TECHNISCHE UNIVERSITÄT WIEN

Nonlinear partial differential equations

Lecture Notes
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Introduction

A huge variety of processes occuring in natural sciences and even economics can be described using partial differential equations (PDE). In realistic modelling these differential equations are in general nonlinear. In contrast to linear partial differential equations, we can not classify nonlinear PDEs in a simple way. Therefore, various specific techniques are necessary to study the well-posedness of boundary value problems and initial boundary value problems and the qualitative behaviour of their solutions. In this lecture we introduce typical strategies for semilinear equations and quasilinear equations. We start with presenting some famous nonlinear equations.

1.1 Examples

Liquid in a porous medium We want to derive an equation that describes the evolution of a liquid in a porous medium. Examples that occur in nature are the penetration of water into a dike or the movement of groundwater in a rock layer. We are looking for an equation for the fluid density $n(x,t)$. Assuming conservation of mass, the fluid density n satisfies the conservation equation

$$n_t + \operatorname{div}(nv) = 0\,,$$

where v denotes the average velocity of the fluid. In the following we are always looking at dimensionfree and scaled equations. The above equation is called a conservation equation, since it implies that the mass $\int_{\mathbb{R}^3} n\mathrm{d}x$ is constant in time and hence conserved. Secondly, we assume that the velocity v is proportional to the gradient of the pressure p. This is known as *Darcy's law*:

$$v = -k\nabla p\,,$$

where the constant $k > 0$ depends on the viscosity of the fluid and the permeability of the medium. Thirdly, we assume that the pressure p satisfies the following equation:

$$p = n^\gamma, \qquad \gamma > 0\,.$$

Combining all three equations, we obtain the porous medium equation

$$n_t - \operatorname{div}\Big(\frac{k\gamma}{\gamma+1}\nabla n^{\gamma+1}\Big) = 0\,.$$

Rewriting the equation

$$n_t - \mathrm{div}(D(n)\nabla n) = 0 \qquad \text{with } D(n) = \kappa\gamma n^\gamma\,,$$

we observe directly that it belongs to the class of *quasilinear parabolic equations*. In general, the equation holds on a domain $\Omega \in \mathbb{R}^n$ and its description is completed by boundary and initial conditions, e.g.

$$n = 0 \qquad \text{on } \partial\Omega,\ t > 0, \qquad n(\cdot, 0) = n_0 \qquad \text{in } \Omega\,.$$

The function $D(n)$ is the pressure dependent diffusion coefficient. Note that the diffusivity gets larger for larger pressure density. The differential operator $L(n) = -\mathrm{div}(D(n)\nabla n)$ is not uniformly elliptic, since the diffusivity vanishes for $n = 0$. Therefore, the porous medium equation belongs to the class of degenerate parabolic differential equations. If the diffusivity is constant, we obtain the heat equation $n_t - D_0\Delta n = 0$. In contrast to this equation, the porous medium equation ($\gamma > 0$) has a finite speed of propagation. Physically, this means that areas saturated by the liquid spread with finite speed in the medium.

Incompressible liquids and gases Consider a liquid or a gas with particle density $n(x, t)$ and medium speed $v(x, t)$. The equation

$$n_t + \mathrm{div}(nv) = 0\,, \tag{1.1}$$

describes the conservation of the particle density. The particle flow is described by the flow density nv satisfying the following equation

$$(nv)_t + \mathrm{div}(nv \otimes v) + \nabla p = F(v)\,, \tag{1.2}$$

where $v \otimes v$ is a matrix with components $v_i v_j$ and p is the pressure of the fluid. If it holds $F(v) = 0$, the two previous equations are *Euler equations*, which describe thin gases such as air. In this case, the differential equations are of first order and belong to the class of hyperbolic conservation equations (both total particle density and total flow density are constant in time), which will not be of covered in this lecture. We rather focus on the equations with describe fluids like water and oil which are viscous and where we expect diffusive terms in the equations of the particle flows. One can show that in this case

$$F(v) = (\lambda + \mu)\nabla(\mathrm{div}v) + \mu\Delta v\,,$$

where λ and $\mu > 0$ are specific constants for the fluid and the Laplace operator is considered componentwise for the vector-valued function v (i.e., Δv is a vector). Then, (1.1) and (1.2) are the Navier-Stokes equations. The mathematical analysis is very challenging and we only consider here the special case of a stationary homogeneous fluid. Thanks to the stationarity the time derivatives vanish. Homogenity means that the particle density is constant in space. Both assumptions imply that $n(x, t) = n_0$ is constant. For simplicity we assume that $n_0 = 1$ and $\mu = 1$. Then, (1.1) and (1.2) simplify to

$$\mathrm{div}(v \otimes v) + \nabla p = \Delta v\,, \qquad \mathrm{div}v = 0\,.$$

Since $\mathrm{div}(v \otimes v) = (\mathrm{div}v)v + (v \cdot \nabla)v = (v \cdot \nabla)v$, we obtain the stationary Navier-Stokes equations

$$(v \cdot \nabla)v + \nabla p = \Delta v\,, \qquad \mathrm{div}v = 0\,,$$

where the components of the vector $(v \cdot \nabla)v$ are given by $\sum_i v_i \partial_i v_j$ with $\partial_i = \partial/\partial x_i$. The term Δv is called the diffusive term and $(v \cdot \nabla)v$ is the convective term. Because of the convective term the Navier-Stokes equations are nonlinear. The equations are solved in a domain Ω. Typical boundary conditions are $v = 0$ on $\partial\Omega$ (the fluid rests on the boundary) or $v \cdot \nu = 0$ on $\partial\Omega$ (the bounary is isolating, no fluid can pass the boundary), where ν is the unit normal vector on $\partial\Omega$. Fluids satisfying $\mathrm{div} v = 0$ are called *incompressible*.

Chemical reactions The concentration of a chemical on a domain Ω can be described via the equation

$$u_t - \Delta u = R_+ - R_- \,.$$

Here, we assumed the diffusivity to be constant. The function R_+ describes the source term, and R_- the sink in a domain. We assume that the chemical is produced with constant rate $R_+ = R_0 > 0$ and the sink term depends quadratically on the density of the chemical, i.e., $R_- = -u^2 \le 0$. This parameter choice models for instance binary reactions, where the chemical is transformed. If the chemical can not pass the boundary of the domain, we use homogeneous Neumann boundary conditions:

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,\ t > 0\,, \qquad u(\cdot, 0) = u_0 \qquad \text{in } \Omega\,.$$

At time $t = 0$ the chemical has density u_0. This initial boundary value problem belongs to the class of semilinear parabolic equation. Equations of the form

$$u_t - \Delta u = f(x, u)$$

are called *reaction diffusion equations*.

Question: What happens for $t \to \infty$? We expect that the concentration does not change for large times, i.e., $u_t = 0$. Then the problem becomes solving the equation

$$-\Delta u = R_0 - u^2 \qquad \text{in } \Omega\,, \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega\,.$$

The solution is given by $u_\infty = \sqrt{R_0}$. In chemistry, one is interested in the speed of convergence of $u(\cdot, t)$ to the stationary solution. In particular, one is interested whether and in which sense the inequality

$$\|u(\cdot, t) - u_\infty\| \le \|u_0\| e^{-\lambda t}$$

holds and how large the convergence rate $\lambda > 0$ is.

Electrons in a semiconductor I The movement of charged particles such as electrons in a semiconductor material can classically be described by a diffusion term and a drift term. The diffusion is caused by changes in the particle concentration n, and the drift is generated by the electric field. The total particle flow density can be described by

$$J = \nabla n - n \nabla \phi\,,$$

where ϕ denotes the electric potential and $-\nabla\phi$ is the electric field. The evolution of the density of electrons can be described by the equation for mass conservation

$$n_t - \mathrm{div} J = 0\,.$$

In the following we set $n_t = 0$, since we are interested in stationary solutions. Then we want to find J solving

$$\mathrm{div} J = 0\,.$$

The electric potential is the solution to the Poisson equation

$$\Delta\phi = n - f(x)\,,$$

where $f(x)$ are given charges in the semiconductor and $n - f$ is the total charge density. Hence, the pair (n, ϕ) is the solution to the system of equations

$$\mathrm{div}(\nabla n - n\nabla\phi) = 0\,, \qquad \Delta\phi = n - f(x)$$

on the domain Ω of the semiconductor. These equations form a system of *nonlinear elliptic equations* which is called stationary drift diffusion model. We assume that the density of the electrons and the electric potential are known at the boundary of the domain:

$$n = g, \quad \phi = \psi \qquad \text{on } \partial\Omega\,.$$

In electrical engineering, the function $\psi \to \int_K J \cdot \nu \mathrm{d}s$ is of particular interest. A voltage is therefore applied and one asks for the current flow through the contact $K \subset \partial\Omega$. This results in the so-called current-voltage characteristic, which characterises the semiconductor device.

Electrons in a semiconductor II Electrons are actually objects from quantum mechanics and they are described by so-called complex-valued wave functions $\psi(x,t)$. The integral

$$\int_\Omega |\psi(x,t)|^2 \mathrm{d}x$$

is considered in quantum mechanics as the probability that the ensemble of electrons is at time t in the domain Ω. It is postulated that the wave function satisfies the *Schrödinger equation*

$$i\psi_t + \Delta\psi - V(x,t)\psi = 0 \qquad \text{in } \mathbb{R}^n\,, \qquad \psi(\cdot,0) = \psi_0\,.$$

Here, $V(x,t)$ is a given real-valued potential. For instance the potential could be the one of the atoms in a semiconductor or the voltage applied to the semiconductor. The Schrödinger equation is neither elliptic nor parabolic nor hyperbolic. We will observe that it has on the one side the properties of a wave equation (electrons are described by a wave in a certain way) and on the other side regularising effects which we know from elliptic and parabolic equations.

There is a simple connection to the fluid-dynamical equations by differentiating the term $|\psi|^2$ in time. We obtain

$$\begin{aligned}\partial_t|\psi|^2 = \partial_t(\bar\psi\psi) &= \partial_t\bar\psi\cdot\psi + \bar\psi\cdot\partial\psi = (-i\Delta\bar\psi + iV\bar\psi)\psi + \bar\psi(i\Delta\psi - iV\psi)\\ &= -i\mathrm{div}(\nabla\bar\psi\psi - \bar\psi\nabla\psi) = -2\mathrm{div}\mathrm{Im}(\bar\psi\nabla\psi)\,,\end{aligned}$$

since $z - \bar z = 2i\mathrm{Im}(z)$ for $z \in \mathbb{C}$. By setting $J = -2\mathrm{Im}(\bar\psi\nabla\psi)$, we obtain

$$\partial_t|\psi|^2 - \mathrm{div} J = 0\,,$$

which is the conservation equation for the particle density $n = |\psi|^2$.

The above introduced Schrödinger equation is linear. In some applications (e.g. in optics) nonlinear equations occur of the form

$$i\psi_t + \Delta\psi \pm |\psi|^\alpha \psi = 0\,,$$

where $\alpha >$ depends on the model. A typical example is the cubic Schrödinger equation with $\alpha = 2$. The above equation is semilinear due to the term $|\psi|^\alpha\psi$. Depending on the sign and on α there exists a solution for all time or the solution is unbounded for finite times. This will studied in Chapter 4.2.

Minimal surfaces Let an area in $\mathbb{R}^3$ be described by the function $u(x_1, x_2)$, i.e., the area is equal to the set $F = \{(x_1, x_2, u(x_1, x_2)) : (x_1, x_2) \in \Omega\}$. By differential geometry it is well known that the surface of F given is by

$$S(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\mathrm{d}x\,.$$

We want to find the surface with minimal volume satisfying the boundary conditions $u = g$ on $\partial\Omega$. A typical example of this surfaces are soap skins. By using variational calculus one can show that the function u satisfies the following minimal surface equation:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \qquad \text{in } \Omega\,, \qquad u = g \qquad \text{on } \partial\Omega\,.$$

This equation is quasi-linear (since the diffusion coefficient depends on u) and elliptic, since we can rewrite the equation to (with $x := x_1$ and $y = x_2$)

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0\,.$$

The coefficient matrix

$$\begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_x u_y & 1 + u_x^2 \end{pmatrix}$$

has positive eigenvalues, and thus the equation is elliptic. Therefore, the equation belongs to the class of *quasilinear elliptic equations* of type

$$\mathrm{div}(a(\nabla u)) = 0\,.$$

In differential geometry the expression $(1/2)\mathrm{div}(\nabla u/\sqrt{1 + |\nabla u|^2})$ correponds to the mean curvature of the surface F. Minimal surfaces have a vanishing mean curvature.

More nonlinear equations The above presented differential equations are all of second order, i.e., the highest derivative is of second order. Equations of first order are for instance *hyperbolic conservation equations* of the form

$$u_t + \mathrm{div}(F(u)) = 0$$

or the *Hamilton-Jacobi equations*

$$u_t + F(\nabla u) = 0\,.$$

Equations of third order are for instance the *Korteweg-de-Vries equation*

$$u_t + uu_x + u_{xxx} = 0\,,$$

for which special techniques are required and which are not considered here. In general, this type of equations differs heavily form second order equations and therefore different analytical tools are needed since the second order equations have some diffusive effects leading to more regularisation of the solution. (An exclusion forms the Schrödinger equation, which regularises only weakly.) However, the diffusive effects are not restricted to equations of second order. For instance, we will consider the *Derrida-Lebowitz-Speer-Spohn equation* which is a fourth order nonlinear equation with a logarithmic nonlinearity,

$$u_t + (u(\ln u)_{xx})_{xx} = 0\,.$$

This equation describes the particle density of electrons in a specific semiconductor or the limit of a random variable, which models a particle system with spin.

Goal For the above presented equations and their generalisations we study the following problems:

- Do we have existence and uniquness of a solution to these problems?
- Does the solution depends continuously on the given data (coefficients, boundary values, initial values)?
- Is the solution regular, if the given data are regular?
- How does the solution behave for large times?

1.2 Sobolev spaces

In this section we give an overview on Sobolev spaces. In the following, let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an open domain.

Definition 1.1. *Let $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is the set of all functions $u \in L^p(\Omega)$ such that*

$$D^\alpha u \in L^p \qquad \text{for all } |\alpha| \leq m\,,$$

where α is a multi index and $D^\alpha u$ is the corresponding partial derivative of u in the sense of distributions.

To be precise, $W^{m,p}(\Omega)$ is a set of equivalence class of functions, which coincide up to a null set.

On $W^{m,p}(\Omega)$ we define the norm by

$$\|u\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \qquad \text{for } p < \infty\,,$$

$$\|u\|_{W^{m,p}(\Omega)} = \max_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \qquad \text{for } p = \infty\,.$$

The space $W^{m,p}(\Omega)$ is a Banach space with respect to this norm. For $m = 0$, it holds $W^{0,p}(\Omega) = L^p(\Omega)$. For $p = 2$ we write $H^m(\Omega) = W^{m,2}(\Omega)$. This space endowed with the scalar product

$$(u, v)_{H^m} = \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_{L^2}$$

is a Hilbert space.

Definition 1.2. *Let $m \in \mathbb{N}_0$ and $1 \le p \le \infty$. The Sobolev space $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.*

The space $W_0^{m,p}(\Omega)$ is also a Banach space. Analogously, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ endowed with the scalar product defined above is a Hilbert space.

Next, we present some properties of $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$.

Theorem 1.3 (Dense sets)**.** *Let $1 \le p < \infty$ and $m \in \mathbb{N}_0$. Then, $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$ and $C_0^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega)$. If $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$ (actually $C^{0,1}$ is sufficient), then $C^\infty(\bar{\Omega})$ is dense in $W^{m,p}(\Omega)$.*

Proof. For the proof we refer to Evans [10, Section 5.3.2 and Section 5.3.3.]. □

Theorem 1.4 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$ (actually $C^{0,1}$ is sufficient) and let $1 \le p, q < \infty$, $k, m \in \mathbb{N}_0$ with $m > k$.*

(i) The embedding $W^{m,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$ is continuous if $m - n/p \ge k - n/q$. Further, the embedding is compact if $m - n/p > k - n/q$.

(ii) The embedding $W^{m,p}(\Omega) \hookrightarrow C^k(\bar{\Omega})$ is compact, if $m - n/p > k$.

For arbitrary bounded domains Ω, both results hold for $W_0^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega)$.

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Proof. A proof can be found in e.g. Alt [2, Theorem 8.9] and Gilbarg-Trudinger [11, Theorem 7.26]. □

The number $m - n/p$ can be seen as a characterization of the regularity of $W^{m,p}$-functions. To give a meaning to a compact embedding of the form $H^1(\Omega) \hookrightarrow L^p(\Omega)$, we introduce the following notation. Consider the intervals

$$N^* = \begin{cases} [1, \frac{2n}{n-2}] & : n \ge 3\,, \\ [1, \infty) & : n = 2\,, \\ [1, \infty] & : n = 1\,, \end{cases} \qquad \text{and} \qquad N_* = \begin{cases} [\frac{2n}{n+2}, \infty) & : n \ge 3\,, \\ (1, \infty) & : n = 2\,, \\ [1, \infty) & : n = 1\,. \end{cases} \tag{1.3}$$

Then the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for all $p \in N^*$. We say that (p, q) is an admissible pair, if $p \in N^*$ and $1/p + 1/q = 1$. Then, $q \in N_*$ by construction. With this notation, for admissible pairs (p, q) the product fg of functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ is integrable by the Hölder inequality: $fg \in L^1(\Omega)$.

Similarly as for H^1 functions we can define the trace $u|_{\partial U}$ for $u \in W^{1,p}(\Omega)$, see e.g. Alt [2, Theorem A 6.6.].

Theorem 1.5 (Trace of Sobolev functions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$ (actually $C^{0,1}$ is sufficient) and $1 \leq p < \infty$. Then, there exists exactly one continuous linear map, called trace, $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ satisfying*

$$\gamma(u) = u|_{\partial\Omega} \qquad \text{for all } u \in W^{1,p}(\Omega) \cap C^0(\bar{\Omega})\,.$$

Now, we can characterize $W_0^{1,p}$-functions similarly as H_0^1-functions.

Theorem 1.6 (Characterization of $W_0^{1,p}$-functions). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$ (actually $C^{0,1}$ is sufficient), $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then,*

$$u \in W_0^{1,p}(\Omega) \qquad \text{if and only if} \qquad \gamma(u) = 0\,.$$

The Poincaré inequality holds in general for $W_0^{1,p}$ functions (see Evans [10, Section 5.8.1]).

Theorem 1.7 (Poincaré inequality). *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$ (actually $C^{0,1}$ is sufficient) and $1 \leq p < \infty$. Then there exists a constant $C_p > 0$, such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq C_p \|\nabla u\|_{L^p(\Omega)}\,.$$

1.3 Convergence and compactness

In finite dimensional spaces bounded sequences have convergent subsequences. In infinite dimensional spaces this result does not hold in general. The lack of compactness often causes problems in the theory of nonlinear partial differential equations. To overcome this issue, one can use the concept of weak convergence. In the following we state several results from functional analysis without proving them. Proofs can be found in Zeidler [24, Kapitel 21].

Let B be a Banach space with norm $\|\cdot\|$. Its dual space is defined by $B' = \{f : B \to \mathbb{R} : f \text{ is linear and continuous}\}$. The norm of B' is given by

$$\|f\|_{B'} = \sup_{\|u\| \leq 1} |f(u)|\,.$$

We often write $\langle f, u\rangle_{B'}$ instead of $f(u)$. For instance the dual space of $L^p(\Omega)$ is given by $L^q(\Omega)$, where $1 \leq p < \infty$ and $1/p + 1/q = 1$. Later, we use the following result.

Proposition 1.8. *Let B and X be Banach spaces with continuous and dense embedding $B \hookrightarrow X$. Then the embedding $X' \hookrightarrow B'$ is continuous. If X is reflexive, then X' is dense in B'. (X is reflexive $\Leftrightarrow$ the canonical embedding of X in its bidual $(X')'$ is a continuous isomorphism.)*

Proof. The proof is given in Zeidler [24, Problem 18.6.]. □

Applying the Proposition to $X = L^p(\Omega)$ and $B = H_0^1(\Omega)$ with $p \in N^*$, we obtain

$$L^q(\Omega) \subset H^{-1}(\Omega) = (H_0^1(\Omega))'\,, \quad q \in N_*\,,\ 1/p + 1/q = 1\,. \tag{1.4}$$

For $n = 3$, we can choose $q \geq 6/5$, whereas for $n = 2$, $q > 1$ and for $n = 1$, $q \geq 1$.

Definition 1.9. *A sequence $(u_k)_k \subset B$ is called weakly convergent to $u \in B$, if for all $F \in B'$*

$$\langle F, u_k\rangle_{B'} \to \langle F, u\rangle_{B'}\,, \quad \text{for } k \to \infty\,.$$

In this case, we write

$$u_k \rightharpoonup u\,, \quad \text{for } k \to \infty\,.$$

Example 1.10. *We claim that the sequence $(u_k)_k$, defined by $u_k(x) = \sin(kx)/\pi$, converges weakly in $L^2(0,2\pi)$ to zero. Let $f \in (L^2(0,2\pi))' = L^2(0,2\pi)$. Then, the Fourier coefficients are given by*

$$\langle f, u_k\rangle_{L^2(0,2\pi)} = \frac{1}{\pi}\int_0^{2\pi} \sin(kx) f(x)\mathrm{d}x\,.$$

By the Riemann-Lebesque theorem it holds that the coefficients converge to zero for $k \to \infty$. Hence, $u_k \rightharpoonup 0$ in $L^2(0,2\pi)$. But, u_k does not converge strongly in $L^2(0,2\pi)$, since

$$\|u_k - 0\|^2_{L^2(0,2\pi)} = \frac{1}{\pi^2}\int_0^{2\pi} \sin^2(kx)\mathrm{d}x = \frac{1}{\pi} \neq 0\,.$$

Hence, weak convergence does not imply strong convergence. Note that the sequence $(u_k)_k$ does neither converge pointwise nor almost everywhere in $(0,2\pi)$.

The concept of weak convergence is very useful. In particular, the following result holds which is a special case of the theorem by Alaoglu (for the proof see Zeidler [24, Theorem 21. D]).

Theorem 1.11 (Eberlein-Šmuljan)**.** *Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.*

We remark that reflexive Banach spaces satisfy $B'' = B$. The Lebesque space $L^p(\Omega)$ is reflexive if and only if $1 < p < \infty$. The space $L^p(\Omega)$ is separable (i.e., it has an atmost countable dense subset) if and only if $1 \leq p < \infty$. So, $L^1(\Omega)$ is separable, but not reflexive; it holds $(L^1(\Omega))' = L^\infty(\Omega)$. The space $L^\infty(\Omega)$ neither separable nor reflexive. The dual $(L^\infty(\Omega))'$ is given by the set of bounded, σ-additive, signed measures that are absolutely continuous with respect to the Lebesque measure. It contains the set $L^1(\Omega)$, i.e., $L^1(\Omega) \subset (L^\infty(\Omega))'$, and the inclusion is strict.

Weakly convergent sequences have the following properties (see Zeidler [24, Proposition 21.23], Emmrich [9, Theorem A.2.17]):

Proposition 1.12 (Properties of weak convergence)**.** *Let B be a Banach space.*

(i) Strong convergence implies weak convergence.

(ii) If $\dim(B) < \infty$, then weak convergence implies strong convergence.

(iii) Theorem of Banach-Steinhaus: If $u_k \rightharpoonup u$ $(k \to \infty)$ for a sequence $(u_k)_k \subset B$, then (u_k) is bounded and

$$\|u\| \leq \liminf_{k\to\infty} \|u_k\|\,.$$

(iv) Let B be reflexive. If all weakly convergent subsequences of a bounded sequence $(u_k)_k \subset B$ have the same limit $u \in B$, then the whole sequence $(u_k)_k$ converges weakly to u.

Remark 1.13. *For part (iv) an even stronger statement holds: Let B be a topological space. Then $(u_k)_k$ converges to $u \in B$ if and only if every subsequecence of $(u_k)_k$ has a subsubsequence which converges to the same limit $u \in B$.*

Remark 1.14. *Let $(u_k)_k$ be a weakly convergent sequence, $u_k \rightharpoonup u$ in B, and $f : B \to B$ be a continuous function. If $(f(u_k))_k$ is (weakly) convergent, is $f(u_k) \rightharpoonup f(u)$ in B? In general the answer is negative. For instance let $u_k(x) = \sin(kx)$, $x \in (0, 2\pi)$ and $f(x) = |x|$. Then by Example 1.10, $u_k \rightharpoonup 0$ in $L^2(0, 2\pi)$. But, $|u_k|$ does not converge weakly to 0 in $L^2(0, 2\pi)$ since for $\phi(x) = 1$,*

$$\int_0^{2\pi} |\sin(kx)| \cdot 1 \mathrm{d}x = \frac{1}{k} \int_0^{2\pi k} |\sin y| \mathrm{d}y = \int_0^{2\pi} |\sin y| \mathrm{d}y = 4\,.$$

Hence, the integral does not converge to zero.

In non-reflexive (but separable) Banach spaces (e.g. $L^1(\Omega)$) a similar result as the theorem by Eberlein-Šmuljan can be proven using the weak* convergence. This is in particular applicable if boundedness in $L^\infty(\Omega)$ holds.

Definition 1.15. *A sequence $(F_k)_k \subset B'$ is called weakly* convergent to $F \in B'$, if for all $u \in B$ it holds*

$$\langle F_k, u\rangle_{B'} \to \langle F, u\rangle_{B'} \qquad \text{for } k \to \infty\,.$$

In this case, we write

$$F_k \rightharpoonup^* F \qquad \text{for } k \to \infty\,.$$

Theorem 1.16. *Let B be a separable Banach space. Every bounded sequence in B' has a weakly* convergent subsequence.*

Proof. For a proof we refer to Zeidler [24, Section 21.8]. □

Let $(u_k)_k$ be a bounded sequence in $L^\infty(\Omega)$ and $B = L^1(\Omega)$. Since, $(L^1(\Omega))' = L^\infty(\Omega)$, there exists a subsequence $(u_{k_n})_n$, such that $u_{k_n} \rightharpoonup^* u$ in $L^\infty(\Omega)$, i.e.,

$$\int_\Omega u_{k_n} v \mathrm{d}x \to \int_\Omega uv \mathrm{d}x \quad \text{for all } v \in L^1(\Omega)\,.$$

Proposition 1.17 (Properties of weak* convergence)**.** *Let B be a separable Banach space.*

(i) Strong convergence implies weak* convergence.

(ii) For reflexive Banach spaces weak convergence and weak* convergence are equivalent.

(iii) Theorem by Banach-Steinhaus: If it holds $u_k \rightharpoonup^* u$ $(k \to \infty)$ for a sequence $(u_k)_k \subset B'$, then $(u_k)_k$ is bounded and

$$\|u\|_{B'} \le \liminf_{k\to\infty} \|u_k\|_{B'}\,.$$

(iv) If $u_k \rightharpoonup^* u$ *in* B' *and* $v_k \to v$ *in* B *or if* $u_k \to u$ *in* B' *and* $v_k \rightharpoonup v$ *in* B*, then*

$$\langle u_k, v_k \rangle_{B'} \to \langle u, v \rangle_{B'} \qquad \text{for } k \to \infty\,.$$

Remark 1.18. *If* $(u_k)_k$ *is a bounded sequence in* $L^1(\Omega)$ *neither Theorem 1.12 nor Theorem 1.17 are applicable. In general it does not hold that a weakly convergent subsequence exists. It holds under additional assumptions. By the theorem of Dunford-Pettis (see. Ekeland/Temam [8, Chapter VII, Theorem 1.3]:*
If the function $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\lim_{x\to\infty} \phi(x)/x = \infty$ *and* $\sup_k \int_\Omega \phi(|u_k|)\mathrm{d}x < \infty$*, then there exists a weakly convergent subsequence of* $(u_k)_k$ *in* $L^1(\Omega)$*. If we choose* $\phi(x) = |x|^\alpha$ *with* $\alpha > 1$*, then boundedness of* $\|u_k\|_{L^\alpha}(\Omega)$ *implies existence of a subsequence* $(u_{k_n})_n$ *with* $u_{k_n} \rightharpoonup u$ *in* $L^1(\Omega)$*.*

Finally, we recall the Dominated Convergence Theorem and its 'reverse':

Theorem 1.19 (Dominated Convergence Theorem)**.** *Let* $\Omega \in \mathbb{R}^n$ *(*$n \geq 1$*) be an open domain and* $(f_k)_k \subset L^1(\Omega)$ *be a sequence.*

(i) If $f_k \to f$ *almost everywhere in* Ω *for* $k \to \infty$ *and if there exists a function* $g \in L^1(\Omega)$ *with* $|f_k| \leq g$ *for all* k*, then* $f \in L^1(\Omega)$ *and* $f_k \to f$ *in* $L^1(\Omega)$*.*

(ii) If $f_k \to f$ *in* $L^1(\Omega)$*, then there exists a subsequence* (f_{k_m}) *with* $f_{k_m} \to f$ *almost everywhere in* Ω *and it exists a function* $g \in L^1(\Omega)$ *such that* $|f_{k_m}| \leq g$ *for all* m*.*

This theorem also holds if we replace $L^1(\Omega)$ by $L^p(\Omega)$ with $1 \leq p < \infty$ in part (i) and with $1 \leq p \leq \infty$ in part (ii). A proof is given in Brézis [4, Théorème IV.9].

Chapter 2

Nonlinear elliptic equations

In this chapter we want to find solutions for nonlinear elliptic differential equations given by

$$-\Delta u = f(x,u) \qquad \text{and} \qquad -\operatorname{div} a(\nabla u) = f(x)\,.$$

The first equation is a *semilinear equation* (since the highest derivative is linear in u), and the second equation is a *quasi-linear equation* since

$$\operatorname{div} a(\nabla u) = \sum_{i=1}^{n} \partial_{x_i} a_i(\partial_{x_1} u, \dots, \partial_{x_n} u) = \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial z_j}(\nabla u)\partial_{x_i}\partial_{x_j} u\,.$$

To prove existence and uniqueness of solutions to these equations, we will use two principal techniques:

- fixed point theorems: The differential equation is formulated as a fixed point equation $S(u) = u$ and the existence of a fixed point u is proven. For this reason, we typically use some compactness results which we get by so-called a-priori estimates.
- a-priori estimates: These are inequalities for $S(u)$, where the right hand side is independent of u. We obtain these estimates for example by the maximum principle or monotonicity of the nonlinearities.

2.1 Semilinear equations

We consider the differential equation

$$L(u) = f(x,u) \qquad \text{in } \Omega, \qquad u = g \qquad \text{on } \partial\Omega\,. \tag{2.1}$$

In the following subsections we suppose the following assumptions.

Assumption 1. *The subset $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$, the differential operator $L(u)$ is defined by*

$$L(u) = -\operatorname{div}(A\nabla u) + cu\,,$$

where $A(x) = (a_{i,j}(x))$ is an $(n \times n)$ matrix and $c(x)$ is some function. Further, A is symmetric and elliptic (resp. positive definit uniformly in x), i.e., there exists $\alpha > 0$ such that $\xi^T A(x)\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$. Furthermore, $a_{i,j} \in L^\infty(\Omega)$ and $c \in L^\infty(\Omega)$ with $c \geq 0$ in Ω. The boundary function g can be extended to Ω such that it holds $g \in H^1(\Omega)$. Finally, let f be a Carathéodory function (see definition below).

Definition 2.1. *We call $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéordory function, if for all $u \in \mathbb{R}$ the function $x \mapsto f(x,u)$ is measurable and for almost every $x \in \Omega$ the function $u \mapsto f(x,u)$ is continuous.*

Goal: Prove existence (and uniqeness) of solutions to (2.1).
Idea of the proof: We define the so-called *fixed point operator*. Let v be some function and u be a solution to

$$L(u) = f(x,v) \qquad \text{in } \Omega \qquad u = g \qquad \text{on } \partial\Omega\,. \tag{2.2}$$

This equation forms a linear differential equation for given v. By the lemma of Lax-Milgram there exists a unique solution u under appropriate assumptions. Equation (2.2) defines a map $S(v) = u$. If we can prove that the map has a fixed point u^*, i.e., $S(u^*) = u^*$, then we showed existence of a solution to (2.1), since the differential equation to the fixed point is given by

$$L(u^*) = f(x,u^*)\,.$$

Next, we state some already known fixed point theorems and study in particular the conditions under which existence of a fixed point holds.

Theorem 2.2 (Fixed point theorem by Banach)**.** *Let B be a Banach space, $M \subset B$ be a closed set and $S : M \to M$ be a contraction, i.e., there exists $k \in (0,1)$ such that for all $x, y \in M$:*

$$\|S(x) - S(y)\| \le k\|x - y\|\,.$$

Then, S has a unique fixed point.

Under which conditions can we expect that the above defined operator is contractive? Let v_1 and v_2 be two functions and u_1 and u_2 be two solutions of

$$L(u_1) = f(x,v_1) \qquad L(u_2) = f(x,v_2)\,.$$

To obtain contraction, we would have to prove

$$\|u_1 - u_2\| \le k\|v_1 - v_2\|$$

for some $k < 1$. One can argue that this is only true if the right hand side of (2.1) is 'small' in a certain sense which does not fit to our setup. Therefore, we will use another fixed point theorem. (the Banach fixed point theorem will play a crucial role for semilinear parabolic equation, see Section 3.2).

Theorem 2.3 (Fixed point theorem by Schauder)**.** *Let B be a Banach space, $K \subset B$ a compact and convex set and $S : K \to K$ a continuous map. Then S has a fixed point.*

Definition. *A set K is compact if every sequence in K has a convergent subsequence in K. A set K is convex if for all $u, v \in K$ and $\lambda \in [0,1]$, $\lambda u + (1-\lambda)v \in K$.*

Note that the fixed point theorem of Schauder only gives existence of a fixed point but no uniqueness.

Proof. We only state the idea of the proof. The complete proof is given in Gilbarg-Trudinger [11, Section 11.1]. The proof is based on the following fixed point theorem:

Theorem 2.4 (Fixed point theorem by Brouwer)**.** *Let $S : \overline{B_1(0)} \to \overline{B_1(0)}$ be a continuous function, where $\overline{B_1(0)}$ is a closed unit ball in $\mathbb{R}^n$. Then S has a fixed point.*

A proof of the fixed point theorem by Brouwer is given for instance in Evans [10, Section 8.1.4].

Since K is compact, there exists for every $j \in \mathbb{N}$ points $x_1, \dots, x_{N_j}$ such that the balls $B_{1/j}(x_i),\ i = 1, \dots, N_j$ cover the set K. Let $K_j \subset K$ be the convex hull of $\{x_1, \dots, x_{N_j}\}$ and F_j be an approximation of the identity with values in K_j. Then by construction the function $F_j \circ S|_{K_j}$ is a continuous map from K_j to K_j. By the fixed point theorem of Brouwer the map has a fixed point u_j, since K_j is finite dimensional. Since K is compact, a subsequence $(u_{j_k})_k$ of $(u_j)_j$ converges to some $u \in K$. Since F_j is an approximation of the identity, it holds

$$\|u_{j_k} - S(u_{j_k})\| = \|F_{j_k} \circ S(u_{j_k}) - S(u_{j_k})\| \to 0 \qquad \text{for } j_k \to \infty\,.$$

Since S is continuous, we obtain $u = S(u)$, which concludes the proof. □

Next, we give the first existence result:

Theorem 2.5 (Existence for semilinear equations)**.** *Suppose Assumption 1 holds. Further, let f be a Carathéodory function with*

$$|f(x,u)| \le h(x) \qquad \text{for } x \in \Omega,\ u \in \mathbb{R},\ \text{where } h \in L^q(\Omega),\ q \in N_*\,,$$

where N_* is defined in (1.3)*. Then there exists a weak solution $u \in H^1(\Omega)$ to* (2.1) *and a constant $C > 0$, which depends only on A, c and Ω such that*

$$\|u\|_{H^1(\Omega)} \le C(\|g\|_{H^1(\Omega)} + \|h\|_{L^q(\Omega)})\,. \tag{2.3}$$

The weak formulation of (2.1) is given by

$$\int_\Omega (\nabla u^T A \nabla v + cuv)\mathrm{d}x = \int_\Omega f(x,u)v\mathrm{d}x \qquad \text{for all } v \in H_0^1(\Omega)\,.$$

The integral on the right hand side is well-defined. By the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in N^*$ it follows $v \in L^p(\Omega)$ and the Hölder inequality (since $1/p + 1/q = 1$) implies $f(\cdot, u)v \in L^1(\Omega)$.

