Multigoal-oriented AFEM with convergence rates

Roland Becker[1], Maximilian Brunner[2], **Paula Hilbert**[2], Michael Innerberger[3], Dirk Praetorius[2]

[1]University of Pau, Laboratoire de Mathématiques et de leurs Applications
[2]TU Wien, Institute of Analysis and Scientific Computing
[3]Janelia Research Campus, Howard Hughes Medical Institute

Motivation

- AFEM resolves singularities of a PDE solution
- goal: target quantity derived from solution
- GOAFEM: balance singularities of PDE and goal
- multiGOAFEM: goals $\{G_j\}_{j=1}^N$ with $2 \le N \in \mathbb{N}$
- new algorithm for $2 \le N$ goals
- aim: solve only **2 FE problems per level**

Setting

- $\Omega \subset \mathbb{R}^d$ Lipschitz, $H_0^1(\Omega)$ equipped with
$$\|v\|^2 := a(v,v) := \langle \boldsymbol{A}v, v\rangle_{L^2(\Omega)}$$
- $\boldsymbol{A}$ SPD, $f, g_j \in L^2(\Omega)$, $\boldsymbol{f}, \boldsymbol{g}_j \in [L^2(\Omega)]^d$
- **primal problem:** find $u^\star \in H_0^1(\Omega)$ such that
$$a(u^\star, v) = \langle f, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{f}, \nabla v\rangle_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega)$$
- **dual problems:** find $z_j^\star \in H_0^1(\Omega)$ such that
$$a(v, z_j^\star) = G_j(v) := \langle g_j, v\rangle_{L^2(\Omega)} + \langle \boldsymbol{g}_j, \nabla v\rangle_{L^2(\Omega)}$$
- solvability is passed on to FE space $X_\ell \subset H_0^1(\Omega)$
- Galerkin solutions $u_\ell^\star, z_{j,\ell}^\star \in X_\ell$ on mesh $\mathcal{T}_\ell$
- residual-based a-posteriori estimators $\eta_\ell, \zeta_{j,\ell}$:
$$\eta_\ell(T; v_\ell) := h_T^2 \|f + \mathrm{div}(\boldsymbol{A}v_\ell - \boldsymbol{f})\|_{L^2(T)}^2 + h_T \|[\![(\boldsymbol{A}\nabla v_\ell - \boldsymbol{f})^\top \boldsymbol{n}]\!]\|_{L^2(\partial T \cap \Omega)}^2 \quad \forall T \in \mathcal{T}_\ell$$
- **multigoal-estimator product** Δ_ℓ satisfies
$$\sum_{j=1}^N |G_j(u^\star) - G_j(u_\ell^\star)| \lesssim \eta_\ell(u_\ell^\star) \sum_{j=1}^N \zeta_{j,\ell}(z_{j,\ell}^\star) =: \Delta_\ell$$

NGO-AFEM algorithm

Multigoal-oriented AFEM

Input: initial mesh $\mathcal{T}_0$, goals $\{G_j\}_{j=1}^N$,
Dörfler parameter $0 < \theta \le 1$,
contraction $0 < \varrho < (N-1)^{-1}$,
initialize $\zeta_i := 0$ for $i \le 0$

for $\ell = 0, 1, \dots$ **repeat**

- SOLVE & ESTIMATE (primal problem)
  compute $u_\ell^\star$ and $\eta_\ell(u_\ell^\star)$
- SOLVE & ESTIMATE (one dual problem)
  compute $z_{j,\ell}^\star, \zeta_\ell := \zeta_{j,\ell}(z_{j,\ell}^\star)$, $j = \mathrm{mod}(\ell, N) + 1$
- MARK determine marked elements $\mathcal{M}_\ell^u, \mathcal{M}_\ell^z$ via θ-Dörfler marking for $\eta_\ell, \zeta_{j,\ell}$
  (1) let $\mathcal{M}_\ell^{uz} \subseteq \mathcal{M}_\ell^u \cup \mathcal{M}_\ell^z$ contain smaller set and $\min\{\#\mathcal{M}_\ell^u, \#\mathcal{M}_\ell^z\}$ additional elements
  (2) **if** active goal has largest estimator, i.e,
  $$\varrho \max_{i=1,\dots,N-1} \zeta_{\ell-i} \le \zeta_\ell$$
  then regular marking:
  $$\mathcal{M}_\ell := \mathcal{M}_\ell^{uz}$$
  else irregular marking: choose
  $$\mathcal{M}_\ell \subseteq \mathcal{M}_\ell^{uz} \text{ with } \#\mathcal{M}_\ell \le \#\mathcal{M}_{\ell-1}$$
- REFINE $\mathcal{T}_{\ell+1}$ by NVB of marked elements $\mathcal{M}_\ell$

