A happy marriage: Adaptive FEM ∞ Newton

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Abstract problem

- X Hilbert space with dual space X'
- find $u^\star \in X$ such that

$$Au^\star = F \quad \text{in } X'$$

- $F \in X'$ functional
- $A\colon X \to X'$ **strongly monotone**

$$\forall\, v, w \in X\colon \alpha \, \|v - w\|_X^2 \le \langle Av - Aw \,,\, v - w \rangle$$

- **Fréchet differentiable** with **locally Lipschitz continuous** derivative

$$\forall\, v, w, z \in X\colon \|\mathrm{d}A[v]z - \mathrm{d}A[w]z\|_{X'} \le L(\|v\|_X, \|w\|_X)\, \|v - w\|_X \, \|z\|_X$$

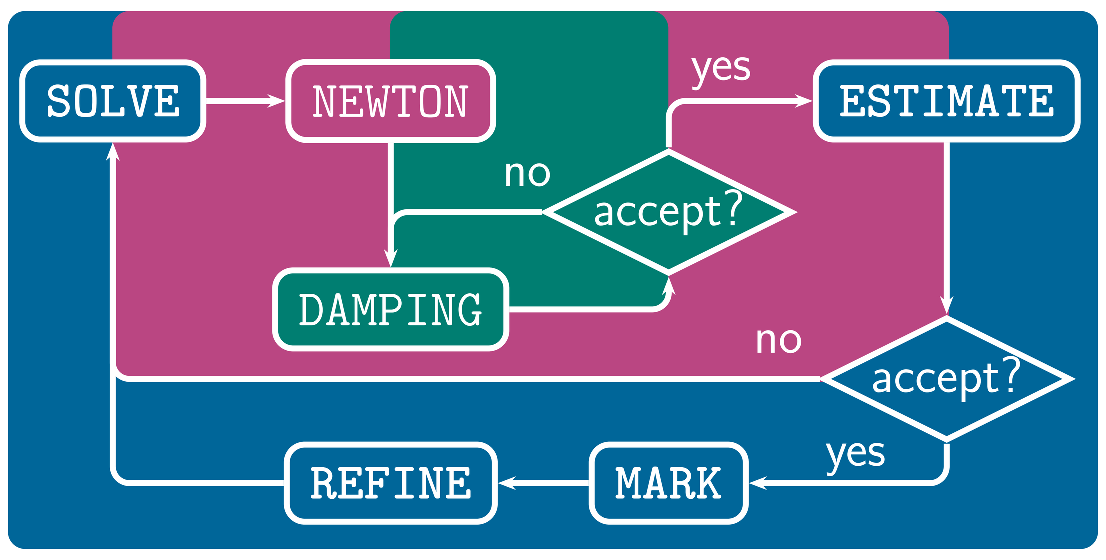
- solvability is passed on to FE space $X_\ell \subset X$
- Galerkin solution $u_\ell^\star \in X_\ell$ associated to mesh $\mathcal{T}_\ell$
- residual-based a-posteriori error estimator η_ℓ

Linearize first, discretize second?

- **classical** approach: infinite-dimensional inexact Newton's method and **mesh enrichment**

! *mesh independence* of Newton's method **holds only** for **quasiuniform meshes** and sufficiently fine initial meshes

Discretize first, linearize second!



- steered by **discrete dual norms**

✓ **adaptive damping** for **global linear** and **local quadratic convergence**

✓ **no energy** or sufficiently fine mesh

Newton's method on fixed mesh

- computable linearization error $\|F - Av_\ell\|_{X_\ell'}$

✓ Newton contraction: exists $0 < q[\delta] < 1$ s.t.

$$\|F - Au_\ell^{k+1}\|_{X_\ell'} \le q[\delta]\, \|F - Au_\ell^k\|_{X_\ell'}$$

NEWTON$(u_\ell^k, \delta_{\mathrm{min}})$ step on level X_ℓ

Input: previous iterate u_ℓ^k, minimal damping δ_{min}

- compute Newton update $\varrho_\ell^k \in X_\ell$ by solving

$$\mathrm{d}A[u_\ell^k]\, \varrho_\ell^k = F - Au_\ell^k \quad \text{in } X_\ell'$$

