that $dt = -\frac{|y|^2}{4\tau^2} d\tau$ and obtain

$$\begin{aligned}\int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt &= \int_0^\infty \int_{\mathbb{R}^d} |y|^{-2s-2}(4\tau)^{s+1}\tau^{d/2}(\pi|y|^2)^{-d/2}e^{-\tau}(u(x-y) - u(x))dy \left(\frac{|y|^2}{4\tau^2}d\tau\right) \\ &= 4^s \pi^{-d/2} \int_0^\infty \tau^{d/2+s-1}e^{-\tau}d\tau \frac{1}{2}\int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2s}} dy.\end{aligned}$$

Now, Lemma 4.2 gives that the integral in y is equal to $-C(d,s)^{-1}(-\Delta)^s u(x)$ and the integral in τ is just $\Gamma(\frac{d}{2}+s)$ by definition of the Gamma-function. Plugging everything together, we obtain

$$\frac{1}{\Gamma(-s)} \int_0^\infty t^{-s-1}(U(x,t) - u(x))\, dt = -\frac{2^{2s}\Gamma(\frac{d}{2}+s)}{\pi^{d/2}\Gamma(-s)C(d,s)}(-\Delta)^s u(x),$$

and by choice of $C(d,s)$ in Definition 4.1 (noting $-s\Gamma(-s) = \Gamma(1-s)$) the constant in front of $(-\Delta)^s u(x)$ is equal to one, which finishes the proof. ■

4.4 The Caffarelli-Silvestre Extension

In this section, we discuss yet another approach to interpret the fractional Laplacian. This time, we study a PDE-approach, where the fractional Laplacian is given as a Dirichlet-to-Neumann operator of a degenerated elliptic PDE in one additional space dimension. In literature, this is often called Caffarelli-Silvestre extension, [CS07].

The Caffarelli-Silvestre extension problem reads as follows: For a given function u and $\alpha \in \mathbb{R}$, we seek a function $\mathcal{U} = \mathcal{U}(x,y) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$\operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) = 0 \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^+, \tag{4.5}$$

$$\mathcal{U}(x,0) = u(x) \quad \text{on } \mathbb{R}^d. \tag{4.6}$$

The first equation can equivalently be written as

$$\Delta_x \mathcal{U} + \frac{\alpha}{y}\partial_y \mathcal{U} + \partial_{y^2}^2 \mathcal{U} = 0. \tag{4.7}$$

The following theorem links the solution of (4.5) with the integral fractional Laplacian.

Theorem 4.8. *Let $s \in (0,1)$, $u \in \mathcal{S}$, $\alpha = 1-2s$ and $\mathcal{U}$ a solution of (4.5) with boundary data u. Then, we have*

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s(-\Delta)^s u,$$

with the constant $d_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

Proof. For (4.5) exists a integral representation of the solution $\mathcal{U}$ (think of it as a double layer potential). We refer to [CS07] for the fact that

$$\mathcal{U}(x,y) = C\int_{\mathbb{R}^d} \frac{y^{2s}}{(|x-z|^2+|y|^2)^{(d+2s)/2}} u(z)dz$$

solves (4.5) with an appropriate normalization constant C.
As we have $\lim_{y\to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot,y) = \lim_{y\to 0^+} \frac{\mathcal{U}(x,y)-\mathcal{U}(x,0)}{y^{1-\alpha}}$ in the sense that the existence of the former limit implies the existence of the latter limit. Using the integral representation of $\mathcal{U}$, we compute

$$\begin{aligned}\lim_{y\to 0^+} \frac{\mathcal{U}(x,y)-\mathcal{U}(x,0)}{y^{1-\alpha}} &= \lim_{y\to 0^+} \frac{C}{y^{1-\alpha}} \int_{\mathbb{R}^d} \frac{y^{2s}}{(|x-z|^2+|y|^2)^{(d+2s)/2}}(u(z)-u(x))dz \\ &= C \lim_{y\to 0^+} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} \frac{u(z)-u(x)}{(|x-z|^2+|y|^2)^{(d+2s)/2}}dz,\end{aligned}$$

where we used Lebesgue dominated convergence. As we have uniform convergence for the limit with respect to y, we may interchange both limits. With dominated convergence, we may pull the limit $y\to 0$ inside the integral and obtain

$$\begin{aligned}\lim_{y\to 0^+} \frac{\mathcal{U}(x,y)-\mathcal{U}(x,0)}{y^{1-\alpha}} &= C \lim_{y\to 0^+} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} \frac{u(z)-u(x)}{(|x-z|^2+|y|^2)^{(d+2s)/2}}dz \\ &= C \lim_{\varepsilon\to 0} \lim_{y\to 0^+} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} \frac{u(z)-u(x)}{(|x-z|^2+|y|^2)^{(d+2s)/2}}dz \\ &= C \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d\setminus B_\varepsilon(x)} \frac{u(z)-u(x)}{|x-z|^{d+2s}}dz = -C(-\Delta)^s u(x),\end{aligned}$$

where the constant C can be explicitly computed to finish the proof. ■

Remark. From the Fourier definition of the fractional Laplacian, we may define

$$(-\Delta)^{-s}u := \mathcal{F}^{-1}(|\zeta|^{-2s}\mathcal{F}(u))$$

for functions, where the right-hand side makes sense (there is an additional singularity at $\zeta = 0$ to be taken care of). Then, we obtain $(-\Delta)^{-s} = ((-\Delta)^s)^{-1}$ directly from the Fourier definition. This, together with the integral definition, motivates that a fundamental solution of the fractional Laplacian is given by $C\frac{1}{|x|^{d-2s}}$. This knowledge can be used to derive the solution formula applied in the previous proof.
In the following chapter, we present an alternative proof for the statement of Theorem 4.8 on bounded domains, which could also be used for the problem posed on the whole space.

4.5 The Fractional Laplacian on Bounded Domains

From now on, we consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. Our goal is to (numerically) solve the equation

$$\begin{aligned}(-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Omega^c.\end{aligned} \tag{4.8}$$

4.5.1 The Integral Definition

For a function $u : \Omega \to \mathbb{R}$, we denote its zero-extension by $\widetilde{u} : \mathbb{R}^d \to \mathbb{R}$. Then, we can easily extend the definition (4.1) by

$$(-\Delta)^s u(x) := (-\Delta)^s \widetilde{u}(x) = C(d,s)\ P.V. \int_{\mathbb{R}^d} \frac{\widetilde{u}(x) - \widetilde{u}(y)}{|x-y|^{d+2s}} dy.$$

Obviously, this definition is well-defined for functions $u \in C_0^\infty(\Omega)$ and can be, by density, extended to functions in $\widetilde{H}^s(\Omega)$.

Moreover, the definition is still coherent with the probabilistic interpretation at the beginning of the chapter with the modification that particles hitting $\partial\Omega$ are destroyed.

Remark. Sometimes in literature, a different definition, the so-called **regional fractional Laplacian**,

$$(-\Delta)^s_\Omega u(x) := C(d,s,\Omega)\ P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{d+2s}} dy,$$

where integration is restricted to Ω, is used. While some of the results presented in the following also hold for the regional fractional Laplacian, there are also some considerable differences, and in the following, we will not discuss the regional definition any more.

4.5.2 The Spectral Definition

The Laplacian $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ is a symmetric, self-adjoint operator with compact inverse. As such, classical spectral theory provides eigenfunctions ϕ_k and eigenvalues λ_k satisfying $-\Delta\phi_k = \lambda_k \phi_k$, $\phi_k|_{\partial\Omega} = 0$ that are an orthonormal basis in $L^2(\Omega)$ as well as an orthogonal basis in $H_0^1(\Omega)$ such that

$$-\Delta u = \sum_{k=1}^\infty \lambda_k u_k \phi_k, \qquad u_k := \int_\Omega u \phi_k dx.$$

Using this expansion, and the spectral theorem, a possible way to define the s-th power of the Laplacian would be to take the s-th power of the eigenvalues, i.e.,

$$(-\Delta)^s_\sigma u := \sum_{k=1}^\infty \lambda_k^s u_k \phi_k, \qquad u_k := \int_\Omega u \phi_k dx$$

for $u \in C_0^\infty(\Omega)$. Extending this definition by density to the space $\widetilde{H}^s(\Omega)$ defines the so called **spectral fractional Laplacian** $(-\Delta)^s_\sigma$.

4.5.3 The Caffarelli-Silvestre extension

Let $u \in \widetilde{H}^s(\Omega)$. Then, the extension problem

$$\begin{aligned} \operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) &= 0 \qquad && \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mathcal{U}(x,0) &= \widetilde{u}(x) && \text{on } \mathbb{R}^d \end{aligned}$$

with data $\widetilde{u} \in H^s(\mathbb{R}^d)$ is well-defined. Moreover, Theorem 4.8 is still valid, and we have

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -d_s(-\Delta)^s \widetilde{u} = -d_s(-\Delta)^s u.$$

Therefore, the integral fractional Laplacian on a bounded domain still has an interpretation of a Dirichlet-to-Neumann operator of the same extension problem as in the case of the whole space.

A different approach for the extension problem would be to restrict the PDE to Ω, i.e., to study the problem

$$\mathcal{L}\mathcal{U} := \operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) = 0 \qquad \text{in } \Omega \times \mathbb{R}^+ \tag{4.9a}$$
$$\mathcal{U} = 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{4.9b}$$
$$\mathcal{U}(x,0) = u(x) \quad \text{in } \Omega. \tag{4.9c}$$

The following proposition shows that the Neumann-data of the extension problem (4.9a) actually gives the spectral fractional Laplacian.

Theorem 4.9. *Let $s \in (0,1)$, $u \in \mathcal{S}$, $\alpha = 1-2s$ and $\mathcal{U}$ a solution of* (4.9a) *with boundary data $u \in C_0^\infty(\Omega)$. Then, we have*

$$\lim_{y \to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -c_s(-\Delta)^s_\sigma u.$$

For the proof of the theorem, we make use of classical spectral theory, which we briefly recall here.

The spectral theorem for (un-)bounded self adjoint operators A states the existence of unique spectral measure $E : \mathcal{A} \to \mathcal{L}(L^2)$, where $\mathcal{A}$ is a sigma-algebra on the spectrum $\sigma(A)$, and $\mathcal{L}(L^2)$ are the linear operators mapping from L^2 to L^2, such that

$$A = \int_{\sigma(A)} \lambda dE(\lambda).$$

Moreover, for $f, g \in L^2(\Omega)$, the spectral measure E weakly defines a complex measure $E_{f,g}$ with total variation bounded by $\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}$ in terms of

$$E_{f,g}(\Delta) := (E(\Delta)f, g) \quad \forall \Delta \in \mathcal{A}.$$

Using this notation as well as the functional calculus, that comes with this setting, we may write for the spectral fractional Laplacian

$$(-\Delta)^s_\sigma = \int_0^\infty \lambda^s dE(\lambda), \tag{4.10}$$

as the spectrum of the Laplacian is contained in $\mathbb{R}^+$.
Moreover, the heat-semigroup has the representation

$$e^{t\Delta} = \int_0^\infty e^{-t\lambda} dE(\lambda).$$

Using this together with formula (4.4), we can formally interchange integrations to write

$$\begin{aligned}(-\Delta)^s_\sigma &= \int_0^\infty \lambda^s dE(\lambda) = \int_0^\infty \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) t^{-1-s} dt\, dE(\lambda) \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\Delta} - I) t^{-1-s} dt.\end{aligned}$$

This formal computation can also be easily justified, and we observe that the heat equation approach from Section 4.3 coincides with the spectral fractional Laplacian on bounded domains.

We are now in the position to proof the preceding theorem.

Proof of Theorem 4.9. We prove that the formula

$$\mathcal{V}(x,y) := \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta}(-\Delta)^s_\sigma u(x)\, e^{-y^2/(4t)} t^{s-1} dt \tag{4.11}$$

defines a solution to (4.9a) in the weak $L^2(\Omega)$ sense, i.e., we have $(\mathcal{L}\mathcal{V}, g)_{L^2(\Omega)} = 0$ for all $g \in L^2(\Omega)$ as well as weak boundary conditions. Then, computing the Neumann-data proves the theorem.

1. step. At first, we show that $\mathcal{V}(\cdot, y) \in L^2(\Omega)$ as well as

$$(\mathcal{V}(\cdot, y)\, ;\, g)_{L^2} = \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u\, ;\, g\right)_{L^2}\, e^{-y^2/(4t)} t^{s-1} dt. \tag{4.12}$$

In order to do so, let $R > 0$ and define $\mathcal{V}_R$ as

$$\mathcal{V}_R(x,y) = \frac{1}{\Gamma(s)} \int_0^R e^{t\Delta}(-\Delta)^s_\sigma u(x)\, e^{-y^2/(4t)} t^{s-1} dt.$$

As $e^{-y^2/(4t)} t^{s-1}$ is integrable and $e^{t\Delta}(-\Delta)^s_\sigma u$ is bounded, we may interchange the L^2-scalar product with the integration in t by standard rules of Bochner integration to obtain

$$\begin{aligned}(\mathcal{V}_R(\cdot, y)\, ;\, g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^R \left(e^{t\Delta}(-\Delta)^s_\sigma u\, ;\, g\right)_{L^2}\, e^{-y^2/(4t)} t^{s-1} dt \\ &\frac{1}{\Gamma(s)} \int_0^R \int_0^\infty e^{-t\lambda} \lambda^s dE_{u,g}(\lambda)\, e^{-y^2/(4t)} t^{s-1} dt,\end{aligned}$$

where the last equality follows from the spectral calculus. Now interchanging the integration with Fubini's theorem, we obtain together with the scaling $\tau = t\lambda$

$$\begin{aligned}(\mathcal{V}_R(\cdot, y)\, ;\, g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^R e^{-t\lambda} \lambda^s e^{-y^2/(4t)} t^{s-1} dt\, dE_{u,g}(\lambda) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^{\lambda R} e^{-\tau} \tau^{s-1} e^{-y^2/(4\tau)\lambda} d\tau\, dE_{u,g}(\lambda) \\ &\leq \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau} \tau^{s-1} d\tau\, dE_{u,g}(\lambda) \\ &\leq \|u\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, for all fixed $y \in \mathbb{R}^+$, we have $\mathcal{V}_R(\cdot, y) \in L^2(\Omega)$, and for any sequence $(R_j)_{j\in\mathbb{N}}$ satisfying $R_j \to \infty$ the sequence $(\mathcal{V}_{R_j}(\cdot, y))_{j\in\mathbb{N}}$ is a Cauchy sequence that converges weakly to $\mathcal{V}(\cdot, y)$, which implies $\mathcal{V}(\cdot, y) \in L^2(\Omega)$. Taking the limit $R \to \infty$ also immediately shows equation (4.12).

2. step. Similar arguments can be used to show that $\mathcal{V}(\cdot, y) \in \mathrm{dom}(-\Delta)$ by showing that the limit $\lim_{\tau\to 0^+} \left(\frac{e^{\tau\Delta}\mathcal{V}(\cdot,y)-\mathcal{V}(\cdot,y)}{\tau} \; ; \, g\right)_{L^2}$ exists for all $g \in L^2(\Omega)$. We refer to [ST10] for details.