Output: goal values $G_1(u_\ell^\star), \dots, G_N(u_\ell^\star)$

- cycles through goals $j = 1, \dots, N$ as ℓ increases

Variants of NGO-AFEM

irregular marking:

- natural strategy: select $\#\mathcal{M}_{\ell-1}$ elements of $\mathcal{M}_\ell$
- $\mathcal{M}_\ell = \emptyset$ is also admissible and entails $\mathcal{T}_{\ell+1} = \mathcal{T}_\ell$

initialization:

- solve **all** dual problems on $\mathcal{T}_0$, renumber so that
$$\zeta_{N,0} \le \zeta_{N-1,0} \le \dots \le \zeta_{1,0}$$

Main results

R-linear convergence:

- for all $0 < \theta \le 1$
- for **sufficiently small** $0 < \rho < (N-1)^{-1}$
- exist $0 < C_{\mathrm{lin}}$ and $0 < q_{\mathrm{lin}} < 1$ such that
$$\Delta_{\ell+n} \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^n\, \Delta_\ell \quad \text{for all } \ell, n \in \mathbb{N}_0$$

optimal rates:

- suppose $u^\star$ is approximable with rate $0 < s$
- suppose **all** $z_j^\star$ are approximable with rate $0 < t$
- for **sufficiently small** $0 < \theta \ll 1$
- for **sufficiently small** $0 < \rho < (N-1)^{-1}$
- there holds
$$\Delta_\ell \le C_{\mathrm{opt}} (\dim X_\ell)^{-(s+t)} \quad \text{for all } \ell \in \mathbb{N}$$
- no irregular marking:
  optimal rates only for $\min_{j=1,\dots,N} \eta_\ell \zeta_{j,\ell}$

Analytical tools

perturbed contraction:

- for **sufficiently small** $0 < \rho < (N-1)^{-1}$
- exist $0 < q < 1$ and C_{sum} such that
$$\Delta_{\ell+2N-1} \le q\,\Delta_\ell + \mathrm{R}_\ell \quad \text{with} \quad \sum_{k=\ell}^\infty \mathrm{R}_k^2 \le C_{\mathrm{sum}} \Delta_\ell^2$$
- let $j = \mathrm{mod}(\ell, N) + 1$ be the active goal
- regular marking $\Rightarrow \eta_\ell \zeta_{j,\ell}$ contracts via θ
- irregular marking $\Rightarrow \eta_\ell \zeta_{j,\ell}$ contracts via ϱ
- combine this for $\ell+N-1, \dots, \ell+2(N-1)$ with stability $\Rightarrow$ contraction of Δ_ℓ every $2N-1$ levels

control of $\#\mathcal{M}_\ell$:

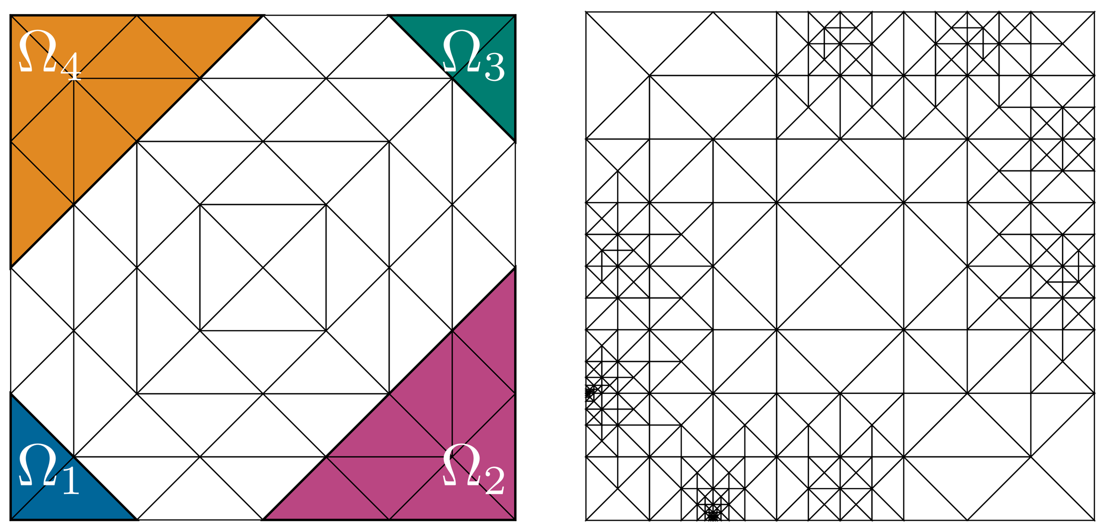
- **Dörfler marking:** control $\#\mathcal{M}_\ell^{uz}$ with $\eta_\ell \zeta_{j,\ell}$
- **regular marking:** $\varrho \max_{i=1,\dots,N-1} \zeta_{\ell-i} \le \zeta_\ell$ gives
$$\Delta_\ell = \sum_{i=1}^N \eta_\ell\, \zeta_{i,\ell} \lesssim \eta_\ell \zeta_{j,\ell}$$
- **irregular marking:** go back to the last level with regular marking via $\#\mathcal{M}_\ell \le \#\mathcal{M}_{\ell-1}$