Proof. The proof is divided in four steps.

Step 1: Definition of the fixed point operator. Let $v \in K = \{v \in L^2(\Omega) : \|v\|_{H^1(\Omega)} \le M\}$ where $M > 0$ will be defined later. The set K is compact in $L^2(\Omega)$, since if we take a sequence $(u_k)_k \subset K$, then by the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ there exists a subsequence $(u_{k_n})_n$ of $(u_k)_k$ with $u_{k_n} \to u$ in $L^2(\Omega)$ and $u_{k_n} \rightharpoonup u$ in $H_0^1(\Omega)$ for $n \to \infty$. The limit function u lies in K, since by Proposition 1.12

$$\|u\|_{H^1(\Omega)} \le \liminf_{n\to\infty} \|u_{k_n}\|_{H^1(\Omega)} \le M\,.$$

Hence, K is compact.

Let $u \in H^1(\Omega)$ be the uniquely determined solution of the linear boundary value problem

$$L(u) = f(x,v) \qquad \text{in } \Omega\,, \qquad u = g \qquad \text{on } \partial\Omega\,. \tag{2.4}$$

The problem is solveable (e.g. by the Lemma of Lax-Milgram), if the right hand side is in $H^{-1}(\Omega)$. This is the case since by assumption and (1.4) it holds $|f(x,v(x))| \le h(x) \in L^q(\Omega) \hookrightarrow H^{-1}(\Omega)$. This defines the operator $S : K \to L^2(\Omega),\ S(v) = u$. It remains to show that S is a self-map (i.e., $S : K \to K$) and it is continuous.

Step 2: S is a self-map. We prove $S(K) \subset K$, which is the key part of the proof. Let $u = S(v) \in S(K)$. We use the test function $u - g \in H_0^1(\Omega)$ in the weak formulation of (2.4):

$$\int_\Omega (\nabla u^T A \nabla (u-g) + cu(u-g))\mathrm{d}x = \int_\Omega f(x,v)(u-g)\mathrm{d}x\,.$$

Using the ellipticity of A and Young's inequality, we bound the left hand side by:

$$\begin{aligned}
&\int_\Omega (\nabla u^T A \nabla (u-g) + cu(u-g))\mathrm{d}x \\
&\quad = \int_\Omega (\nabla (u-g)^T A \nabla (u-g) + \nabla g^T A \nabla (u-g) + c(u-g)^2 + cg(u-g))\mathrm{d}x \\
&\quad \ge \int_\Omega (\alpha |\nabla (u-g)|^2 - \frac{\alpha}{2}|\nabla (u-g)|^2 - \frac{|A|^2}{2\alpha}|\nabla g|^2 + c(u-g)^2 - \frac{c}{2}(u-g)^2 - \frac{c}{2}|g|^2)\mathrm{d}x \\
&\quad \ge \frac{\alpha}{2}\|\nabla (u-g)\|^2_{L^2(\Omega)} - \frac{1}{2\alpha}\|A\|_{L^\infty(\Omega)}\|\nabla g\|^2_{L^2(\Omega)} - \frac{1}{2}\|c\|_{L^\infty(\Omega)}\|g\|^2_{L^2(\Omega)}\,.
\end{aligned} \tag{2.5}$$

For bounding the right hand side we use the assumption on f and the Hölder inequality:

$$\begin{aligned}
\int_\Omega f(x,v)(u-g)\mathrm{d}x &\le \int_\Omega h|u-g|\mathrm{d}x \le \|h\|_{L^q(\Omega)}\|u-g\|_{L^p(\Omega)} \\
&\le \frac{\epsilon}{2}\|u-g\|^2_{L^p(\Omega)} + \frac{1}{2\epsilon}\|h\|^2_{L^q(\Omega)}\,,
\end{aligned}$$

with $1/p + 1/q = 1$. Then, it holds $p \in N^*$, and we deduce $H^1(\Omega) \hookrightarrow L^p(\Omega)$. By the embedding and the Poincaré inequality, it follows

$$\|u-g\|_{L^p(\Omega)} \le C_1\|u-g\|_{H^1(\Omega)} \le C_2\|\nabla (u-g)\|_{L^2(\Omega)}$$

with some constants C_1, C_2. For $\epsilon > 0$ sufficiently small (e.g., $\epsilon = \alpha/2C_2^2$), we can bound the term $\frac{\epsilon}{2}\|u-g\|^2_{L^p(\Omega)} \le \frac{\alpha}{4}\|\nabla (u-g)\|^2_{L^2(\Omega)}$ in (2.5). Hence, the estimates of both sides become

$$\frac{\alpha}{4}\|\nabla (u-g)\|^2_{L^2(\Omega)} \le \frac{1}{2\alpha}\|A\|^2_{L^\infty(\Omega)}\|\nabla g\|^2_{L^2(\Omega)} + \frac{1}{2}\|c\|_{L^\infty(\Omega)}\|g\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|h\|^2_{L^q(\Omega)} =: M_0^2\,.$$

Applying the Poincaré inequality once more, we obtain

$$\|u\|_{H^1(\Omega)} \le \|u-g\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} \le (C_3 M_0 + \|g\|_{H^1(\Omega)}) =: M \tag{2.6}$$

and hence $u \in K$. Hence, we proved that S is a self-map and the a-priori bound (2.3) holds provided the existence of a fixed point is shown.

Step 3: S is continuous. Let $(v_k)_k \subset K$ be a sequence with $v_k \to v$ in $L^2(\Omega)$ for $k \to \infty$. We have to show that $S(v_k) \to S(v)$ in $L^2(\Omega)$. To show this, we take a subsequence of (v_k) which for simplicity we denote again $(v_k)_k$. Let $u_k = S(v_k)$. Then u_k satisfies the weak formulation

$$\int_\Omega (\nabla u_k^T A \nabla w + cu_k w)\mathrm{d}x = \int_\Omega f(x,v_k)w\mathrm{d}x \qquad \text{for } w \in H_0^1(\Omega)$$

and the estimate (2.6), i.e., u_k is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence $(u_{k_n})_n$ such that for $n \to \infty$

$$u_{k_n} \rightharpoonup u \qquad \text{in } H^1(\Omega)\,, \qquad u_{k_n} \to u \qquad \text{in } L^2(\Omega)\,.$$

The weak convergence in $H^1(\Omega)$ holds by Theorem 1.11 of Eberlein-Šmuljan. The convergence in $L^2(\Omega)$ holds, since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. We obtain

$$\int_\Omega (\nabla u_{k_n}^T A \nabla w + c u_{k_n} w \mathrm{d}x \to \int_\Omega (\nabla u^T A \nabla w + cuw)\mathrm{d}x \qquad \text{for } n \to \infty\,.$$

Further, by the continuity of the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ and $\gamma(u_{k_n}) = g$, $u_{k_n} \rightharpoonup u$ in $H^1(\Omega)$ implies $\gamma(u_{k_n}) \rightharpoonup \gamma(u)$ in $L^2(\partial\Omega)$, and hence $\gamma(u) = g$. (Recall: The property that for a weakly convergent sequence also weak convergence of the image sequence holds, is called *weak sequential continuity* and it holds for all linear continuous maps between Banach spaces.)

For the limit $k_n \to \infty$ in $\int_\Omega f(x, v_k) w \mathrm{d}x$ we use the following result for Carathéodory functions:

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) and f be a Carathéodory function with*

$$|f(x,u)| \leq C|u|^r + h(x)\,, \qquad 1 \leq q, qr < \infty\,,\ C \geq 0\,,\ h \in L^q(\Omega)\,,\ h \geq 0\,,$$

for all $x \in \Omega$ and $u \in \mathbb{R}$. We define $F(u) = f(\cdot, u(\cdot))$ for $u \in L^{qr}(\Omega)$ Then, F is a continuous map from $L^{qr}(\Omega) \to L^q(\Omega)$.

The proof of the lemma is given after the proof of Theorem 2.5. The map $v \mapsto f(\cdot, v)$ is by Lemma 2.6 continuous from $L^2(\Omega)$ to $L^q(\Omega)$ (choose: $C = 0$ and $r = 2/q$). Since $v_{k_n} \to v$ in $L^2(\Omega)$, it follows $f(\cdot, v_{k_n}) \to f(\cdot, v)$ in $L^q(\Omega)$. Hence,

$$\int_\Omega f(x, v_{k_n}) w \mathrm{d}x \to \int_\Omega f(x, v) w \mathrm{d}x\,.$$

Hence, u satisfies

$$\int_\Omega (\nabla u^T A \nabla w + cuw)\mathrm{d}x = \int_\Omega f(x,v) w \mathrm{d}x, \qquad w \in H_0^1(\Omega)\,,$$

and thus u is a solution of the problem $L(u) = f(x, v)$ in Ω and $u = g$ on $\partial\Omega$. Due to the unique solveability of the linear problem, it holds $u = S(v)$. We have proved that $S(v_{k_n}) = u_{k_n} \to u = S(v)$ in $L^2(\Omega)$ and the limit u is unique. In particular, we proved that the arbitrary choosen subsequences $(v_k)_k$ contains a subsequence $(v_{k_n})_{k_n}$ which converges to a uniquely determined limit u. By Proposition 1.12 (iv) the whole sequence $(u_k)_k$ converges to u, and continuity of S holds.

Step 4: Existence of a fixed point. The assumptions for the fixed point theorem by Schauder are satisfied. Hence, there exists $u \in K$, such that $S(u) = u$, ie., $u \in H^1(\Omega)$ is a solution to (2.1).

□

Proof of Lemma 2.6. Let $(u_k)_k \subset L^{qr}(\Omega)$ be a sequence with $u_k \to u$ in $L^{qr}(\Omega)$. We have to show that $F(u_k) \to F(u)$ in $L^q(\Omega)$. By the reversal of the dominated convergence (Theorem 1.19) there exists a subsequence $(u_{k_n})_n$ such that $u_{k_m} \to u$ almost everywhere in Ω for $m \to \infty$ and it holds the uniform bound $|u_{k_m}| \leq u^* \in L^{qr}(\Omega)$ for all m. In particular, it holds

$F(u_{k_m}) \to F(u)$ almost everywhere in Ω. Since $|F(u_{k_m})|^q \le C(|u^*|^{qr} + h^q) \in L^1(\Omega)$, $F(u_{k_m})$ is dominated by an L^1-function and we can apply the theorem of dominated convergence. We get $F(u_{k_m}) \to F(u)$ in $L^q(\Omega)$ for $m \to \infty$. Similarly to the argument at the end of the previous proof, convergence holds for the whole sequence (if we have chosen previously a subsequence). This concludes the proof of the Lemma. □

Remark 2.7. *The above calculations simplify enormously provided u satisfies homogeneous boundary conditions. In general, we can deduce the proofs for existence results of nonlinear problems to problems with homogeneous boundary conditions by using the transformation $v = u - g$. If u is a solution to $L(u) = f(x,u)$ in Ω, $u = g$ on $\partial\Omega$, then v solves the homogeneous problem*

$$L(v) = L(u) - L(g) = f(x, v + g(x)) - L(g) =: F(x,v) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega\,.$$

However, F does not satisfy the assumption of Theorem 2.5, since $L(g) \in H^{-1}(\Omega)$. The proof changes only slightly if we relax the conditions for the nonlinearity in the following way: Let $F(\cdot, v) = F_1(\cdot, v) + F_0$, where $F_0 \in H^{-1}(\Omega)$ and F_1 is a Carathéodory-function with $|F_1(\cdot, v)| \le h \in L^q(\Omega)$, $q \in N_*$. In Chapter 3, we assume mainly homogeneous boundary conditions to simplify the proofs.

The weak solution of (2.1) is indeed twice weakly differentiable. To show the statement, we use the following regularity result (see Gilbarg-Trudinger [11, Theorem 9.15] or Troianiello [22, Theorem 3.17(ii)]).

Theorem 2.8 ($W^{2,r}$-regularity of linear elliptic equations)**.** *Suppose Assumption 1 holds for L and Ω. Further, let $\partial\Omega \in C^2$, $a_{i,j} \in C^{0,1}(\bar\Omega)$, $g \in W^{2,r}(\Omega)$ and $f \in L^r(\Omega)$, with $1 < r < \infty$. Then, there exists a unique solution $u \in W^{2,r}(\Omega)$ of*

$$L(u) = f\,, \quad \text{in } \Omega\,, \quad u = g \quad \text{on } \partial\Omega\,,$$

and the a-priori estimate

$$\|u\|_{W^{2,r}(\Omega)} \le C(\|g\|_{W^{2,r}(\Omega)} + \|f\|_{L^r(\Omega)})$$

holds where the constant $C > 0$ is independent of u.

Corollary 2.9. *Assume the conditions of Theorem 2.5 and Theorem 2.8. Then there exists a solution $u \in W^{2,q}(\Omega)$ to (2.1).*

Proof. By Theorem 2.5, we know that there exists a weak solution u to $L(u) = f(x,u)$ in Ω with $u - g \in H_0^1(\Omega)$. The right hand side can be interpreted as a function $\tilde f(x) = f(x, u(x))$. Then u solves the linear equation $L(u) = \tilde f$. Since $|\tilde f| \le h \in L^q(\Omega)$, Theorem 2.8 can be applied which concludes the proof. □

Discussion: How can we extend the existence result (Theorem 2.5)? In particular:
Qu1: Is the solution unique?
Qu2: Does a solution exist for more general non-linearities of the form $|f(x,u)| \le C|u|^\beta + h(x)$?
In general, the answer is negative for both questions.

Example 2.10 (Counter example for **Qu1**). *Consider the boundary value problem*

$$-u'' = cu \quad in\ (0,1)\,, \quad u(0) = u(1) = 0\,,$$

with $c > 0$. (The case $c \leq 0$ is contained in Theorem 2.5.) The null function is for all $c > 0$ a solution. By the theory of ordinary differential equations, we know that this problem has only a non-vanishing solution $u \neq 0$, if $c = (2\pi k)^2$ with $k \in \mathbb{N}$. If $c = (2\pi k)^2$, then there exist two solutions $u_1 = 0$ and $u_2 = \sin(2\pi kx)$. Hence, in general the above boundary value problem does not have a unique solution.

Under certain additional assumptions, the questions **Qu1** and **Qu2** have a positive answer. If the non-linearity f is Lipschitz continuous in the sense

$$|f(x,u) - f(x,v)| \leq f_0|u - v| \quad \text{for } x \in \Omega,\ u, v \in \mathbb{R}$$

and if the Lipschitz constant $f_0 > 0$ is 'sufficiently small', then uniqueness of the solution can be shown (see exercises). More general non-linearities can be considered, if they are *sublinear*, i.e., if for $0 < \beta < 1$, $C > 0$, and $h \in L^q(\Omega)$,

$$|f(x,u)| \leq C|u|^\beta + h(x) \quad \text{for } x \in \Omega,\ u \in \mathbb{R}\,. \tag{2.7}$$

The critical step in the proof of existence are the uniform estimates (to show that the fixed point operator is a self-map). To simplify the calculations, we assume that $u = 0$ on $\partial\Omega$ and use the solution $u \in H_0^1(\Omega)$ of $L(u) = f(x,v)$ for given v as test function in the weak formulation. This leads to

$$\int_\Omega (\alpha|\nabla u|^2 + cu^2)\mathrm{d}x \leq \int_\Omega f(x,v)u\mathrm{d}x \leq C\int_\Omega |v|^\beta |u|\mathrm{d}x + \int_\Omega h|u|\mathrm{d}x\,.$$

We apply Hölder inequality and Young's inequality in both integrals on the right side:

$$\begin{aligned}\int_\Omega (\alpha|\nabla u|^2 + cu^2)\mathrm{d}x &\leq C\|v\|^\beta_{L^{\beta q}(\Omega)}\|u\|_{L^p(\Omega)} + \|h\|_{L^q(\Omega)}\|u\|_{L^p(\Omega)} \\ &\leq \frac{\epsilon}{2}\|u\|^2_{L^p(\Omega)} + \frac{C}{\epsilon}\|v\|^{2\beta}_{L^{\beta q}(\Omega)} + \frac{1}{\epsilon}\|h\|^2_{L^q(\Omega)}\,,\end{aligned}$$

with $q/p + 1/q = 1$ and $q \in N_*$. Then, $p \in N^*$ and $H^1(\Omega) \hookrightarrow L^p(\Omega)$. Using the Poincaré inequality we conclude that for sufficiently small $\epsilon > 0$

$$\|u\|_{H^1(\Omega)} \leq C_1(\|v\|^\beta_{L^{\beta q}(\Omega)} + \|h\|_{L^q(\Omega)}) \leq C_2(M^\beta + \|h\|_{L^q(\Omega)})\,,$$

since $\|v\|_{L^{\beta q}(\Omega)} \leq C_3\|v\|_{H^1(\Omega)} \leq C_3 M$ for $v \in K$. To ensure $H^1(\Omega) \hookrightarrow L^{\beta q}(\Omega)$, we choose for example (for $n \geq 3$) $p = 2n/(n-2)$ and $q = 2n/(n+2)$. Then $H^1(\Omega) \hookrightarrow L^{\beta q}(\Omega)$ holds, if $-n/(\beta q) \leq 1 - n/2$ or $\beta \leq 2n/((n-2)q) = (n+2)/(n-2)$ which holds since $\beta < 1$. If one can find a constant $M > 0$ such that

$$C_2(M^\beta + \|h\|_{L^q(\Omega)}) \leq M\,,$$

then $u \in K$. But, this is the case if and only if $\beta < 1$. This proves $S : K \to K$.

Semilinear equations describe for example stationary states of chemical reactions. In this case, it is possible that the nonlinearity grows superlinear, i.e., (2.7) holds with $\beta > 1$. Is it possible to prove existence results for this kind of nonlinearities? In the case f is not growing too fast, existence of solutions can be proven. The result stated in the next theorem can not be proven with the techniques presented here. We refer to Evans [10, Section 8.5.2] for a proof.

Theorem 2.11 (Existence for semilinear equations II)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathcal{C}^1$ and $1 < p < (n+2)/(n-2)$ ($p < \infty$, if $n \leq 2$). Then there exists a weak solution $u \in H_0^1(\Omega)$, $u \neq 0$ of*

$$-\Delta u = |u|^{p-1}u \quad \text{in } \Omega\,, \quad u = 0 \quad \text{on } \partial\Omega\,.$$

Under the assumptions of the above theorem there exists at least two solutions, since $u = 0$ is always a solution. If $p > (n+2)/(n-2)$, then $u = 0$ is the only (classical) solution of the above equation, if Ω is a star-shaped domain with respect to zero (see Evans [10, Section 9.4.2]). We call $u = 0$ the trivial solution and the number $p = (n+2)/(n-2)$ the critical exponent.

We expect in the case of a monotone decreasing (and possibly superlinear) function $f(x, u)$ that the solution u is bounded and existence of solutions can be proven. In the next section, we will show this result.

2.2 Monotone semilinear equations

As in the Section 2.1, we consider the semilinear equation of the form

$$L(u) = f(x, u) \quad \text{in } \Omega\,, \quad u = g \quad \text{on } \partial\Omega\,. \tag{2.8}$$

Here, we assume that the function $u \mapsto f(x, u)$ is monotonically decreasing. In this case the following existence result holds:

Theorem 2.12 (Existence and uniqueness for monotone semilinear equations I)**.** *We suppose Assumption 1. In addition, let f be a Carathéodory function and for almost all $x \in \Omega$, let $u \mapsto f(x, u)$ be monotonically decreasing. Further, it holds*

$$|f(x, u)| \leq C|u|^{p-1} + h(x) \quad \text{for } x \in \Omega\,,\ u \in \mathbb{R}\,,$$

where $C > 0$, $h \in L^q(\Omega)$, $q \in N_*$, $1 < p < 2n/(n-2)$ ($p < \infty$, if $n \leq 2$) and $1/p + 1/q = 1$. Then, there exists a unique weak solution of (2.8) and a constant $C > 0$, which depends only on A, c and Ω, such that

$$\|u\|_{H^1(\Omega)} \leq C(\|g\|_{H^1(\Omega)} + \|h\|_{L^q(\Omega)})\,. \tag{2.9}$$

The constants p and q are chosen such that the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact and $f(\cdot, u) \in L^q(\Omega)$. This holds by

$$|f(\cdot, u)|^q \leq C(|u|^{(p-1)q} + h^q) = C(|u|^p + h^q) \in L^1(\Omega)\,.$$

The idea of the proof relies again on a fixed point argument. But, we can not apply the fixed point Theorem 2.3 by Schauder which can be seen by the following argument. Set for simplicity $g = 0$. Let v be given and u be the solution of the linearized equation $L(u) = f(x, v)$. This defines the fixed point operator $S(v) = u$. Applying the test function u in the linearized function, we obtain similarly as in the proof of Theorem 2.5

$$\alpha \int_\Omega |\nabla u|^2 \mathrm{d}x \leq \int_\Omega |f(x, v)u| \mathrm{d}x\,. \tag{2.10}$$

Hence, it remains to bound the right hand side. We note that by monotonicity of f it follows

$$f(x,u)u - f(x,0)u = (f(x,u) - f(x,0))(u-0) \leq 0 \tag{2.11}$$

and hence

$$f(x,u)u \leq f(x,0)u \leq \frac{\epsilon}{2}u^2 + \frac{1}{2\epsilon}f(x,0)^2\,.$$

The summand $f(x,0)^2$ is independent of u. By applying the Poincaré inequality, the remaining term u^2 can be absorbed by the left hand side of (2.10). Unfortunately, we have only the term $f(x,v)u$ and not $f(x,u)u$. This fixed point argument does not work with the monotone structure. To solve this problem, we apply a different fixed point theorem.

Theorem 2.13 (Fixed point theorem of Leray-Schauder)**.** *Let B be a Banach space, $S : B \times [0,1] \to B$ be a continuous and compact map with $S(v,0) = 0$ for all $v \in B$ and let there exist a constant $C > 0$ such that for all $u \in B$ and $\sigma \in [0,1]$ with $S(u,\sigma) = u$ it holds*

$$\|u\| \leq C\,.$$

Then $v \mapsto S(v,1)$ has a fixed point.

In contrast to the fixed point theorem of Schauder it is no needed to construct a compact and convex set. But, the fixed point operator has to be compact and all fixed points have to satisfy a uniform estimate. The theorem of Leray-Schauder is a consequence of the theorem of Schauder. See Gilbarg-Trudinger [11, Theorem 11.6] for a proof.

The theorem of Leray-Schauder has the advantage that the estimate has to be satisfied only for all fixed points. In particular, this means that we can take the solution u as test function for our problem (2.8) with $g = 0$. By the calculation above we obtain

$$\alpha \int_\Omega |\nabla u|^2 \mathrm{d}x \leq \int_\Omega f(x,u)u\mathrm{d}x \leq \int_\Omega f(x,0)u\mathrm{d}x \leq \frac{\epsilon}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|f(\cdot,0)\|^2_{L^2(\Omega)}\,.$$

By applying the Poincaré inequality and taking $\epsilon > 0$ sufficiently small, the first term of the right hand side can be absorbed by the left hand side. Hence, it follows

$$\frac{\alpha}{2}\|\nabla u\|^2_{L^2(\Omega)} \leq C(\alpha)\|f(\cdot,0)\|^2_{L^2(\Omega)}$$

which provides a uniform estimate for u.

Proof of Theorem 2.12. The structure of the proof is similiar to the one of Theorem 2.5 and is divided in four steps.

Step 1: Definition of the fixed point operator. First, we define the fixed point operator. Let $v \in L^p(\Omega)$ and $\sigma \in [0,1]$. Let $u \in H^1(\Omega) \hookrightarrow L^p(\Omega)$ be the unique solution to

$$L(u) = \sigma f(x,v) \quad \text{in } \Omega\,, \qquad u = \sigma g \quad \text{on } \partial\Omega\,. \tag{2.12}$$

This defines the operator $S : L^p(\Omega) \times [0,1] \to L^p(\Omega), u = S(v,\sigma)$. If $\sigma = 0$, then the solution of (2.12) is given by $u = 0$, i.e., it holds $S(v,0) = 0$ for all $v \in L^2(\Omega)$.

Step 2: S is continuous and compact. The proof of continuity of S works similarly as in the proof of Theorem 2.5. Let $v_k \to v$ in $L^p(\Omega)$ and $\sigma_k \to \sigma$ for $k \to \infty$. Define $u_k = S(v_k, \sigma_k)$ and apply $u_k - \sigma_k g$ als test function in (2.12):

$$\int_\Omega (\nabla u_k^T A \nabla (u_k - \sigma_k g) + c u_k (u_k - \sigma_k g)) \mathrm{d}x = \sigma_k \int_\Omega f(x, v_k)(u_k - \sigma_k g) \mathrm{d}x \,.$$

Similarly as in the proof of Theorem 2.5 the left hand side can be bounded by using the ellipticity of A by

$$\begin{aligned} \frac{\alpha}{2} \int_\Omega |\nabla u_k|^2 \mathrm{d}x &\le \int_\Omega |f(x, v_k)(u_k - \sigma_k g)| \mathrm{d}x + C(A, c, g) \\ &\le \frac{\epsilon}{2} \|u_k - \sigma_k g\|^2_{L^p(\Omega)} + \frac{1}{2\epsilon} \|f(\cdot, v_k)\|^2_{L^q(\Omega)} + C(A, c, g), \end{aligned} \tag{2.13}$$

where $1/p + 1/q = 1$. The constant $C(A, c, g)$ depends on the L^∞-Normen of A and c and on the H^1-Norm of g. For the last inequality we applied Hölder inequality and Young's inequality. Using the Sobolev and the Poincaré inequality, we obtain for the first integal on the left hand side

$$\frac{\epsilon}{2} \|u_k - \sigma_k g\|^2_{L^p(\Omega)} \le \frac{\epsilon}{2} C_1 \|u_k - \sigma_k g\|^2_{H^1(\Omega)} \le \frac{\alpha}{4} \|\nabla u_k\|^2_{L^2(\Omega)} + C(\sigma) \|g\|^2_{H^1(\Omega)} \,.$$

The first term on the right hand side can be absorbed by the left hand side in (2.13). We bound the second integral in (2.13) by

$$\int_\Omega |f(x, v_k)|^q \mathrm{d}x \le C_2 \Big(\int_\Omega |v_k|^{(p-1)q} \mathrm{d}x + \int_\Omega |h|^q \mathrm{d}x \Big) \,.$$

By $(p-1)q = p$ and the continuous embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ the left hand side is bounded. It follows

$$\|u_k\|_{H^1(\Omega)} \le C_3 \,. \tag{2.14}$$

The sequence $u_k = S(v_k, \sigma_k)$ is also bounded in $H^1(\Omega)$ and compact in $L^p(\Omega)$. Therefore, there exists a subsequence $(k_n)_n$ such that

$$u_{k_n} \rightharpoonup u \quad \text{in } H^1(\Omega) \quad \text{and } u_{k_n} \to u \quad \text{in } L^p(\Omega) \,,$$

for $n \to \infty$. Since $v_k \to v$ in $L^p(\Omega)$, $f(x, v_{k_n})$ converges to $f(x, v)$ in $L^q(\Omega)$ (see Lemma 2.6). For $k_n \to \infty$ the limit of the weak formulation

$$\int_\Omega (\nabla u_{k_n}^T A \nabla w + c u_{k_n} w) \mathrm{d}x = \sigma_{k_n} \int_\Omega f(x, v_{k_n}) w \mathrm{d}x \,, \quad w \in H^1_0(\Omega)$$

yields

$$\int_\Omega (\nabla u^T A \nabla w + c u w) \mathrm{d}x = \sigma \int_\Omega f(x, v) w \mathrm{d}x, \quad w \in H^1_0(\Omega) \,.$$

Further the limit function u satisfies the boundary value. Hence u solves (2.12), i.e., $u = S(v, \sigma)$ and $S(v_{k_n}, \sigma_{k_n}) = u_{k_n} \to u_k = S(v, \sigma)$ in $L^p(\Omega)$. The convergence holds because of the uniqueness of the limit for the entire sequence. Hence S is continuous.

The compactness of S is a consequence of the uniform estimate (2.14). If (v_k) is a bounded sequence in $L^p(\Omega)$, then $(u_k) = (S(v_k, \sigma_k))$ is bounded in $H^1(\Omega)$ and has therefore a convergent subsequence in $L^p(\Omega)$.

Step 3: A-priori-estimate It remains to prove a uniform estimate for all fixed points. Let $\sigma \in [0,1]$ and $u \in H^1(\Omega)$ a fixed point of $S(\cdot, \sigma)$. We use $u - \sigma g \in H_0^1(\Omega)$ as test function and obtain a similar expression as above,

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x &\le \sigma \int_\Omega f(x,u)(u-\sigma g)\mathrm{d}x + C(A,c,g) \\ &= \sigma \int_\Omega (f(x,u) - f(x,\sigma g))(u-\sigma g)\mathrm{d}x + \sigma \int_\Omega f(x,\sigma g)(u-\sigma g)\mathrm{d}x + C(A,c,g).\end{aligned} \tag{2.15}$$

By the monotonicity of $f(x,\cdot)$ and $\sigma \le 1$, it follows

$$\frac{\alpha}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x \le \frac{\epsilon}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2\epsilon}\|f(\cdot,\sigma g)\|^2_{L^q(\Omega)} + C(A,c,g).$$

We apply again the triangle, the Sobolev and the Poincaré inequality to the first term on the right hand side and obtain for the sufficiently small $\epsilon > 0$ that

$$\frac{\epsilon}{2}\|u\|^2_{L^p(\Omega)} \le \frac{\alpha}{4}\|\nabla u\|^2_{L^2(\Omega)} + C\|g\|^2_{H^1(\Omega)}.$$

The first term on the right hand side can be absorbed by the left hand side of (2.15). The second term in (2.15) we bound such that it holds

$$\frac{\alpha}{4}\int_\Omega |\nabla u|^2 \mathrm{d}x \le C_4(\|g\|^{2(p-1)}_{L^{(p-1)q}(\Omega)} + \|h\|^2_{L^q(\Omega)}) + C(A,c,g),$$

which is bounded since $(p-1)q = p$. We conclude the uniform estimate

$$\|u\|_{L^p(\Omega)} \le C_5\|u\|_{H^1(\Omega)} \le C_6,$$

and the constant C_6 depends on g and h. We can now apply the fixed point theorem by Leray-Schauder and obtain existence of a fixed point, i.e., a solution to (2.8).

Step 4: Uniqueness of the solution Let u_1 and u_2 be two weak solutions of (2.8). Then, $u_1 - u_2 \in H_0^1(\Omega)$ is a valid test function and we obtain

$$\int_\Omega (\nabla u_i^T A \nabla(u_1 - u_2) + c u_i(u_1 - u_2))\mathrm{d}x = \int_\Omega f(x,u_i)(u_1-u_2)\mathrm{d}x \qquad i = 1,2.$$

If we substract both equations and use monotonicity of f, we obtain

$$\int_\Omega (\nabla(u_1-u_2)^T A \nabla(u_1-u_2) + c(u_1-u_2)^2)\mathrm{d}x = \int_\Omega (f(x,u_1) - f(x,u_2))(u_1-u_2)\mathrm{d}x \le 0,$$

and hence $u_1 - u_2 = 0$ in Ω.

□

We can avoid the growth condition on f if we are looking for L^∞ solutions. Under the condition on the monotonicity this is possible provided the boundary values are in $L^\infty(\Omega)$. To prove the (essential) boundedness of the solutions, we apply a variant of the maximum principle. For this reason we extend the classical maximum principle to a maximum principle for weak solutions. To this end, we first prove an auxiliary result:

Theorem 2.14 (Stampacchia)**.** *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain, $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then, $u^+ = \max(u, 0) \in W^{1,p}(\Omega)$ and*

$$\nabla u^+ = \chi_{u>0} \nabla u,$$

where $\chi_{u>0}$ denotes the characteristic function on $\{u > 0\} = \{u \in \Omega : u(x) > 0\}$. Further, $\nabla u = 0$ almost everywhere in $\{u = 0\} = \{x \in \Omega : u(x) = 0\}$.

The proof of this theorem is based on the following lemma which will be proven in the exercise class.

Lemma 2.15. *Let $F \in \mathcal{C}^1(\mathbb{R})$ with $F' \in L^\infty(\mathbb{R})$ and $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then, $F \circ u \in W^{1,p}(\Omega)$ and*

$$\nabla(F \circ u) = F'(u) \nabla u.$$

Proof of Theorem 2.14. Since $F(z) = z^+$ is not a C^1-function, we need to consider a regularisation (see Troianiello [22, Theorem 1.56]). Let $F_\epsilon(z) = \sqrt{z^2 + \epsilon^2} - \epsilon$ for $z > 0$ and $F_\epsilon(z) = 0$ otherwise. Then $F_\epsilon \in C^1(\mathbb{R})$ and $F_\epsilon(z) \to z^+$ for $\epsilon \to 0$ and for all $z \in \mathbb{R}$. Further, by Lemma 2.15, $F_\epsilon \circ u \in W^{1,p}(\Omega)$. By partial integration for $v \in C_0^\infty(\Omega)$

$$\int_\Omega (F_\epsilon \circ u) \nabla v \mathrm{d}x = -\int_\Omega F_\epsilon'(u) \nabla u v \mathrm{d}x = -\int_{\{u>0\}} \frac{u \nabla u}{\sqrt{u^2 + \epsilon^2}} v \mathrm{d}x$$

In the limit we obtain

$$\int_\Omega u^+ \nabla v \mathrm{d}x = -\int_{\{u>0\}} \nabla u v \mathrm{d}x.$$

This means $u^+ \in W^{1,p}(\Omega)$ and $\nabla u^+ = \chi_{\{u>0\}} \nabla u$. To prove the second statement we observe that $u = u^+ - (-u)^+$ and therefore

$$\nabla u = \chi_{\{u>0\}} \nabla u + \chi_{\{-u>0\}} \nabla u = 0 \quad \text{almost everywhere in } \{u = 0\},$$

which concludes the proof. □

Next we can formulate the weak maximum principle for weak solutions.

Theorem 2.16 (Weak maximum principle)**.** *Suppose Assumption 1 holds. Let $u \in H^1(\Omega)$ with*

$$L(u) \leq 0 \quad \text{in } \Omega, \quad u \leq 0 \quad \text{on } \partial\Omega.$$

Then, $u \leq 0$ almost everywhere in Ω.

Proof. Since $u \leq 0$ on $\partial\Omega$ it holds $u^+ = 0$ on $\partial\Omega$. Further by the theorem of Stampacchia (Theorem 2.14) $u^+ \in H_0^1(\Omega)$. We can use u^+ as test function in the weak formulation and obtain

$$0 \geq \int_\Omega (\nabla u^T A \nabla u^+ + c u u^+) \mathrm{d}x = \int_{\{u>0\}} (\nabla u^T A \nabla u + c u^2) \mathrm{d}x \geq 0.$$

Hence $u^+ = \text{const}$ in Ω. The function u^+ satisfies the elliptic problem $\Delta u^+ = 0$ in Ω and $u^+ = 0$ on $\partial\Omega$. Therefore, by the uniqueness of the solution $u^+ = 0$ and $u \leq 0$ in Ω. □

Next, we can state the existence result for elliptic equations without a condition on the global growth of the nonlinearity.

Theorem 2.17 (Existence and uniqueness for monotone elliptic equations II)**.** *Supppose Assumption 1 holds. Further, let $g \in L^\infty(\Omega)$. The function f satisfies the following conditions*

- *f is a Carathéodory-function,*
- *$u \mapsto f(x,u)$ is monotone decreasing for almost every $x \in \Omega$,*
- *there exists a constant $M_0 > 0$ such that $f(x, M_0) \leq 0$ and $f(x, -M_0) \geq 0$ for almost all $x \in \Omega$,*
- *it holds $|f(x,u)| \leq h(x) \in L^q(\Omega)$ for almost all $x \in \Omega$ and all $|u| \leq M$, where $q \in N_*$ and $M = \max(M_0, \|g\|_{L^\infty(\Omega)})$.*

Then, there exists a unique weak solution $u \in H^1(\Omega) \cap L^\infty(\Omega)$ of (2.8) and it holds

$$\|u\|_{L^\infty(\Omega)} \leq \max\{M_0, \|g\|_{L^\infty(\Omega)}\}, \quad \|u\|_{H^1(\Omega)} \leq C(\|g\|_{H^1(\Omega)} + \|h\|_{L^q(\Omega)}),$$

and the constant C depends on A, c and Ω.