3. step. $\mathcal{V}$ satisfies the boundary condition at $y = 0$. We use (4.12) and the transformation $\tau = t\lambda$ as in step 1 to compute together with Lebesgue dominated convergence and the definition of the Γ-function

$$
\begin{aligned}
(\mathcal{V}(\cdot, y) \; ; \, g)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau}\tau^{s-1} e^{-y^2/(4\tau)\lambda} d\tau \; dE_{u,g}(\lambda) \\
&\xrightarrow[y\to 0]{} \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty e^{-\tau}\tau^{s-1} d\tau \; dE_{u,g}(\lambda) = (u \; ; \, g)_{L^2} \,.
\end{aligned}
$$

4. step. $\mathcal{V}$ is differentiable and satisfies equation (4.9a). With the help of Lebesgue dominated convergence, we compute

$$
\lim_{h\to 0} \left(\frac{\mathcal{V}(\cdot, y+h) - \mathcal{V}(\cdot, y)}{h} \; ; \, g\right)_{L^2} = \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u \; ; \, g\right)_{L^2} \, \partial_y e^{-y^2/(4t)} t^{s-1} dt.
$$

Thus, $\mathcal{V}$ is differentiable in y, and the y-derivatives are just applied to $e^{-y^2/(4t)}$. An elementary calculation gives

$$
\left(\partial^2_{y^2} + \frac{1-2s}{y}\partial_y\right) e^{-y^2/(4t)} = \partial_t \left(\frac{e^{-y^2/(4t)}}{t^{1-s}}\right).
$$

With this identity and integration by parts, we obtain

$$
\begin{aligned}
\left(\left(\partial^2_{y^2} + \frac{1-2s}{y}\partial_y\right)\mathcal{V}(\cdot, y) \; ; \, g\right)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u \; ; \, g\right)_{L^2} \, \partial_t \left(e^{-y^2/(4t)} t^{s-1}\right) dt \\
&= -\frac{1}{\Gamma(s)} \int_0^\infty \partial_t \left(e^{t\Delta}(-\Delta)^s_\sigma u \; ; \, g\right)_{L^2} \, e^{-y^2/(4t)} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \int_0^\infty \lambda e^{-t\lambda}\lambda^s dE_{u,g}(\lambda) \; e^{-y^2/(4t)} t^{s-1} dt \\
&= ((-\Delta)\mathcal{V}(\cdot, y) \; ; \, g)_{L^2}
\end{aligned}
$$

and $\mathcal{V}$ indeed solves the differential equation.

5. step. Finally, we compute the Neumann-data. With (4.12), and the substitution $\tau = y^2/(4t)$ we obtain

$$
\begin{aligned}
\left(\frac{\mathcal{V}(\cdot, y) - \mathcal{V}(\cdot, 0)}{y^{2s}} \; ; \, g\right)_{L^2} &= \frac{1}{\Gamma(s)} \int_0^\infty \left(e^{t\Delta}(-\Delta)^s_\sigma u \; ; \, g\right)_{L^2} \, \frac{(e^{-y^2/(4t)} - 1)}{y^{2s}} t^{s-1} dt \\
&= \frac{1}{4^s\Gamma(s)} \int_0^\infty \left(e^{\frac{y^2}{4\tau}\Delta}(-\Delta)^s_\sigma u \; ; \, g\right)_{L^2} \, (e^{-\tau} - 1)\tau^{s+1} dt.
\end{aligned}
$$

Now, dominated convergence implies

$$\lim_{y\to 0}\left(e^{\frac{y^2}{4\tau}\Delta}(-\Delta)^s_\sigma u\;;\,g\right)_{L^2} = ((-\Delta)^s_\sigma u\;;\,g)_{L^2}\,,$$

and taking the limit $y \to 0$ in the previous equation gives

$$\lim_{y\to 0}\left(\frac{\mathcal{V}(\cdot,y)-\mathcal{V}(\cdot,0)}{y^{2s}}\;;\,g\right)_{L^2} = \frac{\Gamma(-s)}{4^s\Gamma(s)}\,((-\Delta)^s_\sigma u\;;\,g)_{L^2}\,,$$

which finishes the proof. ■

Remark. The operators $(-\Delta)^s$ and $(-\Delta)^s_\sigma$ on Ω are indeed inherently different. We refer to [SV14] for the following results regarding the eigenvalues and eigenfunctions:

- Let λ_1 denote the smallest eigenvalue. Then,

  $$\lambda_1((-\Delta)^s) < \lambda_1((-\Delta)^s_\sigma).$$

- The eigenfunctions of $(-\Delta)^s_\sigma$ coincide with the eigenfunctions of the Laplacian, the smoothness does only depend on Ω. E.g., on the unit sphere they are $C^\infty(B_1(0))$.

  However, the eigenfunctions for $(-\Delta)^s$ are only Hölder-continuous for some Hölder-exponent β, and [SV14, Pro. 2] provides that $e_1 \notin W^{1,\infty}(B_1(0))$ (for $d > 2s$), where e_1 is the eigenfunction corresponding to $\lambda_1((-\Delta)^s)$.

The differences in the operators can also be seen in the asymptotics of the solutions of the equations $(-\Delta)^s u = f$ and $(-\Delta)^s_\sigma \widetilde{u} = f$. We refer to [Gru15, CS16], where f, Ω are assumed to be sufficiently smooth, for the asymptotic behavior for x close to $\partial\Omega$:

- $u \simeq \operatorname{dist}(x,\partial\Omega)^s + v(x)$, with v smooth;
- $\widetilde{u} \simeq \begin{cases}\operatorname{dist}(x,\partial\Omega)^{2s} + \widetilde{v}(x) & 0 < s < 1/2\\ \operatorname{dist}(x,\partial\Omega) + \widetilde{v}(x) & 1/2 < s < 1\end{cases}$ with $\widetilde{v}$ smooth.

Chapter 5

Numerical Approximation

5.1 The Integral Fractional Laplacian - Weak Formulation

In the following, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We consider the integral fractional Laplacian $(-\Delta)^s$ and study the model problem

$$(-\Delta)^s u = f \quad \text{in } \Omega, \tag{5.1}$$
$$u = 0 \quad \text{on } \Omega^c \tag{5.2}$$

with a given right-hand side $f \in H^{-s}(\Omega)$.

Multiplying the equation with a test-function $v \in \widetilde{H}^s(\Omega)$ and integrating over $\mathbb{R}^d$ gives

$$\int_{\mathbb{R}^d} (-\Delta)^s u \cdot v dx = C(d,s) \int_{\mathbb{R}^d} P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} v(x) dy\, dx.$$

Exploiting the symmetry in x, y and afterwards noticing that the principal value is not needed (compare Lemma 4.2), we obtain

$$\begin{aligned} C(d,s) \int_{\mathbb{R}^d} P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} v(x) dy\, dx &= \frac{C(d,s)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}} dy\, dx \\ &=: a(u,v). \end{aligned}$$

By definition $a(\cdot,\cdot)$ is a symmetric bilinear form.
The weak formulation of our model problem reads as: Finding $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) = \int_\Omega f v dx \quad \forall v \in \widetilde{H}^s(\Omega). \tag{5.3}$$

We recall that the norm on $\widetilde{H}^s(\Omega)$ is defined as the $H^s(\mathbb{R}^d)$-norm of the zero extension, i.e., $\|u\|_{\widetilde{H}^s(\Omega)} = \|\widetilde{u}\|_{H^s(\mathbb{R}^d)}$. In order to show ellipticity of the bilinear form, we employ a fractional Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq C |\widetilde{u}|_{H^s(\mathbb{R}^d)} \quad \forall u \in \widetilde{H}^s(\Omega).$$

As $a(u,u) \sim |u|_{H^s(\mathbb{R}^d)}$, this immediately implies that

$$\|u\|^2_{\widetilde{H}^s(\Omega)} \lesssim a(u,u)$$

and we may apply the Lax-Milgram lemma to prove the following proposition.

Theorem 5.1. *There exists a unique weak solution $u \in \widetilde{H}^s(\Omega)$ of (5.3) that also satisfies*

$$\|u\|_{\widetilde{H}^s(\Omega)} \lesssim \|f\|_{H^{-s}(\Omega)}.$$

In the following, we are interested in approximative numerical solutions of (5.3) by employing a Galerkin finite element scheme. However, compared to the finite element approximation of the Laplace equation, there are some additional difficulties.

Remark.

- The bilinear form $a(\cdot,\cdot)$ is non-local, i.e., for functions $\varphi_i, \varphi_j > 0$ satisfying $\operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset$, we get

$$\begin{aligned} a(\varphi_i,\varphi_j) &= \frac{C(d,s)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\varphi_i(x)-\varphi_i(y))(\varphi_j(x)-\varphi_j(y))}{|x-y|^{d+2s}}dy\,dx \\ &= -\frac{C(d,s)}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(\varphi_i(x)\varphi_j(y)-\varphi_i(y)\varphi_j(x))}{|x-y|^{d+2s}}dy\,dx \\ &= -C(d,s)\int_{\operatorname{supp}\varphi_i}\int_{\operatorname{supp}\varphi_j}\frac{\varphi_i(x)\varphi_j(y)}{|x-y|^{d+2s}}dy\,dx < 0 \end{aligned}$$

  Thinking of φ_i as local basis functions, this computation shows that the system matrices will be densely populated.

- The energy norm ($\widetilde{H}^s(\Omega)$-norm) is non-local, and the seminorm part is not additive. Let $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$. Then,

$$|v|^2_{H^s(\Omega)} = |v|^2_{H^s(\Omega_1)} + |v|^2_{H^s(\Omega_2)} + 2\int_{\Omega_1}\int_{\Omega_2}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}dy\,dx,$$

  and the last term can not be bounded by the first two terms on the right-hand side.

  This leads to problems in the finite element analysis as error estimates are usually derived elementwise. However, Proposition 5.2 gives a way to circumvent this by deriving a localized upper bound for the seminorm.

For the discretization of (5.3), we consider a Galerkin method in a finite-dimensional subspace $X_h \subset \widetilde{H}^s(\Omega)$ described in more detail in Section 3.1. The Galerkin formulation reads as: Finding $u_h \in X_h$ such that

$$a(u_h, v_h) = \int_\Omega f v_h dx \quad \forall v_h \in X_h, \tag{5.4}$$

and the Lax-Milgram lemma gives the existence and uniqueness of u_h. Moreover, the Ceá-Lemma is still valid and provides a best-approximation result

$$\|u-u_h\|_{\widetilde{H}^s(\Omega)} \lesssim \min_{v_h\in X_h}\|u-v_h\|_{\widetilde{H}^s(\Omega)}. \tag{5.5}$$

It remains to choose the space X_h. To that end, we assume that $\mathcal{T} = \{T_i \ : \ i = 1, \dots, N\}$ is a shape regular (with constant σ) triangulation of Ω without any hanging nodes (look at Section 3.2 for the definitions). Moreover, for $T \in \mathcal{T}$, we define the element patch

$$\omega_T := \text{interior} \bigcup \{\overline{T'} \ : \ \overline{T'} \cap \overline{T} \neq \emptyset\}$$

and denote the diameter of T by h_T.
Then, two possible choices of X_h are the piecewise constant functions $\mathcal{P}^0(\mathcal{T}) \subset \widetilde{H}^s(\Omega)$ for $s < 1/2$ or the piecewise affine functions

$$\mathcal{P}^1(\mathcal{T}) := \{v \in C(\Omega) \ : \ v|_T \in P_1(T) \forall T \in \mathcal{T}\} \subset \widetilde{H}^s(\Omega)$$

for $0 < s < 1$. In the following, we only discuss the case of piecewise affine functions.

Before we can prove a-priori estimates, we need to discuss the mentioned localization of the H^s-seminorm, which is also called Faermann localization in literature, [Fae00, Fae02].

Theorem 5.2. *Let $s \in (0,1)$ and $u \in H^s(\Omega)$. Then,*

$$|u|^2_{H^s(\Omega)} \leq \sum_{T \in \mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy \, dx + \frac{C}{s h_T^{2s}} \|u\|^2_{L^2(T)},$$

where the constant $C > 0$ depends only on d and the shape-regularity constant σ.

Proof. Let $T \in \mathcal{T}$ and define $\omega_T^c := \Omega \backslash \omega_T$. We write

$$|u|^2_{H^s(\Omega)} \leq \sum_{T \in \mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy \, dx + \int_T \int_{\omega_T^c} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy \, dx.$$

The first term already has the right form and it remains to estimate the second one. Using Fubini's theorem, we get

$$\begin{aligned}\int_T \int_{\omega_T^c} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} dy \, dx &\leq 2 \int_T |u(x)|^2 \int_{\omega_T^c} |x-y|^{-d-2s} dy \, dx + \int_{\omega_T^c} |u(y)|^2 \int_T |x-y|^{-d-2s} dx \, dy \\ &=: I_{T,1} + I_{T,2}.\end{aligned}$$

In the following, we show that $\sum_{T \in \mathcal{T}} I_{T,1} = \sum_{T \in \mathcal{T}} I_{T,2}$. With the characteristic function $\chi_{\omega_T^c}$ of the set ω_T^c, we write

$$\begin{aligned}\sum_{T \in \mathcal{T}} I_{T,2} &= \sum_{T \in \mathcal{T}} \int_\Omega \chi_{\omega_T^c} |u(y)|^2 \int_T |x-y|^{-d-2s} dx \, dy \\ &= \int_\Omega |u(y)|^2 \underbrace{\sum_{T \in \mathcal{T}} \chi_{\omega_T^c} \int_T |x-y|^{-d-2s} dx}_{=:f(y)} \, dy.\end{aligned}$$

As $\chi_{\omega_T^c}(y) = 0$ for $y \in \widehat{T}$ satisfying $\overline{\widehat{T}} \cap \overline{T} \neq \emptyset$, we get for $y \in \widehat{T}$

$$f(y) = \sum_{T \in \mathcal{T}} \chi_{\omega_T^c} \int_T |x-y|^{-d-2s} dx = \int_{\omega_{\widehat{T}}^c} |x-y|^{-d-2s} dx.$$

This finally leads to

$$\sum_{T\in\mathcal{T}} I_{T,2} = \sum_{\widehat{T}\in\mathcal{T}} \int_{\widehat{T}} |u(y)|^2 \int_{\omega^c_{\widehat{T}}} |x-y|^{-d-2s} dx\, dy = \sum_{\widehat{T}\in\mathcal{T}} I_{\widehat{T},1}.$$

Using this equality gives the estimate

$$|u|^2_{H^s(\Omega)} \leq \sum_{T\in\mathcal{T}} \int_T \int_{\omega_T} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dy\, dx + 4 \sum_{T\in\mathcal{T}} I_{T,1},$$

and we have to estimate $I_{T,1}$. Due to the assumption of shape-regularity of $\mathcal{T}$, there is a constant $c > 0$ depending only on the σ-shape regularity constant such that $\omega^c_T \subseteq \Omega \backslash B_{ch_T}(x)$. This allows to estimate using polar coordinates

$$\begin{aligned} I_{T,1} &= \int_T |u(x)|^2 \int_{\omega^c_T} |x-y|^{-d-2s} dy\, dx \\ &\lesssim \int_T |u(x)|^2 \int_{ch_T}^{\infty} r^{-1-2s} dr\, dx \lesssim \frac{1}{sh_T^{2s}} \|u\|^2_{L^2(T)}, \end{aligned}$$

which finishes the proof. ∎

5.2 The Integral Fractional Laplacian - Regularity

In the following, our goal is to derive a convergence rate of the finite element method applied to the model problem. A key question when doing so, is always the regularity of the given problem, i.e., if the right-hand side f of the equation satisfies $f \in H^r(\Omega)$ for $r > -s$, for which $\ell \geq s$ can we expect that $u \in H^\ell(\Omega)$?