- call DAMPING$(u_\ell^k, \varrho_\ell^k, \delta_{\mathrm{min}})$ for $\delta, \delta_{\mathrm{min}} \in (0, 1]$

Output: new Newton iterate $u_\ell^{k+1} := u_\ell^k + \delta\, \varrho_\ell^k$

✓ fixed minimal number $k_{\mathrm{min}} \in \mathbb{N}$ of Newton steps to tame **local Lipschitz** constants

✓ sufficiently small initial residuum not required

Globalizing damping strategy

- $0 < q[\delta] = 1 - \delta + C\,\delta^2\, \|F - Au_\ell^k\|_{X_\ell'} < 1$

✓ if $\delta = 1$ and $q[\delta] < 1$: **quadratic convergence**

! but constant C in $q[\delta]$ unknown

replace $q[\delta]$ by $1 - \delta^{3/2}/2$, valid for $\delta \searrow 0$

Adaptive DAMPING$(u_\ell^k, \varrho_\ell^k, \delta_{\mathrm{min}})$

Input: previous iterate u_ℓ^k, Newton update ϱ_ℓ^k, current minimal damping δ_{min}. Initialize $\delta := 1$

repeat set $u_\ell^{k+1} := u_\ell^k + \delta\, \varrho_\ell^k$, $\delta := \delta/2$, $\delta_{\mathrm{min}} := \min\{\delta, \delta_{\mathrm{min}}\}$ **until**

$$\|F - A(u_\ell^{k+1})\|_{X_\ell'} \le (1 - \delta_{\mathrm{min}}^{3/2}/2)\, \|F - Au_\ell^k\|_{X_\ell'}$$

Output: damping δ, minimal damping δ_{min}

✓ terminates with uniform $0 < \delta_0 \le \delta_{\mathrm{min}} \le \delta$

NAILFEM algorithm

Adaptive FEM ∞ Newton

Input: initial mesh $\mathcal{T}_0$, start guess $u_0^0 \in X_0$, initial minimal damping $\delta_{\mathrm{min}} := 1/2$, mesh-adaptivity parameter $0 < \theta \le 1$, linearization-adaptivity parameter $0 < \lambda_{\mathrm{lin}}$, fixed minimal number $1 \le k_{\mathrm{min}}$ of Newton steps

for $\ell = 0, 1, \dots$ **repeat**

- **for** $k = 0, 1, \dots$ **repeat**
  - » SOLVE get $u_\ell^{k+1} :=$ NEWTON$(u_\ell^k, \delta_{\mathrm{min}})$ on X_ℓ
  - » ESTIMATE compute estimator $\eta_\ell(u_\ell^{k+1})$
  - **until** $k_{\mathrm{min}} \le k + 1$ **and** $\|F - Au_\ell^{k+1}\|_{X_\ell'} \le \lambda_{\mathrm{lin}}\, \eta_\ell(u_\ell^{k+1})$
- MARK Dörfler marking for η_ℓ with θ
- REFINE $\mathcal{T}_{\ell+1}$ by NVB of marked elements
- employ **nested iteration** $u_{\ell+1}^0 := u_\ell^{k[\ell]}$

Output: sequence of approximations u_ℓ^k, error estimators $\eta_\ell(u_\ell^k)$, and residuals $\|F - Au_\ell^k\|_{X_\ell'}$

Theorem

- quasi-error for indices from NAILFEM reads

$$\mathrm{H}_\ell^k := \|F - Au_\ell^k\|_{X_\ell'} + \eta_\ell(u_\ell^\star)$$

$$= \text{linearization error} + \text{discretization error}$$

- **sufficiently large** k_{min} for both results

- **arbitrary** choice of $0 < \theta \le 1$ and $0 < \lambda_{\mathrm{lin}}$

$\Longrightarrow$ **full R-linear convergence**, i.e., exists $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ such that

$$\mathrm{H}_\ell^k \le C_{\mathrm{lin}}\, q_{\mathrm{lin}}^{|\ell,k| - |\ell',k'|}\, \mathrm{H}_{\ell'}^{k'} \quad \text{if } |\ell',k'| \le |\ell,k|$$

with $|\ell, k|$ the total number of steps up to (ℓ, k)