Note that the condition above on f is only a *local* condition. It holds for example for $f(x,u) = -u^3$ and $f(x,u) = 1 - e^u$.

Proof. We define

$$f_M(x,v) = \begin{cases} f(x,v) & \text{if } -M \leq v \leq M, \\ f(x,M) & \text{if } v > M, \\ f(x,-M) & \text{if } v < -M. \end{cases}$$

Then $v \mapsto f_M(x,v)$ is monotone decreasing, $f_M(x,M) \leq f_M(x,M_0) \leq 0$, $f_M(x,-M) \geq f_M(x,-M_0) \geq 0$ for $M \geq M_0$ and $|f_M(x,v)| \leq \max\{|f(x,M)|, f(x,-M)\}$ for all $v \in \mathbb{R}$. By Theorem 2.5 it follows existence of a weak solution $u \in H^1(\Omega)$ of

$$L(u) = f_M(x,u) \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega. \tag{2.16}$$

We claim that u solves the original problem. To this end, we prove that $|u| \leq M$ and therefore $f_M(x,u) = f(x,u)$ holds.

By definition of M, $(u-M)^+ = 0$ on $\partial\Omega$. We apply this function as test function in the weak formulation of (2.16) annd obtain

$$\begin{aligned} \int_\Omega (\nabla u^T A \nabla (u-M)^+ + cu(u-M)^+)\mathrm{d}x &= \int_\Omega f_M(x,u)(u-M)^+\mathrm{d}x \\ &= \int_\Omega (f_M(x,u) - f_M(x,M))(u-M)^+\mathrm{d}x + \int_\Omega f_M(x,M)(u-M)^+\mathrm{d}x \leq 0, \end{aligned}$$

since f_M monotone decreasing and $f(x,M) \leq 0$. By Theorem 2.14 of Stampacchia, it follows $\nabla(u-M)^+ = \chi_{u>M}\nabla(u-M) = \chi_{u>M}\nabla u$. Therefore, we obtain

$$\int_\Omega ((\nabla(u-M)^+)^T A \nabla(u-M)^+ + c((u-M)^+)^2\mathrm{d}x \leq -\int_\Omega cM(u-M)^+\mathrm{d}x \leq 0$$

and hence $(u-M)^+ = 0$, which implies $u \leq M$. The inequality $u \geq -M$ holds analogously, if we consider the test function $(u+M)^- = -(-u-M)^+$. This shows the claim and concludes the proof. □

2.3 Quasilinear equations

In this section, we want to find solutions to elliptic equations of the form

$$-\text{div } a(\nabla u) = f \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega. \tag{2.17}$$

We follow Evans [10, Section 9.1]. Inhomogeneous boundary problems $u = g$ can also be considered by *homogenize* the problem, i.e., define $v = u - g$ and consider the problem $-\text{div}b(x, \nabla v) = f$ with $b(x, \nabla v) = a(\nabla v + \nabla g(x))$ (see Remark 2.7).

In this section we impose the following assumption:

Assumption 2. *The domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with $\partial\Omega \in \mathcal{C}^1$, $a = (a_1, \ldots, a_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field and $f \in L^2(\Omega)$. Further, the nonlinearity a is monotone.*

The last condition is motivated by the previous section and it is defined in the following way.

Definition 2.18. *Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. We call a to be monotone if*

$$(a(p) - a(q)) \cdot (p - q) \geq 0 \quad \text{for all } p, q \in \mathbb{R}^n.$$

The vector field is called stongly monotone, if there exists a constant $\gamma > 0$ such that

$$(a(p) - a(q)) \cdot (p - q) \geq \gamma |p - q|^2 \quad \text{for all } p, q \in \mathbb{R}^n.$$

Example 2.19. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex, twice continuously differentiable function. We claim that $a = \nabla\phi$ is monotone. This holds, since by the theorem of Taylor for all $p, q \in \mathbb{R}^n$ it holds*

$$\begin{aligned}(a(p) - a(q)) \cdot (p - q) &= \sum_{i=1}^{n} (\partial_i \phi(p) - \partial_i \phi(q))(p_i - q_i) \\ &= \sum_{i,j=1}^{n} \int_0^1 \partial^2_{i,j} \phi(p + \theta(q - p)) \mathrm{d}\theta (p_i - q_i)(p_j - q_j) \geq 0\end{aligned}$$

with $\partial_i = \partial/\partial p_i$ and $\partial^2_{i,j}/(\partial p_i \partial p_j)$, since the Hessian $D^2\phi$ of ϕ is positive definite by the the convexity. If in addition we assume that ϕ is uniform convex, i.e. $p^T D^2 \phi p \geq \gamma |p|^2$ for all $p \in \mathbb{R}^N$ and $\gamma > 0$, then a is strongly monotone.

The function $\phi(p) = \sqrt{1 + |p|^2}$ is a convex function and $a(p) = p/\sqrt{1 + |p|^2}$. This is the equation for minimal surfaces.

We show the existence of a solution to (2.17). The weak formulation is given of the form: Find $u \in H^1_0(\Omega)$ with $a(\nabla u) \in L^2(\Omega)$ such that

$$\int_\Omega a(\nabla u) \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \qquad \text{for all } v \in H^1_0(\Omega).$$

Theorem 2.20 (Existence for quasilinear equations)**.** *Suppose Assumption 2 holds. Further, we assume that a grows at most linear and is coercive, i.e., there exist constants $C > 0$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$|a(p)| \leq C(1 + |p|) \quad \text{and} \quad a(p) \cdot p \geq \alpha |p|^2 - \beta \quad \text{for all } p \in \mathbb{R}^n.$$

Then, there exists a solution $u \in H_0^1(\Omega)$ of (2.17).

The growth condition ensures that $a(\nabla u) \in L^2(\Omega)$. The second condition corresponds to a generalized coercivity condition, since for $a(\nabla u) = \nabla u$ (in case of the Laplace operator in (2.17)) it holds $a(p) \cdot p = |p|^2$, and the inequality holds for $\alpha = 1$ and $\beta = 0$.

The idea of the proof relies again on applying a fixed point theorem. Here, we proceed differently compared to the previous sections by defining first approximated solutions via the Galerkin method. The advantage of these solutions is that they are defined only in a finite dimensional space. Hence, we can apply the fixed point Theorem 2.4 by Brouwer. In this case no kinds of compactness is needed. We show that approximated solutions are uniformly bounded. Then, using compactness arguments we can turn over to the limit in infinitely many dimensions. First, we prove a technical result (see Figure 2.1).

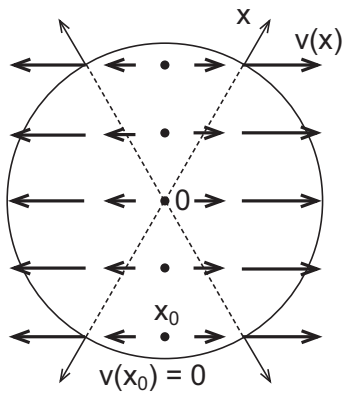

Figure 2.1: Illustration of Lemma 2.21. The vector field $v(x)$ satisfies the condition $v(x) \cdot x \geq 0$ for all x on the circle.

Lemma 2.21. *Let $v : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Further, it holds*

$$v(x) \cdot x \geq 0 \quad \text{for all } |x| = r$$

for some $r > 0$. Then, there exists $x_0 \in \overline{B_r(0)}$, such that $v(x_0) = 0$.

Proof. Assume that for all $x \in \overline{B_r(0)}$ it holds $v(x) \neq 0$. Define the continuous map $w : \overline{B_r(0)} \to \partial B_r(0)$ by

$$w(x) = -\frac{r}{|v(x)|} v(x), \qquad x \in \overline{B_r(0)}.$$

We can interpret w also as a function from $\overline{B_r(0)}$ to $\overline{B_r(0)}$. By the fixed point theorem of Brouwer, there exists $x_0 \in \overline{B_r(0)}$ such that $w(x_0) = x_0$. By construction of w it holds $x_0 \in \partial B_r(0)$ and

$$r^2 = |x_0|^2 = w(x_0) \cdot x_0 = -\frac{r}{|v(x_0)|} v(x_0) \cdot x_0 \leq 0.$$

Since $x_0 \neq 0$, we get a contradiction. □

Proof of Theorem 2.20. We split the proof in three steps.

Step 1: Solution to the finite dimensional problem. Let $(w_k)_k$ be an orthonormal basis of $H_0^1(\Omega)$ with respect to the inner product $(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$. The sequence $(w_k)_k$ can be for example equal to the normalized eigen functions of $-\Delta$ in $H_0^1(\Omega)$. We want to find solutions u_m of

$$\int_\Omega a(\nabla u_m) \cdot \nabla w_k \mathrm{d}x = \int_\Omega f w_k \mathrm{d}x \qquad \text{for all } k = 1, \dots, m, \tag{2.18}$$

which are linear combinations of $w_1, \dots, w_m$. We can present u_m by

$$u_m = \sum_{k=1}^m d_k w_k,$$

i.e., we have to determine the coefficient d_k. For this we use Lemma 2.21.

We define $v = (v_1, \dots, v_m) : \mathbb{R}^m \to \mathbb{R}^m$ by

$$v_k(d) = \int_\Omega \Big(a\Big(\sum_{j=1}^m d_j \nabla w_j \Big) \cdot \nabla w_k - f w_k \Big) \mathrm{d}x, \qquad k = 1, \dots m,\ d = (d_1, \dots, d_m) \in \mathbb{R}^m.$$

If d^* is a zero of v, then $u = \sum_{k=1}^m d_k^* w_k$ is a solution of (2.18). By the coercivity of a it follows

$$\begin{aligned} v(d) \cdot d &= \int_\Omega \Big(a\Big(\sum_{j=1}^m d_j \nabla w_j \Big) \cdot \sum_{k=1}^m d_k \nabla w_k - f \sum_{k=1}^m d_k w_k \Big) \mathrm{d}x \\ &\geq \int_\Omega \Big(\alpha \Big| \sum_{j=1}^m d_j \nabla w_j \Big|^2 - \beta - f \sum_{k=1}^m d_k w_k \Big) \mathrm{d}x \\ &= \alpha |d|^2 - \beta \mathrm{meas}(\Omega) - \sum_{k=1}^m d_k \int_\Omega f w_k \mathrm{d}x, \end{aligned}$$

since $(w_j)_j$ is orthonormal with respect to $(\cdot,\cdot)_{H_0^1(\Omega)}$. By Young's inequality,

$$v(d) \cdot d \geq \frac{\alpha}{2} |d|^2 - \beta \mathrm{meas}(\Omega) - \frac{1}{2\alpha} \sum_{k=1}^m (f, w_k)_{L^2}^2.$$

Hence, it remains to bound the last term. Let $\phi \in H_0^1(\Omega)$ be the weak solution of $-\Delta \phi = f$ in Ω. Then,

$$(\phi, w_k)_{H_0^1(\Omega)} = \int_\Omega \nabla \phi \cdot \nabla w_k \mathrm{d}x = \int_\Omega f w_k \mathrm{d}x$$

and hence,

$$\sum_{k=1}^m (f, w_k)_{L^2}^2 = \sum_{k=1}^m (\phi, w_k)_{H_0^1(\Omega)} \leq \|\phi\|_{H_0^1(\Omega)}^2 \leq C_1 \|f\|_{L^2(\Omega)}^2 \ ,$$

since w_k is normalized with respect to $H_0^1(\Omega)$. In the last step the Sobolev embedding is used. Hence, we obtain

$$v(d) \cdot d \geq \frac{\alpha}{2} |d|^2 - \beta \mathrm{meas}(\Omega) - \frac{C_1}{2\alpha} \|f\|_{L^2(\Omega)}^2.$$

If we choose $r > 0$ sufficiently large, then we can obtain $v(d) \cdot d \geq 0$ for $|d| = r$. By Lemma 2.15, existence of d^* with $v(d^*) = 0$ follows. Then $u_m = \sum_j d_j^* w_j$ solves (2.18).

Step 2: A-priori estimates. The theorem is proven if we can show that the limit $\lim_{m\to\infty} u_m$ exists in an appropriate sense and it solve the boundary value problem. Before taking the limit $m \to \infty$ we show some uniform estimates in m. Multiplying the Galerkin equation (2.18) with d_k^* and summing over $k = 1, \ldots, m$ we obtain

$$\int_\Omega a(\nabla u_m) \cdot \nabla u_m \mathrm{d}x = \int_\Omega f u_m \mathrm{d}x. \tag{2.19}$$

The coercivity of a implies

$$\int_\Omega (\alpha|\nabla u_m|^2 - \beta)\mathrm{d}x \leq \int_\Omega a(\nabla u_m) \cdot \nabla u_m \mathrm{d}x = \int_\Omega f u_m \mathrm{d}x \leq \frac{\epsilon}{2} \int_\Omega |u_m|^2 \mathrm{d}x + \frac{1}{2\epsilon} \int_\Omega |f|^2 \mathrm{d}x.$$

We choose $\epsilon > 0$ sufficiently small and use the Poincaré inequality to bound the first term on the right hand side by the integral on the left hand side. We obtain

$$\|u_m\|_{H^1(\Omega)} \leq C(1 + \|f\|_{L^2(\Omega)}). \tag{2.20}$$

Step 3: Limit $m \to \infty$. By Theorem 1.11 of Eberlein-Šmuljan there exists a subsequence $(u_{m_k})_k$ such that $u_{m_k} \rightharpoonup u$ in $H^1(\Omega)$ for $k \to \infty$. In particular, it holds

$$\nabla u_{m_k} \rightharpoonup \nabla u \qquad \text{in } L^2(\Omega)^n.$$

Unfortunately weak convergence of ∇u is not sufficient to deduce convergence $a(\nabla u_{m_k}) \to a(\nabla u)$ (in any sense). We have actually shown in Remark 1.14 that in general for weakly convergent sequences (u_m) and continuous functions f the sequence $f(u_m)$ is *not* (weakly) convergent. We need an additional information about the nonlinearity to carry out convergence. We claim that monotonicity will be sufficient. The following procedure is called *method of Browder and Minty*.

Since the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, by (2.20) it follows the existence of a subsequence of (u_m) with

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(\Omega).$$

If we choose a common subsequence, then it holds further

$$\nabla u_{m_k} \rightharpoonup \nabla u \qquad \text{in } L^2(\Omega)^n. \tag{2.21}$$

We claim that

$$a(\nabla u_{m_k}) \rightharpoonup a(\nabla u) \qquad \text{in } L^2(\Omega)^n.$$

By the growth condition for a and (2.21) it follows that $(a(\nabla u_m))$ is bounded in $L^2(\Omega)$:

$$\|a(\nabla u_m)\|_{L^2(\Omega)} \leq C(1 + \|\nabla u_m\|_{L^2(\Omega)}) \leq C.$$

Therefore there exists a subsequence of (u_{m_k}) which we denote again by (u_{m_k}) such that

$$a(\nabla u_{m_k}) \rightharpoonup b \qquad \text{in } L^2(\Omega)^n.$$

We remains to show that $b = a(\nabla u)$.

Taking the limit $m_k \to \infty$ in the Galerkin equations (2.18), we obtain

$$\int_\Omega b \cdot \nabla w_k \mathrm{d}x = \int_\Omega f w_k \mathrm{d}x, \quad k \in \mathbb{N}.$$

Since (w_k) is a basis in $H_0^1(\Omega)$, this relation holds for all $v \in H_0^1(\Omega)$:

$$\int_\Omega b \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x. \tag{2.22}$$

We apply the monotonnicity of a and (2.19):

$$\begin{aligned} 0 &\le \int_\Omega (a(\nabla u_m) - a(\nabla v)) \cdot (\nabla u_m - \nabla v) \mathrm{d}x \\ &= \int_\Omega (f u_m - a(\nabla u_m) \cdot \nabla v - a(\nabla v) \cdot (\nabla u_m - \nabla v)) \mathrm{d}x, \end{aligned}$$

with $v \in H_0^1(\Omega)$. The above convergence results allow to take the limit $m_k \to \infty$:

$$0 \le \int_\Omega (f u - b \cdot \nabla v - a(\nabla v) \cdot (\nabla u - \nabla v)) \mathrm{d}x.$$

Note that the trick is given by the fact that we replaced the expression $a(\nabla u_m) \cdot \nabla u_m$ by $f u_m$ using (2.19) since we do not know directly the limit, since $a(\nabla u_m)$ and ∇u_m converge only weakly. If we choose $v = u$ in (2.22), we can replace fu by $b \cdot \nabla u$ and obtain

$$0 \le \int_\Omega (b - a(\nabla v)) \cdot (\nabla u - \nabla v) \mathrm{d}x.$$

Let $w \in H_0^1(\Omega)$ and $v = u \pm \lambda w$ for $\lambda > 0$. Then,

$$0 \le \mp \int_\Omega (b - a(\nabla u \pm \lambda \nabla w)) \cdot \nabla w \mathrm{d}x.$$

The limit $\lambda \to 0$ yields

$$0 \le \mp \int_\Omega (b - a(\nabla u)) \cdot \nabla w \mathrm{d}x,$$

and hence equality

$$0 = \int_\Omega (b - a(\nabla u)) \cdot \nabla w \mathrm{d}x \qquad \text{for all } w \in H_0^1(\Omega).$$

We conclude $b = a(\nabla u)$ which ends the proof.

□

We can prove uniqueness of the solution to (2.17), if a is strongly monotone.

Theorem 2.22 (Uniqueness for quasilinear equations)**.** *Suppose the assumption of Theorem 2.20. Further let a be strongly monotone. Then, there exists a unique solution $u \in H_0^1(\Omega)$ of (2.17).*

Proof. Let u_1 and u_2 be weak solutions of (2.17). Then it holds for all $v \in H_0^1(\Omega)$ that

$$\int_\Omega a(\nabla u_i) \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \qquad i = 1, 2.$$

Wir substract the two equations for $i = 1$ and $i = 2$ and consider $v = u_1 - u_2$ as test function. By strong monotonicity of a, we obtain

$$0 = \int_\Omega (a(\nabla u_1) - a(\nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) \mathrm{d}x \geq \gamma \int_\Omega |\nabla u_1 - \nabla u_2|^2 \mathrm{d}x.$$

This implies that $\nabla(u_1 - u_2) = 0$ in Ω and since $u_1 - u_2 = 0$ on $\partial\Omega$, it holds $u_1 - u_2 = 0$ in Ω. □

By the proof on uniqueness, one can guess that it is possible to also proof a comparison principle. This is indeed the case.

Proposition 2.23 (Comparison principle for quasilinear equations)**.** *Suppose the assumption of Theorem 2.20. Further let a be strongly monotone. We set $L(u) = -\mathrm{div} a(\nabla u)$. If $u, v \in H^1(\Omega)$ are two functions satisfying*

$$L(u) \leq L(v) \quad \text{in } \Omega, \quad u \leq v \quad \text{on } \partial\Omega,$$

then $u \leq v$ in Ω.

Proof of Proposition 2.23. The inequality $L(u) \leq L(v)$ translates in the weak formulation to

$$\int_\Omega a(\nabla u) \cdot \nabla w \mathrm{d}x \leq \int_\Omega a(\nabla v) \cdot \nabla w \mathrm{d}x \qquad \text{for all } w \in H_0^1(\Omega) \text{ with } w \geq 0.$$

This can be rewritten to

$$\int_\Omega (a(\nabla u) - a(\nabla v)) \cdot \nabla w \mathrm{d}x \leq 0 \qquad \text{for all } w \in H_0^1(\Omega) \text{ with } w \geq 0.$$

The function $w = (u - v)^+ = \max(0, u - v)$ satisfies $w = 0$ on $\partial\Omega$, since we imposed $u - v \leq 0$ on $\partial\Omega$. By Theorem 2.14 of Stampacchia, $w \in H_0^1(\Omega)$ is a valid test function. By the strong monotonicity it follows,

$$\begin{aligned} 0 &\geq \int_{\{u>v\}} (a(\nabla u) - a(\nabla v)) \cdot (\nabla u - \nabla v) \mathrm{d}x \\ &\geq \gamma \int_{\{u>v\}} |\nabla u - \nabla v|^2 \mathrm{d}x = \gamma \int_\Omega |\nabla (u - v)^+|^2 \mathrm{d}x. \end{aligned}$$

We conclude that $\nabla(u - v)^+ = 0$. Then, $(u - v)^+ = \text{const}$ in Ω. Since $(u - v)^+ = 0$ on $\partial\Omega$, $(u - v)^+ = 0$. Hence, $u \leq v$ in Ω. □

Remark 2.24. *The previous theorem on existence can not be applied to the minimal surface equation*

$$\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

since $a(p) = p/\sqrt{1 + |p|^2}$ is not coercive. However, uniqueness of the solution can be proven (see exercises).

2.4 Drift-diffusion equations

In this section, we consider the stationary drift-diffusion equation for semiconductors:

$$\operatorname{div}(\nabla u - u\nabla\phi) = 0, \quad \Delta\phi = u - f(x) \quad \text{in } \Omega \tag{2.23}$$

with the boundary conditions

$$u = g, \quad \phi = \psi \quad \text{on } \partial\Omega. \tag{2.24}$$

In physics, u corresponds to the particle density (therefore, we expect $u \geq 0$) and ϕ corresponds to the electrical potential. The equations form a system of nonlinear elliptic equations, since the expression $u\nabla\phi$ is nonlinear. We can show the existence of solutions to this system by applying a fixed point theorem. The key result is given next:

Theorem 2.25 (Existence for the drift-diffusion equations)**.** *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded domain with $\partial\Omega \in C^1$, $f \in L^\infty(\Omega)$ with $0 < f_* \leq f(x) \leq f^*$ for $x \in \Omega$, $g, \psi \in H^1(\Omega) \cap L^\infty(\Omega)$ with $0 < g_* \leq g(x) \leq g^*$ for $x \in \partial\Omega$. Then there exists a solution $(u, \phi) \in (H^1(\Omega) \cap L^\infty(\Omega))^2$ of (2.23)-(2.24) with*

$$\frac{1}{Me^K} \leq u \leq Me^K, \quad -K \leq \phi \leq K \quad \text{in } \Omega,$$

where

$$\begin{aligned} M &= e^{\|\psi\|_{L^\infty(\Omega)}} \max(g^*, 1/g_*), \\ M_0 &= \max(\ln(Mf^*), \ln(M/f_*)) \\ K &= \max(M_0, \|\psi\|_{L^\infty(\Omega)}). \end{aligned}$$

The weak formulation of (2.23)-(2.24) is given of the following form: Search for $(u, \phi) \in H^1(\Omega)^2$ with $u = g$ and $\phi = \psi$ on $\partial\Omega$ such that for all $w \in H_0^1(\Omega)$ it holds:

$$\int_\Omega \nabla u \cdot \nabla w \mathrm{d}x = \int_\Omega u\nabla\phi \cdot \nabla w \mathrm{d}x, \quad \int_\Omega \nabla\phi \cdot \nabla w \mathrm{d}x = -\int_\Omega (u - f(x)) w \mathrm{d}x. \tag{2.25}$$

All integrals are well-defined except one. To ensure that the drift term $\int_\Omega u\nabla\phi \cdot \nabla w \mathrm{d}x$ is well-defined for test functions $w \in H_0^1(\Omega)$, we require $u \in L^\infty(\Omega)$.

The drift term leads to some mathematical difficulties. Therefore, we introduce a variable transformation which transforms the equation (2.23) in a more systematic form. More precisely, we define the so called *Slotboom-variable* $v = e^{-\phi}u$. Then,

$$\nabla v = e^{-\phi}(\nabla u - u\nabla\phi) \quad \text{and} \quad 0 = \operatorname{div}(\nabla u - u\nabla\phi) = \operatorname{div}(e^\phi \nabla v).$$

Then, we solve the system

$$\operatorname{div}(e^\phi \nabla v) = 0, \quad \Delta\phi = e^\phi v - f(x) \quad \text{in } \Omega \tag{2.26}$$

with the boundary conditions

$$v = v_D := e^{-\psi} g, \quad \phi = \psi \quad \text{on } \partial\Omega. \tag{2.27}$$

Every solution of this system $(v, \phi) \in (H^1(\Omega) \cap L^\infty(\Omega))^2$ defines via $u = e^\phi v$ a weak solution of the original problem (2.23)-(2.24). The advantage of (2.26) is that the first equation defines a symmetric differential operator. This fact allows to deduce estimates in $H^1(\Omega)$ and $L^\infty(\Omega)$. Typically, these estimates depend on ϕ (because of the diffusion coefficient e^ϕ). Further, ϕ depends on v. Via a cut-off technique we obtain estimates which do not depend on v and ϕ.

Proof of Theorem 2.25. To obtain appropriate L^∞-estimates, we apply the *cut-off method of Stampacchia*, i.e., we first solve the system

$$\operatorname{div}(e^\phi \nabla v) = 0, \quad \Delta\phi = e^\phi v_M - f(x) \quad \text{in } \Omega, \tag{2.28}$$

with

$$v_M = \begin{cases} 1/M & v \le 1/M \\ v & 1/M < v < M \\ M & v \ge M. \end{cases}$$

This definition ensures that $v_M \in L^\infty(\Omega)$. We divide the proof in four steps:

Step 1: Construction of the fixed point operator. Let $\tilde{v} \in L^2(\Omega)$. Solve

$$-\Delta\phi = -e^\phi \tilde{v}_M(x) + f(x) \quad \text{in } \Omega, \quad \phi = \psi \quad \text{on } \partial\Omega. \tag{2.29}$$

We write $F(x, \phi) = -e^\phi \tilde{v}_M(x) + f(x)$. Then, we observe that the differential equation is semilinear. We claim that the existence theorem 2.17 is applicable. Hence, we have to show:

- F is a Carathéodory function, since $\phi \mapsto e^\phi$ is continuous and both $\tilde{v}_M$ and f are integrable.
- The function $\phi \mapsto -e^\phi \tilde{v}_M + f(x)$ are monotone decreasing
- By construction of v_M and the definition of M_0 it follows

$$F(x, M_0) \le -e^{M_0}\frac{1}{M} + f^* \le 0 \quad \text{and} \quad F(x, -M_0) \ge -e - M_0 M + f_* \ge 0.$$

- For all $|\phi| \le M_0$ it holds: $|F(x, \phi)| \le e^{M_0} M + f^* \in L^\infty(\Omega)$.

The assumptions of Theorem 2.17 are satisfied and we deduce the existence and uniqueness of a solution $\phi \in H^1(\Omega)$ of (2.29) with the a-priori estimates

$$\|\phi\|_{L^\infty(\Omega)} \le K = \max(M_0, \|\phi\|_{L^\infty(\Omega)}), \quad \|\phi\|_{H^1(\Omega)} \le C_1(f, \psi, M). \tag{2.30}$$

Next, we solve for given $\sigma \in [0, 1]$

$$\operatorname{div}(e^\phi \nabla v) = 0 \quad \text{in } \Omega, \quad v = \sigma v_D \quad \text{on } \partial\Omega.$$

This corresponds to an elliptic problem with diffusion coefficient $A(x) = e^{\phi(x)}$. Since $A(x) \ge e^{-K} > 0$, the problem is uniformly elliptic and we obtain a unique solution $v \in H^1(\Omega)$ with

$$\|v\|_{H^1(\Omega)} \le C_2(K)\|v_D\|_{H^1(\Omega)}. \tag{2.31}$$

This defines a fixed point operator $S : L^2(\Omega)\times[0,1] \to L^2(\Omega), S(\tilde{v},\sigma) = v$. It follows directly, $S(\tilde{v},0) = 0$.

Step 2: Continuity of S. Let $\tilde{v}_k \to \tilde{v}$ in $L^2(\Omega)$ and $\sigma_k \to \sigma$ for $k \to \infty$ be two convergent sequences. Define $v_k = S(\tilde{v}_k, \sigma_k)$ Further, let ϕ_k be the solution to the Poisson problem (2.29) with $(\tilde{v}_k)_M$. By the previous estimates, (v_k) and (ϕ_k) are bounded in $H^1(\Omega)$. Further, (ϕ_k) is bounded in $L^\infty(\Omega)$. Therefore, there are subsequences $(k_n)_n$ with

$$\begin{aligned} v_{k_n} &\rightharpoonup v \quad \text{in } H^1(\Omega), \quad v_{k_n} \to v \quad \text{in } L^2(\Omega) \\ \phi_{k_n} &\rightharpoonup \phi \quad \text{in } H^1(\Omega), \quad \phi_{k_n} \to \phi \quad \text{in } L^2(\Omega). \end{aligned}$$

Hence, we have the following lemma, whose proof will be done in the exercises.

Lemma 2.26. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 \le p < \infty$, $(u_k)_k$ be a sequence with $u_k \to u$ in $L^p(\Omega)$ for $k \to \infty$ and $f \in C^0(\mathbb{R})$. Let $(u_k)_k$ be bounded in $L^\infty(\Omega)$ or let f be bounded in $\mathbb{R}$. Then, it holds $f(u_k) \to f(u)$ in $L^p(\Omega)$.*

We deduce, $e^{\phi_{k_n}} \to e^\phi$ in $L^2(\Omega)$. Further, $(\tilde{v}_{k_n})_M \to \tilde{v}_M$ in $L^2(\Omega)$. Then, the product converges in $L^1(\Omega)$, i.e., $e^{\phi_{k_n}}(\tilde{v}_{k_n})_M \to e^\phi \tilde{v}_M$ in $L^1(\Omega)$. Thus, we can take the limit $k_n \to \infty$ in the weak formulation

$$\int_\Omega \nabla\phi_{k_n}\cdot\nabla w \mathrm{d}x = -\int_\Omega (e^{\phi_{k_n}}(\tilde{v}_{k_n})_M - f(x))w\mathrm{d}x, \quad w \in C_0^\infty(\Omega),$$

and we obtain

$$\int_\Omega \nabla\phi\cdot\nabla w \mathrm{d}x = -\int_\Omega (e^{\phi}\tilde{v}_M - f(x))w\mathrm{d}x, \quad w \in C_0^\infty(\Omega).$$

The same limit in

$$\int_\Omega e^{\phi_{k_n}}\nabla v_{k_n}\cdot\nabla w \mathrm{d}x = 0, \quad w \in C_0^\infty(\Omega),$$

yields

$$\int_\Omega e^{\phi}\nabla v\cdot\nabla w \mathrm{d}x = 0, \quad w \in C_0^\infty(\Omega),$$

since $e^{\phi_{k_n}}$ converges strongly in $L^2(\Omega)$ and ∇v_{k_n} converges weakly in $L^2(\Omega)$, hence the product converges weakly in $L^1(\Omega)$. The weak formulation holds only for $w \in C_0^\infty(\Omega)$. By a density argument we can show that the weak formulation holds indeed for all $w \in H_0^1(\Omega)$. Hence, we have shown that $v = S(\tilde{v},\sigma)$ and $S(\tilde{v}_{k_n},\sigma_{k_n}) \to v_{k_n} \to v = S(\tilde{v},\sigma)$ in $L^2(\Omega)$. By uniqueness of the limit, the whole seqence converges. Hence S is continuous. The compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and (2.31) imply compactness of S.

Step 3: A-priori-estimates. Let $v \in H^1(\Omega)$ be a fixed point of $S(\cdot,\sigma)$. By the definition of M, we obtain $M \ge e^{-\psi}g = v_D$ and therefore $(v-M)^+ = 0$ on $\partial\Omega$. Hence, $(v-M)^+ \in H_0^1(\Omega)$ is a valid test function for the first equation in (2.28):

$$0 = \int_\Omega e^\phi \nabla v\cdot\nabla(v-M)^+\mathrm{d}x = \int_\Omega e^\phi |\nabla(v-M)^+|^2\mathrm{d}x.$$

Hence, $(v-M)^+$ is constant. Since $(v-M)^+ = 0$ on $\partial\Omega$, $(v-M)^+ = 0$ and $v \le M$ in Ω. Similarly, by $1/M \le e^{-\psi}g$ and by using the test function $(v-1/M)^-$ we can deduce

$(v - 1/M)^- = 0$ and $v \geq 1/M$ in Ω. Hence, $v_M = v$. Further, we obtain a uniform L^∞ estimate and therefore a L^2 estimate for all fixed points of $S(\cdot, \sigma)$.

Step 4: End of the proof. By the fixed point theorem of Leray-Schauder we obtain existence of a fixed point v of $S(\cdot, 1)$. Then, (v, ϕ) solves the boundary value problem (2.27) and (2.28), where ϕ defines the Poisson equation. Since $1/M \leq v \leq M$, (v, ϕ) solves already (2.26)-(2.27).

□

If the boundary conditions and the boundary of the domain are more regular, we can show the following regularity result:

Proposition 2.27. *Suppose the conditions of Theorem 2.25 holds. Let $\partial\Omega \in C^\infty$, $f, g, \psi \in C^\infty(\bar{\Omega})$. Then, every bounded weak solution to* (2.23)-(2.24) *is a classical solution.*

Proof. The proof illustrates the so-called *boot strapping technique*. A weak solution of (2.23)-(2.24) satisfies

$$\Delta\phi = u - f(x) \in L^\infty(\Omega), \quad \Delta u = \operatorname{div}(u\nabla\phi) = \nabla u \cdot \nabla\phi + u\Delta\phi.$$

By Theorem 2.8, we obtain $\phi \in W^{2,p}(\Omega)$ for every $p < \infty$. This implies $|\nabla\phi| \in L^\infty(\Omega)$ and hence $\Delta u \in L^2(\Omega)$. Further, Theorem 2.8 yields $u \in H^2(\Omega)$. Further, for all $i = 1, \ldots, n$

$$\Delta\partial_i\phi = \partial_i u - \partial_i f \in L^2(\Omega),$$

and Theorem 2.8 implies $\partial_i\phi \in H^2$. Hence $\phi \in H^3(\Omega)$. Further, for sufficiently large $p < \infty$

$$\Delta\partial_i u = \underbrace{\nabla\partial_i u}_{\in L^2(\Omega)} \cdot \underbrace{\nabla\phi}_{\in L^\infty(\Omega)} + \underbrace{\nabla u}_{\in H^1(\Omega)} \cdot \underbrace{\nabla\partial_i\phi}_{\in L^p(\Omega)} + \underbrace{\partial_i u}_{\in H^1(\Omega)} \cdot \underbrace{\Delta\phi}_{\in L^p(\Omega)} + \underbrace{u}_{\in L^\infty(\Omega)} \cdot \underbrace{\Delta\partial_i\phi}_{\in L^2(\Omega)} \in L^2(\Omega)$$

with the boundary conditions $\partial_i u = \partial_i g$ on $\partial\Omega$. Hence, $\partial_i u \in H^2(\Omega)$ and $u \in H^3(\Omega)$. We can iterate this argument and obtain $u, \phi \in H^m(\Omega)$, for m is sufficiently large such that it holds $u, \phi \in C^2(\bar{\Omega})$ by the embedding of Sobolev.

□

Finally, we can ask whether the system (2.23)-(2.24) is uniquely solveable. In general, this is not the case. This is not surprisingly since there exist semiconductor devices (thyristors) whose function is based on the fact that there are several states. For sufficiently small applied voltage (i.e., $|\nabla\psi|$ is 'small') uniqueness of a solution can be proven. In particular, the case $\psi = 0$ is easy and will be considered in the exercises.

Chapter 3

Nonlinear parabolic equations

In this chapter, we present techniques to analyse semilinear equations of the form

$$u_t - \Delta u = f(x, u)$$

and quasilinear equations of the form

$$u_t - \operatorname{div}(a(u)\nabla u) = f(x, t).$$

Due to the nonlinearities, we will use fixed point theorems of Schauder and Leray-Schauder as in the previous chapter. By a-priori estimates we deduce compactness which is required in these theorems. For certain frameworks of these time-dependent equations we can also use fixed point theorems by Banach which do not require compactness. Further, Banach fixed point theorems provide uniqueness of the solutions.

To define weak solutions of parabolic equations, we need to introduce Sobolev spaces in time and space. For this reason, we start with reviewing some statements for these spaces.

3.1 Sobolev spaces in space and time

Solutions of parabolic equations are functions in space and time. Since the regularity of the solution in space can differ from the one in time (and in general it will differ), we need Sobolev spaces, which are different in the space variable and time variable. We interpret a solution $u(x,t)$ for almost all $t > 0$ as a function $u(t) : \Omega \to \mathbb{R}$ which lies in a Banach space B. This leads for example to the definition of spaces of the form $C^k([0,T];B)$, $L^p(0,T;B)$ and $W^{m,p}(0,T;B)$, respectively. These notations mean that $u(t) \in B$ and $t \mapsto u(t)$ are continuously differentiable, integrable and weakly differentiable, respectively. More precisely, we define the following *function spaces with values in a Banach space*.