Comparing to the Laplace equation, the answer of this problem is more involved, and current research is still conducted to derive the correct regularity results on polygonal Lipschitz domains, [BN20]. For smooth domains or Hölder-regularity, we refer to [Gru15, ROS14].

We start with an example highlighting the difference to the integer order Laplacian.

Example 5.3. *Let $\Omega = B_1(0)$, $f \equiv 2^{2s}\Gamma(1+s)^2$. Then, the exact solution of (5.1) is given by*

$$u(x) = (1-|x|^2)^s_+, \qquad \text{where } g_+ := \max\{g, 0\}.$$

As an exercise, the reader may check that $u \in H^{s+1/2-\varepsilon}(\Omega)$ for all $\varepsilon > 0$, but $u \notin H^{s+1/2}(\Omega)$. Therefore, even for smooth Ω, f, one could not expect more additional regularity than $1/2-\varepsilon$, which is vastly different to the case of the integer Laplacian, where the same problem setting would give smooth solutions.

Examining the previous example shows that singularities in the derivatives appear at the whole boundary. In order to capture these, one can introduce weight-functions and weighted Sobolev-spaces.

We recall the definition $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ and define $\rho(x,y) := \min\{\rho(x), \rho(y)\}$ as well as the **weighted Sobolev-spaces** $H^{1+s}_\alpha(\Omega) := \{v \in H^1(\Omega) : \|v\|_{H^{1+s}_\alpha(\Omega)} < \infty\}$ with

$$\|v\|^2_{H^{1+s}_\alpha(\Omega)} := \|v\|^2_{H^1(\Omega)} + \sum_{\beta:|\beta|=1} \int_\Omega \int_\Omega \frac{|D^\beta v(x) - D^\beta v(y)|^2}{|x-y|^{d+2s}} \rho(x,y)^{2\alpha} dy\, dx.$$

Moreover, in the following, we introduce two sets $A, B \subset \Omega \times \Omega$. The idea behind the set A is that for functions satisfying $w(x,y) = w(y,x)$ the integration over $\Omega \times \Omega$ can be reduced to two times the integration over A, which is defined as

$$A := \{(x,y) \in \Omega \times \Omega \;:\; \rho(x,y) = \rho(x)\}.$$

Similarly to previous discussions, we split the set into a part containing the singularity at $x = y$ and the rest in

$$B := \{(x,y) \in A \;:\; |x-y| \geq \rho(x)\}.$$

We now start to answer the question about regularity by citing some results on Hölder-regularity of the solution u proven in [ROS14]. For the domain Ω, we additionally impose a so called exterior ball condition, which means that $\forall x \in \partial\Omega$ there exists some ball $B_{\rho_0}(y) \subset \mathbb{R}^d \backslash \Omega$ with $x \in \partial B_{\rho_0}(y)$, i.e., each point on the boundary can be touched by a ball completely outside of Ω.

This condition either imposes convexity (e.g. for polygonal domains) or additional smoothness of the domain (more than C^1).

Proposition 5.4. *Let Ω be a bounded Lipschitz domain that satisfies the exterior ball condition. Let $f \in L^\infty(\Omega)$ and u solve* (5.1).
Then, $u \in C^s(\mathbb{R}^d)$ and

$$\|u\|_{C^s(\mathbb{R}^d)} \leq C \|f\|_{L^\infty(\Omega)},$$

where the constant $C > 0$ depends only on Ω and s.

Having established the Hölder-regularity, we next state that provided f has additional Hölder-regularity in $C^\beta(\Omega)$, $\beta > 0$, the solution u also gains additional Hölder regularity.

Proposition 5.5.

(i) Let $\beta > 0$ be such that $\beta, \beta + 2s \notin \mathbb{N}$. Assume $f \in C^{\beta}(\Omega)$ and let $u \in C^s(\mathbb{R}^d)$ be the solution of (5.1). Then, $u \in C^{\beta+2s}(\Omega)$.

(ii) Let $0 < s < 1/2$ and $\beta \in (0, 1-2s)$. Then,

$$\sup_{x,y\in\Omega} \rho(x,y)^{\beta+s} \frac{|u(x)-u(y)|}{|x-y|^{\beta+2s}} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|f\|_{C^\beta(\Omega)} \right).$$

Here, the constant $C > 0$ does only depend on Ω and s.

(iii) Let $1/2 < s < 1$ and $\beta \in (0, 2-2s)$. Then,

$$\sup_{x,y\in\Omega} \rho(x,y)^{\beta+s} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\beta+2s-1}} \leq C(\|f\|).$$

Here, the constant $C(\|f\|) > 0$ depends only on Ω, s and as indicated on some norm of f (a weighted Hölder-seminorm).

Now, we can finally start proving Sobolev regularity of the solution u of (5.1). We start with the case $0 < s < 1/2$.

Theorem 5.6. *Let $0 < s < 1/2$, $f \in C^{1/2-s}(\Omega)$ and u solve (5.1). Then, for every $\varepsilon > 0$, we have $u \in H^{s+1/2-\varepsilon}(\Omega)$ with*

$$|u|_{H^{s+1/2-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon} \|f\|_{C^{1/2-s}(\Omega)},$$

where the constant $C > 0$ does only depend on Ω, d, and s.

Proof. Let $\theta \in (s, 1)$ and the sets A, B defined as above. Then, Proposition 5.4 implies

$$\begin{aligned}\int\int_B \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\theta}} dy\, dx &\lesssim \|f\|^2_{L^\infty(\Omega)} \int_\Omega \int_{B_{\rho(x)}(x)^c} |x-y|^{-d-2\theta+2s} dy dx \\ &\lesssim \frac{1}{\theta - s} \|f\|^2_{L^\infty(\Omega)} \int_\Omega \rho(x)^{2(s-\theta)} dx. \end{aligned} \tag{5.6}$$

As the distance function behaves locally like the function x_i (after an affine transformation), one can show that

$$\int_\Omega \rho(x)^{-\alpha} dx \leq \frac{C}{1-\alpha} \qquad \text{for } \alpha > -1.$$

This implies that the right-hand side in (5.6) is finite if and only if $\theta < s + 1/2$.
It remains to estimate the integral over $A \backslash B$. Let $f \in C^\beta(\Omega)$ for $\beta > 0$. Then, Proposition 5.5 (ii) gives in the same way

$$\int\int_{A\backslash B} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\theta}} dy\, dx \lesssim \|f\|^2_{C^\beta(\Omega)} \int_\Omega \rho(x)^{-2(\beta+s)} \int_{B_{\rho(x)}(x)} |x-y|^{-d-2\theta+2\beta+4s} dy dx.$$

In polar coordinates, the inner integral reads as $C \int_0^{\rho(x)} r^{-1-2\theta+2\beta+4s} dr$, which is finite provided $\beta + 2s > \theta$. In that case, we can estimate

$$\int \int_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2\theta}} dy\, dx \lesssim \|f\|^2_{C^\beta(\Omega)} \frac{1}{\beta + 2s - \theta} \int_\Omega \rho(x)^{2(s-\theta)} dx < \infty, \tag{5.7}$$

which holds for $\theta < s + 1/2$ as well.
Choosing $\beta = 1/2 - s$ and $\theta = s + 1/2 - \varepsilon$, we guarantee $\beta + 2s > \theta$ as well as $\theta < s + 1/2$, and the choice of parameters gives

$$\frac{1}{\beta + 2s - \theta} \int_\Omega \rho(x)^{2(s-\theta)} dx = \frac{1}{\varepsilon} \int_\Omega \rho(x)^{-1+2\varepsilon} dx \lesssim \frac{1}{\varepsilon^2}.$$

Summing up the estimates for the integrals over B and $A \setminus B$ proves the theorem. ■

Remark. Estimate (5.7) shows that higher regularity of f does not provide any benefit, as the parameter β does not appear in the integrand on the right-hand side. However, this is not an artifact of the proof as shown in Example 5.3.

Finally, we discuss the regularity for the case $s > 1/2$. We start by proving that, in this case, the solution is in $H^1(\Omega)$.

Lemma 5.7. *Let $1/2 < s < 1$, $f \in L^\infty(\Omega)$ and u solve (5.1). Then, $u \in H^1(\Omega)$ and*

$$|u|_{H^1(\Omega)} \leq \frac{C}{(1-s)(2s-1)} \|f\|_{L^\infty(\Omega)}.$$

Proof. We only sketch the main arguments, for details, we refer to [AB17].
1. step. Local Hölder regularity: For $\gamma \in (0, 2s)$, we have

$$|u|_{C^\gamma(B_{\rho(x)/2}(x))} \leq C\rho(x)^{s-\gamma} \|f\|_{L^\infty(\Omega)} \qquad \forall x \in \Omega,$$

where the constant $C > 0$ blows up only for $\gamma \to 2s$.
2. step. Estimate of the seminorm. Let $\varepsilon \in (0, 1-s)$, $\gamma = 1 - \varepsilon/2$, and the sets A, B defined as above, where $\rho(x)$ is replaced by $\rho(x)/2$. With step 1, we may estimate as in the previous lemma

$$\int \int_{A \setminus B} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2(1-\varepsilon)}} dy\, dx \leq \frac{C}{\varepsilon} \|f\|^2_{L^\infty(\Omega)} \int_\Omega \rho(x)^{2(s-1+\varepsilon)} dx.$$

Using the global Hölder regularity of Proposition 5.4, we obtain

$$\int \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{d+2(1-\varepsilon)}} dy\, dx \leq \frac{C}{1 - s + \varepsilon} \|f\|^2_{L^\infty(\Omega)} \int_\Omega \rho(x)^{2(s-1+\varepsilon)} dx.$$

Adding both estimates and estimating the integral of $\rho(x)$ by $\frac{C}{2s-1+\varepsilon}$, we obtain

$$|u|^2_{H^{1-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon(1 - s + \varepsilon)(2s - 1 + \varepsilon)} \|f\|^2_{L^\infty(\Omega)}.$$

3. step. Taking the limit $\varepsilon \to 0$ gives

$$\lim_{\varepsilon \to 0} \varepsilon |u|^2_{H^{1-\varepsilon}(\Omega)} = C|u|_{H^1(\Omega)},$$

which, together with step 2 proves the lemma. ■

Thus, we have established that for $s > 1/2$ the solution is in $H^1(\Omega)$, which allows us to look at the regularity of ∇u.

Proposition 5.8. *Let $1/2 < s < 1$, $f \in C^\beta(\Omega)$ and u solve (5.1). Then, for every $\varepsilon > 0$, we have $u \in H^{s+1/2-\varepsilon}(\Omega)$ with*

$$|\nabla u|_{H^{s-1/2-\varepsilon}(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}(2s-1)} \|f\|_{C^\beta(\Omega)},$$

where the constant $C > 0$ does only depend on Ω, d, β, and s.

Using weighted Sobolev-spaces, the singular behavior of the derivatives can be more explicitly captured by the weight-function.

Proposition 5.9. *Let $1/2 < s < 1$, $f \in C^{1-s}(\Omega)$ and u solve (5.1). Then, for every $\varepsilon > 0$, we have $u \in H^{s+1-\varepsilon}_{1/2-\varepsilon}(\Omega)$ with*

$$\|\nabla u\|_{H^{s+1-\varepsilon}_{1/2-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon},$$

where the constant $C > 0$ does only depend on Ω, d, s, and $\|f\|_{C^{1-s}(\Omega)}$.

For the proofs of both propositions, we refer to [AB17]. In fact, both can be proven simultaneously similarly to Theorem 5.6 by using the estimates on the weighted Hölder seminorms of Proposition 5.5.

The case $s = 1/2$ can be proven with similar techniques and gives the estimate

$$|u|_{H^{1-\varepsilon}(\Omega)} \leq \frac{C}{\varepsilon} \|f\|_{L^\infty(\Omega)},$$

which only needs L^∞-regularity of f. However, the limit $\varepsilon \to 0$ does not exist in this case.

5.3 FEM for the Integral Fractional Laplacian

Having established the regularity of the exact solutions, we can turn our attention to derive a-priori estimates for the finite element approximation.

Due to the quasi-optimality estimate (5.5) it remains to construct an operator mapping $\widetilde{H}^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$ with optimal approximation properties.
A possible choice hereby is the Scott-Zhang projection, introduced in [SZ90], which is a quasi-interpolation operator based on local averaging. Its definition – given in the following – allows for

a certain freedom in the choice of the averaging set, which can be exploited to deduce additional properties of the operator, such as preservation of homogeneous Dirichlet boundary conditions.

For a given mesh $\mathcal{T}$, we call the set of all vertices of the elements in $\mathcal{T}$ the nodes $\mathcal{N}(\mathcal{T})$. Moreover, we define the set of edges $\mathcal{E}(\mathcal{T})$ as all faces of elements in $\mathcal{T}$.
Let $\mathcal{A}(z)$ be a so called averaging set, which is for a fixed node $z \in \mathcal{N}(\mathcal{T})$ either an edge $E_z \in \mathcal{E}(\mathcal{T})$ with $z \in E_z$ or an element $T_z \in \mathcal{T}$ with $z \in T_z$.
Moreover, for all $z \in \mathcal{N}(\mathcal{T})$, there exists a local dual basis functions, i.e., $\psi_z \in \mathcal{P}^1(\mathcal{A}(z))$ such that

$$\int_{\mathcal{A}(z)} \psi_z \varphi_{z'} = \delta_{zz'} \qquad \forall z' \in \mathcal{N}(\mathcal{T}),$$

where $\varphi_{z'}$ denotes the nodal basis function associated with the node $z' \in \mathcal{N}(\mathcal{T})$. With these notations, we may define the Scott-Zhang projection.

Definition 5.10. *Let $z \in \mathcal{N}(\mathcal{T})$. Choose $\mathcal{A}(z)$ either as*

- $\mathcal{A}(z) = E_z$ *with* $E_z \subseteq \partial\Omega$, *for* $z \in \partial\Omega$
- $\mathcal{A}(z) = T_z$ *with* $z \in \overline{T_z}$ *for* $z \in \Omega$.

Then, the Scott-Zhang projection J_h is defined as

$$J_h v := \sum_{z \in \mathcal{N}(\mathcal{T})} \varphi_z \int_{\mathcal{A}(z)} \psi_z v \, dx.$$

We stress that the choice of averaging set is not unique, as there might be multiple elements with $z \in \overline{T_z}$. However, it can be shown that the value $\int_{\mathcal{A}(z)} \psi_z v \, dx$ does not depend on the choice of the possible averaging sets in this case.

The Scott-Zhang projection is well-defined for $s > 1/2$ and $v \in H^s(\Omega)$ and maps $H^s(\Omega) \to \mathcal{P}^1(\mathcal{T})$. Moreover, as the averaging set is chosen as boundary edge, for nodes at the boundary, it also satisfies $J_h : \widetilde{H}^s(\Omega) \to \mathcal{P}^1_0(\mathcal{T})$, i.e., homogeneous Dirichlet boundary conditions are preserved.

The Scott-Zhang projection indeed is a projection, i.e., $J_h v_h = v_h$ for all $v_h \in \mathcal{P}^1(\mathcal{T})$ and stable in L^2 and H^1 (and consequently in H^s for $s \in (0,1)$). Moreover, it has local approximation properties in H^s, which is stated in the following proposition, c.f., [Cia13].