- suppose $u^\star$ is approximable with rate $0 < r$
- define total computational cost

$$\mathrm{cost}(\ell, k) := \sum\nolimits_{|\ell',k'| \le |\ell,k|} \#\mathcal{T}_{\ell'}$$

- **sufficiently small** $0 < \theta \ll 1$, $0 < \lambda_{\mathrm{lin}} \ll 1$

$\Longrightarrow$ **rate-optimality** with respect to the **total computational cost/complexity**

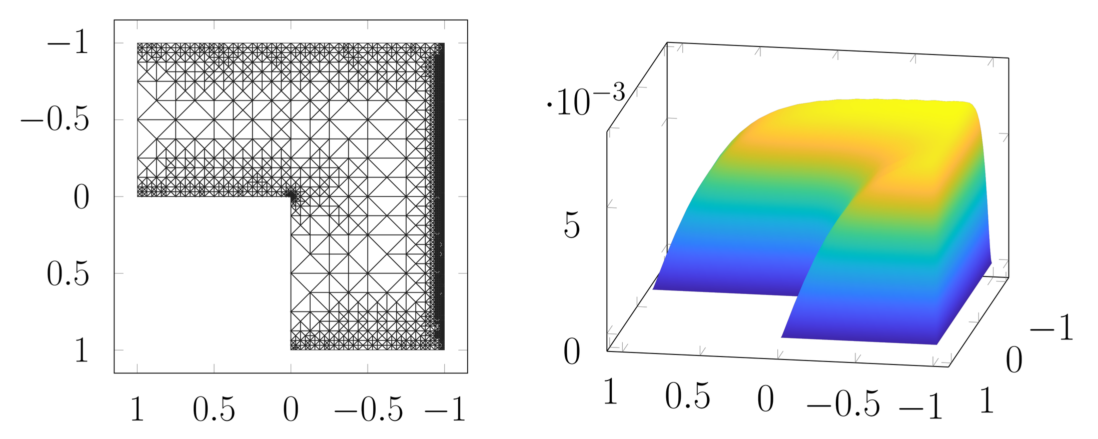
$$\mathrm{H}_\ell^k \le C_{\mathrm{opt}}\, \mathrm{cost}(\ell, k)^{-r} \text{ for all indices } (\ell, k)$$