Definition 3.1. *Let B be a Banach space and $T > 0$. We define:*

(i) The space $C^k([0,T];B)$ is the set of all functions $u : [0,T] \to B$ which are k-times continuously differentiable. The norm is given by

$$\|u\|_{C^k([0,T];B)} = \sum_{i=0}^{k} \max_{0 \le t \le T} \|u^{(i)}(t)\|_B.$$

(ii) The space $L^p(0,T;B)$ is the set of all (equivalence classes of) measurable functions $u : (0,T) \to B$ which satisfy

$$\|u\|_{L^p(0,T;B)} = \Big(\int_0^T \|u(t)\|_B^p \mathrm{d}t \Big)^{1/p} < \infty \quad \text{for } 1 \le p < \infty \text{ and}$$
$$\|u\|_{L^\infty(0,T;B)} = \operatorname{ess\,sup}_{0<t<T} \|u(t)\|_B < \infty.$$

The above defined spaces are all Banach spaces. If H is a Hilbert space, then $L^2(0,T;H)$ is a Hilbert space with the scalar product

$$(u,v)_{L^2(0,T;H)} = \int_0^T (u(t),v(t))_H \mathrm{d}t, \quad u,v \in L^2(0,T;H).$$

For the dual space of $L^p(0,T;B)$ the following result holds (for the proof see [24, Prop 23.7 and exercise 23.12d]):

Proposition 3.2. *Let B be a reflexive and separable Banach space, $1 \le p < \infty$ and $1/p + 1/q = 1$. Then, the dual space of $L^p(0,T;B)$ can be identified with $L^q(0,T;B')$:*

$$(L^p(0,T;B))' = L^q(0,T;B').$$

If $B = L^p(\Omega)$ with $1 \le p < \infty$, then $L^p(0,T;L^p(\Omega))$ can be identified with $L^p(\Omega \times (0,T))$. This does not hold for $p = \infty$ (see Roubíček [17, Example 1.42]). The space $L^\infty(0,T;L^\infty(\Omega))$ is smaller than $L^\infty(\Omega \times (0,T))$. Namely, a function $u \in L^\infty(0,T;L^\infty(\Omega))$ has the property that $u : (0,T) \to L^\infty(\Omega)$ is measurable with respect to $L^\infty(\Omega)$, whereas the function $u \in L^\infty(\Omega \times (0,T))$ is only Lebesque-measurable on $\Omega \times (0,T)$. If $u \in L^\infty(\Omega \times (0,T))$ is $L^\infty(\Omega)$-measurable, then $u \in L^\infty(0,T;L^\infty(\Omega))$.

Next, we recall that we can define weak solutions for parabolic equations. Let u be a classical solution to

$$u_t - \Delta u = f(x,t) \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(\cdot,0) = u_0 \quad \text{in } \Omega,$$

where f is a regular function. Multiply the differential equation with a test function $w \in C_0^\infty(\Omega \times (0,T))$, integrate over $\Omega \times (0,T)$ and integrate by parts:

$$\int_0^T (u_t, w)_{L^2(\Omega)} \mathrm{d}t + \int_0^T \int_\Omega \nabla u \cdot \nabla w \mathrm{d}x \mathrm{d}t = \int_0^T \int_\Omega f w \mathrm{d}x \mathrm{d}t.$$

A weak solution should be weakly differentiable with respect to x, i.e., $u(t) \in H_0^1(\Omega)$ for almost all t. The function $t \mapsto \nabla u(t)$ should be square integrable, i.e., $u \in L^2(0,T;H_0^1(\Omega))$. The time derivative satisfies the equation $u_t(t) = \Delta u(t) + f(t) \in H^{-1}(\Omega)$. On the other side, $t \mapsto u_t(t)$ should be square integrable for almost all t to ensure that the integral over t makes sense. Therefore we impose $u_t \in L^2(0,T;H^{-1}(\Omega))$. Hence, $u(t)$ and $u_t(t)$ are in different Banach spaces. The generalization of this concepts leads to the so called evolution triple.

Let H be a separable Hilbert space and V be a reflexive separable Banach space with a continuous and dense embedding $V \hookrightarrow H$. (A typical example would be $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$.) By Proposition 1.8, the embedding $H' \hookrightarrow V'$ is continuous and dense. By the Riesz mapping we can identify the space H and its dual space H'. In particular it holds,

$$\langle u, v \rangle_{V'} = (u,v)_H \quad \text{for } u \in H,\ v \in V.$$

Then we obtain the following inclusion:

$$V \hookrightarrow H \simeq H' \hookrightarrow V'.$$

We call such a triple (V,H,V') an evolution triple. We sum up:

Definition 3.3. *Let H be a separable Hilbert space and V be a reflexive separable Banach space with continuous and dense embedding $V \hookrightarrow H$. Then, the triple $V \hookrightarrow H \hookrightarrow V'$ is called a Gelfand triple or an evolution triple.*

An example of an evolution triple is given by

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega).$$

Hence in general we look for weak solutions of parabolic equations with $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V')$. Therefore, we define for $1 < p < \infty$:

$$W^{1,p}(0,T;V,H) = \{u \in L^p(0,T;V) : u_t \in L^q(0,T;V')\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

This space has the following properties:

Proposition 3.4. *Let $V \hookrightarrow H \hookrightarrow V'$ be an evolution triple and $1 < p < \infty$.*

1. *The space $W = W^{1,p}(0,T;V,H)$ endowed with the norm*

$$\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^q(0,T;V')}, \quad u \in W,$$

is a Banach space.

2. *The space $C^1([0,T];V)$ is dense in $W^{1,p}(0,T;V,H)$.*

3. *The embedding $W^{1,p}(0,T;V,H) \hookrightarrow C^0([0,T];H)$ is continuous (if required after choosing appropriate representatives).*

4. *Let $u \in W^{1,p}(0,T;V,H)$. Then, the map $t \mapsto \|u(t)\|_H$ is absolutely continuous (in particular also almost everywhere differentiable), und it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 = 2\langle u_t(t), u(t)\rangle_{V'} \quad \text{for almost all } 0 < t < T.$$

Proof. 1. See Evans [10, page 287], and Zeidler [24, Proposition 23.23(iv)].

2. See Emmrich [9, Theorem 8.1.9].

3. We show that there exists a constant $C > 0$ such that for all $u \in W^{1,p}(0,T;V,H)$ it holds

$$\|u\|_{L^\infty(0,T;H)} \le C(\|u\|_{L^p(0,T;V)} + \|u_t\|_{L^q(0,T;V')}).$$

First, let $u \in C^1([0,T];V)$. Then it holds for all $t, t^* \in [0,T]$:

$$\|u(t)\|_H^2 = \|u(t^*)\|_H^2 + 2\int_{t^*}^t (u_t(s), u(s))_H \mathrm{d}s. \tag{3.1}$$

There exists $t^* \in [0,T]$ such that

$$\|u(t^*)\|_H^2 = \frac{1}{T}\int_0^T \|u(s)\|_H^2 \mathrm{d}s. \tag{3.2}$$

Since $u_t(s) \in V \subset H \subset V'$, we can write

$$(u_t(s), u(s))_H = \langle u_t(s), u(s)\rangle_{V'} \leq \|u_t(s)\|_{V'}\|u(s)\|_V.$$

Inserting this inequality and (3.2) in (3.1), we obtain using the Hölder inequality and the Young inequality

$$\begin{aligned}\|u(t)\|_H^2 &\leq \frac{1}{T}\int_0^T \|u(s)\|_H^2 \mathrm{d}s + 2\int_0^T \|u_t(s)\|_{V'}\|u(s)\|_V \mathrm{d}s \\ &\leq \frac{1}{T}\|u\|^2_{L^2(0,T;H)} + \|u\|^2_{L^p(0,T;V)} + \|u_t\|^2_{L^q(0,T;V')}.\end{aligned}$$

If $p \geq 2$, then the embedding $L^p(0,T;V) \hookrightarrow L^2(0,T;V) \hookrightarrow L^2(0,T;H)$ implies the conjecture. Otherwise, we apply the interpolation inequality (which follows by the Hölder inequality):

$$\int_0^T \|u\|_H^2 \mathrm{d}t \leq \sup_{t\in(0,T)} \|u(t)\|_H^{2-p} \int_0^T \|u\|_H^p \mathrm{d}t \leq \epsilon\|u\|^2_{L^\infty(0,T;H)} + C_1(\epsilon)\Big(\int_0^T \|u\|_H^p \mathrm{d}t\Big)^{2/p},$$

which implies

$$\|u(t)\|_H^2 \leq \epsilon\|u\|^2_{L^\infty(0,T;H)} + C_2(\epsilon)\|u\|^2_{L^p(0,T;V)} + \|u_t\|^2_{L^q(0,T;V')}.$$

Taking the supremum over $t \in (0,T)$ and choosing $\epsilon < 1$, the left hand side absorbes the first term on the right hand side. The statement follows, since $C^1([0,T];V)$ is dense in $W^{1,p}(0,T;V,H)$ by 2.

4. Follows by (3.1) after taking the limit $t^* \to t$ and by a density argument.

□

The property 3. of the previous theorem means that a weak solution of a parabolic equation, which is an element of $W^{1,p}(0,T;V,H)$, satisfies the initial value $u(0) = u_0$ in the sense of H.

If (u_k) is a sequence such that (u_k) is bounded in $L^p(0,T;V)$ and $(\partial_t u_k)$ is bounded in $L^q(0,T;V')$ with $1/p+1/q = 1$, then there exists a subsequence (k_n) satisfying the properties

$$u_{k_n} \rightharpoonup u \quad \text{in } L^p(0,T;V), \quad \partial_t u_{k_n} \rightharpoonup \partial_t u \quad \text{in } L^q(0,T;V'), \quad \text{for } k_n \to \infty.$$

In the previous chapter we observed that for nonlinear equations weak convergence is in general not sufficient to take the limit in the nonlinearities. But, under the above conditions the sequence (u_k) is compact in $L^p(0,T;H)$ if $V \hookrightarrow H$ is compact.

Lemma 3.5 (Lemma of Aubin)**.** *Let $V \hookrightarrow H \hookrightarrow V'$ be an evolution triple and $1 < p < \infty$. Let the embedding $V \hookrightarrow H$ be compact. Then the embedding $W^{1,p}(0,T;V,H) \hookrightarrow L^p(0,T;H)$ is compact.*

The proof is given e.g. in Showalter [20, page 106 f.]. Note that the embedding $L^p(0,T;V) \hookrightarrow L^p(0,T;H)$ is in general not compact, even if $V \hookrightarrow H$ is compact. An additional information on the time derivative is necessary. The Lemma of Aubin states that if (u_k) is bounded in $L^p(0,T;V)$ and $(\partial_t u_k)$ is bounded in $L^q(0,T;V')$ with $1 < p < \infty$, then there exists a subsequence with

$$u_{k_n} \to u \quad \text{in } L^p(0,T;H) \quad \text{for } k_n \to \infty.$$

In many cases, strong convergence will be sufficient to take the limit in the nonlinearities.

Remark 3.6. *The lemma of Aubin holds also in the following more general version: Let X, B, Y be Banach spaces with compact embedding $X \hookrightarrow B$ and continuous embedding $B \hookrightarrow Y$. Further, let $U \subset L^p(0,T;X)$ and $\{\partial u/\partial t : u \in U\} \subset L^r(0,T;Y)$ be bounded with either $1 \le p < \infty$, $r = 1$ or $p = \infty$, $r > 1$. Then, U is relatively compact in $L^p(0,T;B)$.*

3.2 Semilinear equations

In this section we are interested in solutions of the semilinear equation of the form

$$u_t + L(u) = f(x,t,u) \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega. \tag{3.3}$$

We impose the following assumptions:

Assumption 3. *The set $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$. The differential operator $L(u)$ is defined by*

$$L(u) = -\mathrm{div}(A\nabla u) + cu,$$

where $A(x) = (a_{ij}(x))$ is a $(n \times n)$-matrix and $c(x)$ is some function. Further, let A be symmetric and elliptic with $a_{ij} \in L^\infty(\Omega)$. Assume $c \in L^\infty(\Omega)$ with $c \ge 0$ and $u_0 \in L^2(\Omega)$. Finally, let f be a Carathéodory function (i.e., measurable in (x,t) and continuous in u) and $(x,t) \mapsto f(x,t,u)$ be integrable for all $u \in \mathbb{R}$.

The following results hold also for time-continuous functions A and c and inhomogeneous Dirichlet boundary conditions under suitable assumptions. We will not state these more general results here to not overload the presentation with the variety of assumptions. First, we give the definition of a weak solution for (3.3):

Definition 3.7. *Let $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V \hookrightarrow H \hookrightarrow V'$ be an evolution triple and $T > 0$. We call u a weak solution to (3.3) if*

(i) $u \in W^{1,2}(0,T;V,H)$ and $f(\cdot,\cdot,u) \in L^2(\Omega \times (0,T))$;

(ii) for all $v \in L^2(0,T;V)$ it holds

$$\int_0^T \langle u_t, v\rangle_{H^{-1}} \mathrm{d}t + \int_0^T \int_\Omega (\nabla u^T A \nabla v + cuv)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega f(x,t,u)v\mathrm{d}x\mathrm{d}t;$$

(iii) $u(\cdot,0) = u_0$ almost everywhere in Ω.

For linear parabolic equations we recall the following existence result:

Theorem 3.8 (Existence for linear equations)**.** *Suppose Assumption 3 holds. Further, let $T > 0$ and $f \in L^2(\Omega \times (0,T))$. Then, there exists a uniquely determined weak solution $u \in W^{1,2}(0,T;V,H)$ of*

$$u_t + L(u) = f(x,t) \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega$$

(in the sense of Definition 3.7).

If the nonlinearity $f(x,t,u)$ is Lipschitz continuous in u, we can show the existence of solutions to (3.3).

Theorem 3.9 (Global existence and uniqueness for semilinear equations)**.** *Suppose Assumption 3 holds. Further, let $T > 0$, $f(\cdot,\cdot,0) \in L^2(0,T;L^2(\Omega))$ and f be Lipschitz continuous in u uniform in (x,t), i.e., there exists $L > 0$ such that for all $x \in \Omega, t \in (0,T)$ and $u, v \in \mathbb{R}$ it holds*

$$|f(x,t,u) - f(x,t,v)| \leq L|x - v|.$$

Then, there exists a uniquely determined weak solution of (3.3) *(in the sense of Definition 3.7).*

Proof. The proof uses the fixed point theorem of Banach (see Section 2.1) with the space $X = C^0([0,T^*];L^2(\Omega))$. It turns out that the fixed point operator is only a contraction for sufficiently small $T^* > 0$. Then, existence and uniqueness of a solution holds in the time interval $[0,T^*]$. Since T^* is chosen independently of u, we can extend the solution on $[0,T]$.

Step 1: Definition of the fixed point operator. Let $v \in X$. Lipschitz continuity of f implies $|f(x,t,v)| \leq C(1+|v|) \in L^2(0,T;L^2(\Omega))$. By Theorem 3.8, there exists a unique weak solution $u \in W^{1,2}(0,T;V,H)$ of

$$u_t + L(u) = f(x,t,v(x,t)) \quad \text{in } \Omega,\ t > 0, \quad u(0) = u_0 \quad \text{in } \Omega.$$

By Proposition 3.4, $u \in X$. This defines the fixed point operator $S : X \to X, S(v) = u$. We need to show that S is a contraction.

Step 2: S is a contraction. Let $v_1, v_2 \in X$ and $u_1 = S(v_1)$, $u_2 = S(v_2)$. The function $u_1 - u_2$ solves the equation

$$(u_1-u_2)_t + L(u_1-u_2) = f(x,t,v_1) - f(x,t,v_2) \quad \text{in } \Omega,\ t > 0, \quad (u_1-u_2)(0) = 0 \quad \text{in } \Omega.$$

If we apply the test function $u_1 - u_2 \in L^2(0,T;H_0^1(\Omega))$ in the weak formulation of this equation and the ellipticity of A, it follows for $t \in (0,T^*)$

$$\begin{aligned}\int_0^t \langle (u_1-u_2)_t, u_1-u_2\rangle_{V'}\mathrm{d}s + \int_0^t\int_\Omega (\alpha|\nabla(u_1-u_2)|^2 + c(u_1-u_2)^2)\mathrm{d}x\mathrm{d}s \\ \leq \int_0^t\int_\Omega (f(x,s,v_1) - f(x,s,v_2))(u_1-u_2)\mathrm{d}x\mathrm{d}s.\end{aligned}$$

By Proposition 3.4 (4.) and since $(u_1-u_2)(0) = 0$ we can write the first integral on the left hand side as $\frac{1}{2}\|(u_1-u_2)\|^2_{L^2(\Omega)}$. The Lipschitz continuity of f and the Poincaré inequality yield

$$\begin{aligned}\frac{1}{2}\|(u_1-u_2)(t)\|^2_{L^2(\Omega)} + \alpha\int_0^t\int_\Omega |\nabla(u_1-u_2)|^2\mathrm{d}x\mathrm{d}s \\ &\leq \int_0^t \|f(\cdot,s,v_1) - f(\cdot,s,v_2)\|_{L^2(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}\mathrm{d}s \\ &\leq L\int_0^t \|v_1-v_2\|_{L^2(\Omega)}\|u_1-u_2\|_{L^2(\Omega)}\mathrm{d}s \\ &\leq C_1\int_0^t \|v_1-v_2\|_{L^2(\Omega)}\|\nabla(u_1-u_2)\|_{L^2(\Omega)}\mathrm{d}s \\ &\leq \frac{\alpha}{2}\int_0^t \|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}\mathrm{d}s + \frac{C_1^2}{2\alpha}\int_0^t \|v_1-v_2\|^2_{L^2(\Omega)}\mathrm{d}s.\end{aligned}$$

We take the maximum over $t \in [0, T^*]$ and deduce

$$\|S(v_1) - S(v_2)\|^2_{L^\infty(0,T^*;H)} = \|u_1 - u_2\|^2_{L^\infty(0,T^*;H)} \le \frac{C_1^2 T^*}{\alpha} \|v_1 - v_2\|^2_{L^\infty(0,T^*;H)}.$$

The constant C_1 depends on the Lipschitz constant L and the Poincaré constant. We choose $T^* > 0$ such that $C_1^2 T^*/\alpha < 1$ holds. Then, the above inequality shows that S on X is a contraction (since the norm in $L^\infty(0, T^*; H)$ is the same as the norm in $C^0([0, T^*]; H)$), and the fixed point theorem deduces the existence of a uniquely determined solution u on $[0, T^*]$.

Step 3: Extension of the local solution. By Proposition 3.4(3), it holds by $u(t) \in L^2(\Omega)$ for all $t \in [0, T^*]$. In particular, it holds $u(T^*) \in L^2(\Omega)$. We can take this function as initial value and repeat the above argument. This gives a solution which is defined on $[T^*, 2T^*]$, since T^* is independent of u. After finitely many steps we construct a solution on $[0, T]$. This global solution is unique, since if there are two solutions u_1 and u_2, then by the estimates of the second step we can deduce directly $u_1 = u_2$.

□

Semilinear parabolic equations play an important role in chemical reactions. The Lipschitz continuity of f is a strong condition. In Section 1.1 we considered for example $f(u) = R_0 - u^2$; this function is not Lipschitz continuous in $\mathbb{R}$. We claim that the problem (3.3) is also solveable for this kind of functions, at least it is solveable local in time.

Theorem 3.10 (Local existence and uniqueness for semilinear equations)**.** *Suppose that Assumption 3 holds. Let $u_0 \in L^\infty(\Omega)$. Further, let f be locally Lipschitz continuous in u uniformly in (x, t), i.e., for all $R > 0$, there exists $L(R) > 0$ such that*

$$|f(x, t, u) - f(x, t, v)| \le L(R)|u - v| \quad \text{for all } |u| \le R,\ |v| \le R,\ x \in \Omega,\ t > 0,$$

and let f grow at most polynomially, i.e. there exist constants $K > 0$ and $r > 0$ such that

$$|f(x, t, u)| \le K(1 + |u|^r) \quad \text{for all } u \in \mathbb{R}.$$

Then, there exists $T^* > 0$ such that on the interval $[0, T^*]$ (3.3) has a uniquely determined solution $u \in W^{1,2}(0, T^*; V, H)$ in the sense of Definition 3.7. Further, u satisfies the estimate

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} e^{\lambda t}, \quad 0 \le t \le T^*,$$

where $\lambda > 0$ is independent of u.

The proof of Theorem 3.10 is based on the fixed point theorem of Banach and the weak maximum principle. We will apply a test function of type $(u - Me^{\lambda t})^+$. Then, to calculate the integral of the form

$$\int_0^T \langle u_t, (u - Me^{\lambda t})^+ \rangle_{H^{-1}} \mathrm{d}t$$

we formulate the following lemma. The proof is part of the exercises.

Lemma 3.11. *Let the function $z : [0,T] \to [0,\infty)$ be continuously differentiable and $u \in W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$. Then, it holds for $0 \le \tau \le T$:*

$$\int_0^\tau \langle u_t, (u-z)^+\rangle_{H^{-1}} \mathrm{d}t = \frac{1}{2}\int_\Omega ((u-z)^+(\tau)^2 - (u-z)^+(0)^2)\mathrm{d}x + \int_0^\tau \int_\Omega z_t(u-z)^+ \mathrm{d}x\mathrm{d}t.$$

Proof of Theorem 3.10. Let $M = \|u_0\|_{L^\infty(\Omega)}$, $T^* > 0$ and $\lambda > 0$. We define the set

$$B = \{v \in C^0([0,T^*];L^2(\Omega)) : \|v(t)\|_{L^\infty(\Omega)} \le Me^{\lambda t}, 0 \le t \le T^*\}.$$

Further, let $v \in B$ and $u \in W^{1,2}(0,T^*;V,H)$ be the unique solution of the linear problem

$$u_t + L(u) = f(x,t,v(x,t)) \quad \text{in } \Omega,\ t>0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega.$$

The solution exists by Theorem 3.8, since

$$|f(x,t,v)| \le K(1+|v|^r) \le K(1+M^r e^{r\lambda t}) \in L^2(\Omega \times (0,T^*)). \tag{3.4}$$

We claim that the solution u is an element of the set B. To show this claim, we observe that $(u - Me^{\lambda t})^+ \in L^2(0,T^*;V)$ is a test function and $(u-Me^{\lambda t})^+(0) = (u_0 - M)^+ = 0$ holds. We obtain

$$\begin{aligned}\int_0^\tau \langle u_t, (u-Me^{\lambda t})^+\rangle_{H^{-1}}\mathrm{d}t + \int_0^\tau\int_\Omega (\underbrace{(\nabla(u-Me^{\lambda t})^+)^T A\nabla u}_{=\chi_{u>Me^{\lambda t}}\nabla u^T A\nabla u \ge 0} + \underbrace{cu(u-Me^{\lambda t})^+}_{\ge 0})\mathrm{d}x\mathrm{d}t \\ = \int_0^\tau\int_\Omega f(x,t,v)(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We apply Lemma 3.11, the ellipticity of A and the estimate (3.4):

$$\begin{aligned}\frac{1}{2}\int_\Omega (u-Me^{\lambda t})^+(\tau)^2\mathrm{d}x + \int_0^\tau\int_\Omega (Me^{\lambda t})_t(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t \\ \le \int_0^\tau\int_\Omega f(x,t,v)(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t\end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{2}\int_\Omega (u-Me^{\lambda t})^+(\tau)^2\mathrm{d}x &\le \int_0^\tau\int_\Omega (f(x,t,v)-\lambda Me^{\lambda t})(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t \\ &\le \int_0^\tau\int_\Omega (K(1+M^re^{r\lambda t})-\lambda Me^{\lambda t})(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t.\end{aligned}$$

We choose $\lambda = K(2M^r+1)/M$ and if $r > 1$, $T^* \le (\ln 2)/(\lambda(r-1))$. Then, it follows $e^{(r-1)\lambda t} \le e^{(r-1)\lambda T^*} \le 2$ for $0 \le t \le T^*$. If $r \le 1$, we obtain $e^{(r-1)\lambda t} \le 1$ independent of the choice of T^*. Then,

$$\begin{aligned}\frac{1}{2}\int_\Omega (u-Me^{\lambda t})^+(\tau)^2\mathrm{d}x &\le \int_0^\tau\int_\Omega (K(1+M^re^{r\lambda t})-K(2M^r+1)e^{\lambda t})(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t \\ &\le \int_0^\tau\int_\Omega KM^re^{\lambda t}\underbrace{(e^{(r-1)\lambda t}-2)}_{\le 0}(u-Me^{\lambda t})^+\mathrm{d}x\mathrm{d}t \le 0.\end{aligned}$$

This implies directly $u(\cdot,t) \le Me^{\lambda t}$ in Ω, $t \in [0,T^*]$. Applying the test function $(-u - Me^{\lambda t})^+$, we obtain by a similar calculation $-u(\cdot,t) - Me^{\lambda t} \le 0$. Thus, we obtain $|u(\cdot,t)| \le$

$Me^{\lambda t}$ and hence $u \in B$. This defines the fixed point operator $S : B \to B, S(v) = u$. It remains to show that S is a contraction.

Let $v_1, v_2 \in B$ and $u_1 = S(v_1)$, $u_2 = S(v_2)$. Using the test function $u_1 - u_2$ in the difference of the weak formulation for u_1 and u_2, it follows

$$\int_0^\tau \langle (u_1-u_2)_t, u_1-u_2\rangle_{H^{-1}} \mathrm{d}t + \int_0^\tau \int_\Omega (\nabla(u_1-u_2)^T A \nabla(u_1-u_2) + c(u_1-u_2)^2)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^\tau \int_\Omega (f(x,t,v_1) - f(x,t,v_2))(u_1-u_2)\mathrm{d}x\mathrm{d}t.$$

We apply the ellipticity of A and the Poincaré inequality and the Young inequality:

$$\begin{aligned}\frac{1}{2}\int_\Omega (u_1-u_2)(\tau)^2\mathrm{d}x + \alpha \int_0^\tau \int_\Omega |\nabla(u_1-u_2)|^2 \mathrm{d}x\mathrm{d}t \\ &\le \int_0^\tau \|f(x,t,v_1)-f(x,t,v_2)\|_{L^2(\Omega)} \|u_1-u_2\|_{L^2(\Omega)}\mathrm{d}t \\ &\le C_1 \int_0^\tau \|f(x,t,v_1)-f(x,t,v_2)\|_{L^2(\Omega)} \|\nabla(u_1-u_2)\|_{L^2(\Omega)}\mathrm{d}t \\ &\le \frac{C_1^2}{2\alpha}\int_0^\tau \|f(x,t,v_1)-f(x,t,v_2)\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{\alpha}{2}\int_0^\tau \|\nabla(u_1-u_2)\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

By the local Lipschitz continuity of f it follows

$$\begin{aligned}\frac{1}{2}\|(u_1-u_2)(\tau)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\int_0^\tau \int_\Omega |\nabla(u_1-u_2)|^2\mathrm{d}x\mathrm{d}t \\ &\le \frac{C_1^2 L(Me^{\lambda t})\tau}{2\alpha}\|v_1-v_2\|^2_{L^\infty(0,T^*;L^2(\Omega)}.\end{aligned}$$

We take the supremum over $\tau \in [0, T^*]$:

$$\|u_1-u_2\|_{L^\infty(0,T^*;L^2(\Omega))} \le \frac{C_1^2 L^* T^*}{\alpha}\|v_1-v_2\|_{L^\infty(0,T^*;L^2(\Omega))},$$

where $L^* := \sup_{t\in(0,T^*)} L(Me^{\lambda t})$. If $C_1^2 L^* T^*/\alpha < 1$, then we obtain contraction. Hence, we have to choose $T^* > 0$ such that $T^* < T$ and $C_1^2 L^* T^* < \alpha$. □

Theorem 3.10 does not exclude the possible global existence of a solution. The following result shows that in general this is not the case. To this end, we consider as in Evans [10, Section 9.4.1] the specific equation

$$u_t - \Delta u = u^2 \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega. \tag{3.5}$$

Since the mapping $u \to u^2$ is locally Lipschitz, there exists a solution local in time. We claim that for sufficiently large $T > 0$ and $u_0 > 0$ there can not exist a solution of (3.5). This result can be interpreted in the following way. If diffusive forces are negligible, we obtain the differential equation

$$u_t = u^2, \quad t > 0, \quad , u(0) = u_0 > 0.$$

The solution is given by

$$u(t) = \frac{1}{1/u_0 - t}, \quad t \ge 0,$$

and exists only for $t < 1/u_0$. The diffusion term $-\Delta u$ has a smoothing effect such that the reaction term u^2 and the diffusion term Δu are acting in contrast to each other. Since $u \to u^2$ grows super-linear, we expect that the reactions appear more rapidly than the diffusions smooth out. This means that the solution 'explodes' after finite time.

Proposition 3.12 (Non-existence of global solutions)**.** *Let u be a weak solution of* (3.5) *with $u_0 > 0$ and let $w_1 > 0$ be the eigen function of $-\Delta$ on $H_0^1(\Omega)$ corresponding to the smallest eigen value $\lambda_1 > 0$ with $\int_\Omega w_1 \mathrm{d}x = 1$. If the initial values are chosen such that $\int_\Omega u_0 w_1 \mathrm{d}x > \lambda_1$, then there exists a $t^* > 0$ such that*

$$\lim_{t \to t^*} \int_\Omega u(x,t) w_1(x) \mathrm{d}x = \infty.$$

Hence, we say that the equation (3.5) *has not a weak solution which exists globally in time.*

Proof of Proposition 3.12. The eigenvalue equation

$$-\Delta w = \lambda w \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

has the eigen value $\lambda_1 > 0$ with eigen function $w_1 \in H^2(\Omega)$. By the maximum principle it is possible to choose w_1 such that $w_1 > 0$ in Ω and $\int_\Omega w_1 \mathrm{d}x = 1$. Let u be a weak solution of (3.5). Since $u_t - \Delta u \geq 0$, $u(0) \geq 0$ and $u = 0$ on $\partial\Omega$, it follows by the weak maximal principle that $u \geq 0$ in Ω, $t > 0$. (Choose $u^- = \min(0, u)$ as test function and calculate as in the proof of Theorem 3.10). We define

$$z(t) = \int_\Omega u(t) w_1 \mathrm{d}x, \quad t \geq 0.$$

Since $u \in C^0([0,T]; L^2(\Omega))$, z is defined for all $t \geq 0$. It follows (similarly as in Proposition 3.4)

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle u, w_1\rangle_{H^{-1}} = \langle u_t, w_1\rangle_{H^{-1}} = \langle \Delta u + u^2, w_1\rangle_{H^{-1}} = \int_\Omega (u\Delta w_1 + u^2 w_1)\mathrm{d}x.$$

In the last step we applied twice integration by parts. Since w_1 is a eigen function, we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}t}(t) = -\lambda_1 z + \int_\Omega u^2 w_1 \mathrm{d}x.$$

In the proof we show next that the second integral can be bounded from below by z^2. By the Hölder inequality,

$$z = \int_\Omega u\sqrt{w_1}\sqrt{w_1}\mathrm{d}x \leq \Big(\int_\Omega u^2 w_1 \mathrm{d}x\Big)^{1/2}\Big(\int_\Omega w_1 \mathrm{d}x\Big)^{1/2} = \Big(\int_\Omega u^2 w_1 \mathrm{d}x\Big)^{1/2},$$

since $\int_\Omega w_1 \mathrm{d}x = 1$. Therefore, $z^2 \leq \int_\Omega u^2 w_1 \mathrm{d}x$ and

$$\frac{\mathrm{d}z}{\mathrm{d}t} \geq -\lambda_1 z + z^2, \quad t \geq 0.$$

This differential inequality is solveable. The function $y(t) = e^{\lambda_1 t} z(t)$ solves

$$\frac{\mathrm{d}y}{\mathrm{d}t} = e^{\lambda_1 t}\frac{\mathrm{d}z}{\mathrm{d}t} + \lambda_1 e^{\lambda_1 t} z \geq e^{\lambda_1 t} z^2 = e^{-\lambda_1 t} y^2.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{1}{y}\Big) = \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}t} \geq e^{-\lambda_1 t}.$$

Integrating from 0 to t yields

$$-\frac{1}{y(t)} + \frac{1}{y(0)} \geq -\frac{1}{\lambda_1}(e^{-\lambda_1 t} - 1).$$

Solving the inequality for $y(t)$, we obtain

$$y(t) \geq \frac{y(0)\lambda_1}{\lambda_1 - y(0)(1 - e^{-\lambda_1 t})}.$$

provided the denominator is not zero. Next, we choose u_0 such that $z(0) = y(0) = \int_\Omega u_0 w_1 \mathrm{d}x > \lambda_1$. Then, there exists $t^* > 0$ such that the above denominator is zero, i.e., $y(t) \to \infty$ and $z(t) = e^{-\lambda_1 t} y(t) \to \infty$ for $t \to t^*$. This shows the claim.

□

The result of Proposition 3.12 makes sense for chemical reactions. The right hand side of the equation

$$u_t - \Delta u = u^2$$

models a binary reaction in which molecules are generated. Indeed a chain reaction is described and the number of molecules is growing across the board. A solution can not exists for all times. By the reaction equation

$$u_t - \Delta u = R_0 - u^2$$

a binary reaction with a constant source term is described in which molecules are deleted. If molecules can not flow through the boundary of the domain, it is plausible that the particle density converges to the value $u = \sqrt{R_0}$ (since $R_0 - u^2 = 0$) and a global solution exists. However, this case is not covered in Theorem 3.10. Similarly as for the elliptic equations the monotonicity of $u \mapsto R_0 - u^2$ is important for the global existence. The following result holds for monotone semilinear equations.

Theorem 3.13 (Global existence and uniqueness for monotone semilinear equations)**.** *Suppose Assumption 3 holds. Let $u_0 \in L^\infty(\Omega)$ and $T > 0$. Further, let $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ be a function with the following properties*

- *f is a Carathéodory function*
- *$u \mapsto f(x,t,u)$ is monotone decreasing for almost all $x \in \Omega$, $t > 0$,*
- *there exists a constant $M_0 > 0$ such that $f(x,t,M_0) \leq 0$ and $f(x,t,-M_0) \geq 0$ for almost all $x \in \Omega$, $t > 0$,*
- *for all $M > 0$ it holds $|f(x,t,u)| \leq h_M(x,t) \in L^2(\Omega \times (0,T))$ for almost all $x \in \Omega$, $t > 0$ for all $|u| \leq M$.*

Then, there exists a unique weak solution $u \in W^{1,2}(0,T;V,H)$ of (3.3) (in sense of

Definition 3.7). Further, u fulfils the estimate

$$|u| \le \max(M_0, \|u_0\|_{L^\infty(\Omega)}) \quad \text{in } \Omega \times (0,T).$$

The condition $h_M \in L^2(\Omega)$ can be weakened (see Theorem 2.17). This condition simplifies the proof of the theorem. We use the following auxiliary result which can be proven similarly as Lemma 2.26.