Proposition 5.11. *Let $T \in \mathcal{T}$, $\max\{1/2, s\} < \ell \leq 2$, $v \in H^\ell(\Omega)$ and J_h be the Scott-Zhang projection of Definition 5.10. Then,*

$$\int_T \int_{\omega_T} \frac{|(v - J_h v)(x) - (v - J_h v)(y)|^2}{|x-y|^{d+2s}} dy dx \leq C h_T^{2(\ell - s)} |v|^2_{H^\ell(\omega_T)},$$

where the constant $C > 0$ depends only on Ω, d, s, ℓ, and the shape-regularity of $\mathcal{T}$ and blows up for $s \to 1$.

Using the quasi-optimality (5.5), the Faermann localization (Theorem 5.2), the approximation properties of the Scott-Zhang projection (Proposition 5.11) as well as the regularity results of Section 5.2, we immediately prove the a-priori estimates of the following theorem.

Theorem 5.12. *Let Ω be a bounded Lipschitz domain that satisfies the exterior ball condition and $\mathcal{T}$ be a uniform mesh. Let u solve (5.3), and u_h solve (5.4) with right-hand side $f \in L^\infty(\Omega)$ satisfying $f \in C^{1/2-s}(\Omega)$ for $0 < s < 1/2$ and $f \in C^\beta(\Omega)$ for some $\beta > 0$ and $1/2 < s < 1$. Then,*

$$
\begin{aligned}
\|u - u_h\|_{\widetilde{H}^s(\Omega)} &\leq \frac{C}{\varepsilon} h^{1/2-\varepsilon} \|f\|_{C^{1/2-s}(\Omega)} && 0 < s < 1/2 \\
\|u - u_h\|_{\widetilde{H}^s(\Omega)} &\leq \frac{C}{\varepsilon} h^{1/2-\varepsilon} \|f\|_{L^\infty(\Omega)} && s = 1/2 \\
\|u - u_h\|_{\widetilde{H}^s(\Omega)} &\leq \frac{C}{\sqrt{\varepsilon}(2s-1)} h^{1/2-\varepsilon} \|f\|_{C^\beta(\Omega)} && 1/2 < s < 1.
\end{aligned}
$$

The previous theorem is formulated in the energy norm. However, using a classical Aubin-Nitsche duality argument, one can also deduce estimates in L^2.

Corollary 5.13. *Let Ω be smooth, $r \geq 0$ and $f \in H^r(\Omega)$ additionally to the assumptions of Theorem 5.12. Then,*

$$
\|u - u_h\|_{L^2(\Omega)} \lesssim h^{\alpha+\beta} \|f\|_{H^r(\Omega)},
$$

with $\alpha = \min\{s + r, 1/2 - \varepsilon\}$, $\beta = \min\{s, 1/2 - \varepsilon\}$ for all $\varepsilon > 0$.

The result of Theorem 5.12 states that, provided f and Ω are regular enough, we obtain convergence of the FEM-solution to the exact solution with rate $h^{1/2-\varepsilon}$. Comparing this to the FEM for the classical Laplacian, we see that this rate is sub-optimal due to the reduced regularity of the exact solution.
One way to derive an approximation which converges faster to the exact solution is to use the regularity in weighted Sobolev spaces of Proposition 5.9 as well as graded meshes, defined as follows:

Let h be a global mesh-size parameter (think about a mesh-size of a uniform triangulation), and $\mu > 1$ be a grading parameter. Then, we call the shape-regular mesh $\mathcal{T}_h$ **graded**, if every element $T \in \mathcal{T}_h$ satisfies

- $h_T \leq C h^\mu$ if $T \cap \partial\Omega \neq \emptyset$,
- $h_T \leq C h \operatorname{dist}(T, \partial\Omega)^{1-1/\mu}$ if $T \cap \partial\Omega = \emptyset$.

where the constants depend only on the shape-regularity constant.

Proposition 5.14. *Let Ω be a bounded Lipschitz domain that satisfies the exterior ball condition and $\mathcal{T}_h$ be a graded mesh with grading parameter $\mu = \frac{2}{1+2\varepsilon}$. Let $s \in (1/2, 1)$, $f \in C^{1-s}(\Omega)$, u solve (5.3), and u_h solve (5.4). Then,*

$$
\|u - u_h\|_{\widetilde{H}^s(\Omega)} \leq \frac{C}{2s-1} h^{1-2\varepsilon} \|f\|_{C^{1/2-s}(\Omega)}.
$$

Thus, using graded meshes, we gained a convergence rate of $h^{1/2-\varepsilon}$ compared to uniform meshes.

5.4 FEM for the Extension Problem - Spectral Fractional Laplacian

In the previous chapter, we established that the integral fractional Laplacian and the spectral fractional Laplacian are indeed different operators. Therefore, deriving a numerical approximation to the spectral fractional Laplacian is also of interest.
Recalling the definition $(-\Delta)^s_\sigma u := \sum \lambda_k^s u_k \varphi_k$ with eigenvalues λ_k and eigenfunctions φ_k of the Dirichlet-Laplacian $(-\Delta)$ and the Fourier coefficients $u_k = \int u\varphi_k$, one can see that deriving a numerical method directly is rather hard.
Fortunately, Theorem 4.9, provides us with a different interpretation by

$$\lim_{y\to 0^+} y^\alpha \partial_y \mathcal{U}(\cdot, y) = -c_s(-\Delta)^s_\sigma u,$$

i.e., as Neumann-data of the solution $\mathcal{U}$ of the Caffarelli-Silvestre extension problem

$$\begin{aligned} \operatorname{div}(y^\alpha \nabla \mathcal{U}(x,y)) &= 0 && \text{in } \Omega \times (0,\infty) \\ \mathcal{U} &= 0 && \text{on } \partial\Omega \times (0,\infty) \\ \mathcal{U}(x,0) &= u(x) && \text{in } \Omega, \end{aligned}$$

where $\alpha = 1 - 2s \in (-1, 1)$.
As this is a degenerated elliptic problem, we can propose a finite element formulation to derive a numerical approximation.

For simplicity, from now on, we assume that Ω is convex. A suitable function space for the extension problem is given by weighted Sobolev spaces in the extended variable.
For a subset $D \subset \mathbb{R}^d \times \mathbb{R}^+$, we define the space $L^2(D; y^\alpha)$ as $L^2(D)$-space with measure $y^\alpha d\lambda(x,y)$ and norm

$$\|v\|^2_{L^2(D;y^\alpha)} = \int_D |v|^2 y^\alpha dxdy < \infty.$$

Imposing this weighted integrability also for the gradient, we define the weighted Sobolev space $H^1(D; y^\alpha)$ as

$$H^1(D; y^\alpha) := \{w \in L^2(D; y^\alpha) \,:\, |\nabla w| \in L^2(D; y^\alpha)\}$$

with norm

$$\|w\|^2_{H^1(D;y^\alpha)} := \|w\|^2_{L^2(D;y^\alpha)} + \|\nabla w\|^2_{L^2(D;y^\alpha)}.$$

These spaces satisfy:

- For $\alpha = 0$, we have the standard $H^1(D)$-space.
- $H^1(D; y^\alpha)$ is a Hilbert space.
- $C^\infty(D) \cap H^1(D; y^\alpha)$ is dense in $H^1(D; y^\alpha)$.

- Let Ω be bounded and $\mathcal{Y} > 0$, then

$$
\begin{aligned}
H^1(\Omega \times (0, \mathcal{Y})) &\hookrightarrow H^1(\Omega \times (0, \mathcal{Y}); y^\alpha) \qquad && \alpha \in (0,1) \\
H^1(\Omega \times (0, \mathcal{Y})) &\hookleftarrow H^1(\Omega \times (0, \mathcal{Y}); y^\alpha) \qquad && \alpha \in (-1,0)
\end{aligned}
$$

In order to shorten notation, we write $\mathcal{C} := \Omega \times (0, \infty)$ in the following.

As we want to derive a weak formulation for the extension problem, we need spaces with zero boundary conditions, defined by

$$\mathring{H}^1(\mathcal{C}; y^\alpha) := \{w \in H^1(\mathcal{C}; y^\alpha) \ : \ w = 0 \text{ on } \partial\Omega \times (0, \infty)\}.$$

On this space, similarly to the non-weighted Sobolev space, holds a weighted Poincaré inequality

$$\|w\|_{L^2(\mathcal{C}; y^\alpha)} \lesssim \|\nabla w\|_{L^2(\mathcal{C}; y^\alpha)} \qquad \forall w \in \mathring{H}^1(\mathcal{C}; y^\alpha).$$

Moreover, we have a trace inequality for the boundary at $y = 0$. Denoting the trace onto $\Omega \times \{0\}$ by tr_Ω, we have that $\operatorname{tr}_\Omega \mathring{H}^1(\mathcal{C}; y^\alpha) = \widetilde{H}^s(\Omega)$ as well as

$$\|\operatorname{tr}_\Omega w\|_{\widetilde{H}^s(\Omega)} \lesssim \|w\|_{H^1(\mathcal{C}; y^\alpha)}.$$

5.4.1 Weak Formulation

We multiply equation (4.9a) by a test-function $w \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ and integrate by parts to obtain

$$
\begin{aligned}
0 = -\int_{\mathcal{C}} \operatorname{div}(y^\alpha \nabla \mathcal{U}) \cdot w dx dy &= \int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w dx dy - \int_{\partial \mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot n w \, dx dy \\
&= \int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w \, dx dy + \int_{\Omega \times \{0\}} y^\alpha \partial_y \mathcal{U} w \, dx dy,
\end{aligned}
$$

where we used that the test-function vanishes on $\partial\Omega \times (0, \infty)$. Using Theorem 4.9 as well as the fractional PDE, we obtain

$$\int_{\Omega \times \{0\}} y^\alpha \partial_y \mathcal{U} w \, dx dy = \langle -c_s (-\Delta)^s_\sigma u \, ; \operatorname{tr}_\Omega w \rangle = -c_s \langle f \, ; \operatorname{tr}_\Omega w \rangle.$$

Inserting this in the equation above gives the weak formulation of finding $\mathcal{U} \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ such that

$$\int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla w \, dx dy = c_s \langle f \, ; \operatorname{tr}_\Omega w \rangle \qquad \forall w \in \mathring{H}^1(\mathcal{C}; y^\alpha). \tag{5.8}$$

5.4.2 Regularity

Before studying a finite element method applied to the extension problem, we need to discuss the regularity of solutions of the extension problem. The main question hereby is, what regularity one can expect in the extended variable y.

As a first step, we note that solutions of $(-\Delta)^s_\sigma u = f$ can be written as $u = \sum u_k \varphi_k$, where φ_k are the eigenfunctions of $(-\Delta)$, and u_k denote the Fourier-coefficients of u. As the inverse operator is given by $(-\Delta)^{-s}_\sigma$ the Fourier coefficients of u and f satisfy the relation

$$u_k = \lambda_k^{-s} f_k \qquad \text{with } u_k = \int u \varphi_k, \text{ and } f_k = \int f \varphi_k,$$

where λ_k is the k-th eigenvalue of the negative Laplacian.
The representation of u cane be used to derive a representation of $\mathcal{U}$ as

$$\mathcal{U}(x,y) = \sum_{k=1}^{\infty} u_k \varphi_k(x)\psi_k(y), \tag{5.9}$$

see, e.g., [NOS15], where ψ_k solves a so called Bessel-ODE

$$\begin{aligned}
\psi_k'' + \frac{\alpha}{y}\psi_k' &= \lambda_k \psi_k \quad \text{ in } (0,\infty)\\
\psi_k(0) &= 1\\
\lim_{y\to\infty} \psi_k(y) &= 0,
\end{aligned}$$

which directly follows from plugging (5.9) into the equation (4.9a).

For $s = 1/2$, we have $\alpha = 0$ and the Bessel-ODE can be easily solved as

$$\psi_k(y) = e^{-\sqrt{\lambda_k}y}.$$

For $s \in (0,1)\backslash\{1/2\}$ solving the Bessel-ODE is considerable harder. However, the solution can be expressed by means of Bessel functions of second kind K_s as

$$\psi_k(y) = \frac{2^{1-s}}{\Gamma(s)}(\sqrt{\lambda_k}y)^s K_s(\sqrt{\lambda_k y}).$$

These Bessel functions of second kind are given by

$$K_s(z) = \frac{\mathcal{J}_s(z)\cos(s\pi) - \mathcal{J}_{-s}(z)}{\sin(s\pi)},$$

where $\mathcal{J}_m$ are Bessel functions of first kind, which are given by the power-series

$$\mathcal{J}_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2^{2\ell+m}\ell!(m+\ell)!} x^{2\ell+m}.$$

Using some properties of the Bessel-functions of second kind, we can derive asymptotic estimates. We refer to [AS64, Sec. 9.6] for:

- K_s is positive for $s > -1$.
- $K_s(z) = K_{-s}(z)$.
- $(z^s K_s(z))' = -z^s K_{1-s}(z)$.
- $\lim_{z\to 0^+} K_s(z)z^s = C_s$.
- For $z > 0$, we have the $z^{\min\{s,1/2\}} e^z K_s(z)$ is decreasing.

Using the second, third and fourth property gives

$$\psi_k'(y) \sim y^{-\alpha}, \qquad \psi_k''(y) \sim y^{-\alpha-1} \quad \text{for } y \to 0^+.$$

Employing (5.9) theses properties directly transfer to $\partial_y \mathcal{U}$ and $\partial_{yy}^2 \mathcal{U}$.
With this asymptotics for $y \to 0^+$, we can estimate the weighted Sobolev-norms of $\mathcal{U}$. We start with

$$\int_{\Omega\times(0,\mathcal{Y})} y^\alpha |\partial_y \mathcal{U}|^2 dxdy \lesssim \int_0^{\mathcal{Y}} y^{\alpha-2\alpha} dy \leq C < \infty$$

as $\alpha < 1$. However, the same argument shows that dropping the weight function y^α would lead to

$$\int_{\Omega\times(0,\mathcal{Y})} |\partial_y \mathcal{U}|^2 dxdx \lesssim \int_0^{\mathcal{Y}} y^{-2\alpha} dy = \infty$$

for $s \leq 1/4$. Thus, we can expect that $\partial_y \mathcal{U} \in L^2(\mathcal{C}; y^\alpha)\backslash L^2(\mathcal{C})$.
Looking at the second derivatives, we obtain even worse behavior concerning the weight for the weighted Sobolev-spaces, as

$$\int_{\Omega\times(0,\mathcal{Y})} y^\beta |\partial_{y^2}^2 \mathcal{U}|^2 dxdy \lesssim \int_0^{\mathcal{Y}} y^{\beta-2-2\alpha} dy \leq C < \infty$$

if we have $\beta - 2 - 2\alpha > -1$ or $\beta > 2\alpha + 1$, which is not valid for $\beta = \alpha$. Therefore, we expect $\partial_{y^2}^2 \mathcal{U} \in L^2(\mathcal{C}; y^\beta)\backslash L^2(\mathcal{C}; y^\alpha)$, which is made precise in the following theorem.

Theorem 5.15. *Let $s \in (0,1)$ and $f \in \widetilde{H}^{1-s}(\Omega)$. Let $\mathcal{U} \in \mathring{H}^1(\mathcal{C}; y^\alpha)$ solve (5.8). Then, for $s \in (0,1)\backslash\{1/2\}$ and $\beta > 2\alpha + 1$ we have*

$$\|\Delta_x \mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)}^2 + \|\partial_y \nabla_x \mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)}^2 \lesssim \|f\|_{\widetilde{H}^{1-s}(\Omega)}^2$$
$$\|\partial_{y^2}^2 \mathcal{U}\|_{L^2(\mathcal{C};y^\beta)} \lesssim \|f\|_{L^2(\Omega)}.$$

For $s = 1/2$ we have

$$\|\mathcal{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\widetilde{H}^{1/2}(\Omega)}.$$

Proof. [NOS15]. ■

Since Ω is assumed to be convex, we obtain a classical shift theorem, i.e., $\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)}$ for $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Using the first statement of the previous theorem provides

$$\|D_x^2 \mathcal{U}\|_{L^2(\mathcal{C};y^\alpha)}^2 \lesssim \|f\|_{L^2(\Omega)}^2$$

for all second derivatives in x as well as all mixed second order derivatives in x and y. However, looking at the result for the second derivative in y-direction shows much worse regularity that has to be compensated by a higher power y^β.