Nonsymmetric semilinear PDEs

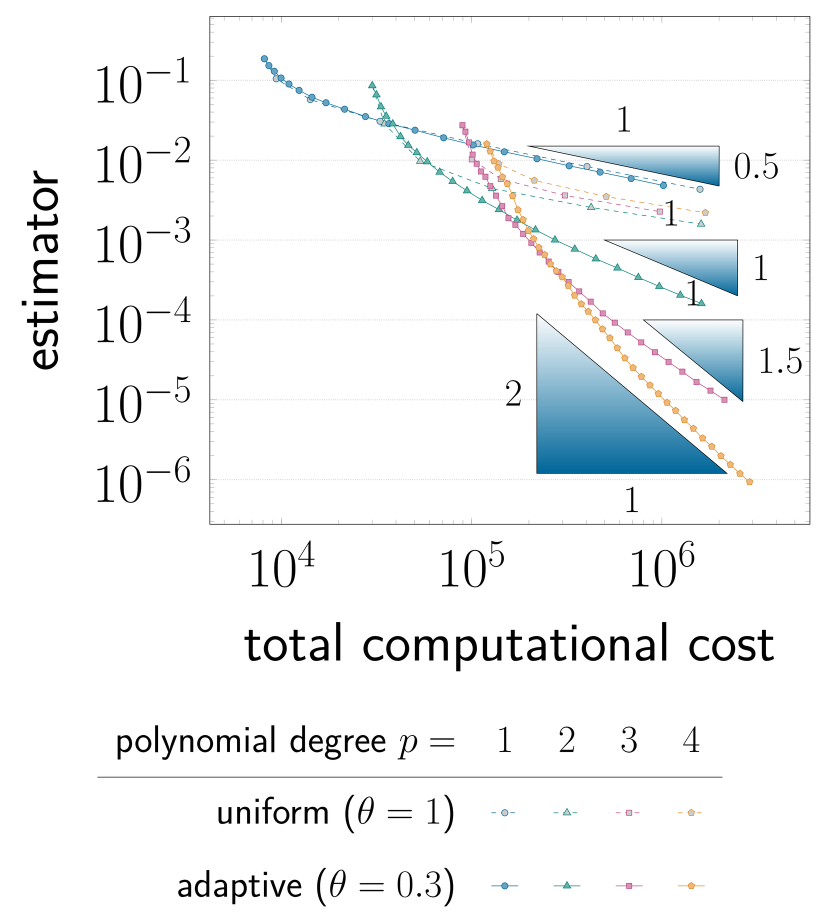
- find $u^\star \in H_0^1(\Omega)$ to

$$-\Delta u^\star + \boldsymbol{b} \cdot \nabla u^\star + c(u^\star) = f \in H^{-1}(\Omega) \text{ with}$$

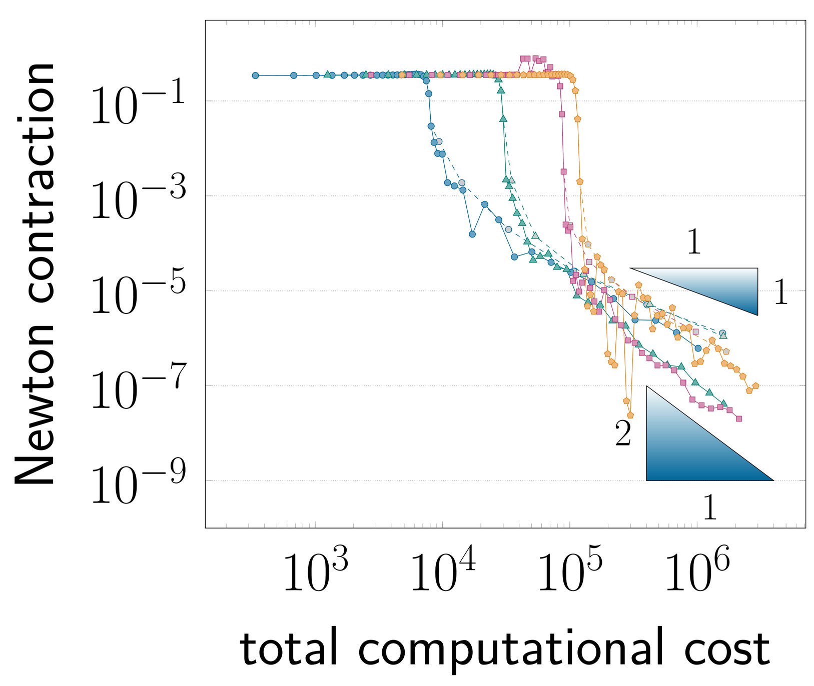
- $\boldsymbol{b} = (-50, 0)^\intercal$, $c(u) = \sum_{n=0}^{40} (40u)^n/n! \approx \exp(40u)$
- conforming finite elements of degree $p \in \{1, 2, 3, 4\}$, right-hand side $f \equiv 2$, $k_{\mathrm{min}} = 1$



optimal convergence



quadratic convergence



✓ optimal rate $-p/2$ of η for degree p wrt. cost

✓ quadratic conv. $\|F - Au_\ell^k\|_{X_\ell'} / \|F - Au_\ell^{k-1}\|_{X_\ell'}$

Conclusions

✓ robust damping strategy: fully computable, asymptotically quadratic convergence

✓ first cost-optimal AFEM beyond nonlinear PDEs with energy structure

Reference

P. Bringmann, M. Brunner, D. Praetorius: *Newton's method in adaptive iteratively linearized FEM*, arXiv: 2512.19357, 2025.

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