Lemma 3.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 \le p < \infty$, $(u_k)_k$ a sequence with $u_k \to u$ in $L^p(\Omega)$ for $k \to \infty$ and f a Carathéodory function with $|f(\cdot,u)| \le h \in L^p(\Omega)$ for all $u \in \mathbb{R}$. Then, it follows $f(\cdot,u_k) \to f(\cdot,u)$ in $L^p(\Omega)$.*

Proof. By the reverse of the theorem of the dominated convergence there exists a subsequence $(u_{k_n})_n$ with $u_{k_n} \to u$ almost everywhere in Ω for $k_n \to \infty$. Since f is continuous in u, we obtain $f(x,u_{k_n}) \to f(x,u)$ almost everywhere. Further, $|f(\cdot,u_{k_n})|^p \le h^p \in L^1(\Omega)$. By the theorem of the dominated convergence it follows that $f(\cdot,u_{k_n}) \to f(\cdot,u)$ in $L^p(\Omega)$. Since u is uniquely determined, the whole sequence converges. □

Proof of Theorem 3.13. To use the monotonicity of f, we apply the fixed point theorem of Leray-Schauder. Since f is only locally bounded with respect to u, we use additionally the cut-off method of Stampacchia. Let $v \in L^2(\Omega \times (0,T))$, $\sigma \in [0,1]$ and $u \in W^{1,2}(0,T;V,H)$ be the unique solution of the linear problem

$$u_t + L(u) = \sigma f(x,t,v_M(x,t)) \quad \text{in } \Omega,\ t>0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = \sigma u_0 \quad \text{in } \Omega, \tag{3.6}$$

with $v_M = \max(-M, \min(M,v))$ and $M = \max(M_0, \|u_0\|_{L^\infty(\Omega)})$. The solution exists, since $f(x,t,v_M(x,t))$ is an L^2-function in (x,t). This defines the fixed point operator $S : L^2(\Omega \times (0,T)) \times [0,1] \to L^2(\Omega \times (0,T))$, $S(v,\sigma) = u$. It holds $S(v,0) = 0$ for all v. Further, S is continuous, since if $v_k \to v$ in $L^2(\Omega \times (0,T))$, $\sigma_k \to \sigma$ and $u_k = S(v_k,\sigma_k)$, it follows by the conditions on f and Lemma 3.14 that

$$f(x,t,(v_k)_M) \to f(x,t,v_M) \in L^2(\Omega \times (0,T)).$$

Since the right hand side of the linear equation is bounded in L^2 for all k, it follows by a-priori estimates that $(u_k)_k$ is bounded in $W^{1,2}(0,T;V,H)$ and - by the Lemma of Aubin (Lemma 3.5)- $u_{k_n} \to u$ in $L^2(\Omega \times (0,T))$ for a subsequence $(k_n)_n$. Note that this lemma is applicable since V is compactly embedded in H. We can take the limit $k_n \to \infty$ in the weak formulation of u_{k_n} and conclude that u is a weak solution of (3.6). Thus, we have shown $u = S(v,\sigma)$ and continuity of S.

To show compactness of S , we take u as a test function in (3.6):

$$\begin{aligned}\frac{1}{2}\int_\Omega u(\tau)^2 \mathrm{d}x - \frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x + \alpha \int_0^\tau \int_\Omega |\nabla u|^2 \mathrm{d}x\mathrm{d}t &\le \sigma \int_0^\tau \int_\Omega f(x,t,v_M) u \mathrm{d}x\mathrm{d}t \\ &\le \int_0^\tau \int_\Omega h_M |u| \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\|h_M\|^2_{L^2(\Omega\times(0,T))} + \frac{1}{2}\|u\|^2_{L^2(\Omega\times(0,T))}.\end{aligned}$$

By the Grönwall's lemma we conclude that $\|u\|_{L^2(0,T;V)} \le C_1$. Then, also u_t is bounded, since

$$\|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \le \|A\nabla u\|_{L^2(0,T;L^2(\Omega))} + \| - cu + \sigma f(x,t,v_M)\|_{L^2(0,T;L^2(\Omega))} \le C_2.$$

Then by the lemma of Aubin it follows that $u = S(v, \sigma)$ lies in a compact subset of $L^2(0,T;L^2(\Omega))$, i.e., S is compact.

The important steps of the proof are the uniform estimates for all fixed points of $S(\cdot,\sigma)$. Let $u \in W^{1,2}(0,T;V,H)$ be such a fixed point. We use the test function $(u-M)^+$, the ellipticity of L and Lemma 3.11:

$$\frac{1}{2}\int_\Omega (u-M)^+(\tau)^2 \mathrm{d}x + \int_0^\tau \int_\Omega cu(u-M)^+ \mathrm{d}x\mathrm{d}t \le \int_0^\tau \int_\Omega f(x,t,u_M)(u-M)^+ \mathrm{d}x\mathrm{d}t.$$

The second term on the left hand side can be written as

$$\int_0^\tau \int_\Omega c(u-M)(u-M)^+ \mathrm{d}x\mathrm{d}t + M\int_0^\tau \int_\Omega c(u-M)^+ \mathrm{d}x\mathrm{d}t \ge 0.$$

By the conditions on f the right hand side can be bounded from above by

$$\int_0^\tau \int_{\{u>M\}} f(x,t,M)(u-M)^+ \mathrm{d}x\mathrm{d}t \le \int_0^\tau \int_{\{u>M\}} f(x,t,M_0)(u-M)^+ \mathrm{d}x\mathrm{d}t \le 0,$$

since $f(x,t,M_0) \le 0$. We conclude that $(u-M)^+(\tau) = 0$, hence $u \le M$ in $\Omega \times (0,T)$.

The test function $(u+M)^-$ yields

$$\frac{1}{2}\int_\Omega (u+M)^-(\tau)^2 + \int_0^\tau \int_\Omega cu(u+M)^- \mathrm{d}x\mathrm{d}t \le \int_0^\tau \int_\Omega f(x,t,u_M)(u+M)^- \mathrm{d}x\mathrm{d}t.$$

Then, we bound:

$$\int_0^\tau \int_\Omega cu(u+M)^- \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega c((u+M)^-)^2 - M\int_0^\tau \int_\Omega c(u+M)^- \mathrm{d}x\mathrm{d}t \ge 0.$$

Further, since $f(x,t,-M_0) \ge 0$

$$\begin{aligned}\int_0^\tau \int_\Omega f(x,t,u_M)(u+M)^- \mathrm{d}x\mathrm{d}t &\le \int_0^\tau \int_{\{u<-M\}} f(x,t,-M)(u+M)^- \mathrm{d}x\mathrm{d}t \\ &\le \int_0^\tau \int_{\{u<-M\}} f(x,t,-M_0)(u+M)^- \mathrm{d}x\mathrm{d}t \le 0.\end{aligned}$$

This yields $(u+M)^- = 0$ and hence $u \ge -M$. Thus, we obtain a uniform bound for $u \in L^\infty(\Omega)\times(0,T))$ and by the boundedness of Ω in $L^2(\Omega\times(0,T))$. In particular, we have shown $f(x,t,u_M) = f(x,t,u)$, since $|u| \le M$. The conditions for the fixed point theorem of Leray-Schauder are satisfied. There exists a solution to (3.3).

It remains to show uniqueness of solutions. Let u_1 and u_2 be two weak solutions of (3.3). We use the test function $u_1 - u_2$ in the difference of the weak solutions for u_1 and u_2, respectively. Because of the ellipticity of L and the non-negativity of c we obtain

$$\frac{1}{2}\int_\Omega (u_1-u_2)(\tau)^2 \mathrm{d}x \le \int_0^\tau \int_\Omega (f(x,t,u_1) - f(x,t,u_2))(u_1-u_2)\mathrm{d}x\mathrm{d}t \le 0,$$

where we used the monotonicity of f. Therefore $u_1 = u_2$ and uniqueness follows.

□

3.3 Positivity and long-time behaviour

Theorem 3.13 provides existence and uniqueness of solutions to the reaction diffusion equation

$$u_t - \Delta u = R_0 - u^2 \quad \text{in } \Omega,\ t > 0, \quad u(0) = u_0 \quad \text{in } \Omega \tag{3.7}$$

with boundary conditions $u = 0$ on $\partial\Omega$, with $R_0 > 0$. It is more realistic for chemical applications to assume homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

For positive initial conditions we expect that the concentration $u(t)$ is positive and converges to the stationary state $\sqrt{R_0}$ for $t \to \infty$. We will prove this conjecture in this section.

We study the more general monotone semilinear equations of the form

$$u_t + L(u) = f(u) \quad \text{in } \Omega,\ t > 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega. \tag{3.8}$$

The conditions for Ω and $L(u)$ are the same as at the beginning of Section 3.2 (see Assumption 3). The function $f : \mathbb{R} \to \mathbb{R}$ is continuous and monotone decreasing. The weak formulation is given in the following: Let $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Then $V \hookrightarrow H \hookrightarrow V'$ is an evolution triple. We search for $u \in W^{1,2}(0,T;V,H)$, such that for all $v \in L^2(0,T;V)$

$$\int_0^T \langle u_t, v\rangle_{V'} \mathrm{d}t + \int_0^T \int_\Omega (\nabla u^T A \nabla v + cuv) \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega f(u) v \mathrm{d}x\mathrm{d}t.$$

The existence results of Section 3.2 are not directly applicable since here we impose Neumann boundary conditions. However, the weak formulation is the same as the one for the Dirichlet problem except for the definition of the space V. The proof of the theorems of the previous subsection work analogously for the above weak formulation except when we use the properties of the space V. In the case we consider here this is only the case when the Poincaré inequality is applied. If $V = H_0^1(\Omega)$, then by the uniform bound for ∇u in L^2 it follows a bound for u in H^1. This argument does not hold if $V = H^1(\Omega)$. However, the standard estimates in the previous section provide besides the uniform estimate for ∇u in L^2 also estimates for u in $L^\infty(0,T;H^1(\Omega))$ and in particular in $L^2(0,T;L^2(\Omega))$. This yields an estimate for u in $L^2(0,T;H^1(\Omega))$ without applying the Poincaré inequality. We can directly carry over the proof of Theorem 3.13 to the above Neumann boundary value problem. We obtain the following existence result:

Theorem 3.15 (Global existence and uniqueness for semilinear equations with Neumann boundary)**.** *Suppose Assumption 3 holds. Further, let $T > 0$ and $u_0 \in L^\infty(\Omega)$. Let the function $f : \mathbb{R} \to \mathbb{R}$ be continuous and monotone decreasing and let there exist $M_0 > 0$ such that $f(M_0) \le 0$ and $f(-M_0) \ge 0$. Then there exists a unique solution $u \in W^{1,2}(0,T;V,H)$ of (3.8) with $|u| \le \max\{M_0, \|u_0\|_{L^\infty(\Omega)}\}$ in $\Omega \times (0,T)$.*

We claim that the solution to (3.8) is positive if $u_0 > 0$.

Proposition 3.16. *Let $u_0 \in L^\infty(\Omega)$ with $u_0 \ge u_* > 0$ in Ω. There exists a constant $m_0 > 0$ such that $f(m_0) \ge 0$. Further let $u \in W^{1,2}(0,T;V,H)$ be a weak solution to (3.8). Then,*

$$u(\cdot,t) \ge \min(m_0, u_*) e^{-\lambda t} \quad \text{in } \Omega,\ t > 0,$$

with $\lambda = \sup_\Omega c$.

The function $f(u) = R_0 - u^2$ satisfies the conditions of the proposition with $m_0 = \sqrt{R_0}$. Thus, the solution of the reaction-diffusion equation (3.7) satisfies $u \geq \min(\sqrt{R_0}, u_*)$.

Proof. The proof illustrates a variant of the minimum principle. In the first approach the test function $(u-m)^-$ with $0 < m = \min(m_0, u_*)$ is applied in the weak formulation of (3.8). Using the ellipticity of L and $f(u) \geq f(m) \geq f(m_0) \geq 0$ we obtain for $u \leq m$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u-m)^-)^2 \mathrm{d}x + \int_\Omega (c(u-m)(u-m)^- + cm(u-m)^-)\mathrm{d}x \\ \leq \int_{\{u<m\}} f(u)(u-m)^- \mathrm{d}x \leq 0.$$

Since $c \geq 0$ it follows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u-m)^-)^2 \mathrm{d}x \leq -\int_\Omega c(u-m)^- \mathrm{d}x.$$

We observe the problem that the integral on the right hand side is non-negative and can not be bounded without any additional requirements. Therefore, the test function $(u-m)^-$ does not give the right result. This lies on the fact that $\inf u$ is positive for finite times but becomes smaller for larger times. Therefore the constant $m > 0$ might be not below the lower threshold for arbitrary $T > 0$.

Therefore, we take the test function $(u - me^{-\lambda t})^-$ with a constant $\lambda > 0$ which has to be determined (see the proof of Theorem 3.10). By Lemma 3.11 and ellipticity of L, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u-me^{-\lambda t})^-)^2 \mathrm{d}x + \int_\Omega c(((u-me^{-\lambda t})^-)^2 + me^{-\lambda t}(u-me^{-\lambda t})^-)\mathrm{d}x \\ \leq \int_\Omega f(u)(u-me^{-\lambda t})^- \mathrm{d}x - (me^{-\lambda t})_t \int_\Omega (u-me^{-\lambda t})^- \mathrm{d}x.$$

The first integral on the right hand side is non-positive, since $f(u) \geq f(m) \geq 0$ for $u < me^{-\lambda t} \leq m$. Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega ((u-me^{-\lambda t})^-)^2 \mathrm{d}x \leq \int_\Omega m(\lambda - c)e^{-\lambda t}(u-me^{-\lambda t})^- \mathrm{d}x.$$

If we choose $\lambda = \sup_\Omega c$, then the integral on the right hand side is non-positive and we obtain $(u - me^{-\lambda t})^- = 0$ which concludes the proof.

□

Finally, we show that for $t \to \infty$ the solution to (3.8) converges to the weak solution of the stationary problem

$$L(u_\infty) = f(u_\infty) \quad \text{in } \Omega, \quad \frac{\partial u_\infty}{\partial \nu} = 0, \quad \text{on } \partial\Omega. \tag{3.9}$$

The problem is solveable if c is strictly positive, since in this case an estimate for u_∞ in L^2 can be obtained. The diffusion term $\mathrm{div}(A\nabla u_\infty)$ yields a bound for ∇u_∞ in L^2, hence a bound for u_∞ in H^1 holds. This bound guarantees the compactness of the fixed point operator. By the maximum principle it even holds $u_\infty \in L^\infty(\Omega)$. The detailed proof works similar to the proof of Theorem 2.17 and is not carried out here.

Theorem 3.17. *Let $-f$ be (strongly) monotone function, i.e., there exists $\gamma \geq 0$ ($\gamma > 0$) such that*

$$-(f(u) - f(v))(u - v) \geq \gamma(u - v)^2 \quad \text{for all } u, v \in \mathbb{R}.$$

Further, let $u \in W^{1,2}(0, T; V, H)$ be a weak solution to (3.8). Then,

$$\|u(t) - u_\infty\|_{L^\infty(\Omega)} \leq \|u_0 - u_\infty\|_{L^\infty(\Omega)} e^{-\lambda t}, \quad t \geq 0,$$

with $\lambda = \inf_\Omega c + \gamma$.

Proof of Theorem 3.17. The proof uses a variant of the Moser iteration technique. We show the estimates for $u - u_\infty$ in L^p and take then the limit $p \to \infty$. Let p be an odd number. The difference solves the equation

$$(u - u_\infty)_t - L(u - u_\infty) = f(u) - f(u_\infty).$$

Since both u and u_∞ are in $L^\infty(\Omega)$, we can take the test function $(u - u_\infty)^p$ in the weak formulation of the above equation and obtain

$$\begin{aligned}\langle (u - u_\infty)_t, (u - u_\infty)^p \rangle_{V'} + p \int_\Omega (u - u_\infty)^{p-1} \nabla (u - u_\infty)^T A \nabla (u - u_\infty) \mathrm{d}x \\ + \int_\Omega c(u - u_\infty)^{p+1} \mathrm{d}x = \int_\Omega (f(u) - f(u_\infty))(u - u_\infty)^p \mathrm{d}x.\end{aligned}$$

The first integral on the left hand side can be written as

$$\langle (u - u_\infty)_t, (u - u_\infty)^p \rangle_{V'} = \frac{1}{p+1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (u - u_\infty)^{p+1} \mathrm{d}x.$$

This connection holds for regular function and by a density argument also for functions in $W^{1,2}(0, T; V, H)$. A detailed proof works similarly as the one for Lemma 3.11. The second integral can be bounded in the following way:

$$\begin{aligned}p \int_\Omega (u - u_\infty)^{p-1} \nabla (u - u_\infty)^T A \nabla (u - u_\infty) \mathrm{d}x \\ = \frac{4p}{(p+1)^2} \int_\Omega \nabla ((u - u_\infty)^{(p+1)/2})^T A \nabla ((u - u_\infty)^{(p+1)/2}) \mathrm{d}x \\ \geq \frac{4p\alpha}{(p+1)^2} \int_\Omega |\nabla ((u - u_\infty)^{(p+1)/2})|^2 \mathrm{d}x.\end{aligned}$$

By the strong monotonicity of $-f$ it follows

$$\begin{aligned}\int_\Omega (f(u) - f(u_\infty))(u - u_\infty)^p \mathrm{d}x = \int_\Omega (f(u) - f(u_\infty))(u - u_\infty)(u - u_\infty)^{p-1} \mathrm{d}x \\ \leq -\gamma \int_\Omega (u - u_\infty)^{p+1} \mathrm{d}x \leq 0.\end{aligned}$$

Note that we used that $(u - u_\infty)^{p-1} \geq 0$, since $p - 1$ is even. Hence, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (u - u_\infty)^{p+1} \mathrm{d}x + \frac{4p\alpha}{1+p} \int_\Omega |\nabla ((u - u_\infty)^{(p+1)/2})|^2 \mathrm{d}x \\ + (p+1) \int_\Omega (c + \gamma)(u - u_\infty)^{p+1} \mathrm{d}x \leq 0. \qquad (3.10)\end{aligned}$$

Writing $F(t) = \|u(t) - u_\infty\|^{p+1}_{L^{p+1}(\Omega)}$, we obtain the differential equation

$$F'(t) + (p+1)\lambda F(t) \leq 0$$

with $\lambda = \int_\Omega c + \gamma$. Next, we apply the following version of Grönwall's Lemma

Lemma 3.18. *Let $\lambda_0 \in \mathbb{R}$ and $F : [0,T] \to [0,\infty)$ be a absolute continuous function (hence almost everywhere differentiable function) with*

$$F'(t) + \lambda_0 F(t) \leq 0 \quad \textit{for almost all } t \in [0,T].$$

Then, $F(t) \leq F(0)e^{-\lambda_0 t}$ for $t \in [0,T]$.

We obtain with $\lambda_0 = (p+1)\lambda$ after taking the $(p+1)$-th root:

$$\|u(t) - u_\infty\|_{L^{p+1}(\Omega)} \leq \|u_0 - u_\infty\|_{L^{p+1}(\Omega)} e^{-\lambda t}.$$

Since $\|v\|_{L^p(\Omega)} \to \|v\|_{L_\infty(\Omega)}$ for $p \to \infty$ and for all $v \in L^\infty(\Omega)$ (see exercises), we conclude the proof.

□

Remark 3.19 (Neumann vs Dirichlet boundary conditions)**.** *We have not used the integral*

$$\frac{4p\alpha}{p+1} \int_\Omega |\nabla((u-u_\infty)^{(p+1)/2})|^2 \mathrm{d}x$$

in (3.10)*, since for the Neumann problem the Poincaré inequality is not applicable. In the case of the Dirichlet problem we can bound this integral from below by*

$$\frac{4p\alpha}{p+1} C_P^2 \int_\Omega |((u-u_\infty)^{(p+1)/2})|^2 \mathrm{d}x = \frac{4p\alpha}{p+1} C_P^2 \int_\Omega ((u-u_\infty)^{p+1}) \mathrm{d}x,$$

where $C_p > 0$ is the Poincaré constant. In this case the differential equation becomes

$$F'(t) + \Big((p+1)\lambda + \frac{4p\alpha}{p+1} C_P^2\Big) F(t) \leq 0$$

and Grönwall's Lemma yields

$$\|u(t) - u_\infty\|_{L^{p+1}(\Omega)} \leq \|u_0 - u_\infty\|_{L^{p+1}(\Omega)} \exp\Big(- \Big(\lambda + \frac{4p\alpha}{(p+1)^2} C_p^2\Big) t\Big).$$

In the limit $p \to \infty$ we obtain the same estimate as in Theorem 3.17. For finite values of p the rate is in this case larger than in the case with Neumann boundary conditions.

Remark 3.20 (Reaction diffusion equation)**.** *We observe that Theorem 3.17 is not directly applicable for the reaction diffusion equation* (3.7)*, since $f(u) = R_0 - u^2$ is not strongly monotone:*

$$-(f(u) - f(v))(u - v) = (u^2 - v^2)(u - v) = (u + v)(u - v)^2.$$

Under the assumptions of Proposition 3.16, the solution u is positive. To be more precisely, $u \geq m = \min(\sqrt{R_0}, u_*)$. Hence

$$-(f(u) - f(v))(u - v) = 2m(u - v)^2, \quad \textit{for all } u, v \geq 0.$$

Hence the function $-f$ is strongly monotone on the interval $[m, \infty)$ and Theorem 3.17 is applicable. The solution to the boundary value problem

$$-\Delta u_\infty = R_0 - u_\infty^2 \quad \text{in } \Omega, \quad \frac{\partial u_\infty}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

is constant and is given by $u_\infty = \sqrt{R_0}$. The monotonicity on the right hand side implies directly the unique solveability, i.e., $u_\infty = \sqrt{R_0}$ is the unique solution. By Theorem 3.17, it holds

$$\|u(t) - \sqrt{R_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{for } t \to \infty.$$

Hence, we have shown that the concentration converges to the constant stationary concentration.

3.4 Quasilinear equations

In this section we want to analyse the quasi-linear equation of the form

$$u_t - \operatorname{div}(a(u)\nabla u) = 0.$$

This type of equations describes for example the evolution of a concentration u with a density dependent diffusion coefficient $a(u)$. In this section, we impose that $a(u)$ is strictly positive. The case $a(u) \geq 0$ is treated for the case $a(u) = u^m$ in Section 3.5. Indeed we can handle more general equations by formulating the initial boundary value problem in an abstract way. In the following we follow mainly Showalter [20, Section III.4].

Let $V \hookrightarrow H \hookrightarrow V'$ be an evolution triple, i.e., V is a separable reflexive Banach space and H is a separable Hilbert space with a continuous and dense embedding $V \hookrightarrow H$ (see Definition 3.7). Further, let $A : V \to V'$ be an operator. We search for a solution $u \in W^{1,p}(0, T; V, H)$ of the *abstract Cauchy-problem*

$$u_t(t) + A(u(t)) = f(t), \quad t > 0, \quad u(0) = u_0, \tag{3.11}$$

with $1 < p < \infty$. The function $f \in X'$ is given with $X = L^p(0, T; V)$ and $X' = L^q(0, T; V')$, $1/p + 1/q = 1$. By Proposition 3.4, the space $W^{1,p}(0, T; V, H)$ is embedded continuously in $C^0([0, T]; H)$, such that the initial value $u(0) = u_0$ holds in the sense of H. Note that the boundary conditions are formulated in the space V.

Example 3.21. *1. A typical example is given by $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $A = -\Delta$. The Cauchy problem is then*

$$u_t - \Delta u = f(t) \quad \text{in } \Omega,\ t > 0, \quad u(0) = u_0, \tag{3.12}$$

and u satisfies homogeneous Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$. This example can be generalized: The operator A, defined by $u \mapsto -\operatorname{div}(a(u)\nabla u)$, maps $V = H_0^1(\Omega)$ to $V' = H^{-1}(\Omega)$, since if $0 \leq a(u) \leq a^*$ for all $u \in \mathbb{R}$,

$$\begin{aligned}\|A(u)\|_{V'} &= \sup_{\|v\| \leq 1} |\langle A(u), v\rangle_{V'}| = \sup_{\|v\| \leq 1} \left| \int_\Omega a(u)\nabla u \cdot \nabla v \mathrm{d}x \right| \leq a^* \sup_{\|v\| \leq 1} \|u\|_V \|v\|_V \\ &\leq a^* \|u\|_V.\end{aligned}$$

2. *Another example is the operator* $A : V \to V'$ *with* $V = W_0^{1,p}(\Omega)$*, defined by* $A(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$*,* $p \geq 2$*. This leads to th so-called p-Laplace equation*

$$u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

which is another generalization of the heat equation (3.12)*. We obtain* (3.12) *if we take* $p = 2$*. We show that* A *is well-defined. Let* $u, v \in V$*. By the Hölder inequality and* $1/p + 1/q = 1$ *it follows*

$$\begin{aligned}\langle A(u), v\rangle_{V'} &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x \leq \Big(\int_\Omega |\nabla u|^{(p-1)q}\mathrm{d}x\Big)^{1/q}\Big(\int_\Omega |\nabla v|^p \mathrm{d}x\Big)^{1/p} \\ &= \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla v\|_{L^p(\Omega)}.\end{aligned}$$

The supremum over all $v \in W_0^{1,p}(\Omega)$ *with* $\|v\|_V \leq 1$ *yields*

$$\|A(v)\|_{V'} \leq \|\nabla u\|_{L^p(\Omega)}^{p-1} \leq \|u\|_V^{p-1}.$$

Hence, $A : V \to V'$*.*

We can interpret the abstract Cauchy problem in two different ways. The *weak formulation* is given by

$$\langle u_t(t), v\rangle_{V'} + \langle A(u(t)), v\rangle_{V'} = \langle f(t), v\rangle_{V'} \quad \text{for } v \in V,\ t > 0, \quad u(0) = u_0.$$

Note that the formulation makes only sense, if $f(t) \in V'$. In this formulation the time dependence is explicitely written. If $f \in X' = L^q(0, T; V')$ holds, then we can formulate (3.11) as *operator equation*

$$u' + A(u) = f \quad \text{in } X', \quad u(0) = u_0.$$

Here, we interpret A as an operator $A : X \to X'$, i.e., the time dependence is hidden in the definition of the spaces X and X'. Hence, we distinguish the operators $A : V \to V'$ and $A : X \to X'$.

In the following we introduce some notation, which we use for the existence of the solution to the abstract Cauchy-Problem. We recall that the operator $A : V \to V'$ is bounded, if bounded sets in V are mapped to bounded sets in V'. Note that the operators defined in Example 3.21 are bounded.

Definition 3.22. *Let* V *be a reflexive Banach space and* $A : V \to V'$ *be an operator.*

(i) A *is called demi-continuous if for all* $(u_k)_k \subset V$ *with* $u_k \to u$ *in* V *it follows that* $A(u_k) \rightharpoonup A(u)$ *for* $k \to \infty$*.*

(ii) A *is called hemi-continuous if for all* $u, v, w \in V$ *the real-valued function* $t \mapsto \langle A(u + tv), w\rangle_{V'}$*,* $t \in [0, 1]$ *is continuous.*

(iii) A *is called of type M if for all* $(u_k)_k \subset V$ *satisfying*

$$u_k \rightharpoonup u \text{ in } V, \quad A(u_k) \rightharpoonup f \text{ in } V' \quad \text{and} \quad \limsup_{k\to\infty}\langle A(u_k), u_k\rangle_{V'} \leq \langle f, u\rangle_{V'}$$

it follows that $A(u) = f$*.*

(iv) A is called coercive if for all $u \in V$ it holds

$$\frac{\langle A(u), u\rangle_{V'}}{\|u\|_V} \to \infty, \quad \text{if } \|u\|_V \to \infty.$$

(v) A is called monotone if for all $u, v \in V$ it holds

$$\langle (A(u) - A(v), u - v\rangle_{V'} \geq 0.$$

The type-M property is used to identify the limit $f = A(u)$ in nonlinear problems for only weakly convergent sequences $u_k \rightharpoonup u$ and $A(u_k) \rightharpoonup f$.

Example 3.23. *We consider the operators of Example 3.21.*

1. *Let $V = H_0^1(\Omega)$ and $A(u) = -\mathrm{div}(a(u)\nabla u)$ where $a \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $a(u) \geq a_* > 0$. We claim that A is hemi-continuous and coercive. The coercivity can be proven easily: By the Poincaré inequality it follows for $u \in V$*

$$\frac{\langle A(u), u\rangle_{V'}}{\|u\|_V} = \frac{1}{\|u\|_V}\int_\Omega a(u)|\nabla u|^2 \mathrm{d}x \geq a_* C\|u\|_V \to \infty \quad \text{for } \|u\|_V \to \infty.$$

To prove hemi-continuity we consider for given $u, v, w \in V$ the map

$$F(t) = \langle A(u + tv), w\rangle_{V'} = \int_\Omega a(u + tv)\nabla u \cdot \nabla w \mathrm{d}x + t\int_\Omega a(u + tv)\nabla v \cdot \nabla w \mathrm{d}x.$$

Let $t_0 \in [0, 1]$. By the dominated convergence theorem it holds

$$\lim_{t\to t_0}\int_\Omega a(u + tv)\nabla u \cdot \nabla w \mathrm{d}x = \int_\Omega a(u + t_0 v)\nabla u \cdot \nabla w \mathrm{d}x.$$

This implies $F(t) \to F(t_0)$ for $t \to t_0$. Hence A is hemi-continuous.

The operator A is in general not monotone (but of type M).

2. *Let $V = W_0^{1,p}(\Omega)$ with $p \geq 2$ and $A(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. Then A is coercive, monotone and demi-continuous. Let $u, v \in V$. The coercivity is a consequence of*

$$\frac{\langle A(u), u\rangle_{V'}}{\|u\|_V} = \frac{1}{\|u\|_V}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u \mathrm{d}x = \frac{1}{\|u\|_V}\|\nabla u\|^p_{L^p(\Omega)} \geq C\|u\|_V^{p-1}.$$

To show monotonicity we note that the function $x \to \frac{1}{p}|x|^p$, $x \in \mathbb{R}^n$ is convex. Thus by Example 3.21 the map $x \to \nabla\frac{1}{p}|x|^p = |x|^{p-2}x$ is monontone:

$$(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^n.$$

Therefore, we obtain

$$\langle A(u) - A(v), u - v\rangle_{V'} = \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v)\mathrm{d}x \geq 0.$$

Hence, A is monotone. Next, let $u_k \to u$ in V. Then

$$|\nabla u_k|^{p-2}\nabla u_k \to |\nabla u|^{p-2}\nabla u \quad \text{in } L^{p/(p-1)}(\Omega).$$

By $\nabla v \in L^p(\Omega)$ this implies

$$\langle A(u_k), v\rangle_{V'} = \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla v \mathrm{d}x \to \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x.$$

Hence, $A(u_k) \rightharpoonup A(u)$ in V' and A is demi-continuous.

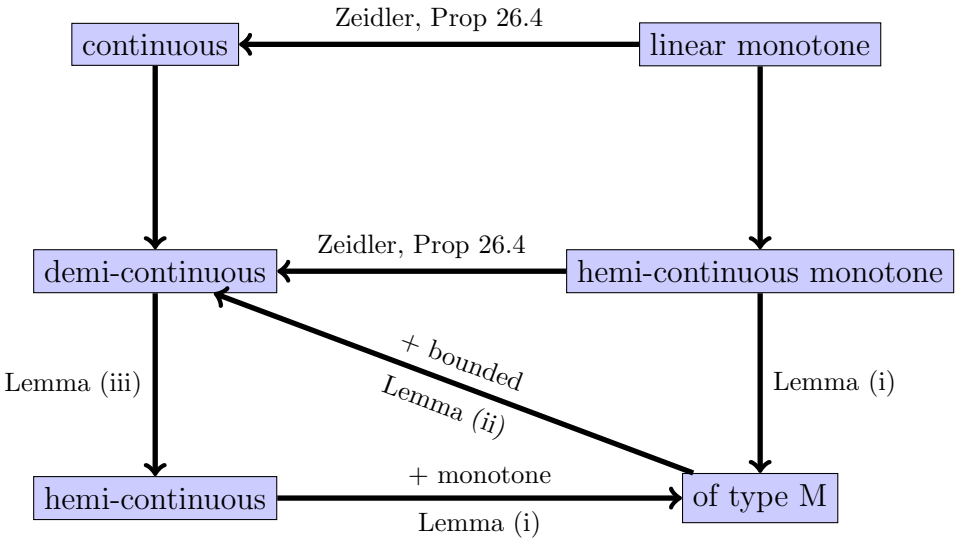

Figure 3.1: Relations between the properties: (demi-/hemi-)continuous, linear monotone, and of type M

The following relations hold between the above notations:

Lemma 3.24. *Let $A : V \to V'$ be an operator. Then:*

(i) If A is hemi-continuous and monotone, then A is of type M.

(ii) If A is bounded and of type M, then A is demi-continuous.

(iii) If A is demi-continuous, then A is also hemi-continuous.

The proof of this lemma is an exercise. The different relations are illustrated in Figure 3.1, where each arrow stands for a implication. The proof that linear, monotone operators are continuous and hemi-continuous, monotone operators are demi-continuous is given in Zeidler [24, Prop. 26.4]. Continuous operators are obviously demi-continuous, since strong convergence implies weak convergence by Proposition 1.12 (i). The main result is stated in the next theorem:

Theorem 3.25 (Existence for the abstract Cauchy problem)**.** *Let $V \hookrightarrow H \hookrightarrow V'$ be an evolution triple, $1 < p < \infty$, $X = L^p(0,T,V)$ and $A : X \to X'$ a bounded coercive operator of type M with*

$$\int_0^t \langle A(u), u\rangle_{V'} \mathrm{d}s \geq \alpha \int_0^t \|u\|_V^p \mathrm{d}s \quad \text{for all } u \in X, 0 < t < T \tag{3.13}$$

for some $\alpha > 0$. Further, let $u_0 \in H$ and $f \in X'$. Then, there exists a weak solution $u \in W^{1,p}(0,T;V,H)$ of (3.11).

Proof of Theorem 3.25. To construct a solution we apply the Galerkin method.

Step 1: Solution to a finite dimensional problem Since V is a separable Banach space there exists by Zeidler [24, Prop 21.49, p. 272] a basis $(v_k)_k$ of V in the sense that $(v_1, ..., v_m)$ are linear independent and the closure of $\cup_m \mathrm{span}\{v_1, \ldots, v_m\}$ is equal to V. Define the finite

dimensional space $V_m = \operatorname{span}\{v_1, \dots, v_m\}$. We consider the following Galerkin equation: Find $u_m(t) \in V$ such that

$$\langle u_m'(t), v_k\rangle_{V'} + \langle A(u_m(t)), v_k\rangle_{V'} = \langle f(t), v_k\rangle_{V'}, \quad k = 1, \dots, m, \tag{3.14}$$

with the initia value $u_m(0) = u_m^0$ where u_m^0 is the orthogonal projection in V_m of u_0. Then, $u_m^0 \to u_0$ in H for $m \to \infty$. The above equation form a nonlinear system of ordinary differential equations. By a variant of the theorem of Peano this system is solveable, if $F(u,t) = \langle f(t) - A(u), v_k\rangle_{V'}$ satisfies the following conditions:

(i) $|f(u,t)| \le g(t)$ for all $\|u\| \le c$, $t \in [0,T]$ and

(ii) F is a Carathéodory function, i.e., $t \mapsto F(u,t)$ is measurable for all $\|u\| \le c$ and $u \mapsto F(u,t)$ is continuous for all $t \in [0,T]$

(see Evans [10, Problem 30.2-30.3]).

We only have to show that the restriction $A : V_m \to V_m'$ is continuous. By Lemma 3.24(ii) $A : X \to X'$ is demi-continuous. We claim the then also $A : V \to V'$ is demi-continuous. Choose $(w_k)_k \subset V$ with $w_k \to w$ in V. Since $(w_k)_k$ is constant in time it even holds $w_k \to w$ in $X = L^p(0,T;V)$. Thus, $A(w_k) \rightharpoonup A(w)$ in X'. Since w_k does not depend on the time, we obtain directly $A(w_k) \rightharpoonup A(w)$ in V', i.e., $A : V \to V'$ is demi-continuous. It follows that the restriction $A : V_m \to V_m'$ is continuous, since

$$\begin{aligned}\lim_{k\to\infty} \|A(w_k) - A(w)\|_{V_m'} &= \lim_{k\to\infty} \sup_{\|v\|_{V_m}\le 1} |\langle A(w_k) - A(w), v\rangle| \\ &= \sup_{\|v\|_{V_m}\le 1} \lim_{k\to\infty} |\langle A(w_k) - A(w), v\rangle| = 0.\end{aligned}$$

Note that we can change limit and supremum since the space V_m' is finite dimensional. Hence, there exists a solution $u_m : [0, T_m] \to V_m$ of (3.14), which we can write of the form $u_m(t) = \sum_{k=1}^m d_k^m(t) v_k$.