5.4.3 Truncation in y-Direction

Before we can make a finite element formulation for the extension problem, we have to take care of the problem that the extension problem is formulated in an unbounded domain ($(0,\infty)$ in y-direction).

The most basic idea hereby is to truncate the domain and estimate the truncation error. In order to formulate this, let $\mathcal{C}_\mathcal{Y} := \Omega \times (0, \mathcal{Y})$ denote the truncated domain and we impose zero-Dirichlet conditions at $y = \mathcal{Y}$. Defining the space

$$\mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha) := \{w \in H^1(\mathcal{C}_\mathcal{Y}; y^\alpha) \ : \ w = 0 \text{ on } \partial\Omega \times (0,\infty) \cup \Omega \times \{\mathcal{Y}\}\}$$

we can derive a weak formulation in the same way as (5.8), which reads: Find $\mathcal{V} \in \mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha)$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla\mathcal{V} \cdot \nabla w dx\, dy = c_s\langle f\ ;\ \mathrm{tr}_\Omega\, w\rangle \qquad \forall w \in \mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha). \tag{5.10}$$

Our goal for this section is to prove the following theorem, which states exponential decay in the cut-off parameter.

Theorem 5.16. *Let $\mathcal{U}$ solve* (5.8) *and $\mathcal{V}$ denote the zero-continuation of the solution* (5.10) *with data $f \in H^{-s}(\Omega)$. Let $\mathcal{Y} > 0$. Then,*

$$\|\nabla(\mathcal{U} - \mathcal{V})\|_{L^2(\mathcal{C}; y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4} \|f\|_{H^{-s}(\Omega)},$$

where λ_1 is the smallest eigenvalue of the negative Laplacian with homogeneous Dirichlet boundary conditions.

The prove of the theorem makes use of the following lemma, which states exponential decay of the solution $\mathcal{U}$ to the non-truncated domain.

Lemma 5.17. *Let $\mathcal{U}$ solve* (5.8) *with data $f \in H^{-s}(\Omega)$. Let $\mathcal{Y} > 0$. Then,*

$$\|\nabla\mathcal{U}\|_{L^2(\Omega\times(\mathcal{Y},\infty); y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{H^{-s}(\Omega)},$$

where λ_1 is the smallest eigenvalue of the negative Laplacian with homogeneous Dirichlet boundary conditions.

Proof. We use the representation (5.9) and the explicit formulas for ψ_k.
1. step. Case $s = 1/2$: Here, we have $\alpha = 0$ and $\psi_k = e^{-\sqrt{\lambda_k}y}$. As the eigenfunctions φ_k are an orthogonal system in $H_0^1(\Omega)$ with $\|\nabla_x\varphi_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, we get

$$\begin{aligned}\int_\mathcal{Y}^\infty \int_\Omega |\nabla\mathcal{U}|^2 dx\, dy &= \int_\mathcal{Y}^\infty \int_\Omega |\nabla_x\mathcal{U}|^2 + |\partial_y\mathcal{U}|^2 dx\, dy \\ &= 2\sum_{k=1}^\infty \sqrt{\lambda_k}|u_k|^2 e^{-2\sqrt{\lambda_k}\mathcal{Y}} = 2\sum_{k=1}^\infty (\lambda_k)^{-1/2}|f_k|^2 e^{-2\sqrt{\lambda_k}\mathcal{Y}} \\ &\lesssim e^{-2\sqrt{\lambda_1}\mathcal{Y}} \|f\|_{H^{-1/2}(\Omega)}.\end{aligned}$$

2. step. Case $s \in (0,1)\backslash\{1/2\}$: With the Bessel-functions of second kind, we have $\psi_k(y) = C_s(\sqrt{\lambda_k}y)^s K_s(\sqrt{\lambda_k}y)$.

$$\begin{aligned}\int_{\mathcal{Y}}^{\infty} y^\alpha \int_\Omega |\nabla \mathcal{U}|^2 dx\, dy &= \int_{\mathcal{Y}}^{\infty} y^\alpha \int_\Omega |\nabla_x \mathcal{U}|^2 + |\partial_y \mathcal{U}|^2 dx\, dy \\ &= \sum_{k=1}^{\infty} |u_k|^2 \int_{\mathcal{Y}}^{\infty} y^\alpha (\lambda_k \psi_k(y)^2 + \psi_k'(y)^2) dy.\end{aligned}$$

Multiplying the Bessel ODE with y^α and integration by parts gives

$$(y^\alpha \psi_k(y)\psi_k'(y))' = y^\alpha(\lambda_k \psi_k(y)^2 + \psi_k'(y)^2)$$

and using that in the previous formula implies

$$\int_{\mathcal{Y}}^{\infty} y^\alpha \int_\Omega |\nabla \mathcal{U}|^2 dx\, dy = \sum_{k=1}^{\infty} |u_k|^2 y^\alpha \psi_k(y)\psi_k'(y)\Big|_{\mathcal{Y}}^{\infty}. \tag{5.11}$$

With the property that $z^{\min\{s,1/2\}} e^z K_s(z)$ is decreasing for $z > 0$, we may estimate using the positivity as well as the formula for the derivative of the Bessel functions

$$y^\alpha \psi_k(y)\psi_k'(y) \lesssim (\sqrt{\lambda_k}y)^s K_s(\sqrt{\lambda_k}y)(\sqrt{\lambda_k}y)^{1-s} K_{1-s}(\sqrt{\lambda_k}y)\lambda_k^s \lesssim e^{-\sqrt{\lambda_k}y}\lambda_k^s$$

for $y > 0$. Inserting that in (5.11) gives

$$\begin{aligned}\int_{\mathcal{Y}}^{\infty} y^\alpha \int_\Omega |\nabla \mathcal{U}|^2 dx\, dy &= \sum_{k=1}^{\infty} |u_k|^2 y^\alpha \psi_k(y)\psi_k'(y)\Big|_{\mathcal{Y}}^{\infty} \lesssim \sum_{k=1}^{\infty} \lambda_k^s |u_k|^2 e^{-\sqrt{\lambda_k}\mathcal{Y}} \\ &\lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}} \|f\|_{H^{-s}(\Omega)},\end{aligned}$$

where the last estimate follows as in step 1. Taking the square-root proves the lemma. ■

Proof of Theorem 5.16. We first note, that due to the homogeneous boundary conditions at $y = \mathcal{Y}$, the zero extension of $\mathcal{V}$ is indeed in $\mathring{H}^1(\mathcal{C}; y^\alpha)$.

Using a test-function w in (5.10) and its zero-extension in (5.8), subtracting both equations leads to

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla(\mathcal{U} - \mathcal{V}) \cdot \nabla w dx\, dy = 0 \qquad \forall w \in \mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha),$$

which in turn gives the best-approximation property

$$\|\nabla(\mathcal{U} - \mathcal{V})\|_{L^2(\mathcal{C}_\mathcal{Y}; y^\alpha)} = \inf_{W \in \mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha)} \|\nabla(\mathcal{U} - W)\|_{L^2(\mathcal{C}_\mathcal{Y}; y^\alpha)}.$$

As on $\mathcal{C}\backslash\mathcal{C}_\mathcal{Y}$ Lemma 5.17 gives the correct decay estimate, it remains to construct a function W such that the same decay estimate holds on $\mathcal{C}_\mathcal{Y}$.

Here, we use $W(x,y) = \eta(y)\mathcal{U}(x,y)$ with the piecewise affine cut-off function η given by

$$\eta(y) := \begin{cases} 1, & 0 \leq y \leq \frac{\mathcal{Y}}{2} \\ \frac{2}{\mathcal{Y}}(\mathcal{Y} - y), & \frac{\mathcal{Y}}{2} < y < \mathcal{Y} \\ 0, & \mathcal{Y} \leq y, \end{cases}$$

which satisfies $|\eta| \leq 1$, $|\eta'| \leq \frac{2}{\mathcal{Y}}$ and therefore

$$|\nabla((1-\eta)\mathcal{U})|^2 \leq 2\left(|\eta'|^2|\mathcal{U}|^2 + (1-\eta)^2|\nabla\mathcal{U}|^2\right) \leq 2\left(\frac{4}{\mathcal{Y}^2}\mathcal{U}^2 + |\nabla\mathcal{U}|^2\right).$$

With the weighted Poincaré inequality, we estimate

$$\begin{aligned} \|\nabla(\mathcal{U} - W)\|^2_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} &\lesssim \frac{1}{\mathcal{Y}^2}\int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha \mathcal{U}^2 dx\, dy + \int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha |\nabla\mathcal{U}|^2 dx\, dy \\ &\lesssim \int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha|\nabla\mathcal{U}|^2 dx\, dy. \end{aligned}$$

In the same way as in the previous lemma, we can exploit (5.9) as well as the properties of the Bessel-functions to write

$$\begin{aligned} \int_{\mathcal{Y}/2}^{\mathcal{Y}}\int_\Omega y^\alpha|\nabla\mathcal{U}|^2 dx\, dy &= \sum_{k=1}^\infty |u_k|^2 y^\alpha \psi_k(y)\psi_k'(y)\Big|_{\mathcal{Y}}^\infty \\ &\lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|f\|^2_{H^{-s}(\Omega)}, \end{aligned}$$

which finishes the proof. ■

5.4.4 FEM – a-priori Analysis

Again, we employ a Galerkin discretization of (5.10). In order to do that, we have to introduce a grid and discrete space on $\mathcal{C}_\mathcal{Y}$.

Let $\mathcal{T}_\Omega$ be a regular, shape-regular mesh on Ω, and $\mathcal{P}^1_0(\mathcal{T}_\Omega)$ the already introduced space of piecewise affine functions with zero boundary conditions.
Moreover, we decompose the interval $[0,\mathcal{Y}] = \bigcup_{m=0}^{M-1} I_m$ where $I_m := [y_m, y_{m+1}]$ are subintervals with $y_j < y_{j+1}$ for all $j \in \{0,\dots,M\}$, and we denote the set of all intervals by $I_\mathcal{Y} := \{I_m \ : \ m = 0,\dots,M-1\}$. We note, that we – on purpose – have not specified the points y_j and that anisotropy in y-direction will be allowed.

Now, a grid on $\mathcal{C}_\mathcal{Y}$ can be defined in a tensor-product fashion, i.e., $\mathcal{T}_\mathcal{Y} = \mathcal{T}_\Omega \otimes I_\mathcal{Y}$, which means that all elements $T \in \mathcal{T}_\mathcal{Y}$ have the form $T = K \times I$ with $K \in \mathcal{T}_\Omega$ and $I \in I_\mathcal{Y}$.

On the grid $\mathcal{T}_\mathcal{Y}$ we define the discrete FEM-space

$$V(\mathcal{T}_\mathcal{Y}) := \left\{W \in C^0(\overline{\mathcal{C}_\mathcal{Y}}) \ : \ W|_T \in \mathcal{P}^1(K)\otimes\mathcal{P}^1(I)\ \forall T = K\times I \in \mathcal{T}_\mathcal{Y}, W|_{\partial\Omega\times(0,\infty)} = 0 = W|_{\Omega\times\{\mathcal{Y}\}}\right\}.$$

For the discrete space $V(\mathcal{T}_\mathcal{Y})$, we immediately have the following properties:

- $\mathrm{tr}_\Omega V(\mathcal{T}_\mathcal{Y}) = \mathcal{P}_0^1(\mathcal{T}_\Omega)$;
- $V(\mathcal{T}_\mathcal{Y}) \subset \mathring{H}^1(\mathcal{C}_\mathcal{Y}; y^\alpha)$;
- $\#\mathcal{T}_\mathcal{Y} = M\#\mathcal{T}_\Omega$, so if $\#\mathcal{T}_\Omega \sim M^d$, we have CM^{d+1} elements and degrees of freedom.

The discrete Galerkin formulation reads as: finding $V_h \in V(\mathcal{T}_\mathcal{Y})$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_h \cdot \nabla W_h dx\, dy = c_s \langle f\ ;\ \mathrm{tr}_\Omega W_h\rangle \qquad \forall W_h \in V(\mathcal{T}_\mathcal{Y}). \tag{5.12}$$

Existence and uniqueness again follow from the Lax-Milgram Lemma. Moreover, the Galerkin-orthogonality again gives the Ceá-Lemma

$$\|\mathcal{V} - V_h\|_{\mathring{H}^1(\mathcal{C}_\mathcal{Y};y^\alpha)} \lesssim \inf_{W_h \in \mathring{H}^1(\mathcal{C}_\mathcal{Y};y^\alpha)} \|\mathcal{V} - W_h\|_{\mathring{H}^1(\mathcal{C}_\mathcal{Y};y^\alpha)}.$$

Therefore, the a-priori convergence depends on the approximation properties of the discrete space $V(\mathcal{T}_\mathcal{Y})$, which is discussed in the following proposition from [NOS15].

Proposition 5.18 (anisotropic interpolation). *There exists a quasi-interpolation operator $\Pi_{\mathcal{T}_\mathcal{Y}} : L^2(\mathcal{C}_\mathcal{Y}; y^\alpha) \to V(\mathcal{T}_\mathcal{Y})$ that satisfies forall $j = 1, \dots, d+1$ and $T = K \times I \in \mathcal{T}_\mathcal{Y}$ that*

$$\|W - \Pi_{\mathcal{T}_\mathcal{Y}} W\|_{L^2(T;y^\alpha)} \lesssim h_K \|\nabla_x W\|_{L^2(\omega_T;y^\alpha)} + h_I \|\partial_y W\|_{L^2(\omega_T;y^\alpha)}$$
$$\|\partial_{x_j}(W - \Pi_{\mathcal{T}_\mathcal{Y}} W)\|_{L^2(T;y^\alpha)} \lesssim h_K \|\nabla_x \partial_{x_j} W\|_{L^2(\omega_T;y^\alpha)} + h_I \|\partial_y \partial_{x_j} W\|_{L^2(\omega_T;y^\alpha)}$$

provided W is smooth enough that the right-hand side exists. Here, $x_{d+1} = y$ denotes the extended variable and h_K, h_I the diameters of K and I, respectively.

We use this proposition for two different types of meshes to derive a-priori rates of convergence. In the following, we only sketch the ideas of the proof, for details, we refer to [NOS15].