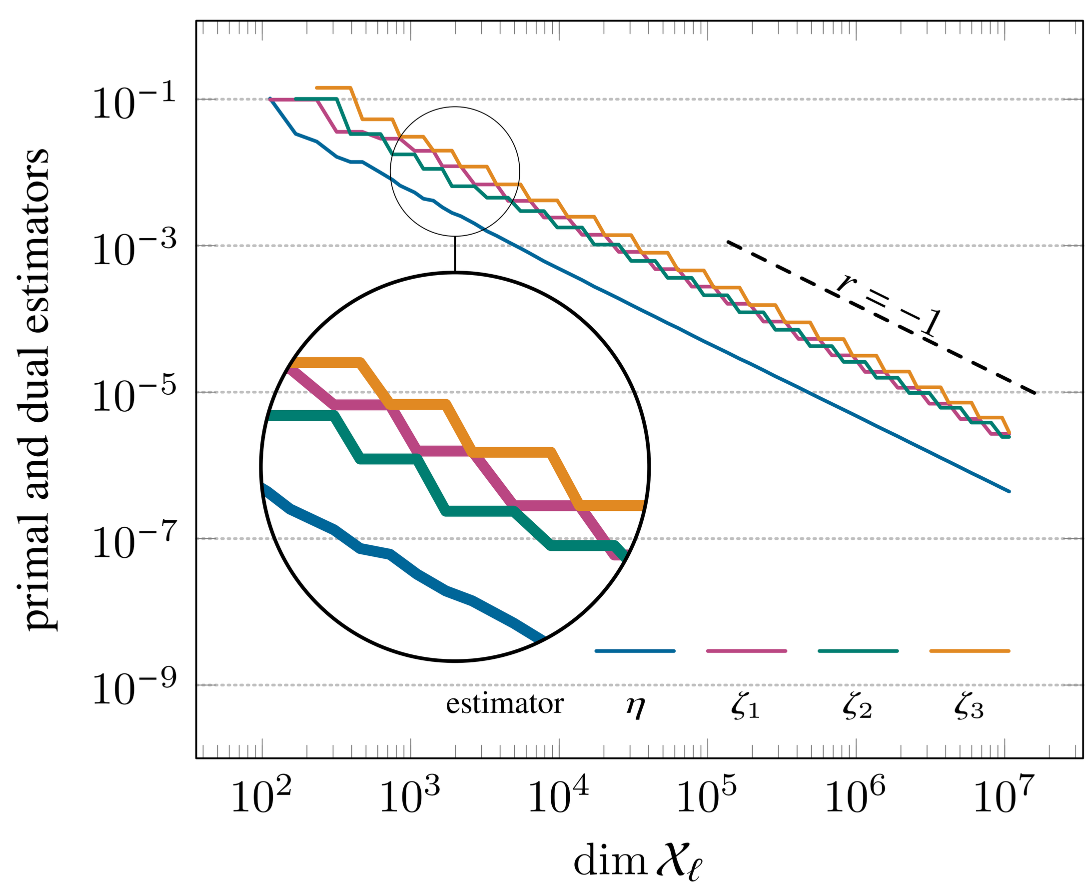
Conclusions

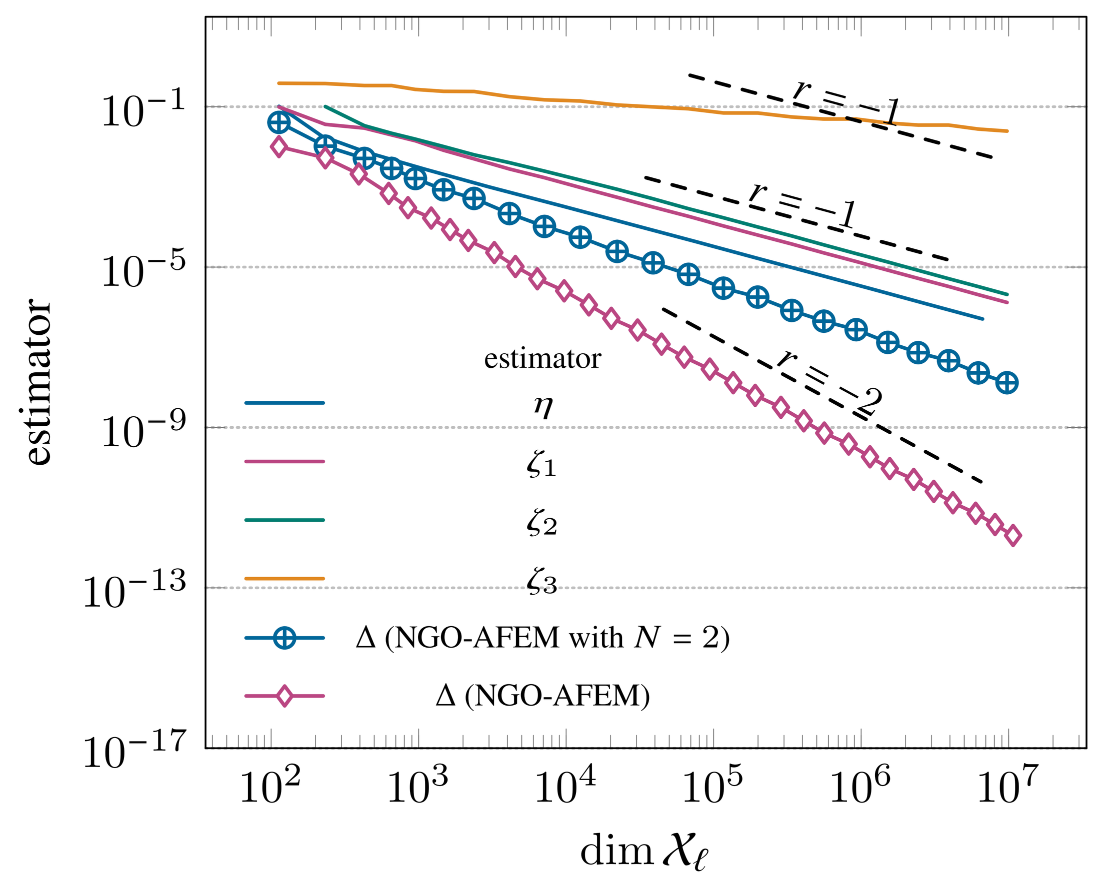
✔ optimal convergence rates for the multigoal error
$$\sum_{j=1}^N |G_j(u^\star) - G_j(u_\ell^\star)| \lesssim (\dim X_\ell)^{-(s+t)}$$
✔ NGO-AFEM algorithm computes only 2 FE solutions per level
✔ remove strong saturation assumption

Numerical experiments

- $\Omega = (0,1)^2$, $\boldsymbol{A} = 1$, $f = 0$, $\boldsymbol{f} = (-1,0)^\top \chi_{\Omega_1}$
- $g_1 = g_2 = g_3 = 0$, $\boldsymbol{g_1} = (1,0)^\top \chi_{\Omega_2}$, $\boldsymbol{g_2} = (1,0)^\top \chi_{\Omega_3}$, $\boldsymbol{g_3} = (0, 1.5)^\top \chi_{\Omega_4}$
- conforming finite elements of degree $p = 2$







✔ optimal rates $-p/2$ for η_ℓ and $\zeta_{j,\ell}$ wrt. $\dim X_\ell$
✔ optimal rates $-p$ for Δ_ℓ wrt. $\dim X_\ell$
✖ suboptimal rates if not all goals are considered

Reference

R. Becker, M. Brunner, P. Hilbert, M. Innerberger, D. Praetorius: *Multigoal-oriented adaptive finite element methods with convergence rates*, arXiv: 2601.01965, 2026.

This research was funded in whole or in part by the Austrian Science Fund (FWF) [10.55776/F65, 10.55776/I6802, 10.55776/PAT3699424, and 10.55776/PAT3446525]. Michael Innerberger is supported by HHMI Janelia.