Theoretically, it could be that $T_m \to 0$ for $m \to \infty$. Then, in the limit $m \to \infty$ we do not get a useful solution. We claim, that we can choose $T_m = T$, i.e., we can maximally extend the interval on which existence holds. To show this claim, we deduce the following a-priori estimates. We multiply (3.14) by d_k^m and sum over all $k = 1, \dots, m$:

$$\langle u_m', u_m\rangle_{V'} + \langle A(u_m), u_m\rangle_{V'} = \langle f, u_m\rangle_{V'} \tag{3.15}$$

By Proposition 3.4, the first term can be written as time derivative, and the term on the right hand side can be bounded using the norm of f in V':

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|_H^2 + \langle A(u_m), u_m\rangle_{V'} \le \|f\|_{V'}\|u_m\|_V, \quad 0 \le t \le T_m.$$

Integrating over $(0,t)$, applying the coercivity inequatlity (3.13) and Proposition 3.4 (iv) yields

$$\begin{aligned}\frac{1}{2}\|u_m(t)\|_H^2 + \frac{\alpha}{2}\int_0^t \|u_m\|_V^p \mathrm{d}s &\le \frac{1}{2}\|u_m^0\|_H^2 + \int_0^t \|f(s)\|_{V'}\|u_m(s)\|_V \mathrm{d}s - \frac{\alpha}{2}\int_0^t \|u_m\|_V^p \mathrm{d}s \\ &\le \frac{1}{2}\|u_0\|_H^2 + \int_0^t \|f(s)\|_{V'}\|u_m(s)\|_V \mathrm{d}s - \frac{\alpha}{2}\|u_m\|_X^p. \end{aligned} \tag{3.16}$$

Since $p > 1$ the right hand side is bounded with respect to m and we obtain

$$\|u_m(t)\|_H \le C(u_0, T, f), \quad 0 \le t \le T_m. \tag{3.17}$$

Hence, we can continue the solution $u_m(t)$ up to $t = T$. This implies $T_m = T$.

Step 2: A-priori estimates and limit $m \to \infty$ The estimate (3.16) shows that (u_m) is bounded in $L^\infty(0,T;H)$ and $X = L^p(0,T;V)$. Hence, we can deduce existence of a subsequence $(u_{m_k})_k$ with

$$u_{m_k} \rightharpoonup u \quad \text{in } X, \quad u_{m_k}(T) \rightharpoonup u^* \quad \text{in } H. \tag{3.18}$$

We recall that $W^{1,p}(0,T;V,H)$ is continuously embedded in $C^0([0,T];H)$, hence $u_m(T) \in H$ is defined. Since A is bounded, the sequence $(A(u_m))$ is bounded in X' and there exists a subsequence with

$$A(u_{m_k}) \rightharpoonup b \quad \text{in } H, \tag{3.19}$$

where wlog the two subsequences in (3.18) and (3.19) are identical (otherwise we first take a subsequence for (3.18) and take then a subsequence of this subsequence for (3.19)).

We multiply the Galerkin equation (3.14) by $\phi \in C^1([0,T])$, integrate over $(0,T)$ and integrate by parts with respect to t:

$$\begin{aligned} -\int_0^T (u_m(s), \phi'(s)v_k)_H \mathrm{d}s + \int_0^T \langle A(u_m(s)) - f(s), \phi(s)v_k\rangle_{V'} \mathrm{d}s \\ = (u_m(0), \phi(0)v_k)_H - (u_m(T), \phi(T)v_k)_H, \quad k = 1, \dots, m. \end{aligned}$$

By (3.18) and (3.19), we can take the limit $m_n \to \infty$ in this equation and obtain

$$\begin{aligned} -\int_0^T (u(s), \phi'(s)v_k)_H \mathrm{d}s + \int_0^T \langle b(s) - f(s), \phi(s)v_k\rangle_{V'} \mathrm{d}s \\ = (u(0), \phi(0)v_k)_H - (u^*, \phi(T)v_k)_H, \quad k \in \mathbb{N}. \end{aligned}$$

Since $(v_k)_k$ forms a basis of V and $C^1([0,T])$ is dense in $L^p(0,T)$, the set of all functions of the form ϕv_k (or their linear combinations) with $\phi \in C^1([0,T])$ dense in $X = L^p(0,T;V)$. This implies

$$\begin{aligned} -\int_0^T (u(s), v_t)_H \mathrm{d}s + \int_0^T \langle b(s) - f(s), v(s)\rangle_{V'} \mathrm{d}s \\ = (u_0, v(0))_H - (u^*, v(T))_H, \end{aligned}$$

for all $v \in W^{1,p}(0,T;V,H)$. This is the weak formulation of the operator equation

$$u_t + b = f \quad \text{in } X', \quad u(0) = u_0, \quad u(T) = u^*. \tag{3.20}$$

It remains to show $b = A(u)$.

Step 3: Identification of the limit To identify $b = A(u)$ we use the type M property of A. We integrate the special Galerkin equation (3.15) over $(0,T)$ and use Proposition 3.4:

$$\int_0^T \langle A(u_m), u_m\rangle_{V'} \mathrm{d}s - \int_0^T \langle f, u_m\rangle_{V'} \mathrm{d}s = \frac{1}{2}\|u_m(0)\|_H^2 - \frac{1}{2}\|u_m(T)\|_H^2.$$

Note that the expression $\int_0^T \langle \cdot, \cdot \rangle_{V'} \mathrm{d}s$ equals $\langle \cdot, \cdot \rangle_{X'}$. Thus,

$$\langle A(u_m), u_m \rangle_{X'} = \langle f, u_m \rangle_{X'} + \frac{1}{2}\|u_m(0)\|_H^2 - \frac{1}{2}\|u_m(T)\|_H^2.$$

Applying $\limsup_{m' \to \infty}$ on both sides we obtain

$$\limsup_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} = \limsup_{m' \to \infty} \Big(\langle f, u_{m'} \rangle_{X'} + \frac{1}{2}\|u_{m'}^0\|_H^2 - \frac{1}{2}\|u_{m'}(T)\|_H^2 \Big). \tag{3.21}$$

Since $(u_{m'})$ converges in X to u, the first term on the right hand side becomes

$$\limsup_{m' \to \infty} \langle f, u_{m'} \rangle_{X'} = \langle f, u \rangle_{X'}.$$

The sequence $(u_{m'}^0)$ converges strongly in H to u_0 such that

$$\limsup_{m' \to \infty} \|u_{m'}^0\|_H^2 = \|u^0\|_H^2.$$

The sequence $(u_{m'}(T))$ converges weakly in H to $u^* = u(T)$ (see (3.18)). In general, the norm $\|u_{m'}(T)\|_H$ does not converge, but it holds

$$\|u(T)\|_H^2 \leq \liminf_{m' \to \infty} \|u_{m'}(T)\|_H^2$$

or after multiplying with (-1)

$$\limsup_{m' \to \infty} (-\|u_{m'}(T)\|_H^2) \leq -\|u(T)\|_H^2,$$

respectively. Hence, by (3.21)

$$\limsup_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} = \langle f, u \rangle_{X'} + \frac{1}{2}\|u_0\|_H^2 - \frac{1}{2}\|u(T)\|_H^2. \tag{3.22}$$

By 3.4 (iv) and (3.20) it follows

$$\frac{1}{2}\|u(T)\|_H^2 - \frac{1}{2}\|u_0\|_H^2 = \int_0^T \langle u_t, u \rangle_{V'} \mathrm{d}t = \int_0^T \langle f - b, u \rangle_{V'} \mathrm{d}t = \langle f - b, u \rangle_{X'}.$$

Inserting this in (3.22), we obtain

$$\limsup_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} \leq \langle f, u \rangle_{X'} - \langle f - b, u \rangle_{X'} = \langle b, u \rangle_{X'}.$$

Hence, we proved

$$u_{m'} \rightharpoonup u \quad \text{in } X, \quad A(u_{m'}) \rightharpoonup b \quad \text{in } X' \quad \text{and} \quad \limsup_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} \leq \langle b, u \rangle_{X'}.$$

Since A is of type M, we obtain $A(u) = b$. This proves that u solves the operator equation $u_t + A(u) = f$ in X' with $u(0) = u_0$ in H. □

The assumptions of Theorem 3.25 are imposed for the space-time operator $A : X \to X'$. In general, only the space operator $A : V \to V'$ is available (see e.g. Example 3.21). Therefore, we have to find out which assumptions the operator $A : V \to V'$ has to fulfil to apply the existence theorem 3.25.

Lemma 3.26. *Let $A : V \to V'$ be a hemi-continuous and monotone operator which is bounded and coercive in the following sense:*

$$\|A(u)\|_{V'} \le C\|u\|_V^{p-1}, \quad \langle A(u), u\rangle_{V'} \ge \alpha\|u\|_V^p \quad \text{for all } u \in V,$$

where $1 < p < \infty$. Then, the space-time operator $A : X \to X'$ is monotone, of type M and coercive with

$$\|A(u)\|_{X'} \le C\|u\|_X^{p-1}, \quad \langle A(u), u\rangle_{X'} \ge \alpha\|u\|_X^p \quad \text{for all } u \in X,$$

(where $C > 0$ is the same constant as above).

Proof of Lemma 3.26. By Hölder inequality, $X' = L^q(0, T; V')$ and $q(p-1) = p$, the boundedness of $A : X \to X'$ follows by

$$\|A(u)\|_{X'} = \Big(\int_0^T \|A(u)\|_{V'}^q \mathrm{d}t\Big)^{1/q} \le \Big(\int_0^T C^q \|u\|_V^{q(p-1)} \mathrm{d}t\Big)^{1/q} = C\|u\|_X^{p-1}.$$

The monotonicity and coercivity of $A : V \to V'$ carry directly over to $A : X \to X'$. We show next, that $A : X \to X'$ is hemi-continuous. In this case, by Lemma 3.24 (i), A is of type M. Let $t_k \to t$ and $u, v, w \in X$. Then, it holds

$$\langle A(u + t_k v), w\rangle_{X'} = \int_0^T \langle A(u + t_k v), w\rangle_{V'} \mathrm{d}s. \tag{3.23}$$

Since $A : V \to V'$ hemi-continuous, it holds $\langle A(u + t_k v), w\rangle_{V'} \to \langle A(u + tv), w\rangle_{V'}$. The boundedness yields

$$\|A(u + t_k v)\|_{V'} \le C\|u + t_k v\|_V^{p-1} \le C'(\|u\|_V^{p-1} + |t_k|^{p-1}\|v\|_V^{p-1}).$$

By the dominated convergence theorem, we can take the limit $t_k \to t$ in (3.23) and obtain hemi-continuity of $A : X \to X'$.

□

Theorem 3.25 yields only existence but no uniqueness of a solution. The solution is unique if the operator is additionally monotone.

Theorem 3.27 (Uniquenss of an abstract Cauchy problem)**.** *Suppose that the same assumptions as in Theorem 3.25 hold. Further, assume that the operator $A : V \to V'$ is monotone. Then, there exists a unique weak solution $u \in W^{1,p}(0, T; V, H)$ of* (3.11)*.*

Proof of Theorem 3.27. Let u_1 and u_2 be two weak solutions. Take $u_1 - u_2$ as a test function in the weak formulation of the difference of the equations for u_1 and u_2. Then,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u_1 - u_2)(t)\|_H^2 &= \langle (u_1 - u_2)_t(t), (u_1 - u_2)(t)\rangle_{V'} \\ &= -\langle A(u_1(t)) - A(u_2(t)), u_1(t) - u_2(t)\rangle_{V'} \le 0.\end{aligned}$$

Since $(u_1 - u_2)(0) = 0$ it holds $(u_1 - u_2)(t) = 0$ for all $t \ge 0$. Hence, we obtain uniqueness. □

Example 3.28. *We consider the p-Laplace equation*

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{3.24}$$

with $p \geq 2$. We have already shown in Example 3.21 that the operator $A : V \to V'$ defined by $A(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $V = W_0^{1,p}(\Omega)$ is monotone, demi-continuous and coercive with

$$\langle A(u), u\rangle_{V'} \geq C\|u\|_V^p.$$

By Lemma 3.24(iii) demi-continuity implies hemi-continuity of $A : V \to V'$. Further, A is bounded, since by the Hölder inequality we obtain

$$|\langle A(u), v)\rangle_{V'}| = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \mathrm{d}x \leq \|\nabla u\|_{L^p(\Omega)}^{p-1}\|\nabla v\|_{L^p(\Omega)} \leq \|u\|_V^{p-1}\|v\|_V,$$

hence, $\|A(u)\|_{V'} \leq \|u\|_V^{p-1}$. Hence the assumptions of Lemma 3.26 are satisfied and the operator $A : X \to X'$ is bounded, coercive and of type M. By Theorem 3.27, there exists a unique weak solution $u \in W^{1,p}(0,T;V,H)$ (mit $H = L^2(\Omega)$)of (3.24).

The Theorem 3.27 on existence and uniqueness is unfortunately not applicable to the second example

$$u_t - \operatorname{div}(a(u)\nabla u) = 0,$$

since the operator $A(u) = -\operatorname{div}(a(u)\nabla u)$, $u \in V = H_0^1(\Omega)$ is only bounded and coercive but in general not of type M (except if $a(u)$ is monotone or bounded). The techniques in the proof of Theorem 3.25 can be modified to adapt the proof to the above quasi-linear equation. This situation is typical for nonlinear equations: There is no general existence theorem which covers all nonlinear equations and often the existence of solutions needs to be proven explicitly for a given nonlinear problem. To do this, the previously presented techniques (i.e., maximum principle, fixed point theorems, monotonicity properites, ...) can be used.

Theorem 3.29 (Existence and uniqueness for quasi-linear diffusion equations)**.** *Let $a \in C^0(\mathbb{R})$ with $0 < a_* \leq a(u) \leq a^*$ for all $u \in \mathbb{R}$ and $u_0 \in L^2(\Omega)$. Then there exists a unique solution $u \in W^{1,2}(0,T;V,H)$ of*

$$u_t - \operatorname{div}(a(u)\nabla u) = 0 \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3.25}$$

with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

Proof of Theorem 3.29. The proof is a modification of the proof of Theorem 3.25.

Step 1: Solution of the Galerkin equation Let (v_k) be the orthonormal basis of H, which is orthogonal with respect to the scalar product $(u,v)_V = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$. Such a basis can be constructed from the eigen functions of $-\Delta$ in $H_0^1(\Omega)$. The eigen functions form a orthonormal basis of $L^2(\Omega)$ and satisfy

$$(v_j, v_k)_V = \int_\Omega \nabla v_j \cdot \nabla v_k \mathrm{d}x = -\int_\Omega \Delta v_j v_k \mathrm{d}x = \lambda_j \int_\Omega v_j v_k \mathrm{d}x = \lambda_j \delta_{jk}.$$

We consider the Galerkin equations

$$(u_m'(t), v_k)_H + \langle A(u_m(t)), v_k\rangle_{V'} = 0, \quad k = 1, \dots, m,$$

with $A(u) = -\mathrm{div}(a(u)\nabla u)$. We look for $u_m(t) = \sum_{k=1}^m d_k^m(t)v_k$ in the space $V_m = \mathrm{span}\{v_1, \dots, m\}$. The initial values are given by $u_m(0) = u_m^0$, with $u_m^0 \to u_0$ in H. The Galerkin equations form a system of nonlinear ordinary differential equations with coefficients d_k^m. Since the operator A is continuous, existence of a local solution in $[0, T_m]$ follows.

Step 2: A-priori estimates If we multiply the Galerkin equations by d_k^m and sum over $k = 1, \dots, m$, it follows

$$\begin{aligned} 0 &= (u_m'(t), u_m(t))_H + \langle A(u_m(t)), u_m(t)\rangle_{V'} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_H^2 + \int_\Omega a(u_m(t))\nabla u_m(t)\cdot\nabla u_m(t)\mathrm{d}x \\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_H^2 + a_*\int_\Omega |\nabla u_m(t)|^2\mathrm{d}x. \end{aligned}$$

We integrate this inequality over $(0, t)$ and obtain

$$\|u_m(t)\|_H^2 + 2a_*\|u_m\|_X^2 \leq \|u_m(0)\|_H^2 \leq \|u_0\|_H^2, \tag{3.26}$$

with $X = L^2(0, T; H_0^1(\Omega))$. By this estimate we can deduce the weak convergence of a subsequence of (u_m) to u. To take the limit $m \to \infty$ in the Galerkin equations, we have to show that $a(u_m)$ converges to $a(u)$. Since (u_m) converges only weakly, we can not deduce this directly. In the proof of Theorem 3.25, we had a similiar problem which we overcame by assuming that A is of type M. This allowed us to identify the limit of the nonlinearity. Here, we have to argue differently. First, we show that also $(\partial_t u_m) = (u_m')$ is bounded.

Let $v \in V$ with $v = v_0 + v^\perp$ and $v_0 \in V_m$. Then $v^\perp = v - v_0$ is orthogonal to v_0 with respect to H. By

$$\langle u_m'(t), v\rangle_{V'} = (u_m'(t), v)_H = (u_m'(t), v_0)_H = -\langle A(u_m(t)), v_0\rangle_{V'},$$

it follows

$$\|\partial_t u_m(t)\|_{V'} = \sup_{\|v\|_V\leq 1} |\langle u_m'(t), v\rangle_{V'}| \leq \sup_{\|v\|_V\leq 1} \|A(u_m(t))\|_{V'}\|v_0\|_V \leq \|A(u_m(t))\|_{V'}, \tag{3.27}$$

since $\|v\|_V \leq 1$ implies $\|v_0\|_V \leq 1$. Further, we obtain

$$\langle A(u_m(t)), v\rangle_{V'} = \int_\Omega a(u_m(t))\nabla u_m(t)\cdot\nabla v\mathrm{d}x \leq a^*\|u_m(t)\|_V\|v\|_V.$$

Hence, $\|A(u_m(t))\|_{V'} \leq a^*\|u_m(t)\|_V$. Therefore, integrating (3.27) over $(0, T)$ yields

$$\|\partial_t u_m\|_{X'}^2 = \int_0^T \|\partial_t u_m(t)\|_{V'}^2\mathrm{d}t \leq (a^*)^2\int_0^T \|u_m(t)\|_V^2\mathrm{d}t = (a^*)^2\|u_m\|_X^2 \leq C, \tag{3.28}$$

where we applied (3.26) in the last step. The above uniform estimates show that there exists a subsequence with

$$u_{m'} \rightharpoonup u \quad \text{in } L^2(0, T; V), \quad \partial_t u_{m'} \rightharpoonup \partial_t u \quad \text{in } L^2(0, T; V'). \tag{3.29}$$

Step 3: Limit $m' \to \infty$ We want to take the limit $m' \to \infty$ in the Galerkin equation

$$\int_0^T \langle \partial_t u_m, v\rangle_{V'} \mathrm{d}t + \int_0^T \int_\Omega a(u_m)\nabla u_m \cdot \nabla v \mathrm{d}x\mathrm{d}t = 0 \quad \text{for all } v \in V_m. \tag{3.30}$$

To be able to take the limit we have to show that $a(u_{m'})$ converges. We show this by appling the Lemma of Aubin (Lemma 3.5) which says that bounded sequences in $W^{1,2}(0,T;V,H)$ have a strongly convergent subsequence in $L^2(0,T;H)$. By (3.29), existence of a subsquence follows, which we denote again by $(u_{m'})$, such that

$$u_{m'} \to u \quad \text{in } L^2(0,T;H).$$

Since the function a is bounded in $\mathbb{R}$, we obtain by Lemma 3.14

$$a(u_{m'}) \to a(u) \quad \text{in } L^2(0,T;H).$$

Hence, we can take the limit $m' \to \infty$ in (3.30) and obtain

$$\int_0^T \langle \partial_t u, v\rangle_{V'} \mathrm{d}t + \int_0^T \int_\Omega a(u)\nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}t = 0 \quad \text{for all } v \in C_0^\infty(\Omega \times (0,T)).$$

This equation holds also for all $v \in X = L^2(0,T;V)$ due to density arguments. This proves the existence of a weak solution of (3.25), if we can show that $u(0) = u_0$ holds. By the uniform estimate (3.28) it follows that (u_m) is bounded in $H^1(0,T;V')$. Since in one spatial dimension the space H^1 is continuously embedded in C^0, (u_m) is bounded in $C^0([0,T],V')$ and in particular $(u_m(0))$ is bounded in V'. Therefore there exists a subsequence $u_{m'}(0) \rightharpoonup w$ in V' for some $w \in V'$. It remains to show that $w = u(0)$. This proof is a bit technically; see proof of Theorem 6.26, Step 4, in the lecture notes 'Partielle Differentialgleichungen'. The idea of the proof is to integrate by parts in time: $\langle u_t, v\rangle = -(u, v_t)_{L^2} + (u(0), v(0))_{L^2}$ for some test function v with $v(T) = 0$.

Step 4: Uniqueness of a solution Let $u_1, u_2 \in W^{1,2}(0,T;V,H)$ be two weak solutions of (3.25). Then, the difference satisfies the weak formulation

$$\int_0^t \langle (u_1 - u_2)_t, v\rangle_{V'} \mathrm{d}s + \int_0^t \int_\Omega (a(u_1)\nabla u_1 - a(u_2)\nabla u_2) \cdot \nabla v \mathrm{d}x\mathrm{d}s = 0.$$

Applying the test function $u_1 - u_2$, we obtain since $(u_1 - u_2)(0) = 0$

$$\frac{1}{2}\int_\Omega (u_1 - u_2)(t)^2 \mathrm{d}x + \int_0^t \int_\Omega (a(u_1)\nabla u_1 - a(u_2)\nabla u_2) \cdot \nabla (u_1 - u_2) \mathrm{d}x\mathrm{d}s = 0.$$

We can not easily bound the second integral. We apply here the so-called dual method or H^{-1}-method.

We define $b(u) = \int_0^u a(s)\mathrm{d}s$, $u \in \mathbb{R}$. Then, $\nabla b(u) = a(u)\nabla u$ and the differential equation (3.25) can be written as

$$u_t - \Delta b(u) = 0 \tag{3.31}$$

For given $t \in [0,T]$ let $w(t) \in H_0^1(\Omega)$ be the unique solution of

$$\Delta w(t) = (u_1 - u_2)(t) \in L^2(\Omega), \quad w(t) = 0 \quad \text{on } \partial\Omega.$$

Since the right hand side is an $L^2(0,T)$ function, w is an $L^2(0,T;H_0^1(\Omega))$ function. Further, $\Delta w_t = (u_1-u_2)_t \in L^2(0,T;V')$. We can apply w as test function in the difference of the equations (3.31) for u_1 and u_2:

$$\int_0^t \langle \Delta w_t, w\rangle_{V'} \mathrm{d}s + \int_0^t \int_\Omega \nabla(b(u_1)-b(u_2))\cdot\nabla w \mathrm{d}x\mathrm{d}s = 0.$$

We integrate by parts in both integrals:

$$\int_0^t (\nabla w_t, \nabla w)_H \mathrm{d}s + \int_0^t \int_\Omega (b(u_1)-b(u_2))\Delta w \mathrm{d}x\mathrm{d}s = 0$$

The boundary terms vanish, since $b(0)=0$ and $u_1-u_2=0$ on $\partial\Omega$ imply $b(u_1)-b(u_2)=0$ on $\partial\Omega$. By Proposition 3.4, it holds

$$\int_0^t (\nabla w_t, \nabla w)_H \mathrm{d}s = \frac{1}{2}\|\nabla w(t)\|_H^2 - \frac{1}{2}\|\nabla w(0)\|_H^2 = \frac{1}{2}\|\nabla w(t)\|_H^2,$$

since $(u_1-u_2)(0)=0$ and hence $w(0)=0$. Thus,

$$\frac{1}{2}\|\nabla w(t)\|_H^2 = -\int_0^t \int_\Omega (b(u_1)-b(u_2))(u_1-u_2)\mathrm{d}x\mathrm{d}s \le 0,$$

since the function b is monotone increasing. Hence, $\nabla w(t)=0$ and $(u_1-u_2)(t)=0$ for almost all $t\in(0,T)$. □

Remark 3.30. *In the above proof of uniquenss the dual method is mentioned, since the boundary value problem $\Delta w = u_1-u_2$ in Ω, $w=0$ on $\partial\Omega$ is the dual problem to* (3.31)*. The name H^{-1} comes from the fact that the norm $\|\nabla w\|_H$ can be interpreted as the H^{-1} norm of u_1-u_2. On one hand side the elliptic estimate yields*

$$\|\nabla w\|_{L^2(\Omega)} \le \|u_1-u_2\|_{H^{-1}(\Omega)},$$

if we take the expression $\|\nabla(\cdot)\|_{L^2(\Omega)}$ as the H_0^1 norm. On the other

$$\begin{aligned}\|u_1-u_2\|_{H^{-1}(\Omega)} = \|\Delta w\|_{H^{-1}(\Omega)} &= \sup_{\|v\|_{H_0^1(\Omega)}\le 1} |\langle \Delta w, v\rangle_{H^{-1}}| \\ &= \sup_{\|v\|_{H_0^1(\Omega)}\le 1} |(\nabla w, \nabla v)_{L^2}| \le \|\nabla w\|_{L^2(\Omega)}.\end{aligned}$$

Putting both inequalities together yields

$$\|\nabla w\|_{L^2(\Omega)} = \|u_1-u_2\|_{H^{-1}(\Omega)},$$

which explains the notion.

3.5 Porous media equation

In this section, we analyse the quasilinear parabolic equation

$$u_t - \Delta(u^m) = 0 \quad \text{in } \Omega,\ t>0, \quad u^m = 0 \quad \text{on } \partial\Omega, \quad u(0)=u_0 \quad \text{in } \Omega, \tag{3.32}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^1$, and $m > 1$. Then, by the maximum principle we expect that $u(t) \geq 0$ for all $t > 0$. The above equation is similar to (3.25), if we rewrite it to

$$u_t - \operatorname{div}(mu^{m-1}\nabla u) = 0.$$

However, the theory of Section 3.4 is not applicable sine the function $a(u) = mu^{m-1}$ is not bounded from below by a positive constant, but only $a(u) \geq 0$ holds. We call (3.32) a *degenerate* parabolic equation. But, we can still show existence and uniqueness of solutions to (3.32). The degeneracy at $u = 0$ causes that we have to weaken the notation of a solution. We can not expect $u(t) \in H^1(\Omega)$ but only $u(t)^m \in H^1(\Omega)$. Therefore, we define

Definition 3.31. *We call a function $u \geq 0$ a weak solution of (3.32), if*

$$u^m \in L^2(0,T;H_0^1(\Omega)), \quad u_t \in L^2(0,T;H^{-1}(\Omega)),$$

and for all $v \in L^2(0,T;H_0^1(\Omega))$

$$\int_0^T \langle u_t, v\rangle_{H^{-1}} \mathrm{d}t + \int_0^T \int_\Omega \nabla(u^m)\cdot\nabla v \mathrm{d}x\mathrm{d}t = 0$$

and $u(0) = u_0$ in the sense of $H^{-1}(\Omega)$.

Theorem 3.32. *Let $T > 0$, $m > 1$ and $u_0 \in L^{m+1}(\Omega)$ with $u_0 \geq 0$ in Ω. Then there exists a unique weak solution of (3.32) in the sense of Definition 3.31.*

Proof of Theorem 3.32. Step 1: Uniqueness of solutions Uniqueness is shown as in Theorem 3.29 by setting $b(u) = u^m$.

Step 2: Solution of an approximated problem To circumvent the degeneracy of the quation, we regularise the problem, i.e., we introduce the parameter $\epsilon > 0$ such that the regularized differential equation becomes elliptic. We obtain a solution of the original problem by taking the limit $\epsilon \to 0$. This approach is typical for nonlinear differential equations which can not be proven by standard techniques.

We regularize twice. We assume that the initial data are bounded. Further, we add a constant $\epsilon \in (0,1)$ to the diffusion coefficient such that it is strictly positive. Let $u_0 \in L^\infty(\Omega)$ with $u_0 \geq 0$ and $\epsilon \in (0,1)$. We consider the regularized problem

$$\partial_t u_\epsilon - \operatorname{div}(a_\epsilon(u_\epsilon)\nabla u_\epsilon) = 0 \quad \text{in } \Omega,\ t > 0, \quad u_\epsilon = 0 \quad \text{on } \partial\Omega, \quad u_\epsilon(0) = u_0 \quad \text{in } \Omega, \tag{3.33}$$

with $a_\epsilon(u) = mu_M^{m-1} + \epsilon$ and $u_M = \max\{0, \min\{u, M\}\}$. Then a_ϵ satisfies the inequality $0 < \epsilon \leq a_\epsilon(u) \leq mM^{m-1} + \epsilon$. By Theorem 3.29 there exists a unique solution $u_\epsilon \in W^{1,2}(0,T;V,H)$.

We prove in the following some estimates for u_ϵ, which are uniform in ϵ. If we use $(u_\epsilon - M)^+ \in L^2(0,T;H_0^1(\Omega))$ as test function with $M = \sup_\Omega u_0$ in the weak formulation of (3.33), it follows

$$\frac{1}{2}\int_\Omega (u_\epsilon - M)^+(t)^2 \mathrm{d}x + \int_0^t \int_\Omega a_\epsilon(u_\epsilon)|\nabla(u_\epsilon - M)^+|^2 \mathrm{d}x\mathrm{d}s = 0.$$

Therefore $u_\epsilon \leq M$. Similarly using the test function u_ϵ^- we can prove $u_\epsilon \geq 0$. To get this result we need the property $u_0 \geq 0$, to ensure $u_\epsilon(0)^- = 0$. Hence, the function u_ϵ is bounded.

Therefore $\nabla(u_\epsilon^m) = m u_\epsilon^{m-1}\nabla u_\epsilon$ is a function in L^2. Further $u_\epsilon = 0$ and $u_\epsilon^m = 0$ holds on $\partial\Omega$ in the weak sense. Therefore, we can apply $u_\epsilon^m \in L^2(0,T;H_0^1(\Omega))$ as test function and obtain

$$\int_0^t \langle \partial_t u_\epsilon, u_\epsilon^m\rangle_{H^{-1}} \mathrm{d}s + \int_0^t \int_\Omega (|\nabla u_\epsilon^m|^2 + m\epsilon u_\epsilon^{m-1}|\nabla u_\epsilon|^2)\mathrm{d}x\mathrm{d}s = 0$$

Similarly as in Lemma 3.11 the first integral can be reformulated such that

$$\frac{1}{m+1}\int_\Omega (u_\epsilon(t)^{m+1} - u_\epsilon(0)^{m+1})\mathrm{d}x + \int_0^t \int_\Omega (|\nabla u_\epsilon^m|^2 + m\epsilon u_\epsilon^{m-1}|\nabla u_\epsilon|^2)\mathrm{d}x\mathrm{d}s = 0$$

Taking the test function u_ϵ we obtain analogously

$$\frac{1}{2}\int_\Omega (u_\epsilon(t)^2 - u_\epsilon(0)^2)\mathrm{d}x + \int_0^t \int_\Omega (m u_\epsilon^{m-1}|\nabla u_\epsilon|^2 + \epsilon|\nabla u_\epsilon|^2)\mathrm{d}x\mathrm{d}s = 0.$$

We obtain the uniform estimates

$$\|u_\epsilon\|_{L^\infty(0,T;L^{m+1}(\Omega))} + \|u_\epsilon^{(m+1)/2}\|_{L^2(0,T;H^1(\Omega))} + \sqrt{\epsilon}\|u_\epsilon\|_{L^2(0,T;H^1(\Omega))} \le C\|u_0\|_{L^{m+1}(\Omega)}. \quad (3.34)$$

The bound for $\sqrt{\epsilon}u_\epsilon$ becomes useless in the limit $\epsilon \to 0$, but is used in taking the limit. By boundedness of u_ϵ we obtain

$$\begin{aligned}\int_0^T \int_\Omega |\nabla u_\epsilon^m|^2 \mathrm{d}x\mathrm{d}t &= \frac{4m^2}{(m+1)^2}\int_0^T \int_\Omega \underbrace{u_\epsilon^{m-1}}_{\le M^{m-1}} |\nabla u_\epsilon^{(m+1)/2}|^2 \mathrm{d}x\mathrm{d}t \\ &\le C(M)\int_0^T \int_\Omega |\nabla u_\epsilon^{(m+1)/2}|^2 \mathrm{d}x\mathrm{d}t \le C.\end{aligned}$$

Hence by Poincaré inequality

$$\|u_\epsilon^m\|_{L^2(0,T;H^1(\Omega))} \le C.$$

For the time derivative we can deduce the following estimate:

$$\begin{aligned}\|\partial_t u_\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} &= \|\mathrm{div}((m u_\epsilon^{m-1} + \epsilon)\nabla u_\epsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\le \|m u_\epsilon^{m-1}\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))} + \epsilon\|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))} \\ &= \|\nabla u_\epsilon^m\|_{L^2(0,T;L^2(\Omega))} + \epsilon\|\nabla u_\epsilon\|_{L^2(0,T;L^2(\Omega))} \le C\|u_0\|_{L^{m+1}(\Omega)}.\end{aligned}$$

Step 3: Limit $\epsilon \to 0$ By (3.34), there exists a subsequence such that for $\epsilon' \to 0$,

$$u_{\epsilon'} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^{m+1}(\Omega)), \quad u_{\epsilon'}^m \rightharpoonup z \quad \text{in } L^2(0,T;H^1(\Omega)).$$

The embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Therefore, the embedding of the dual spaces $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact (see Zeidler [24, Prop. 21.35]). By the lemma of Aubin, the boundedness of (u_ϵ) in $L^2(0,T;L^2(\Omega))$ and of $(\partial_t u_\epsilon)$ in $L^2(0,T;H^{-1}(\Omega))$ imply (u_ϵ) is compact in $L^2(0,T;H^{-1}(\Omega))$. Hence for a subsequence (ϵ') it holds

$$u_{\epsilon'} \to u \quad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

We have to show that $z = u^m$. This holds, if we can show that (u_ϵ) converges pointwise almost everywhere to u. However, we can not deduce pointwise concergence from neither

weak convergence in L^2 nor strong convergence in H^{-1}. But, we can show the identification $z = u^m$ by using the monotonicity of the map $x \mapsto x^m$. To do this we use again the trick by Browder and Minty. Since the product $\langle u_{\epsilon'}, u_{\epsilon'}^m \rangle_{H^{-1}}$ converges to $\langle u, z \rangle_{H^{-1}} = \int_\Omega uz \mathrm{d}x$, the estimate

$$0 \le \int_0^T \int_\Omega (u_{\epsilon'} - v)(u_{\epsilon'}^m - v^m) \mathrm{d}x\mathrm{d}t = \int_0^T \langle u_{\epsilon'} - v, u_{\epsilon'}^m - v^m \rangle_{H^{-1}} \mathrm{d}x$$

for all $v \in L^2(0,T;C_0^\infty(\Omega))$ implies in the limit $\epsilon' \to 0$

$$0 \le \int_0^T \langle u - v, z - u^m \rangle_{H^{-1}} \mathrm{d}t = \int_0^T \int_\Omega (u - v)(z - v^m) \mathrm{d}x\mathrm{d}t.$$

By density arguments the inequality also holds for $v \in L^\infty(0,T;L^\infty(\Omega))$. Choosing $v = u \pm \lambda w$ with $\lambda > 0$ and $w \in L^\infty(0,T;L^\infty))$, we obtain as in the previous sections $z = u^m$.

We can now take the limit $\epsilon' \to 0$ in the weak formulation

$$\int_0^T \langle \partial_t u_{\epsilon'}, v \rangle_{H^{-1}} \mathrm{d}t + \int_0^T \int_\Omega \nabla(u_{\epsilon'}^m + \epsilon' u_{\epsilon'}) \cdot \nabla v \mathrm{d}x\mathrm{d}t = 0, \quad v \in L^2(0,T;H_0^1(\Omega)).$$

Since by (3.34) $\sqrt{\epsilon}\nabla u_\epsilon$ is bounded in $L^2(\Omega \times (0,T))$, it holds

$$\Big| \int_0^T \int_\Omega \epsilon' \nabla u_{\epsilon'} \cdot \nabla v \mathrm{d}x\mathrm{d}t \Big| \le \sqrt{\epsilon'} \|\sqrt{\epsilon'}\nabla u_{\epsilon'}\|_{L^2(\Omega\times(0,T))} \|\nabla v\|_{L^2(\Omega\times(0,T))} \to 0.$$

We obtain

$$\int_0^T \langle u_t, v \rangle_{H^{-1}} \mathrm{d}t + \int_0^T \int_\Omega \nabla(u^m) \cdot \nabla v \mathrm{d}x\mathrm{d}t = 0, \quad v \in L^2(0,T;H_0^1(\Omega)).$$

The boundary condition $u^m = 0$ holds in the weak sense on $\partial\Omega$, since $z(t) = u(t)^m \in H_0^1(\Omega)$. Since $u_\epsilon \in H^1(0,T;H^{-1}(\Omega)) \hookrightarrow C^0([0,T];H^{-1}(\Omega))$, the initial condition holds in the sense of $H^{-1}(\Omega)$.