We start with a quasi-uniform mesh, i.e., every element $T \in \mathcal{T}_\mathcal{Y}$ satisfies $h_T \sim h$. Then, for the solution $\mathcal{V}$ of (5.10) and $y \geq 2h$, the previous proposition implies

$$\int_{2h}^{\mathcal{Y}} y^\alpha \|\partial_y(\mathcal{V} - \Pi_{\mathcal{T}_\mathcal{Y}}\mathcal{V})\|^2_{L^2(\Omega)} dy \lesssim h^2 \int_h^{\mathcal{Y}} y^\alpha \left(\|\partial_{yy}\mathcal{V}\|^2_{L^2(\Omega)} + \|\nabla_x \partial_y \mathcal{V}\|^2_{L^2(\Omega)} \right) dy. \tag{5.13}$$

Using the regularity estimates of Section 5.4.2, we can bound the first term on the right-hand side by

$$h^2 \int_h^{\mathcal{Y}} y^\alpha \|\partial_{yy}\mathcal{V}\|^2_{L^2(\Omega)} dy \leq h^2 \sup_{h \leq y \leq \mathcal{Y}} y^{\alpha-\beta} \int_0^{\mathcal{Y}} y^\beta \|\partial_{yy}\mathcal{V}\|^2_{L^2(\Omega)} dy \lesssim h^{2+\alpha-\beta} \|f\|^2_{L^2(\Omega)}$$

as $\beta > 2\alpha + 1 > \alpha$ allows to estimate $y^{\alpha-\beta}$ by powers of h. For the second term in (5.13), we can directly imply the regularity estimate to obtain

$$h^2 \int_h^{\mathcal{Y}} y^\alpha \|\nabla_x \partial_y \mathcal{V}\|^2_{L^2(\Omega)} dy \leq h^2 \int_0^{\mathcal{Y}} y^\alpha \|\nabla_x \partial_y \mathcal{V}\|^2_{L^2(\Omega)} dy \lesssim h^2 \|f\|^2_{\widetilde{H}^{1-s}(\Omega)}.$$

On the elements in $\Omega \times [0,h]$ and $\Omega \times [h,2h]$ one can use the stability of $\Pi_{\mathcal{T}_\mathcal{Y}}$ as well as $\partial_y \mathcal{V} \sim y^{-\alpha}$ to estimate

$$\int_0^{2h} y^\alpha \|\partial_y(\mathcal{V} - \Pi_{\mathcal{T}_\mathcal{Y}}\mathcal{V})\|^2_{L^2(\Omega)} dy \lesssim \int_0^{2h} y^{-\alpha} dy \simeq h^{1-\alpha}.$$

Since $2+\alpha-\beta < 1-\alpha$, choosing $\beta = 2\alpha+1+\varepsilon$, we arrive at the following a-priori estimate

$$\|\nabla(\mathcal{V} - V_h)\|_{L^2(\mathcal{C}_\mathcal{Y};y^\alpha)} \lesssim h^{s-\varepsilon}\|f\|_{\widetilde{H}^{1-s}(\Omega)}.$$

This estimate indeed is sharp, but sub-optimal as only a rate of h^s is achieved compared to a rate of h for the FEM for the classical Laplacian.
The problem hereby lies in the missing regularity of $\partial_{yy}\mathcal{V}$. However, this can be compensated by using graded meshes only in y-direction. With a grading parameter γ, we choose the intersection points

$$y_m = \left(\frac{m}{M}\right)^\gamma \mathcal{Y} \qquad m = 0,\dots,M.$$

Using the Ceá-Lemma together with the anisotropic interpolation result as well as the exponentially decaying truncation, we arrive at the following proposition.

Proposition 5.19. *Let $\mathcal{T}_\mathcal{Y} = \mathcal{T}_\Omega \otimes I_\mathcal{Y}$, where $I_\mathcal{Y}$ is a graded mesh with parameter $\gamma > \frac{3}{2s}$. Let $\mathcal{U}$ be the solution of* (5.8) *and $V_h \in V(\mathcal{T}_\mathcal{Y})$ the FEM-approximation of the truncated problem at $\mathcal{Y} > 0$. Then,*

$$\|\nabla(\mathcal{U} - V_h)\|_{L^2(\mathcal{C};y^\alpha)} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\widetilde{H}^{1-s}(\Omega)} + \mathcal{Y}^s(\#\mathcal{T}_\mathcal{Y})^{-1/(d+1)}\|f\|_{\widetilde{H}^{1-s}(\Omega)}.$$

Choosing $\mathcal{Y} \simeq \log(\#\mathcal{T}_\mathcal{Y})$ balances both terms and gives

$$\|\nabla(\mathcal{U} - V_h)\|_{L^2(\mathcal{C};y^\alpha)} \lesssim |\log(\#\mathcal{T}_\mathcal{Y})|^s(\#\mathcal{T}_\mathcal{Y})^{-1/(d+1)}\|f\|_{\widetilde{H}^{1-s}(\Omega)},$$

which resembles – up to the logarithmic factor – the classical a-priori estimate for the FEM applied to the Laplacian in $d+1$-dimension. However, the usual problem was posed on $\Omega \subset \mathbb{R}^d$, i.e. in d dimensions, the exponent is still suboptimal due to the added "artificial" dimension. Doing some more advanced hybrid hp-FEM this can also be corrected.

Chapter 6

Dunford-Taylor Approach

6.1 The Dunford-Taylor Definition

The starting point for the Dunford-Taylor calculus is Cauchy's integral formula, which states that

$$f(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{D}} \frac{f(z)}{z - \zeta} dz,$$

where the contour $\mathcal{D}$ is a rectifiable Jordan curve oriented such that f is holomorphic on the right of $\mathcal{D}$ and ζ is also on the right of $\mathcal{D}$.
Now, the Dunford-Taylor calculus formalizes using this representation as an operator valued integral for T being a linear operator between two Hilbert-spaces

$$f(T) := \frac{1}{2\pi i} \int_{\mathcal{D}} \frac{f(z)}{z - T} dz,$$

where the spectrum of T lies on the right of $\mathcal{D}$ and does not touch the contour. For more details about the Dunford-Taylor calculus, we refer to [Yos80].

6.1.1 The Spectral Fractional Laplacian

Using this approach for the negative Laplacian $(-\Delta)$, we may define the solution operator for $(-\Delta)^s_\sigma u = f$ as

$$u = (-\Delta)^{-s}_\sigma f = \frac{1}{2\pi i} \int_{\mathcal{D}} z^{-s}(z + \Delta)^{-1} f dz,$$

where the contour is chosen such that the negative real axis (as well as a neighborhood of the origin) is on the left-side. Deforming the contour onto the real axis gives the so-called Balakrishnan formula

$$(-\Delta)^{-s}_\sigma f = \frac{\sin(s\pi)}{\pi} \int_0^\infty \mu^{-s}(\mu - \Delta)^{-1} f d\mu. \tag{6.1}$$

Inserting an eigenfunction φ of $(-\Delta)$ into this representation and using the transformation $\mu = \lambda t$ indeed gives

$$(-\Delta)_\sigma^{-s}\varphi = \frac{\sin(s\pi)}{\pi}\varphi \int_0^\infty \mu^{-s}\frac{1}{\mu+\lambda}d\mu = \frac{\sin(s\pi)}{\pi}\varphi\lambda^{-s}\int_0^\infty t^{-s}\frac{1}{t+1}dt = \lambda^{-s}\varphi,$$

which coincides with the spectral definition.

In the following, we present a numerical method based on the Balakrishnan formula, which consists of two parts:

- Quadrature for the integral in the variable μ;
- FEM in the variables x to approximate $(\mu - \Delta)^{-1}$ in the quadrature points on the same mesh for all quadrature points.

Using the transformation $\mu = e^y$ in (6.1), we get

$$u = (-\Delta)_\sigma^{-s} f = \frac{\sin(s\pi)}{\pi}\int_{-\infty}^\infty e^{(1-s)y}(e^y I - \Delta)^{-1} f dy. \tag{6.2}$$

In order to approximate the integral, we use so-called **sinc-Quadrature**, which we briefly introduce.
Let f be an entire function that additionally satisfies for fixed $k > 0$ $|f(z)| \lesssim K\exp(\frac{\pi|z|}{k})$ (which allows the application of the Paley-Wiener theorem), and additionally assume $f \in L^1(\mathbb{R})$. Then, f can be expanded in a sinc-series

$$f(z) = \sum_{j=-\infty}^{\infty} f(jk)\,\mathrm{sinc}\left(\frac{z-jk}{k}\right) \qquad \mathrm{sinc}(z) = \frac{\sin(z)}{z}.$$

Using this expansion, we can integrate over the real-axis, interchange integration and summation (since $f \in L^1(\mathbb{R})$) to obtain

$$\begin{aligned}\int_{-\infty}^\infty f(t)dt &= \sum_{j=-\infty}^{\infty} f(jk)\int_{-\infty}^\infty \mathrm{sinc}\left(\frac{t-jk}{k}\right)dt = \sum_{j=-\infty}^{\infty} f(jk)\int_{-\infty}^\infty \mathrm{sinc}\left(\frac{s}{k}\right)ds \\ &= k\sum_{j=-\infty}^{\infty} f(jk).\end{aligned}$$

Truncating the sum gives the sinc-quadrature approximation.
More precisely, for $N \in \mathbb{N}$ set $k := \frac{1}{\sqrt{N}}$ and $y_j := jk$ and applying the formula to (6.2) gives the approximation

$$U^N = Q^N f = \frac{\sin(s\pi)}{\pi} k \sum_{j=-N}^{N} e^{(1-s)y_j}(e^{y_j} I - \Delta)^{-1} f.$$

The most important aspect of this quadrature approximation is that – in our setting – we obtain exponential convergence, see, e.g., [BP15].

Proposition 6.1. *Let $r \in [0,1]$, $f \in \widetilde{H}^r(\Omega)$, and u solve $(-\Delta)^s_\sigma u = f$. Then,*

$$\|u - U^N\|_{\widetilde{H}^r(\Omega)} \lesssim e^{-c\sqrt{N}} \|f\|_{\widetilde{H}^r(\Omega)}.$$

Proof. The proof uses the decay properties for $|x| \to \infty$ as well as the holomorphy of the integrand $z^{-s}(z-\Delta)^{-1}$. For details, we refer to [BP15]. ■

The FEM part is done similarly to Chapter 5. Let $\mathcal{T}$ be a regular, shape-regular triangulation of Ω and $\mathcal{P}^1_0(\mathcal{T})$ the space of piecewise affine functions on $\mathcal{T}$. We define the discrete Laplacian $-\Delta_\mathcal{T} : \mathcal{P}^1_0(\mathcal{T}) \to \mathcal{P}^1_0(\mathcal{T})$ by

$$\int_\Omega (-\Delta_\mathcal{T}) v_h w_h dx := \int_\Omega \nabla v_h \cdot \nabla w_h dx \qquad \forall w_h \in \mathcal{P}^1_0(\mathcal{T}).$$

In order to be able to apply the discrete Laplacian to the function f, we need to first project this function onto the discrete space by using the L^2-orthogonal projection $\Pi_\mathcal{T} : L^2(\Omega) \to \mathcal{P}^1_0(\mathcal{T})$ defined as

$$(\Pi_\mathcal{T} v \,;\, w_h)_{L^2(\Omega)} = (v \,;\, w_h)_{L^2(\Omega)} \qquad \forall w_h \in \mathcal{P}^1_0(\mathcal{T}).$$

With the discrete Laplacian, we can define the FEM-approximation of (6.2) by

$$u_\mathcal{T} := \frac{\sin(s\pi)}{\pi} \int_{-\infty}^{\infty} e^{(1-s)y} (e^y I - \Delta_\mathcal{T})^{-1} \Pi_\mathcal{T} f dy.$$

Combining both the sinc-quadrature approximation and the FEM approximation, we arrive at a fully discrete approximation

$$U^N_\mathcal{T} = \frac{\sin(s\pi)}{\pi} k \sum_{j=-N}^{N} e^{(1-s)y_j} (e^{y_j} I - \Delta_\mathcal{T})^{-1} \Pi_\mathcal{T} f,$$

which approximates the solution of $(-\Delta)^s_\sigma u = f$ with rates specified in the following theorem.

Proposition 6.2. *Let Ω be convex, $s \in (0,1)$ and $f \in \widetilde{H}^{2-2s}(\Omega)$. Then,*

$$\|u - U^N_\mathcal{T}\|_{\widetilde{H}^s(\Omega)} \leq C|\log h| \left(h^{2-s} \|f\|_{\widetilde{H}^{2-2s}(\Omega)} + e^{-c\sqrt{N}} \|f\|_{\widetilde{H}^s(\Omega)} \right).$$

Proof. The proof combines the approximation results of the sinc-quadrature and FEM by using some eigenspace decomposition, which can be found in [BP15]. ■

Remark.

- Choosing $N \simeq |\log(h)|$ equilibrates both error terms. In fact, using a quasi-uniform mesh with mesh-size $h \sim (\#\mathcal{T})^{-1/d}$ gives the error bound (up to a constant depending polynomially on $|\log h|$)

$$\|u - U_{\mathcal{T}}^N\|_{\widetilde{H}^s(\Omega)} \lesssim h^{2-s}\|f\|_{\widetilde{H}^{2-2s}(\Omega)} \simeq (\#\mathcal{T})^{-(2-s)/d}.$$

  Comparing that to the FEM for the Caffarelli-Silvestre extension problem, where we had rates of $(\#\mathcal{T})^{-1/(d+1)}$ or $(\#\mathcal{T})^{-1/d}$ with more involved methods, we obtain better rates here. However, higher regularity $f \in \widetilde{H}^{2-2s}(\Omega)$ was imposed, which translates to $u \in \widetilde{H}^2(\Omega)$, which is not generic regularity. Nonetheless, compared to the FEM for the extension problem higher rates with linear polynomials are possible, whereas additional regularity for the extension problem does not give better rates than $(\#\mathcal{T})^{-1/d}$.

- Assuming only $f \in \widetilde{H}^{1-s}(\Omega)$, the Balakrishnan sinc-quadrature approach also reproduces convergence order $(\#\mathcal{T})^{-1/d}$.

- The construction of the approximation is such that in each quadrature point y_j, $j = 1, \dots, N$ a FEM approximation U_j solving $e^{y_j}(I - \Delta_{\mathcal{T}})U_j = \Pi_{\mathcal{T}} f$ has to be computed. In practice, often $N = 20$ is used, and standard FEM-codes can be used to compute U_j. Moreover, these problems are completely decoupled of each other and can be perfectly parallelized.

6.1.2 The Integral Fractional Laplacian

A direct definition like (6.1) is not possible for the integral fractional Laplacian, as the spectrum does not need to be positive and the Dunford-Taylor calculus can not be justified.
However, using the Fourier definition of the fractional Laplacian, we can derive a different representation. We have with Plancherel's formula

$$a(u,w) = C(d,s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{d+2s}}dx\,dy = \int_{\mathbb{R}^d}(-\Delta)^s u(x)w(x)dx \tag{6.3}$$

$$= \int_{\mathbb{R}^d}\mathcal{F}((-\Delta)^s u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta = \int_{\mathbb{R}^d}|\zeta|^{2s}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta. \tag{6.4}$$

Using Parseval's theorem, we can write

$$\int_{\mathbb{R}^d}\frac{|\zeta|^2}{1+\mu^2|\zeta|^2}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta = \int_{\mathbb{R}^d}(-\Delta)(I-\mu^2\Delta)^{-1}u(x)\,w(x)dx,$$

which leads, using the transformation $t = \mu|\zeta|$ to

$$\begin{aligned}
C_s\int_0^\infty \mu^{1-2s}\int_{\mathbb{R}^d}(-\Delta)(I-\mu^2\Delta)^{-1}u(x)\,w(x)dx\,d\mu &= C_s\int_{\mathbb{R}^d}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}\int_0^\infty\frac{|\zeta|^2\mu^{1-2s}}{1+\mu^2|\zeta|^2}d\mu\,d\zeta\\
&= C_s\int_{\mathbb{R}^d}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}\int_0^\infty\frac{|\zeta|^{2s}t^{1-2s}}{1+t^2}dt\,d\zeta\\
&= C_s\frac{\pi}{2\sin(\pi s)}\int_{\mathbb{R}^d}|\zeta|^{2s}\mathcal{F}(u)(\zeta)\overline{\mathcal{F}w(\zeta)}d\zeta.
\end{aligned}$$

Choosing $C_s = \frac{2\sin(\pi s)}{\pi}$ and inserting this into (6.3), we obtain a different representation for the bilinear form $a(\cdot,\cdot)$

$$a(u,w) = \frac{2\sin(\pi s)}{\pi} \int_0^\infty \mu^{1-2s} \int_{\mathbb{R}^d} (-\Delta)(I - \mu^2\Delta)^{-1} u(x)\, w(x) dx\, d\mu.$$

Again, sinc-quadrature can be employed to approximate the integral on the right-hand side. However, this leads to a non-conforming method with approximative bilinear forms, which can be analyzed by Strang-type estimates. For details, we refer to [BLP19].

Part III

Appendix

Appendix A

Some Facts from other Lectures

In this appendix we collect some results from other courses which are used throughout.

A.1 Abstract Existence & Uniqueness Theorems

This section concludes the mathematical framework which allows the prove the unique existence of solutions of partial differential equations and integral equations. Everything is stated in the context of elliptic problems. We start with the so-called Lax-Milgram lemma which states that an elliptic operator $A \in L(X; X^*)$ is an isomorphism. We prove the lemma for reflexive Banach spaces X and show that the reflexive Banach space X is isomorphic to a Hilbert space provided A is elliptic and symmetric.

Theorem A.1 (Lax-Milgram Lemma). *Let X be a reflexive Banach space and $A \in L(X; X^*)$ be an* **elliptic operator***, i.e.*

$$\|x\|_X^2 \lesssim \langle Ax\,;\,x\rangle \quad \text{for all } x \in X. \tag{1.1}$$

Then, A is an isomorphism. In particular, given $x^* \in X^*$, there is a unique $x \in X$ such that $Ax = x^*$. Moreover, if A is a **symmetric operator***, i.e.*

$$\langle Ax\,;\,y\rangle = \langle Ay\,;\,x\rangle \quad \text{for all } x, y \in X, \tag{1.2}$$

the bilinear form $\langle\!\langle x\,;\,y\rangle\!\rangle := \langle Ax\,;\,y\rangle$ is a scalar product, and the induced norm $|\!|\!|x|\!|\!| := \langle\!\langle x\,;\,x\rangle\!\rangle^{1/2}$ is an equivalent norm on X.

Throughout this section, we are going to use the following observation.

Lemma A.2.

(i) Let X and Y be normed spaces and $A \in L(X;Y)$ be an operator with $\alpha := \inf_{x\in X} \|Ax\|_Y/\|x\|_X > 0$. Then, A is injective and the well-defined operator $A^{-1} : \mathrm{range}(A) \to X$ is continuous with $\|A^{-1}\| \leq \alpha^{-1}$, i.e. A is an isomorphism between X and $\mathrm{range}(A) \leq Y$.

(ii) If X and Y are Banach spaces, A is an isomorphism if and only if $\alpha > 0$ and $A^* \in L(Y^*;X^*)$ is injective.

Proof. (i) Note that non-injectivity of A implies $\alpha = 0$. Thus, A is injective and $A^{-1} : \mathrm{range}(A) \to X$. The equality $\|A^{-1}\| = \alpha^{-1}$ follows from elementary calculations:

$$\alpha = \inf_{x\in X} \frac{\|Ax\|_Y}{\|x\|_X} = \inf_{y\in\mathrm{range}(A)} \frac{\|y\|_Y}{\|A^{-1}y\|_X} = \Big(\sup_{y\in\mathrm{range}(A)} \frac{\|A^{-1}y\|_X}{\|y\|_Y} \Big)^{-1} = \|A^{-1}\|^{-1}.$$

(ii) Let A be an isomorphism. First, there holds $\alpha^{-1} = \|A^{-1}\| > 0$. Moreover, A^* is an isomorphism as well. Conversely, $\alpha > 0$ implies that A is an isomorphism between X and $\mathrm{range}(A) \leq Y^*$. In particular, $\mathrm{range}(A)$ is a closed subspace of Y^*. Therefore, we may apply Banach's closed range theorem to obtain

$$\mathrm{range}(A) = \ker(A^*)^\circ.$$

Thus, $\ker(A^*) = \{0\}$ implies $\mathrm{range}(A) = Y$. ■

Proof of Theorem A.1. **1. step.** A is an isomorphism: The ellipticity of A implies

$$\alpha := \inf_{x\in X} \frac{\|Ax\|_{X^*}}{\|x\|_X} > 0.$$

It thus remains to prove that $\ker(A^*) = \{0\}$ by use of the reflexivity of X. Let $x^{**} \in \ker(A^*)$ and $x \in X$ with $x^{**} = I_X x$, where $I_X : X \hookrightarrow X^{**}$ is the Hahn-Banach embedding. Then,

$$\|x\|_X^2 \lesssim \langle Ax\,;\,x\rangle = \langle x^{**}\,;\,Ax\rangle = \langle A^* x^{**}\,;\,x\rangle = 0,$$

which yields $x = 0$ and thus $x^{**} = I_X x = 0$. Therefore, $\mathrm{range}(A) = X^*$, which concludes the first step.

2. step. $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ is a scalar product: Linearity and symmetry of $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ are obvious. It remains to prove that $\langle\!\langle \cdot\,;\,\cdot \rangle\!\rangle$ is definite, i.e. $\langle\!\langle x\,;\,x \rangle\!\rangle = 0$ implies $x = 0$, which obviously follows from the ellipticity of A.

3. step. $|\!|\!| \cdot |\!|\!|$ is an equivalent norm on X: From the continuity of A, we derive $|\!|\!| x |\!|\!|^2 = \langle Ax\,;\,x\rangle \leq \|Ax\|_{X^*}\|x\|_X \leq \|A\|\|x\|_X^2$. The converse estimate follows from the ellipticity of A, namely $\|x\|_X^2 \lesssim \langle Ax\,;\,x\rangle = |\!|\!| x |\!|\!|^2$. ■

Before we proceed, we state the Lax-Milgram lemma for bilinear forms. We stress that — with respect to the Riesz theorem for Hilbert spaces X — the Lax-Milgram lemma does not need the symmetry of the bilinear form $a(\cdot,\cdot)$.

Corollary A.3 (Lax-Milgram). *Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a continuous bilinear form on a reflexive Banach space X which is reflexive, i.e. $\|x\|_X^2 \lesssim a(x,x)$ for all $x \in X$. Then, given $f \in X^*$, there is a unique $x \in X$ such that $a(x,\cdot) = f$.*

Proof. We consider the operator $A \in L(X; X^*)$ defined by $Ax := a(x,\cdot)$ which is continuous and elliptic. Thus, the claim follows from Theorem A.1. ■

One generalization of the Lax-Milgram lemma is to consider linear problems with side-constraint. This leads to so-called saddle point problems. The following theorem states the unique solvability of a saddle point problem. There are more general formulations proven by Brezzi. However, this formulation should be strong enough for the lecture.

Theorem A.4 (Brezzi). *Let X be a Hilbert space, Y a reflexive Banach space, $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ a continuous bilinear forms. Define $X_0 := \{x \in X \,|\, b(x,\cdot) = 0 \in Y^*\}$ and assume that $a(\cdot,\cdot)$ is X_0-elliptic, i.e.*

$$\|v\|_X^2 \lesssim a(v,v) \quad \text{for all } v \in X_0. \tag{1.3}$$

Moreover, we assume

$$\forall y^* \in Y^* \exists x \in X \quad b(x,\cdot) = y^*. \tag{1.4}$$

Then, given $(x^*, y^*) \in X^* \times Y^*$, there is a unique solution $(x,y) \in X \times Y$ of the saddle point problem

$$\begin{array}{lcl} a(x,\cdot) \;+\; b(\cdot,y) & = & x^* \in X^* \\ b(x,\cdot) & = & y^* \in Y^*. \end{array} \tag{1.5}$$

In particular, the element $x \in X$ satisfies the weak form

$$a(x,\cdot) = x^* \in X_0^*. \tag{1.6}$$

Remark. With $B_1 \in L(X; Y^*)$ defined by $B_1 x := b(x,\cdot)$, the assumptions (1.3)–(1.4) read

- $a(\cdot,\cdot)$ is elliptic on $X_0 = \ker B_1$,
- B_1 is surjective.

We hope that the reader may keep this (abstract) formulation in mind much easier. □

Sketch of Proof. Let (x,y) be a solution of (1.5). We decompose $x = x_1 + x_2$ with $x_1 \in X_0$ and $x_2 \in X_0^\perp$. Note that there hold

$$b(x_2,\cdot) = b(x,\cdot) = y^* \in Y^*, \quad a(x_1,\cdot) = x^* - a(x_2,\cdot) \in X_0^*, \quad \text{and} \quad b(\cdot,y) = x^* - a(x,\cdot) \in X^*.$$

Thus, we first prove that

$$\exists! x_2 \in X_0^\perp \quad b(x_2,\cdot) = y^* \in Y^*. \tag{1.7}$$

The next step is to prove that

$$\exists! x_1 \in X_0 \quad a(x_1, \cdot) = x^* - a(x_2, \cdot) \in X_0^*. \tag{1.8}$$

Defining $x = x_1 + x_2$, we thus obtain a solution of (1.6) and $b(x, \cdot) = y^* \in Y^*$. Moreover, x is in fact the unique solution which solves these two equations. Finally, it remains to prove

$$\exists! y \in Y \quad b(\cdot, y) = x^* - a(x, \cdot) \in X^*. \tag{1.9}$$

This last step is slightly involved and is based on operator techniques. ■

Proof of Theorem A.4. The proof is split into several steps:
1. step. We first prove (1.7): According to (1.4), there is an element $\widetilde{x} \in X$ with $b(\widetilde{x}, \cdot) = y^* \in Y^*$. Since X_0 is a closed subspace of the Hilbert space X, there holds $X = X_0 \oplus X_0^\perp$. Thus, there are $\widetilde{x}_1 \in X_0$ and $\widetilde{x}_2 \in X_0^\perp$ with $\widetilde{x} = \widetilde{x}_1 + \widetilde{x}_2$. With the definition of X_0, there holds $b(\widetilde{x}_2, \cdot) = y^*$. If $x_2 \in X_0^\perp$ also satisfies $b(\widetilde{x}_2, \cdot) = y^*$, there holds $x_2 - \widetilde{x}_2 \in X_0^\perp \cap X_0 = \{0\}$. Altogether, this concludes the proof of (1.7).
2. step. Second, we prove (1.8): We may apply the Lax-Milgram lemma to the operator $A_1 \in L(X_0, X_0^*)$ defined by $A_1 x_1 := a(x_1, \cdot)$. Therefore, there is a unique $x_1 \in X_0$ such that

$$a(x_1, \cdot) = x^* - a(x_2, \cdot) \in X_0^*.$$

3. step. We define $x := x_1 + x_2$ and observe

$$a(x, \cdot) - x^* \in (X_0)^\circ := \big\{x^* \in X^* \,\big|\, \forall v \in X_0 \quad x^*(v) = 0\big\}.$$

4. step. We now prove (1.9): To that end, we consider the operators $B_1 \in L(X, Y^*)$ and $B_2 \in L(Y, X^*)$ defined by $B_1 x := b(x, \cdot)$ and $B_2 y := b(\cdot, y)$, respectively. We now prove that

$$B_1^* \text{ is injective} \quad \text{and} \quad B_2 = B_1^* \circ I_Y \text{ is bijective onto } (X_0)^\circ.$$

According to (1.4), the operator B_1 is surjective, and, in particular, the range of B_1 is closed. Consider the adjoint operator $B_1^* \in L(Y^{**}, X^*)$. Now, let $y^{**} \in \ker(B_1^*)$. For $x \in X$, we have $y^{**}(B_1 x) = (B_1^* y^{**})(x) = 0$, whence $\ker(B_1^*) \subseteq (\mathrm{range}(B_1))^\circ = \{0\}$ as $\mathrm{range}(B_1) = Y^*$, i.e. B_1^* is injective. By definition of the adjoint operator, we have

$$B_1^* y^{**}(x) = y^{**}(B_1 x) = (B_1 x)(y) = b(x, y) = (B_2 y)(x)$$

for any $y \in Y$ and $y^{**} = I_Y(y)$ with $I_Y : Y \hookrightarrow Y^{**}$ the Hahn-Banach embedding. As Y is reflexive, I_Y is an (isometric) isomorphism. Thus, $B_2 = B_1^* \circ I_Y$ is injective and $\mathrm{range}(B_2) = \mathrm{range}(B_1^*)$. As the range of B_1 is closed, we may apply Banach's closed range theorem to infer

$$\mathrm{range}(B_2) = \mathrm{range}(B_1^*) = (\ker B_1)^\circ = (X_0)^\circ.$$

Altogether, $B_2 : Y \to (X_0)^\circ$ is an isomorphism, and there is a unique $y \in Y$ such that

$$b(\cdot, y) = x^* - a(x, \cdot) \in (X_0)^\circ \le X^*. \tag{1.10}$$

5. step. The constructed $(x, y) \in X \times Y$ solves (1.5): The equality $a(x, \cdot) + b(\cdot, y) = x^* \in X^*$ follows from (1.10). The side constraint follows from $y^* = b(x_2, \cdot) = b(x, \cdot)$. We have therefore proven the solvability of (1.5), and it remains to prove the uniqueness of solutions.

6. step. Let $(\widetilde{x}, \widetilde{y}) \in X \times Y$ solve (1.5). We decompose $\widetilde{x} = \widetilde{x}_1 + \widetilde{x}_2$ with $\widetilde{x}_1 \in X_0$ and $\widetilde{x}_2 \in X_0^\perp$. Then, $y^* = b(\widetilde{x}, \cdot) = b(\widetilde{x}_2, \cdot)$ and therefore $x_2 = \widetilde{x}_2$ according to step 1. As $b(\cdot, \widetilde{y}) = 0 \in X_0^*$, we have $a(\widetilde{x}_1, \cdot) = x^* - a(x_2, \cdot) \in X_0^*$ and thus $x_1 = \widetilde{x}_1$ from step 2. Finally, we obtain $x^* - a(x, \cdot) = b(\cdot, \widetilde{y})$. Since the left-hand side is in $(X_0)^\circ$, we obtain $y = \widetilde{y}$ from step 4. ■

Remark. In general, step 2 of the proof only needs that the operator $A_1 \in L(X; X^*)$ defined by $A_1 x := a(x, \cdot)$ is an isomorphism $A_1 \in \mathrm{Iso}(X_0; X_0^*)$. This is usually stated in the so-called LBB-condition, cf. Exercise 3.

Moreover, the theorem holds if X only is a reflexive Banach space as was proven by Brezzi in 1974. However, the proof is then much more involved since one may not use the orthogonal decomposition $X = X_0 \oplus X_0^\perp$. □

Exercise 3. Let $a(\cdot, \cdot)$ be a continuous bilinear form on a reflexive Banach space X. Prove that $A_1 \in L(X; X^*)$ defined by $A_1 x := a(x, \cdot)$ is an isomorphism if and only if it satisfies the *inf-sup condition*

$$\alpha := \inf_{v \in X \setminus \{0\}} \sup_{w \in X \setminus \{0\}} \frac{a(v, w)}{\|v\|_X \|w\|_X} > 0 \tag{1.11}$$

and the *non-degeneracy condition*

$$\forall w \in X \setminus \{0\} \exists v \in X \setminus \{0\} \quad a(v, w) \neq 0. \tag{1.12}$$

(The combination of both conditions is also called LBB-condition and named after Ladyshenskaja, Babuška, and Brezzi.) □

A.2 Lebesgue Spaces

In this section, we recall the most fundamental result for the Lebesgue integral, the dominated convergence theorem.