Step 4: General initial data We assumed $u_0 \in L^\infty(\Omega)$. We consider now general initial data $u_0 \in L^{m+1}(\Omega)$. If $(u_{0,k}) \subset L^\infty(\Omega)$ is a sequence with $u_{0,k} \to u_0$ in $L^{m+1}(\Omega)$ for $k \to \infty$, then there exists a solution sequence (u_k), which solves (3.32) with $u_k(0) = u_{0,k}$. The sequence (u_k) is not bounded from above, since the bound depends on $\sup_\Omega u_{0,k}$. But we do not need the bound to take the limit $k \to \infty$. The bounds (3.34) are sufficient to take the limit $k \to \infty$ in the weak formulation

$$\int_0^T \langle \partial_t u_k, v \rangle_{H^{-1}} \mathrm{d}t + \int_0^T \int_\Omega \nabla(u_k^m) \cdot \nabla v \mathrm{d}x\mathrm{d}t = 0, \quad v \in L^2(0,T;H_0^1(\Omega)).$$

□

Remark 3.33. *The theorem on existence for the porous media equation can be generalized in several ways.*

(i) The condition $u_0 \in L^{m+1}(\Omega)$ is due to the energy estimate (3.34). This condition can be weakened. By Vazquez [23], $u_0 \in L^1(\Omega)$ is sufficient. In this case there exists exactly one solution u of (3.32) with $u \in C^0([0,\infty);L^1(\Omega))$. It is possible to show that u is indeed infinitely many times continuously differentiable on the set $\{(x,t) \in \Omega \times (0,\infty) : u(x,t) > 0\}$.

Close to the set $\{u = 0\}$ the regularity decreases. Further, it is possible to show existence and uniqueness of weak solutions for the more general equation

$$u_t - \Delta f(u) = 0$$

with corresponding boundary and initial conditions provided f is strictly monotonically increasing, satisfies some growth condition and $f(0) = 0$. The proof is nearly identically to the proof of Theorem 3.32.

(ii) The porous media equation on the whole space has an explicit solution, namely the Barenblatt function

$$u(x,t) = t^{-\alpha}\Big(C - \beta\frac{m-1}{2m}\frac{|x|^2}{t^{2\beta}}\Big)_+^{1/(m-1)}, \quad x \in \mathbb{R}^n,\ t > 0,$$

where $z_+ = \max(z, 0)$ is the positive part of z,

$$\alpha = \frac{n}{n(m-1)+2}, \quad \beta = \frac{1}{n(m-1)+2}$$

and the constant C is given by the total mass which is determined by the mass conservation $\int_{\mathbb{R}^n} u(x,t)\mathrm{d}x = \int_{\mathbb{R}^n} u(x,t_0)\mathrm{d}x$, if the mass $M = \int_{\mathbb{R}^n} u(x,t_0)\mathrm{d}x$ is known at t_0. The Barenblatt solution has the shape of an 'upside down paraboloid' which is cut off when it crosses zero. For $t \to 0$ the Barenblatt solution converges to a multiple of the Delta distribution, i.e., $M\delta_0$. As time t increases, the paraboloid flattens and expands. The solution has for all $t > 0$ a compact support which means that there exists a free boundary $\{u = 0\}$ and the positive part of the solution $\{u > 0\}$ extends with finite speed. This is in contradiction to the infinite speed of spreading for solutions of uniform parabolic equations and comes from the fact that the diffusion coefficient mu^{m-1} is so small for small u that it prevents spreading.

(iii) Instead of the regularizing of the differential operator, one can choose another regularization. To this end the porous media equation is solved first with positive initial data and boundary data

$$u_\epsilon = \epsilon \quad \text{on } \partial\Omega, \quad u_\epsilon(0) = u_0 + \epsilon \quad \text{in } \Omega.$$

One can show that the regularized probem satisfies a comparison principle for smooth solutions, i.e., if u_1 and u_2 are two solutions of (3.33) with $u_1 \le u_2$ on $\partial\Omega$ and $u_1(0) \le u_2(0)$ in Ω, then $u_1 \le u_2$ in $\Omega \times (0,T)$. If $\epsilon \le \epsilon'$, the solution u_ϵ in the second step of the above proof satisfies $u_\epsilon = \epsilon \le \epsilon' = u_{\epsilon'}$ on $\partial\Omega$ and $u_\epsilon(0) = u_0 + \epsilon \le u_0 + \epsilon' = u_{\epsilon'}(0)$ and we can conclude $u_\epsilon \le u_{\epsilon'}$. The sequence (u_ϵ) is pointwise monotonically decreasing and bounded from below (since non-negative). By the dominated convergence theorem there exists a pointwise limit $u = \lim_{\epsilon\to 0} u_\epsilon$. Then, $u_\epsilon^m \rightharpoonup z$ in L^2 implies by the uniqueness of the limit, $z = u^m$. The complete proof is given in Schweizer [19, Section 11].

Remark 3.34. *It is possible to show strong convergence of the sequence $(u_{\epsilon'})$ in L^2. This provides an alternative proof of the identity $z = u^m$. The idea of this proof is to show that (u_ϵ) is not in H^1 but in a space with less regularity which is compactly embedded in L^2. To show this we need some parts of the theory of Sobolev spaces with real index. Let $s > 0$ and $1 < p < \infty$. We define the space*

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) : s = k + \theta,\ k \in \mathbb{N}_0,\ \theta \in [0,1),\ \|u\|_{W^{s,p}(\Omega)} < \infty\},$$

where

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|\le k} \int_{\Omega^2} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p\theta}} \mathrm{d}x\mathrm{d}y.$$

These spaces are reflexive Banach spaces. We write $H^s(\Omega) = W^{s,2}(\Omega)$. For $s = 1$, the space $W^{s,p}(\Omega)$ equals the Sobolev space $W^{1,p}(\Omega)$ defined in Section 1.2, see Adams [1, p. 204]. If $\partial\Omega \in C^1$, then the embedding

$$W^{s,p}(\Omega) \hookrightarrow L^r(\Omega)$$

is continuous if $p \le r \le np/(n-sp)$ (if $n > sp$), $p \le r < \infty$ (if $n = sp$) and $p \le r \le \infty$ (if $n < sp$), see Adams [1, Theorem 7.57]. Further, the embedding

$$W^{s,p}(\Omega) \hookrightarrow W^{s',p}(\Omega)$$

is compact, if Ω is bounded with $\partial\Omega \in C^1$ and $s > s' > 0$ (see Grisvard [12, Theorem 1.4.3.2]).

We apply the following lemma by Chavent-Jaffre. The proof of the lemma is an exercise.

Lemma 3.35 (Chavent-Jaffre)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain, $0 < s < 1$ and $1 < p < \infty$. Further, let $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ be a Hölder-continuous function, i.e., $|f(x) - f(y)| \le c_H|x-y|^\theta$ for all $x, y \in \mathbb{R}$ with $c_H > 0$ and $\theta \in (0,1)$ and $u \in W^{2,p}(\Omega)$. Then,*

$$\|f(u)\|_{W^{\theta s, p/\theta}(\Omega)} \le c_H \|u\|^\theta_{W^{s,p}(\Omega)}.$$

We apply the lemma to $f(x) = x^{1/m}$ with $\theta = 1/m$ and $p = 2$. Then it holds

$$\|u_\epsilon\|_{W^{s/m,2m}(\Omega)} \le c_H \|u^m_\epsilon\|^{1/m}_{W^{s,2}(\Omega)} \le C_0 \|u^m_\epsilon\|^{1/m}_{W^{1,2}(\Omega)}.$$

By (3.34)*, we obtain the bound*

$$\|u_\epsilon\|^{2m}_{L^{2m}(0,T;W^{s/m,2m}(\Omega))} = \int_0^T \|u_\epsilon\|^{2m}_{W^{s/m,2m}(\Omega)} \mathrm{d}t \le C_0^{2m} \int_0^T \|u^m_\epsilon\|^2_{H^1(\Omega)} \mathrm{d}t \le C_1. \tag{3.35}$$

The above embeddings imply that for $s' < s$ the embedding

$$W^{s/m,2m}(\Omega) \hookrightarrow W^{s'/m,2m}(\Omega) \hookrightarrow L^{2m}(\Omega)$$

is compact. Hence, we can apply the Lemma of Aubin and conclude that (u_ϵ) is relatively compact in $L^2(\Omega \times (0,T))$:

$$u_{\epsilon'} \to u \quad \text{in } L^2(\Omega \times (0,T)).$$

The subsequence converges pointwise almost everwhere to u. This implies $u^m_{\epsilon'} \to u^m$ converges pointwise almost everywhere, hence, $z = u^m$.

Chapter 4

Further nonlinear equations

In this chapter we illustrate the variety of possible nonlinearities and present further nonlinear equations. We restrict us to a few examples which are important in applications or whose analysis relies on the use of specific techniques. Due to the complexity of the problems we will not prove all presented results.

4.1 Stationary Navier-Stokes equation

In this section we analyse the stationary incompressible Navier-Stokes equation

$$-\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \mathrm{div} u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{4.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with $\partial\Omega \in C^1$. The vector valued function u describes the velocity of a fluid, p describes the pressure and f is an external force. We will deduce the weak formulation of this problem and present some existence result. We follow Ružička [18, Section 3.2.3].

The difficulty of a solution to (4.1) is given by the fact that in both differential equations the corresponding term with the highest derivative is applied to u. If we solve the first equation of (4.1) for given p, then the solution does not necessarily have to solve the second equation. If we solve the second equation of u and the first for p, it is difficult to show compactness, since the divergence operator increases the order of differentiation only by one whereas the Laplace operator decreases the order by two. The idea to solve the equation is to eliminate the pressure in the first equation. If we multiply the first equation in (4.1) with a vector valued test function v (in the sense of a scalar product) with $v = 0$ on $\partial\Omega$ and integrate by parts, we obtain

$$\int_\Omega \nabla u : \nabla v \mathrm{d}x + \int_\Omega (u \cdot \nabla)u \cdot v \mathrm{d}x - \int_\Omega p \mathrm{div} v \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x.$$

Here, ∇u is the derivative matrix of u and

$$\nabla u : \nabla v = \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}.$$

The expression $(u \cdot \nabla)u \cdot v$ reads as the sum

$$\sum_{i,j=1}^{3} u_i \frac{\partial u_j}{\partial x_i} v_j.$$

If $\mathrm{div} v = 0$, the integral containing the pressure vanishes and it remains to solve the equation for u. Hence, we define the space having exactly this property

$$X = \{v \in H_0^1(\Omega)^3 : \mathrm{div} v = 0 \text{ in } \Omega\}.$$

It is equipped with the norm

$$\|v\|_X^2 = \|\nabla v\|_{L^2(\Omega)^{3\times 3}}^2 = \sum_{i,j=1}^{3} \Big\| \frac{\partial v_i}{\partial x_j} \Big\|_{L^2(\Omega)}^2.$$

Hence, X equipped with this norm is a reflexive, separable Banach space (see exercises).

Now we deduce the weak formulation. Let u be a classical solution of (4.1). We multiply (4.1) with $v \in X$, integrate over Ω, use $\mathrm{div} v = 0$ and integrate by parts:

$$\int_\Omega \nabla u : \nabla v \mathrm{d}x + \int_\Omega (u \cdot \nabla) u \cdot v \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x, \quad v \in X. \tag{4.2}$$

To formulate the boundary value problem we define the following operators

$$\begin{aligned} A_1 : X \to X', \quad & \langle A_1 u, v\rangle_{X'} = \int_\Omega \nabla u : \nabla v \mathrm{d}x, \\ A_2 : X \to X', \quad & \langle A_2 u, v\rangle_{X'} = \int_\Omega (u \cdot \nabla) u \cdot v \mathrm{d}x, \\ b \in X', \quad & \langle b, v\rangle_{X'} = \int_\Omega f \cdot v \mathrm{d}x. \end{aligned}$$

Then (4.2) is equivalent to the operator equation

$$A_1 u + A_2 u = b \quad \text{in } X'. \tag{4.3}$$

The operators A_1 and A_2 satisfy the following properties.

Lemma 4.1. *The operator A_1 is linear, continuous, coercive and monotone. The operator A_2 is bounded, strongly continuous (i.e., $u_k \rightharpoonup u$ implies $A_2 u_k \to A_2 u$). The sum $A = A_1 + A_2$ is bounded, coercive and of type M.*

We note that strongly continuous operators are also compact. If the operator is linear, compactness implies strong continuity (this can be calculated directly).

Proof of Lemma 4.1. The properties of A_1 are easy to prove, since A_1 is a vector valued variant of $A = -\Delta$ in $H_0^1(\Omega)$. First, we show boundedness for A_2. Let $u, v \in X$. Then, by Hölder inequality,

$$|\langle A_2 u, v\rangle_{X'}| \le \int_\Omega |u||\nabla u||v| \mathrm{d}x \le \|u\|_{L^4(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^4(\Omega)}.$$

The space $H^1(\Omega)$ embedds continuously in the space $L^4(\Omega)$, provided the space dimension $n \le 4$ (which we assumed). Hence, it follows

$$|\langle A_2 u, v\rangle_{X'}| \le C \|u\|_{H^1(\Omega)}^2 \|v\|_{H^1(\Omega)}.$$

We obtain $\|A_2 u\|_{X'} \le C\|u\|_X^2$, i.e., A_2 is bounded.

To prove strong continuity, let $u_k \rightharpoonup u$ in X. By the compact embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, there exists a subsequence $u_{k'} \to u$ in $L^4(\Omega)$. We have to show that $A_2 u_k \to A_2 u$ in X', i.e.,

$$\|A_2 u_k - A_2 u\|_{X'} = \sup_{\|v\|_X \leq 1} |\langle A_2 u_k - A_2 u, v\rangle| \to 0 \quad (k \to \infty).$$

Suppose this does not hold. Then, there exist $\epsilon > 0$ and a sequence $(v_k) \subset X$ with $\|v_k\|_X \leq 1$ such that for all $k \in \mathbb{N}$ it holds

$$|\langle A_2 u_k - A_2 u, v_k\rangle| \geq \epsilon. \tag{4.4}$$

Since (v_k) is bounded in X by construction, there exists a subsequence with $v_{k'} \rightharpoonup v$ in X and $v_{k'} \to v$ in $L^4(\Omega)$. We bound in the following way

$$\begin{aligned}
&|\langle A_2 u_{k'} - A_2 u, v_{k'}\rangle| \\
&= \Big| \int_\Omega (((u_{k'} - u)\cdot\nabla)u_{k'} \cdot v_{k'} + (u\cdot\nabla)(u_{k'} - u)\cdot(v_{k'} - v) + (u\cdot\nabla)(u_{k'} - u)\cdot v)\mathrm{d}x \Big| \\
&\leq \|u_{k'} - u\|_{L^4(\Omega)} \|\nabla u_{k'}\|_{L^2(\Omega)} \|v_{k'}\|_{L^4(\Omega)} + \|u\|_{L^4(\Omega)} \|\nabla(u_{k'} - u)\|_{L^2(\Omega)} \|v_{k'} - v\|_{L^4(\Omega)} \\
&\quad + \Big| \int_\Omega (u\cdot\nabla)(u_{k'} - u)\cdot v \mathrm{d}x \Big|.
\end{aligned}$$

The first summand converges to zero, since $u_{k'} \to u$ in $L^4(\Omega)$. The second one converges to zero, since $v_{k'} \to v$ in $L^4(\Omega)$, and the third one, since $\nabla(u_{k'} - u) \rightharpoonup 0$ in $L^2(\Omega)$. This contradicts (4.4). Hence A_2 is strongly continuous.

The boundedness of A follows from the boundedness of the operators A_1 and A_2. For showing coercivity, let $u \in X$. then $\mathrm{div} u = 0$ and

$$\langle A_2 u, u\rangle_{X'} = \int_\Omega \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} u_i u_j \mathrm{d}x = \frac{1}{2} \int_\Omega \sum_{j=1}^3 u_j \frac{\partial |u|^2}{\partial x_j} \mathrm{d}x = -\frac{1}{2} \int_\Omega (\mathrm{div} u)|u|^2 \mathrm{d}x = 0.$$

By coercivity of A_1 it holds $\langle Au, u\rangle_{X'} = \langle A_1 u, u\rangle_{X'} \geq \alpha \|u\|_X^2$.

It remains to show that A is of type M. Let $u_k \rightharpoonup u$ in X, $Au_k \rightharpoonup b$ in X' and $\limsup_{k\to\infty} \langle Au_k, u_k\rangle_{X'} \leq \langle b, u\rangle_{X'}$. Since A_2 is strongly continuous, it follows $A_2 u_k \to A_2 u$ in X'. Hence, $A_1 u_k \rightharpoonup b - A_2 u$ in X'. Further, we know that $\langle A_2 u_k, u_k\rangle_{X'} = 0$ which implies

$$\limsup_{k\to\infty} \langle A_1 u_k, u_k\rangle_{X'} \leq \langle b, u\rangle_{X'} = \langle b - A_2 u, u\rangle_{X'}.$$

By Lemma 3.24(i), A_1 is of type M, which yields $A_1 u = b - A_2 u$. This implies $Au = (A_1 + A_2)u = b$. Hence, A is also of type M. □

We can prove an analogous result as Theorem 3.25 for abstract operator equations for bounded coercive operators of type M.

Theorem 4.2. *Let X be a reflexive, separable Hilbert space, $A_1 : X \to X'$ a monotone and hemi-continuous operator and $A_2 : X \to X'$ a strongly continuous operator. Let the sum $A = A_1 + A_2$ be bounded, coercive and of type M. Then, there exists for every $b \in X'$ a solution to the operator equation*

$$(A_1 + A_2)(u) = b.$$

Proof. We apply the Galerkin method. Let $(v_k) \subset X$ be an orthonormal basis. We look for a solution $u_m = \sum_{k=1}^m d_k^m v_k$ of the system of equations

$$0 = \langle Au_m - b, v_k \rangle_{X'} = \Big\langle A\Big(\sum_{j=1}^m d_j^m v_j\Big) - b, v_k \Big\rangle_{X'} =: g_k(d^m), \quad k = 1, \dots, m,$$

with $d^m = (d_1^m, \dots, d_m^m)$. First, we have to ensure that the system of equations is solveable. We note that due to the strong continuity of A_2 and the hemi-continuity and monotonicity of A_1 the sum $A = A_1 + A_2$ is demi-continuous by Lemma 3.24. This implies the continuity of $g = (g_1, \dots, g_m) : \mathbb{R}^m \to \mathbb{R}^m$. Let $R = 2\|b\|_{X'}/\alpha$. Then, it follows for all $\|u_m\|_X = |d^m| = R$ by coercivity of A:

$$g(d^m) \cdot d^m = \sum_{k=1}^m g_k(d^m) d_k^m = \langle Au_m - b, u_m \rangle_{X'} \geq \alpha \|u_m\|_X^2 - \|b\|_{X'} \|u_m\|_X = \frac{2}{\alpha} \|b\|_{X'}^2 > 0,$$

if $b \neq 0$. If $b = 0$, we obtain $g(d^m) \cdot d^m = \alpha \|u_m\|_X^2 > 0$. Hence, the assumptions of Lemma 2.21, which is a consequence of the fixed point theorem of Brouwer, are satisfied and we obtain existence of $d^m \in \mathbb{R}^m$ with $g(d^m) = 0$. Further, the a-priori estimate

$$\|u_m\|_X \leq R$$

holds and there exists a subsequence with $u_{m'} \rightharpoonup u$ in X. We claim that u satisfies the operator equation $Au = b$.

To show this claim we use the type-M property of A. Since A is bounded, it follows by $\|u_m\|_X \leq R$ that the inequality $\|A(u_m)\|_{X'} \leq C(R)$ holds. Hence there exists a subsequence with $u_{m'} \rightharpoonup u$ in X and $A(u_{m'}) \rightharpoonup z$ in X'. Then, by the Galerkin equations it follows

$$0 = \lim_{m' \to \infty} \langle A(u_{m'}) - b, v \rangle_{X'} = \langle z - b, v \rangle_{X'} \quad \text{for all } v \in \operatorname{span}\{v_1, v_2, \dots\}.$$

Hence, $z = b$. If we set $v = u_{m'}$ in the Galerkin equations, we obtain

$$0 = \lim_{m' \to \infty} \langle A(u_{m'}) - b, u_{m'} \rangle_{X'},$$

and by the weak convergence of $(u_{m'})$,

$$\lim_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} = \lim_{m' \to \infty} \langle b, u_{m'} \rangle_{X'} = \langle b, u \rangle_{X'}.$$

We have shown that

$$u_{m'} \rightharpoonup u \quad \text{in } X, \quad , A(u_{m'}) \rightharpoonup b \quad \text{in } X', \quad \lim_{m' \to \infty} \langle A(u_{m'}), u_{m'} \rangle_{X'} = \langle b, u \rangle_{X'}.$$

Since A is of type M, it follows $Au = b$. □

We can show the following result on existence.

Theorem 4.3 (Existence for stationary Navier-Stokes equations)**.** *Let the assumptions from the beginning of this section hold. Let $f \in L^2(\Omega)^3$. Then there exists a weak solution $(u, p) \in X \times L^2(\Omega)$ of (4.1).*

Proof of Theorem 4.3. By Lemma 4.1 and Theorem 4.2, it follows existence of a function u which solves the weak formulation (4.2). It remains to show that some function p exists such that for all $v \in H_0^1(\Omega)^3$

$$\int_\Omega \nabla u : \nabla v \mathrm{d}x + \int_\Omega (u \cdot \nabla) u \cdot v \mathrm{d}x - \int_\Omega p \ \mathrm{div}\ v \mathrm{d}x = \int_\Omega f \cdot v \mathrm{d}x.$$

This is essentially a consequence of the fact that differentiable vector fields can be be written as direct sum of a divergence-free vector field and a gradient, i.e., for all v there exists functions u and p such that

$$v = u + \nabla p, \quad \text{with } \mathrm{div} u = 0.$$

In other words: If a vector field F is orthogonal to the space X, then F is a gradient, i.e., $F = -\nabla p$ for some function p. This statement is given more precisely in the following theorem, which we will not prove.

Theorem 4.4 (De Rham)**.** *Let $F \in (H_0^1(\Omega)^3)'$ be such that $\langle F, v\rangle = 0$ for all $v \in X$ (X as above). Then there exists some $p \in L^2(\Omega)$ with $\int_\Omega p \mathrm{d}x = 0$ such that*

$$\langle F, v\rangle = \int_\Omega p \ \mathrm{div}\ v \mathrm{d}x \quad \textit{for all } v \in H_0^1(\Omega)^3.$$

We apply this theorem for the functional

$$\langle F, v\rangle = \int_\Omega \nabla u : \nabla v \mathrm{d}x + \int_\Omega (u \cdot \nabla) u \cdot v \mathrm{d}x - \int_\Omega f \cdot v \mathrm{d}x,$$

which satisfies by construction of the space X the assumptions of De Rham's theorem (Theorem 4.4). This concludes the proof. □

Remark 4.5. *The proof of uniqueness of a weak solution of a stationary Navier-Stokes equation is quite delicate. Uniqueness follows if the norm $\|f\|_{X'}$ is sufficiently small. Let u and v be two weak solutions of (4.2). Taking the difference and using the test function $u - v$, it follows*

$$\begin{aligned}\int_\Omega |\nabla(u-v)|^2 \mathrm{d}x &= -\int_\Omega ((u\cdot\nabla)u\cdot(u-v) - (v\cdot\nabla)v\cdot(u-v))\mathrm{d}x \\ &= -\int_\Omega (((u-v)\cdot\nabla)u\cdot(u-v) + (v\cdot\nabla)(u-v)\cdot(u-v))\mathrm{d}x. \qquad (4.5)\end{aligned}$$

The second integral vanishes, since by $\mathrm{div}(v) = 0$ *it follows*

$$\int_\Omega (v\cdot\nabla)(u-v)\cdot(u-v))\mathrm{d}x = \frac{1}{2}\int_\Omega (v\cdot\nabla)|u-v|^2\mathrm{d}x = -\frac{1}{2}\int_\Omega \mathrm{div}(v)|u-v|^2\mathrm{d}x = 0.$$

For the first integral we bound by Hölder, Sobolev and Poincaré inequality

$$-\int_\Omega (((u-v)\cdot\nabla)u\cdot(u-v)\mathrm{d}x \le \|u-v\|^2_{L^4(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le C\|\nabla(u-v)\|^2_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

Using the test function u in (4.2) it follows

$$\int_\Omega |\nabla u|^2 \mathrm{d}x = \int_\Omega f \cdot u \mathrm{d}x \le \|f\|_{X'}\|u\|_X = \|f\|_{X'}\|\nabla u\|_{L^2(\Omega)}$$

and hence $\|\nabla u\|_{L^2(\Omega)} \le \|f\|_{X'}$. Then by (4.5), we obtain

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} \le C\|\nabla(u-v)\|^2_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{X'}\|\nabla(u-v)\|^2_{L^2(\Omega)},$$

and if $C\|f\|_{X'} < 1$, it holds $\|\nabla(u-v)\|^2_{L^2(\Omega)} \le 0$ and hence $u = v$ in Ω. The uniqueness of a weak solution for general f is an open problem.

4.2 Schrödinger equation

We consider a quantum mechanical particle in a bounded domain Ω, whose probability to stay at the boundary of the domain vanishes (i.e., the particle can not cross the boundary). Then, the particle can be described by the Schrödinger equation

$$iu_t + \Delta u = 0 \quad \text{in } \Omega,\ t > 0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u(0) = u_0 \quad \text{in } \Omega. \tag{4.6}$$

This equation is *not* of parabolic type. Let u_0 be an eigenfunction of

$$\Delta v = \lambda v \quad \text{in } \Omega, \quad v = 0, \text{on } \partial\Omega,$$

with eigenvalue λ. Then

$$u(x,t) = e^{i\lambda t}u_0(x)$$

is a special solution to (4.6). In the formula for the solution the space and the time variable are separated. Due to the expression $e^{i\lambda t}$ the solution shows oscillations with frequency λ. For more general initial functions the solution is given as a superposition of oscillations with several frequencies. In this sense, the Schrödinger equation is made up of several waves. This behaviour we will analyse later in this section.

First, we will prove the theorem of existence and uniqueness for the linear equation. We consider a more general problem and follow Dautray and Lions [7, Section 18.7]. Let $V \hookrightarrow H \hookrightarrow V'$ be a an evolution triple (with complex valued Hilbert spaces). and $a : V \times V \to \mathbb{C}$ a continuous sesquilinear form such that

(i) for all $u, v \in V : a(u,v) = \overline{a(v,u)}$ (4.7)

(ii) there exist $\lambda \in \mathbb{R}, \alpha > 0$ such that for all $v \in V :\ a(v,v) = +\lambda\|v\|^2_H \ge \alpha\|v\|^2_V$. (4.8)

The first property is called *skew-symmetry*. The second property is a generalization of the *coercivity*. An example of an skew-symmetric and coercive sesquilinear form is given by

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\mathrm{d}x, \quad u, v \in V = H^1_0(\Omega). \tag{4.9}$$

With $H = L^2(\Omega)$, it holds $a(u,u) + \|u\|^2_H = \|u\|^2_V$. We solve the following problem:

$$\langle u_t(t), v\rangle_{V'} + ia(u(t),v) = (f(\cdot,t),v)_H, \quad t > 0, \quad u(0) = u_0, \tag{4.10}$$

for all $v \in L^2(0,T;V)$.

With the above sesquilinear form and $f = 0$ we obtain the Schrödinger equation (4.6). We show next:

Theorem 4.6 (Existence and uniqueness for a linear abstract Schrödinger problem)**.** *Let $f \in L^2(0,T;H)$ with $f_t \in L^2(0,T;V')$ and $u_0 \in V$. Then, there exists exactly one solution $u \in W^{1,2}(0,T;V,H)$ of (4.10). The solution satisfies $u \in L^\infty(0,T;V)$ and $u_t \in L^\infty(0,T;V')$.*

Proof of Theorem 4.6. The proof of uniqueness is an exercise. We apply the Galerkin method to show existence of a solution. Let $(v_k)_k$ be a orthonormal basis of V. Define the finite dimensional space $V_m = \operatorname{span}\{v_1, \dots, v_m\}$. There exists a sequence (u_m^0) with $u_m^0 \in V_m$ such that $u_m^0 \to u_0$ in V for $m \to \infty$ and $\|u_m^0\|_V \le \|u_0\|_V$. We consider the Galerkin equations

$$(u_m'(t), v_k)_H + ia(u_m(t), v_k) = (f(\cdot,t), v_k)_H, \quad t > 0, \quad u_m(0) = u_m^0.$$

We look for a solution in the space V_m, i.e., $u_m(t) = \sum_{k=1}^m d_k^m(t) v_k$. Hence, the system of Galerkin equations is a system of linear ordinary differential equations for the coefficient d_k^m, which is uniquely solveable on $(0,T)$. Next, we want to deduce appropriate a-priori estimates and take the limit $m \to \infty$.

We show three estimates. First, we multiply the Galerkin equations with d_k^m and sum over $k = 1, \dots, m$:

$$(u_m'(t), u_m(t))_H + ia(u_m(t), u_m(t)) = (f(t), u_m(t))_H.$$

Here we used that the scalar product is sesquilinear on H. Adding the complex conjugated equation to this one, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_H^2 = \operatorname{Re}(f(t), u_m(t))_H.$$

We integrate over t and use Young's inequality to obtain

$$\|u_m(t)\|_H^2 - \|u_m(0)\|_H^2 = 2\int_0^t \operatorname{Re}(f(s), u_m(s))_H \mathrm{d}s \le \int_0^t \|f(s)\|_H^2 \mathrm{d}s + \int_0^t \|u_m(s)\|_H^2 \mathrm{d}s.$$

Grönwall's lemma implies

$$\|u_m(t)\|_H^2 \le e^t \Big(\|u_m(0)\|_H^2 + \int_0^T \|f(s)\|_H^2 \mathrm{d}s \Big), \quad t \in [0,T]. \tag{4.11}$$

This yields an estimate for u_m in $L^\infty(0,T;H)$.

In the next step we multiply the Galerkin equations with $(d_k^m)'$ and sum over $k = 1, \dots, m$:

$$\|u_m'(t)\|_H^2 + ia(u_m(t), u_m'(t)) = (f(t), u_m'(t))_H.$$

We substract the complex conjugated equation:

$$ia(u_m(t), u_m'(t)) + \overline{ia(u_m(t), u_m'(t))} = (f(t), u_m'(t))_H - \overline{(f(t), u_m'(t))_H} = 2i\operatorname{Im}(f(t), u_m'(t))_H,$$

which due to skew-symmetry of a is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} a(u_m(t), u_m(t)) = a(u_m(t), u_m'(t)) + a(u_m'(t), u_m(t)) = 2\operatorname{Im}(f(t), u_m'(t))_H.$$

By integrating over $(0,t)$ and integration by parts with respect to t, we obtain

$$\begin{aligned} a(u_m(t),u_m(t)) - a(u_m(0),u_m(0)) &= 2\int_0^t \mathrm{Im}\langle f(s),u_m'(s)\rangle_{V'}\mathrm{d}s \\ &= 2\mathrm{Im}\langle f(t),u_m(t)\rangle_{V'} - 2\mathrm{Im}\langle f(0),u_m(0)\rangle_{V'} - 2\int_0^t \mathrm{Im}\langle f'(s),u_m(s)\rangle_{V'}\mathrm{d}s. \end{aligned}$$

The coercivity of a and the Young's inequality yield

$$\begin{aligned} &\alpha\|u_m(t)\|_V^2 - a(u_m(0),u_m(0)) \le a(u_m(t),u_m(t)) + \lambda\|u_m(t)\|_H^2 - a(u_m(0),u_m(0)) \\ &\le 2\|f(t)\|_{V'}\|u_m(t)\|_V + 2\|f(0)\|_{V'}\|u_m^0\|_V + 2\int_0^t \|f'(s)\|_{V'}\|u_m(s)\|_V\mathrm{d}s + |\lambda|\|u_m(t)\|_H^2 \\ &\le \frac{\alpha}{2}\|u_m(t)\|_V^2 + \frac{2}{\alpha}\|f(t)\|_{V'}^2 + 2\|f(0)\|_{V'}\|u_m^0\|_V + \int_0^t \|u_m(s)\|_V^2\mathrm{d}s + \int_0^t \|f'(s)\|_{V'}^2\mathrm{d}s \\ &\quad + |\lambda|\|u_m(t)\|_H^2. \end{aligned}$$

The first term on the right hand side can be absorbed by the left hand side. By assumption f is an element in the space $H^1(0,T;V')$ which is continuously embedded in $C^0([0,T],V')$. Therefore, the second term $\|f(t)\|_{V'}$ on the right hand side is uniformly bounded for $t \in (0,T)$. This is also the case for the third term, since $\|u_m^0\|_V \le \|u_0\|_V$. The fifth term is bounded, since we assumed $f' \in L^2(0,T;V')$. The estimate (4.11) shows that the last term $\|u_m(t)\|_H$ is uniformly bounded with respect to $t \in (0,T)$ and $m \in \mathbb{N}$. Hence, we can apply Grönwall's lemma and obtain

$$\|u_m(t)\|_V^2 \le C(T)(\|u_0\|_V^2 + \|f\|_{L^2(0,T;H)}^2 + \|f'\|_{L^2(0,T;V')}^2), \quad t \in (0,T), \tag{4.12}$$

where $C(T)$ is a constant which depends only on α and T. Thus, we have shown that (u_m) is bounded in $L^\infty(0,T;V)$.

By (4.12) it follows that (u_m') is bounded in $L^\infty(0,T;V')$ since by continuity of a it holds for all $v \in V_m$

$$|\langle u_m'(t),v\rangle_{V'}| \le |a(u_m(t),v)| + |(f(t),v)_H| \le C\|u_m(t)\|_V\|v\|_V + \|f(t)\|_{V'}\|v\|_V.$$

Similarly as shown above it follows by (4.12) and $u_m' \in V_m$ that the estimate

$$\|u_m'\|_{L^\infty(0,T;V')} = \sup_{(0,T)}\sup_{\|w\|_V=1} |\langle u_m',w\rangle_{V'}| = \sup_{(0,T)}\sup_{\|w\|_V=1} |(u_m',v)_H| \le C \tag{4.13}$$

holds where we write $w = v + v^\perp$ with $v \in V_m$, $v^\perp \in V_m^\perp$ and $V_m \oplus V_m^\perp = H$ and use $(u_m',w)_H = (u_m',v)_H$.

The bounds (4.12) and (4.13) show that there exists a subsequence with

$$u_{m'} \rightharpoonup^* u \quad \text{in } L^\infty(0,T;V), \quad u_{m'}' \rightharpoonup^* u' \quad \text{in } L^\infty(0,T;V').$$

We can take the limit $m \to \infty$ in the Galerkin equation

$$\int_0^T \langle u_m',\chi v\rangle_{V'}\mathrm{d}t + i\int_0^T a(u_m,\chi v)\mathrm{d}t = \int_0^T (f,v)_H\chi\mathrm{d}t, \quad v \in V_m,$$

with a piecewise constant function $\chi : (0,T) \to \mathbb{C}$ (since a is linear) and we obtain

$$\int_0^T \langle u',\chi v\rangle_{V'}\mathrm{d}t + i\int_0^T a(u,\chi v)\mathrm{d}t = \int_0^T (f,v)_H\chi\mathrm{d}t$$

for all $\chi : (0,T) \to \mathbb{C}$ and $v \in V$. We can approximate each function in $L^2(0,T;V)$ by $v_N = \sum_{j=1}^N \chi_j v$. Therefore we obtain the claim of the theorem by a density argument. □

We consider the sesquilinear form (4.9) for the Schrödinger equation (4.6). Theorem 4.6 yields the following corollary.

Corollary 4.7 (Existence and uniqueness for the linear Schrödinger equation)**.** *Let $f \in L^2(\Omega \times (0,T))$ with $f_t \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in H_0^1(\Omega)$. Then, there exists a unique solution u of (4.10) with $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u_t \in L^\infty(0,T;H^{-1}(\Omega))$.*

Remark 4.8. *The above theorem on existence for the Schrödinger equation can be generalized.*

(i) The Schrödinger equation is not only solveable on $[0,\infty)$, but on whole $\mathbb{R}$. This can be seen by transforming the time by $t \mapsto -t$ and analysing the equation

$$iu_t - \Delta u = 0.$$

The abstract formulation is

$$-\langle u_t, v\rangle_{V'} + ia(u,v) = 0.$$

By the proof of Theorem 4.6 we observe that the sign of $\langle u_t, v\rangle_{V'}$ is irrelevant. We obtain therefore also a solution on the interval $(-\infty, 0)$.