Theorem A.5 (Lebesgue Dominated Convergence Theorem). *Let Ω be a measurable subset of $\mathbb{R}^d$ and (f_n) a sequence in $L^1(\Omega)$ which converges to a function $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ pointwise almost everywhere in Ω. Provided there is an integrable function $g \in L^1(\Omega)$ with $|f_n| \leq g$ pointwise almost everywhere, there holds $f \in L^1(\Omega)$ and convergence of the integrals*

$$\lim_{n \to \infty} \int_\Omega f_n \, dx = \int_\Omega f \, dx, \tag{1.13}$$

i.e. one may interchange integral and limit. ■

Exercise 4. Let Ω be a measurable set in $\mathbb{R}^d$ and $f \in L^1(\Omega)$. For fixed $x \in \Omega$, define $\Omega_\varepsilon := \Omega \backslash B_\varepsilon(x)$. Then, there holds $\lim\limits_{\varepsilon \to 0} \int_{\Omega_\varepsilon} f\,dx = \int_\Omega f\,dx$. □

In the following, we prove the Lebesgue differentiation theorem which implies that an L^p-function is uniquely defined by its integral means, cf. (1.15). We state the theorem for $L^1_{\ell oc}$-functions, but we stress the inclusion $L^p(\Omega) \subseteq L^p_{\ell oc}(\Omega) \subseteq L^1_{\ell oc}(\Omega)$ for any $1 \le p \le \infty$, which follows from the Hölder inequality.

Theorem A.6 (Lebesgue Differentiation Theorem). *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $u \in L^1_{\ell oc}(\Omega)$. Then, there holds*

$$\lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} |u(x) - u(y)|\,dy = 0 \quad \textit{for almost every } x \in \Omega. \tag{1.14}$$

In particular, we have

$$u(x) = \lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} u(y)\,dy \quad \textit{for almost every } x \in \Omega. \tag{1.15}$$

We remark that the points $x \in \Omega$, for which (1.14) holds, are called **Lebesgue points** of u.

A.3 Convolution

For measurable functions $u, v : \mathbb{R}^d \to \mathbb{R}$ on the entire space, we define the **convolution**

$$(u * v)(x) := \int_{\mathbb{R}^d} u(x-y)v(y)\,dy \quad \text{pointwise for } x \in \mathbb{R}^d, \tag{1.16}$$

if the integral exists. The substitution $z = x - y$ yields

$$(u * v)(x) = \int_{\mathbb{R}^d} u(z)v(x-z)\,dz = (v * u)(x), \tag{1.17}$$

i.e. convolution is a commutative operation. Moreover, convolution is associative

$$\big((u * v) * w\big)(x) = \big(u * (v * w)\big)(x) \tag{1.18}$$

as follows from the same kind of direct calculation as the commutativity. Throughout this section, the integration domain will be $\mathbb{R}^d$ and is hence omitted. In particular, we abbreviate $L^p = L^p(\mathbb{R}^d)$, $C_0 = C_0(\mathbb{R}^d)$ (for continuous functions with compact support), etc. For $1 \le p \le \infty$, we denote with $p' := p/(p-1) \in [1, \infty]$ the conjugate index, i.e. $1/p + 1/p' = 1$.

Theorem A.7 (Young Inequality). *For $1 \le p, q, r \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and functions $u \in L^p$ and $v \in L^q$, there holds $u * v \in L^r$ with $\|u * v\|_{L^r} \le \|u\|_{L^p}\|v\|_{L^q}$.*

Proof. The case $r = \infty$ has to be treated separately: For $q = p'$, there holds

$$|u * v(x)| = \Big|\int u(x-y)v(y)\,dy\Big| \le \|v\|_{L^q}\Big(\int |u(x-y)|^p\,dy\Big)^{1/p} = \|u\|_{L^p}\|v\|_{L^q}$$

according to the Hölder inequality and the translation invariance of the Lebesgue integral. Therefore, we may restrict to $r < \infty$. According to the Hahn-Banach theorem, we have to prove that

$$|\langle w\,;\,u * v\rangle| \le \|w\|_{L^{r'}}\|u\|_{L^p}\|v\|_{L^q} \quad \text{for all } w \in L^{r'} \tag{1.19}$$

as $L^{r'}$ is the dual space of L^r for $1 \le r < \infty$. For $w \in L^{r'}$ define $\widetilde{w}(x) := w(-x)$ and observe

$$\widetilde{w} * u * v(x) = \int \widetilde{w}(x-y)u * v(y)\,dy.$$

Thus,

$$\langle w\,;\,u * v\rangle = \int \widetilde{w}(-y)u * v(y)\,dy = \widetilde{w} * u * v(0).$$

Define $f_1 := |u|^p/\|u\|_{L^p}^p$, $f_2 := |v|^q/\|v\|_{L^q}^q$, $f_3 := |\widetilde{w}|^{r'}/\|\widetilde{w}\|_{L^{r'}}^{r'}$ and observe that $f_j \in L^1$ is non-negative with norm $\|f_j\|_{L^1} = 1$. Using $\|w\|_{L^{r'}} = \|\widetilde{w}\|_{L^{r'}}$, we have

$$\frac{|\langle w\,;\,u * v\rangle|}{\|u\|_{L^p}\|v\|_{L^q}\|w\|_{L^{r'}}} = \frac{|\widetilde{w} * u * v(0)|}{\|u\|_{L^p}\|v\|_{L^q}\|\widetilde{w}\|_{L^{r'}}} \le f_1^{1/p} * f_2^{1/q} * f_3^{1/r'}(0), \tag{1.20}$$

and it remains to show that the right-hand side is bounded by 1: To that end, we consider the function $g : [0,1]^3 \to \mathbb{R}$,

$$g(\lambda) := f_1^{\lambda_1} * f_2^{\lambda_2} * f_3^{\lambda_3}(0) = \int\!\!\int f_1^{\lambda_1}(-y)f_2^{\lambda_2}(y-z)f_3^{\lambda_3}(z)\,dz\,dy \quad \text{for } \lambda = (\lambda_1,\lambda_2,\lambda_3).$$

For scalars $a_j > 0$, there holds

$$a_1^{\lambda_1}a_2^{\lambda_2}a_3^{\lambda_3} = \exp\Big(\sum_{j=1}^3 \lambda_j \log(a_j)\Big),$$

and the exponential function is convex. We apply the convexity estimate pointwise for the integrand to see, that g is a convex function. We now consider $\lambda := (1/p, 1/q, 1/r') \in \mathbb{R}^3$. Note that

$$1 - \lambda_j \ge 0 \quad \text{and} \quad \sum_{j=1}^3 (1-\lambda_j) = 3 - \sum_{j=1}^3 \lambda_j = 1.$$

We define $\widetilde{e}_j := (1,1,1) - e_j$, where $e_j \in \mathbb{R}^3$ is the standard unit vector. Note that

$$g(\widetilde{e}_j) = 1 \quad \text{as well as} \quad \sum_{j=1}^3 (1-\lambda_j)\widetilde{e}_j = (1,1,1) - \sum_{j=1}^3 (1-\lambda_j)e_j = \lambda.$$

From convexity of g, we infer

$$g(\lambda) \le \sum_{j=1}^3 (1-\lambda_j)g(\widetilde{e}_j) = 1,$$

which proves that the right-hand side of (1.20) is bounded by 1. ∎

Remark. We shall later see as an exercise that $q = p'$ and thus $r = \infty$ does not only imply $u * v \in L^\infty$ but also uniform continuity of $u * v$. $\square$

Theorem A.8. *Let $1 \le p \le \infty$, $u \in L^p$, $k \in \mathbb{N}_0$, and $v \in C_0^k$. Then, $u*v \in L^q$ for all $p \le q \le \infty$ and $u * v \in C^k$ with $\partial^\alpha(u * v) = u * \partial^\alpha v$, where $\alpha \in \mathbb{N}_0^d$ is a multi-index with $|\alpha| \le k$. Moreover, $u * v$ and its partial derivatives up to order k are uniformly continuous on $\mathbb{R}^d$.*

Proof. **1. step.** The claim $u * v \in L^r$ for all $p \le r \le \infty$ follows from Young's inequality as $v \in L^q$ for all $1 \le q \le \infty$: Given $r \in [p, \infty]$, one may choose $q \in [1, \infty]$ such that $1 - 1/q = 1/p - 1/r$ as the right-hand side is non-negative. Now, $1/p + 1/q = 1 + 1/r$, whence $u * v \in L^r$.
2. step. Next, we prove that each function $v \in C_0$ is uniformly continuous on $\mathbb{R}^d$, i.e.

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in \mathbb{R}^d \quad \big(|x - y| \le \delta \Rightarrow |v(x) - v(y)| \le \varepsilon\big).$$

Let $R > 0$ be large enough such that $\operatorname{supp}(v) \subseteq B_R(0)$ and let $\varepsilon > 0$. As v is uniformly continuous on the compact set $B_{3R}(0)$, we may choose $\delta > 0$ such that

$$\forall x, y \in B_{3R}(0) \quad \big(|x - y| \le \delta \Rightarrow |v(x) - v(y)| \le \varepsilon\big).$$

Without loss of generality, we assume $\delta \le R$. For $x, y \in \mathbb{R}^d$ with $|x - y| \le \delta$, there holds either $|x| \le 2R$ and thus $|y| \le |x| + |x - y| \le 3R$, whence $|v(x) - v(y)| \le \varepsilon$, or $|x| > 2R$ and thus $|y| \ge |x| - |x - y| > R$, whence $v(x) = 0 = v(y)$.
3. step. For $v \in C_0$, the convolution $u * v$ is uniformly continuous, i.e.

$$\exists C > 0 \forall \varepsilon > 0 \exists \delta > 0 \forall x, x_0 \in \mathbb{R}^d \quad \big(|x - x_0| \le \delta \Rightarrow |u * v(x) - u * v(x_0)| \le C\varepsilon\big).$$

For $x, x_0 \in \mathbb{R}^d$, the Hölder inequality yields

$$\begin{aligned} |u * v(x) - u * v(x_0)| &= \Big| \int u(y)\big(v(x - y) - v(x_0 - y)\big)\, dy \Big| \\ &\le \|u\|_{L^p} \Big(\int |v(x - y) - v(x_0 - y)|^{p'}\, dy \Big)^{1/p'}. \end{aligned} \tag{1.21}$$

For $y \notin B_{2R}(0)$, there holds $|x_0 - y| > 2R \ge R$, whence $v(x_0 - y) = 0$. Moreover, provided $|x - x_0| \le R$, there holds $|x - y| \ge |x_0 - y| - |x_0 - x| \ge R$, whence $v(x - y) = 0$. Therefore, the integrand on the right-hand side of (1.21) has compact support $\subseteq B_{2R}(x_0)$. — For $\varepsilon > 0$, choose $\delta > 0$ according to the uniform continuity of v on $\mathbb{R}^d$ and assume that $\delta \le R$. Provided $|x - x_0| \le \delta$, we obtain $|v(x - y) - v(x_0 - y)| \le \varepsilon$ and therefore

$$|u * v(x) - u * v(x_0)| \le \|u\|_{L^p} |B_{2R}(x_0)|^{1/p'} \varepsilon = \|u\|_{L^p} |B_{2R}(x_0)|^{1/p'} \varepsilon,$$

i.e. $u * v$ is uniformly continuous.
4. step. We prove the theorem for $k = 1$: According to the last step $u * \partial_j v \in C$, and it remains to prove that $u * v$ is differentiable with $\partial_j(u * v) = u * \partial_j v$. Consider the difference operator Δ_h defined by

$$\Delta_h w(x) := \frac{w(x + he_j) - w(x)}{h} \quad \text{for scalars } h \ne 0.$$

We then have to show

$$\exists C > 0 \forall \varepsilon > 0 \exists \delta > 0 \forall h \in \mathbb{R} \quad \big(|h| \leq \delta \Rightarrow |\Delta_h(u * v)(x) - u * \partial_j v(x)| \leq C\varepsilon\big).$$

First, note that $\Delta_h v \in C_0$ for fixed $h > 0$. Let $x \in \mathbb{R}^d$ be fixed. Then,

$$\begin{aligned} |\Delta_h(u * v)(x) - u * \partial_j v(x)| &= |u * (\Delta_h v - \partial_j v)(x)| \\ &\leq \|u\|_{L^p} \Big(\int |\Delta_h v(x-y) - \partial_j v(x-y)|^{p'} \, dy \Big)^{1/p'}. \end{aligned} \tag{1.22}$$

With the same arguments as in step 3, the support of the integrand on the right-hand side is contained in the compact ball $\overline{B_{2R}(x)}$ for $|h| \leq R$. For fixed $y \in B_{2R}(x)$, there holds

$$\lim_{h \to 0} \Delta_h v(x-y) = \partial_j v(x-y).$$

Thus, for given $\varepsilon > 0$, there holds

$$\forall y \in B_{2R}(0) \exists \delta_y > 0 \forall h \in \mathbb{R} \quad \big(|h| \leq \delta_y \Rightarrow |\Delta_h v(x-y) - \partial_j v(x-y)| \leq \varepsilon\big).$$

Without loss of generality, we assume $\delta_y \leq \min\{\delta_{\Delta_h v}, \delta_{\partial_j v}\}$, where $\delta_{\Delta_h v}, \delta_{\partial_j v} > 0$ are chosen accordingly to the uniform continuity of $\Delta_h v$ and $\Delta_j v$, respectively. By compactness, there is a finite set $F \subset B_{2R}(x)$ such that $B_{2R}(x) \subseteq \bigcup \big\{B_{\delta_y}(y) \,\big|\, y \in F\big\}$. Choose $\delta := \min\big\{\delta_y \,\big|\, y \in F\big\}$. For arbitrary $y \in B_{2R}(x)$, there is an $\widetilde{y} \in F$ with $|y - \widetilde{y}| \leq \delta_{\widetilde{y}}$. Thus, for $|h| \leq \delta$, there holds

$$\begin{aligned} &|\Delta_h v(x-y) - \partial_j v(x-y)| \\ &\quad \leq |\Delta_h v(x-y) - \Delta_h v(x-\widetilde{y})| + |\Delta_h v(x-\widetilde{y}) - \partial_j v(x-\widetilde{y})| + |\partial_j v(x-\widetilde{y}) - \partial_j v(x-y)| \\ &\quad \leq 3\varepsilon, \end{aligned}$$

where the first and the last term are estimated by the uniform continuity. Now, (1.22) becomes

$$|\Delta_h(u * v)(x) - u * \partial_j v(x)| \leq 3\|u\|_{L^p} |B_{2R}(x)|^{1/p'} \varepsilon.$$

5. step. The case of arbitrary $k \in \mathbb{N}$ now follows from induction. ■

We finish this section with a slightly different formulation of the previous result, which can be proven with similar techniques.

Lemma A.9. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, $f \in L^\infty(\Omega)$, and $g \in L^1_{\ell oc}(\mathbb{R}^d)$. Then, $g * f \in C(\mathbb{R}^d)$. Moreover, if $g \in C^k(\mathbb{R}^d)$, there holds $g * f \in C^k(\mathbb{R}^d)$ with $\partial^\alpha(g * f) = (\partial^\alpha g) * f$.*

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