(ii) It is sufficient to assume initial data $u_0 \in L^2(\Omega)$. Then, there exists a solution $u \in C^0(\mathbb{R};L^2(\Omega))$ with $u_t \in C^0(\mathbb{R},D')$ and $D = H^2(\Omega) \cap H_0^1(\Omega)$. For a proof see Cazenave [5, Proposition 3.1.1].

The Schrödinger equation has some remarkable properties. We have already seen that the solution is defined on whole $\mathbb{R}$. Therefore, the Schrödinger equation is not a diffusion equation; but behaves similarly to a wave equation. To discuss is it more in detail, we consider the Schrödinger equation on the space $\mathbb{R}^n$:

$$iu_t + \Delta u = 0 \quad \text{in } \mathbb{R}^n,\ t \in \mathbb{R}, \quad u(0) = u_0. \tag{4.14}$$

For smooth initial data we can solve the problem explicitely.

Proposition 4.9. *Let $u_0 \in C_0^\infty(\mathbb{R}^n)$. Then, the unique solution of (4.14) is*

$$u(x,t) = \Big(\frac{1}{4\pi i t}\Big)^{n/2} e^{i|x|^2/(4t)} \int_{\mathbb{R}^n} e^{-ix\cdot y/(2t)} e^{i|y|^2/(4t)} u_0(y)\mathrm{d}y,$$

and satisfies the inequality

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \le (4\pi|t|)^{-n/2} \|u_0\|_{L^1(\mathbb{R}^n)}.$$

Proof. The proposition follows by Fourier-Transformation of (4.14). We recall that the Fourier transformation of a function $f \in L^1(\Omega)$ is defined by

$$\hat f(k) = \int_{\mathbb{R}^n} f(x) e^{ik\cdot x}\mathrm{d}x, \quad k \in \mathbb{R}^n.$$

It has the property that $\widehat{\nabla f}(k) = ik\widehat f(k)$ and hence

$$\widehat{\Delta f}(k) = -|k|^2 \widehat f(k).$$

The calculation shows that the Fourier transform of the function

$$K_t(x) = \frac{1}{(4\pi it)^{n/2}} e^{i|x|^2/(4t)}$$

is given by $\widehat{K}_t(k) = \exp(-i|k|^2 t)$.

The Fourier transform of the Schrödinger equation (4.14)

$$i\widehat{u}_t - |k|^2 \widehat{u} = 0, \quad \widehat{u}(0) = \widehat{u}_0, \quad k \in \mathbb{R}^n,$$

can be solved explicitely:

$$\widehat{u}(k,t) = e^{-i|k|^2 t} \widehat{u}_0(k), \quad k \in \mathbb{R}^n. \tag{4.15}$$

This can be written as

$$\widehat{u}(t) = \widehat{K}_t \widehat{u}_0 = \widehat{K_t * u_0},$$

where $K_t * u_0$ is the convolution of K_t and u_0. The inverse of the Fourier transform yields

$$u(x,t) = (K_t * u_0)(x) = \int_{\mathbb{R}^n} K_t(x-y) u_0(y) \mathrm{d}y = \frac{1}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{i|x-y|^2/(4t)} u_0(y) \mathrm{d}y,$$

which shows the first claim. The bound is a consequence of

$$|u(x,t)| \leq \frac{1}{(4\pi|t|)^{n/2}} \int_{\mathbb{R}^n} |u_0(y)| \mathrm{d}y.$$

This concludes the proof of the proposition. □

We remark that the solution of the Schrödinger equation (4.14) is not vanishing in the L^2-norm, but remains preserved:

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \quad \text{for all } t \in \mathbb{R}.$$

(Multiply (4.14) with $\bar{u}$ and integrate, see also the proof of Lemma 4.11.) The solution $|u|^2$ satisfies a conservation equation (similarly as the solution of the wave equation), but decays exponentially fast in the L^∞-norm (similarly as the solution to the heat equation). The conservation property is expressed in the *formal* writting $u(t) = e^{it\Delta} u_0$ where the operator $A = e^{it\Delta} : D(A) \to L^2(\mathbb{R}^n)$ is defined via the spectral representation similarly as the operator $e^{-t\Delta}$ for parabolic equation.

The solution of the wave equation is in general not more regular than the initial data - in contrast to the solution to the parabolic equations, which regularize strongly. Theorem 4.6 shows that the solution $u(t) \in V$ is as regular as the initial data $u_0 \in V$. In this sense, the Schrödinger equation is very similar to the wave equation. But, the Schrödinger equation has a remarkable regularizing property which is expressed by the so-called Strichartz inequalities.

Theorem 4.10 (Strichartz inequalities)**.** *Let $u_0 \in L^2(\mathbb{R}^n)$ and u be a solution to* (4.14)*. Then,*

$$\|u\|_{L^{2+4/n}(\mathbb{R};L^{2+4/n}(\mathbb{R}^n))} \leq C_1 \|u_0\|_{L^2(\mathbb{R}^n)},$$
$$\|u\|_{L^r(\mathbb{R};L^p(\mathbb{R}^n))} \leq C_2 \|u_0\|_{L^2(\mathbb{R}^n)},$$

where $r = 4p/(n(p-2))$, $2 \le p < 2n/(n-2)$ (if $n \le 2$, then $p \le \infty$ is possible) and $C_1, C_2 > 0$.

For a proof we refer to Strauss [21], [16, Section X] and Cazenave [5, Theorem 3.2.5]. The theorem says that the solution satifies for almost all $t \in \mathbb{R}$ the property $u(t) \in L^p$ for certain $p > 2$, although u_0 is only in $L^2(\mathbb{R}^n)$. But, this property holds only for almost all $t \in \mathbb{R}$; in general it can not hold for all $t \neq 0$. Indeed the solution has even a better regularity. If $u_0 \in L^2(\mathbb{R}^n)$, then $u(t) \in H^{1/2}_{\text{loc}}(\mathbb{R}^n)$ for almost all $t \in \mathbb{R}$ (see Cazenave [5, Remark 3.4.7]). Hence, we gain for almost all $t \in \mathbb{R}$ 'half' a derivative local in space.

The existence and uniqueness of solutions can also be proven for the semilinear Schrödinger equation under certain conditions on the non-linearity:

$$iu_t + \Delta u - f(u) = 0 \quad \text{in } \mathbb{R}^n,\ t \in \mathbb{R}, \quad u(0) = u_0. \tag{4.16}$$

In physical applications,

$$f(u) = g(|u|^2)u \quad \text{with } g : \mathbb{R} \to \mathbb{R}$$

is mainly of interest. The real valued function $g(|u|^2)$ can be seen as some kind of self-interaction potential. A typical example is the cubic function $f(u) = \pm|u|^2u$.

The analysis of (4.16) is based on so-called *energy estimates*. We call the functional

$$E(u) = \int_{\mathbb{R}^n} \Big(\frac{1}{2}|\nabla u|^2 + F(u)\Big)\mathrm{d}x, \quad \text{where } F(u) = \frac{1}{2}G(|u|^2),\ G' = g,\ G(0) = 0,$$

the *energy* of the system. The function F is defined such that $F'(u) = g(|u|^2)u = f(u)$. In the case $f(u) = \pm|u|^2u$ it holds $F(u) = \pm\frac{1}{4}|u|^4$. We show next for smooth solutions that the energy is constant.

Lemma 4.11. *Let u be a classical solution of (4.16). Then, it holds*

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}, \quad E(u(t)) = E(u_0) \quad \textit{for all } t \in \mathbb{R}.$$

Proof. We multiply (4.16) by $\bar{u}$ and the complex conjugated equation with u:

$$iu_t\bar{u} + \Delta u\bar{u} - g(|u|^2)|u|^2 = 0, \quad -i\bar{u}_t u + \Delta\bar{u}u - g(|u|^2)|u|^2 = 0,$$

and take the difference of these both equation:

$$i\partial_t|u|^2 + \Delta u\bar{u} - \Delta\bar{u}u = 0.$$

By integration over $\mathbb{R}^n$ and integration by parts it follows

$$i\partial_t \int_{\mathbb{R}^n} |u|^2\mathrm{d}x = 0,$$

which yields the first equation of the lemma.

To show the second equation we multiply (4.16) with $\bar{u}_t$ and the complex conjugated equation with u_t:

$$iu_t\overline{u_t} + \Delta u\overline{u_t} - g(|u|^2)u\overline{u_t} = 0, \quad -i\overline{u_t}u_t + \Delta\overline{u}u_t - g(|u|^2)\overline{u}u_t = 0.$$

Adding both equations and integrating over $\mathbb{R}^n$, we obtain:

$$-\int_{\mathbb{R}^n}(\nabla u\cdot\overline{\nabla u_t}+\overline{\nabla u}\cdot\nabla u_t+g(|u|^2)(u\overline{u_t}+\overline{u}u_t))\mathrm{d}x=0$$

and since $\partial_t F(u)=\frac{1}{2}g(|u|^2)\partial_t|u|^2$ it follows

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x+2\int_{\mathbb{R}^n}F(u)\mathrm{d}x\Big)=2\frac{\mathrm{d}E}{\mathrm{d}t}(u).$$

This yields the claim. □

The energy equation of Lemma 4.11 provides an a-priori estimate, if $F(u)$ is non-negative or at least can be bounded in a suitable way.

Lemma 4.12. *We assume that for all $u\in\mathbb{C}$*

$$F(u)\geq -C_1(|u|^2+|u|^q),\quad 2\leq q<2+4/n,\ C_1>0. \tag{4.17}$$

Further, let u is a classical solution of (4.16). Then, it holds

$$\|\nabla u\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}^n))}\leq C_2,$$

where $C_2>0$ depends on $\|u_0\|_{L^2(\mathbb{R}^n)}$ and $E(u_0)$.

Proof. We use the following special case of the *Gagliardo-Nirenberg inequality*:

$$\|v\|_{L^r(\mathbb{R}^n)}\leq K\|v\|^\theta_{L^2(\mathbb{R}^n)}\|\nabla v\|^{1-\theta}_{L^2(\mathbb{R}^n)},\quad \theta=1-\frac{n(r-2)}{2r},\quad v\in H^1(\mathbb{R}^n),$$

and $r\geq 2$ is such that the embedding $H^1(\mathbb{R}^n)\hookrightarrow L^r(\mathbb{R}^n)$ is continuous. (This ensures $0\leq\theta\leq 1$.)

By Lemma 4.11, (4.17) and the above ineqality, it follows for $r=q$

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)}&=E(u_0)-\int_{\mathbb{R}^n}F(u(t))\mathrm{d}x\leq E(u_0)+C_1(\|u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u(t)\|^q_{L^q(\mathbb{R}^n)})\\&\leq E(u_0)+C_1\|u_0\|^2_{L^2(\mathbb{R}^n)}+K^qC_1\|u_0\|^{\theta q}_{L^2(\mathbb{R}^n)}\|\nabla u(t)\|^{(1-\theta)q}_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The condition $q<2+4/n$ implies $(1-\theta)q=n(q-2)/2<2$. Hence, we can apply the Young inequality:

$$\|u_0\|^{\theta q}_{L^2(\mathbb{R}^n}\|\nabla u(t)\|^{(1-\theta)q}_{L^2(\mathbb{R}^n)}\leq\epsilon\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)}+C_3(\epsilon,\|u_0\|_{L^2(\mathbb{R}^n)}).$$

The first term on the right hand side can be absorbed for sufficiently small $\epsilon>0$:

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^n)}\leq C_4(\epsilon,\|u_0\|_{L^2(\mathbb{R}^n)}).$$

Define the right hand side as C_2 we obtain the claim. □

Using the Strichartz inequalities, the Banach fixed point theorem and the semi group theory, the following theorem can be proven (see Strauss [21]).

Theorem 4.13 (Existence and uniqueness for semilinear Schrödinger equations)**.** *Suppose that the assumptions on F of Lemma 4.12 holds. Further, let $f \in C^1$ with $|f'(u)| \leq C_0|u|^p$ for all $u \in \mathbb{C}$, where $0 < p < 4/(n-2)$ (and $0 < p < \infty$, if $n \leq 2$). Further, let $u_0 \in H^1(\mathbb{R}^n)$ with $E(u_0) < \infty$. Then, there exists a unique solution $u \in C^0(\mathbb{R}; H^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R}^n))$ of (4.16).*

Example 4.14. *We check for an example under which conditions the assumptions of Theorem 4.13 are satisfied.*

(i) Let $f(u) = |u|^2u$. Then, $F(u) = (1/4)|u|^4$. Since F is non-negative, F satisfies the inequality (4.17) for all $n \geq 1$. Whereas the inequality, $p = 2 < 4/(n-2)$ is satisfied for $n \leq 3$.

(ii) Let $f(u) = -|u|^2u$. Then, $F(u) = -(1/4)|u|^4$ and $q = 4$ in (4.17). It holds $4 = q < 2 + 4/n$ if and only if $n = 1$.

The question remains what happens if the conditions of 4.13 are not satisfied. Consider the case $f(u) = -|u|^{4/n}u$. Then, $q = 2+4/n$ and hence the condition (4.17) does never hold. One can show that the function

$$u(x,t) = v\Big(\frac{|x|}{t^* - t}\Big)(4\pi i(t^* - t))^{-n/2}e^{i|x|^2/(4(t^*-t))}, \quad x \in \mathbb{R}^n, 0 \leq t < t^*,$$

is a special solution of (4.16), where $u_0(x) = v(|x|)$ and v is a solution of

$$-\Delta v - (4\pi)^{-2}|v|^{4/n}v = 0 \quad \text{in } \mathbb{R}^n.$$

Conservation of the solution holds in L^2 norm, $\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$, but $u(0,t^*)$ is unbounded in L^∞ and H^1 norm. Therefore, we expect that there does not exists a solution for all non-linearities. In fact, the following result holds (see Cazenave [5, Remark 6.5.1]).

Theorem 4.15 (Existence and Non-existence for nonlinear Schrödinger equations)**.** *Let $f(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{R}$ and $0 < \alpha < 4/(n-2)$ (and $\alpha < \infty$ if $n \leq 2$). Then:*

(i) If $\lambda > 0$, then all solutions of (4.16) are global in time.

(ii) If $\lambda < 0$ and $\alpha < 4/n$, then all solutions of (4.16) are global in time.

(iii) If $\lambda < 0$ and $\alpha \geq 4/n$, then all solutions of (4.16) are only global in time, if $\|u_0\|_{H^1(\mathbb{R}^n)}$ is sufficiently small. For large $\|u_0\|_{H^1(\mathbb{R}^n)}$ solutions exist only local in time.

Note that the value $4/n$ corresponds in (ii) and (iii) of the theorem to the bound $q = 2 + 4/n$ for $F(u)$.

4.3 Hamilton-Jacobi equations

To motivate Hamilton-Jacobi equations we consider the Euler equations which were stated in the first chapter

$$n_t + \operatorname{div}(nv) = 0, \quad (nv)_t + \operatorname{div}(nv \otimes v) + \nabla p = 0.$$

We recall that n is the particle density, v the velocity and p the pressure of the liquid. We assume that the liquid is homogeneous and incompressible, i.e., $n = \text{const.} = 1$, and the pressure is constant, $\nabla p = 0$. Further, we consider *laminar flow*, i.e., the velocity can be written als a gradient of a function S, $v = \nabla S$. Then, we obtain

$$0 = \operatorname{div}(\nabla S) = \Delta S, \quad 0 = (\nabla S)_t + \operatorname{div}(\nabla S \otimes \nabla S).$$

We claim that $\operatorname{div}(\nabla S \otimes \nabla S) = (1/2)\nabla|\nabla S|^2$. This holds by

$$\begin{aligned}\operatorname{div}(\nabla S \otimes \nabla S)_i &= \sum_j \frac{\partial}{\partial x_j}\Big(\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j}\Big) = \sum_j \Big(\frac{\partial^2 S}{\partial x_i \partial x_j}\frac{\partial S}{\partial x_j} + \frac{\partial S}{\partial x_i}\frac{\partial^2 S}{\partial x_j^2}\Big) \\ &= \frac{1}{2}\frac{\partial}{\partial x_i}\sum_j \Big(\frac{\partial S}{\partial x_j}\Big)^2 + \frac{\partial S}{\partial x_i}\Delta S = \frac{1}{2}(\nabla|\nabla S|^2)_i.\end{aligned}$$

Hence,

$$\nabla\Big(S_t + \frac{1}{2}|\nabla S|^2\Big) = 0.$$

The velocity potential S is a solution of the equation

$$S_t + \frac{1}{2}|\nabla S|^2 = \text{const. in } x.$$

This equation is a special form of the *Hamilton-Jacobi equation*

$$u_t + H(x, \nabla u) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \quad u(0) = u_0, \tag{4.18}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous. To solve these equations, one could search for weak solutions, i.e., solutions $u \in L^2(0,T; W^{1,\infty}(\mathbb{R}^n))$ of

$$\int_0^T \langle u_t, v\rangle \mathrm{d}t + \int_0^T \int_{\mathbb{R}^n} H(x, \nabla u) v \mathrm{d}x \mathrm{d}t = 0$$

for all $v \in C_0^\infty(\mathbb{R}^n \times (0,T))$ with $u(0) = u_0$ in $\mathbb{R}^n$. We use functions of $W^{1,\infty}(\mathbb{R}^n)$ instead of $H^1(\mathbb{R}^n)$ as in the previous chapters, to ensure that $H(\nabla u, x)$ is defined. For this notion of solution, there are in general several such solutions. We consider the following example (see Evans [10, Example in Section 3.3.3]).

Example 4.16. *The Hamilton-Jacobi equation*

$$u_t + u_x^2 = 0 \quad \textit{in } \mathbb{R},\ t > 0, \quad u(0) = 0,$$

has the solutions $u_1 = 0$ *and*

$$u_2(x,t) = \min\{0, |x| - t\} = \begin{cases} 0 & : |x| \geq t \\ x - t & : 0 \leq x \leq t \\ -x - t & : -t \leq x \leq 0. \end{cases}$$

Both solutions are Lipschitz continuous and in particular elements in $L^2(0,T; W^{1,\infty}(\mathbb{R}))$. *One can prove that there are infinitely many Lipschitz continuous solutions. Hence, our concept of solutions is too weak so that we loose uniqueness.*

In the following we introduce a new concept for a solution, the so-called *viscosity solution*, which guarantees uniqueness of solutions. We follow Evans [10, Section 10.1]. A possible idea to show existence of solutions to the Hamilton-Jacobi equation (4.18) is to regularize the problem

$$\partial_t u_\epsilon + H(x, \nabla u_\epsilon) - \epsilon \Delta u_\epsilon = 0 \quad \text{in } \mathbb{R}^n,\ t > 0, \quad u_\epsilon(0) = u_0. \tag{4.19}$$

Hence, we obtain a quasilinear parabolic equation which we can solve under appropriate conditions on H. If we can deduce appropriate estimates for u_ϵ which are independent of ϵ, then we can take the limit $\epsilon \to 0$ and obtain in the limit the solution of (4.18). This technique is called *method of the viscosity limit*.

We assume that we can not easily find uniform estimates for ∇u_ϵ (but at most for $\sqrt{\epsilon}\nabla u_\epsilon$). We hope that we can find uniform estimates for u_ϵ in L^∞. If the sequence is in addition uniformly continuous, then we can apply Arzelà-Ascoli and deduce the convergence

$$u_{\epsilon'} \to u \quad \text{uniformly in } K \times (0, T)$$

for a subsequence $(u_{\epsilon'})$ of (u_ϵ) where $K \subset \mathbb{R}^n$ is a compact subset. The limit $u \in C^0(\mathbb{R}^n \times [0, \infty))$ is a possible canditate for a solution of (4.18). Of course, we have to figure out in which sense u solves the equation, since ∇u may not be defined. To do this, we use mainly the maximum principle. This leads to the concept of *viscosity solution*. In the following we motivate this concept.

Let u be a smooth solution of (4.18) and $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$ a test function having the property that $u - v$ has a strictly local maximum at (x_0, t_0), i.e., in the neighbourhood of (x_0, t_0) it holds

$$(u - v)(x_0, t_0) > (u - v)(x, t) \quad \text{for all } (x, t) \neq (x_0, t_0). \tag{4.20}$$

We assume that we have found a sequence of solutions u_ϵ of (4.19) such that $u_\epsilon \to u$ locally uniform in $\mathbb{R}^n \times (0, \infty)$. We claim that there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the difference $u_\epsilon - v$ has a local maximum at (x_ϵ, t_ϵ) and $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ for $\epsilon \to 0$.

To prove this, we note that by (4.20) there exists a ball $B = B_r(x_0, t_0)$ such that

$$\max_{(x,t) \in \partial B} (u - v)(x, t) < (u - v)(x_0, t_0).$$

By the local uniform convergence there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ with

$$\max_{(x,t) \in \partial B} (u_\epsilon - v)(x, t) < (u_\epsilon - v)(x_0, t_0).$$

This means that $u_\epsilon - v$ takes the local maximum at a point (x_ϵ, t_ϵ) in the interior of B. Replacing r by a sequence $r_\epsilon \to 0$ we obtain the claim.

Since $u_\epsilon - v$ has a local maximum at (x_ϵ, t_ϵ), the equation

$$\nabla u_\epsilon(x_\epsilon, t_\epsilon) = \nabla v(x_\epsilon, t_\epsilon), \quad \partial_t u_\epsilon(x_\epsilon, t_\epsilon) = \partial_t v(x_\epsilon, t_\epsilon)$$

and the inequality

$$-\Delta u_\epsilon(x_\epsilon, t_\epsilon) \geq -\nabla v(x_\epsilon, t_\epsilon)$$

have to be satisfied. Hence, we obtain the regularized Hamilton-Jacobi equation (4.19)

$$\begin{aligned}\partial_t v(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla v(x_\epsilon, t_\epsilon)) &= \partial_t u_\epsilon(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla u_\epsilon(x_\epsilon, t_\epsilon)) \\ &= \epsilon \Delta u_\epsilon(x_\epsilon, t_\epsilon) \leq \epsilon \Delta v(x_\epsilon, t_\epsilon).\end{aligned}$$

We take the limit $\epsilon \to 0$ and use $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$:

$$v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \leq 0. \tag{4.21}$$

This inequality is a consequence of the assumption that $u - v$ has a strict local maximum at (x_0, v_0). If this maximum is not strict, we consider the function

$$w_\delta(x, t) = v(x, t) + \delta(|x - x_0|^2 + |t - t_0|^2), \quad \delta > 0.$$

Then, the difference $u - w_\delta$ has a strict local maximum at (x_0, t_0) and we can deduce at above that $\partial_t w_\delta(x_0, t_0) + H(x_0, \nabla w_\delta(x_0, t_0)) \leq 0$. The limit $\delta \to 0$ yields again (4.21).

We have shown:

$$u - v \text{ has a local maximum at } (x_0, t_0) \ \Rightarrow v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \leq 0.$$

Analogously, we can show

$$u - v \text{ has a local minimum at } (x_0, t_0) \ \Rightarrow v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \geq 0.$$

This motivates the following definition:

Definition 4.17. *A bounded uniformly continuous function u is a viscosity solution of the Hamilton-Jacobi equation* (4.18)*, if $u(0) = u_0$ in $\mathbb{R}^n$ is satisfied and if for all $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$ it holds*

(i) If $u - v$ has a local maximum at $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then $v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \leq 0$.

(ii) If $u - v$ has a local minimum at $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$, then $v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \geq 0$.

Remark 4.18. *In a similar way we can define viscosity solutions for stationary equations of the form $H(x, u, \nabla u) = 0$. A bounded uniformly continuous function u is a viscosity solution of this equation if for all $v \in C^\infty(\mathbb{R}^n)$ with $u - v$ having a local maximum (minimum) at x_0 it follows*

$$H(x_0, u(x_0), \nabla v(x_0)) \leq 0 \quad (\geq 0).$$

In the literature often only continuity of u is required, see e.g. Bressan [3]. The Hamilton-Jacobi equation can also be considered on bounded domains with appropriate boundary conditions; the definition of a viscosity solution is then accordingly. It is even possible to define the viscosity solutions for equations of the form

$$H(x, u, \nabla u, D^2 u) = 0,$$

see the work by Crandall, Ishii and Lions[6].

Example 4.19. *As an example we consider the function $H(x,p) = |p| - 1$, and hence the equation*

$$|u_x| - 1 = 0, \quad x \in (-1,1), \quad u(-1) = u(1) = 0$$

(see Evans [10, Problem 10.4]). We claim that $u(x) = 1 - |x|$ is a viscosity solution. For $x_0 \neq 0$ is u continuously differentiable, and the Hamilton-Jacobi equation is exactly solved. Hence, let $x_0 = 0$. We have to show that if $u - v$ has a maximum (minimum) at $x_0 = 0$ for a function $v \in C^\infty(-1,1)$, then $|v_x(0)| - 1 \leq 0$ (≥ 0) is satisfied.

If $u - v$ has a local maximum at $x_0 = 0$, then it holds for all x in a neighbourhood of $x_0 = 0$

$$1 - v(0) = (u - v)(0) \geq (u - v)(x) = 1 - |x| - v(x).$$

This is equivalent to

$$\frac{v(x) - v(0)}{|x|} \geq -1.$$

For $x \to 0$, $x > 0$ it follows $v_x(0) \geq -1$; for $x \to 0$, $x < 0$ we obtain $v_x(0) \leq 1$. Hence in total we obtain $|v_x(0)| \leq 1$.

If $u - v$ has an local minimum at $x_0 = 0$, it follows for sufficiently small $|x|$

$$1 - v(0) = (u - v)(0) \leq (u - v)(x) = 1 - |x| - v(x)$$

or

$$\frac{v(x) - v(0)}{|x|} \leq -1.$$

In the limit $x \to 0$, $x > 0$ (resp. $x < 0$) it holds $v_x(0) \leq -1$ (resp. $v_x(0) \geq 1$), hence $|v_x(0)| \geq 1$. This proves the claim.

We note $u(x) = |x| - 1$ is not a viscosity solution of $|u_x| - 1 = 0$ since $v(x) = -x^2 + 1$. Then, $(u-v)(x) = |x| + x^2 - 2$ has a local minimum at $x_0 = 0$, but $|v_x(x_0)| - 1 = 2|x_0| - 1 = -1$ gives a contradiction. One can show that the function $u(x) = |x| - 1$ is a viscosity solution of $1 - |u_x| = 0$, $x \in (-1,1)$.

If the solution of the Hamilton-Jacobi equation is obtained via the method of the viscosity limit, then it is a viscosity solution by the above arguments. But, we can also obtain the viscosity solution via other techniques. In this case we have to check the property (i) and (ii) of Definition 4.17.

The above definition raises the following questions:

- Are classical solutions of the Hamilton-Jacobi equation also viscosity solutions?
- Is a viscosity solution unique?
- Does a viscosity solution exist for the Hamilton-Jacobi solution?

Under appropriate assumptions all three question can be confirmed. We prove the statement for the first two questions. The proof of existence is very technical and can be found in Evans [10, Section 10.3].

First, we answer the first question. Let $u \in C^1(\mathbb{R}^n \times [0,\infty))$ be a classical solution of (4.18) which is bounded and uniformly continuous. We claim that u is a viscosity solution of (4.18). If v is a smooth function and if $u - v$ has a local maximum at (x_0, t_0), then

$$\nabla u(x_0,t_0) = \nabla v(x_0,t_0), \quad u_t(x_0,t_0) = v_t(x_0,v_0).$$

Then,

$$v_t(x_0,t_0) + H(x_0,\nabla v(x_0,t_0)) = u_t(x_0,t_0) + H(x_0,\nabla u(x_0,t_0)) = 0.$$

A similar argument holds for the local minima. Hence, this proves that classical solutions are viscosity solutions. For the reversed claim we need the following lemma.

Lemma 4.20. *Let $u \in C^0(\mathbb{R}^n)$ be differentiable at $x_0 \in \mathbb{R}^n$. Then, there exists a function $v \in C^1(\mathbb{R}^n)$ with the property*

$$u(x_0) = v(x_0), \quad u - v \text{ has a strict local maximum at } x_0.$$

Proof. The proof is taken from Evans [10, Lemma in Section 10.1.2]. We can assume without restriction that $x_0 = 0$, $u(0) = 0$ and $\nabla u(0) = 0$. Otherwise we consider the function $\tilde{u}(x) = u(x + x_0) - u(x_0) - \nabla u(x_0) \cdot x$ instead of u, which satisfies $\tilde{u}(0) = 0$ and $\nabla \tilde{u}(0) = 0$. A Taylor expansion of u, $u(x) = u(0) + \nabla u(0) \cdot x + o(|x|) = o(|x|)$ for $x \to 0$, shows that there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$u(x) = |x| f(x), \quad f(0) = 0.$$

The function

$$F(r) = \max_{x \in B_r(0)} |f(x)|, \quad r \geq 0,$$

is continuous (since f is continuous), monotonically increasing (since the balls $B_r(0)$ become larger for increasing for radius r), and it holds $F(0) = \max_{x=0} |f(x)| = |f(0)| = 0$.

Using this function we construct the test function

$$v(x) = \int_{|x|}^{2|x|} F(r)\mathrm{d}r + |x|^2, \quad x \in \mathbb{R}^n.$$

We claim that $u - v$ has a strict local maximum at $x_0 = 0$. By the monotonicity of F we can bound v by $|v(x)| \leq F(2|x|)(2|x| - |x|) + |x|^2$. Then, we can deduce $v(0) = 0$ and by $|v(x)|/|x| \leq F(2|x|) + |x|$ also $\nabla v(0) = 0$. By $\nabla |x| = x/|x|$ it holds

$$\nabla v(x) = \frac{2x}{|x|} F(2|x|) - \frac{x}{|x|} F(|x|) + 2x, \quad x \neq 0.$$

Hence v is continously differentiable on $\mathbb{R}^n$. The desired property follows for $x \neq 0$ by

$$\begin{aligned} u(x) - v(x) = |x| f(x) - \int_{|x|}^{2|x|} F(r)\mathrm{d}r - |x|^2 \leq \quad & \underbrace{|x| F(|x|) - \int_{|x|}^{2|x|} F(r)\mathrm{d}r}_{\leq 0, \text{ since } F \text{ is monotonically increasing}} \quad - |x|^2 \\ & \leq -|x|^2 < 0 = u(0) - v(0). \end{aligned}$$

Hence, $u - v$ has a strict local maximum at 0. □

The following theorem shows that differentiable viscosity solutions are solutions in the classical sense.

Theorem 4.21. *Let u be a viscosity solution of (4.18) and let u be differentiable at the point $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Then,*

$$u_t(x_0, t_0) + H(x_0, \nabla u(x_0, t_0)) = 0.$$

Proof. We apply the above lemma to $(x_0, t_0) \in \mathbb{R}^{n+1}$ instead of $x_0 \in \mathbb{R}^n$: There exists a function $v \in C^1(\mathbb{R}^{n+1})$, such that $u(x_0, t_0) = v(x_0, t_0)$ and $u - v$ has a strict local maximum at (x_0, t_0). Let v^ϵ be a C^∞ regularization of v such that $v_\epsilon \to v$ in $C^1(\mathbb{R}^{n+1})$ uniformly in a neighbourhood of (x_0, t_0). Then, $u - v_\epsilon$ has a local maximum at (x_ϵ, t_ϵ) and $(x_\epsilon, t_\epsilon) \to (x_0, t_0)$ for $\epsilon \to 0$. Since u is a viscosity solution, it follows

$$\partial_t v_\epsilon(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla v(x_\epsilon, t_\epsilon)) \leq 0.$$

By the uniform convergence of $\partial_t v_\epsilon$ and ∇v_ϵ it holds in the limit $\epsilon \to 0$

$$v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \leq 0. \tag{4.22}$$

Since $u - v$ has a local maximum at (x_0, t_0) and u is differentiable at this point, it has to hold:

$$\nabla u(x_0, t_0) = \nabla v(x_0, t_0), \quad u_t(x_0, t_0) = v_t(x_0, t_0).$$

Inserting this in (4.22) it follows

$$u_t(x_0, t_0) + H(x_0, \nabla u(x_0, t_0)) \leq 0.$$

Finally we apply the above lemma to $-u$ and we obtain existence of a C^1 function f such that $u - v$ has a local minimum at (x_0, t_0). Similarly as above it follows

$$u_t(x_0, t_0) + H(x_0, \nabla u(x_0, t_0)) \geq 0$$

and hence the claim. □

For the uniqueness result we consider the Hamilton-Jacobi equation (4.18) on $\mathbb{R}^n \times (0, T)$, i.e., on a finite time interval. In the definition of viscosity solutions we have to replace he interval $(0, \infty)$ by $(0, T)$. First, we prove the following lemma.

Lemma 4.22. *Let u be a viscosity solution of (4.18) on $\mathbb{R}^n \times (0, T)$. Let $u - v$ have a local maximum (minimum) at $(x_0, t_0) \in \mathbb{R}^n \times (0, T]$. Then,*

$$v_t(x_0, t_0) + H(x_0, \nabla v(x_0, t_0)) \leq 0 \quad (\geq 0).$$

Proof. If $t_0 < T$, there is nothing to prove, since the inequality corresponds to the definition of the viscosity solution. Let $t_0 = T$ and let $u - v$ have a strict local maximum at (x_0, T). We define

$$w(x, t) = v(x, t) + \frac{\epsilon}{T - t}, \quad x \in \mathbb{R}^n,\ 0 < t < T.$$

For sufficiently small $\epsilon > 0$, $u - w$ has a local maximum at (x_ϵ, t_ϵ) and $(x_\epsilon, t_\epsilon) \to (x_0, T)$ for $\epsilon \to 0$, where $0 < t_\epsilon < T$. Since u is a viscosity solution it follows

$$w_t(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla w(x_\epsilon, t_\epsilon)) \leq 0.$$

This means

$$\begin{aligned} v_t(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla v(x_\epsilon, t_\epsilon)) &\leq v_t(x_\epsilon, t_\epsilon) + \frac{\epsilon}{(T - t_\epsilon)^2} + H(x_\epsilon, \nabla v(x_\epsilon, t_\epsilon)) \\ &= w_t(x_\epsilon, t_\epsilon) + H(x_\epsilon, \nabla w(x_\epsilon, t_\epsilon)) \leq 0. \end{aligned}$$

In the limit $\epsilon \to 0$ we obtain

$$v_t(x_0, T) + H(x_0, \nabla v(x_0, T)) \leq 0$$

and hence the claim. The statement with the local minimum implies the reversed inequality. □

The following theorem is the main statement of our observations.

Theorem 4.23. *Let the function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous in the following sense: There exists a constant $L \geq 0$ such that for all $x, y, p, q \in \mathbb{R}^n$*

$$|H(x,p) - H(x,q)| \leq L|p - q|, \quad |H(x,p) - H(y,p)| \leq L(1 + |p|)|x - y|.$$

Then, there exists at most one viscosity solution of the Hamilton-Jacobi equation (4.18).

Proof. The proof is based on the *method of variable doubling*, i.e., instead of (x_0, t_0) we consider (x_0, y_0, t_0, s_0). Let u and w be two different viscosity solutions of (4.18) with the same initial values. Then,

$$\sigma = \sup_{(x,t) \in \mathbb{R}^n \times [0,T]} (u - w)(x, t) > 0.$$

(We assume without restriction that $\sigma > 0$.) We define for $\epsilon, \lambda \in (0, 1)$

$$f(x, y, t, s) = u(x, t) - w(y, s) - \lambda(t + s) - \epsilon^{-2}(|x - y|^2 + |t - s|^2) - \epsilon(|x|^2 + |y|^2).$$

Using Lipschitz continuity of H we want to obtain a contradiction to the assumption $\lambda > 0$. We need a couple of technical estimates.

1^{st} *estimate:* By definition viscosity solutions are bounded. Therefore there exists (x_0, y_0, t_0, s_0) with

$$f(x_0, y_0, t_0, s_0) = \max_{\mathbb{R}^{2n} \times [0,T]^2} f(x, y, t, s).$$

We can choose $\epsilon, \lambda \in (0, 1)$ small enough such that

$$f(x_0, y_0, t_0, s_0) \geq \max_{\mathbb{R}^n \times [0,T]} f(x, x, t, t) \geq \frac{\sigma}{2}. \tag{4.23}$$

In particular it holds $f(0, 0, 0, 0) \leq f(x_0, y_0, t_0, s_0)$ which is equivalent to

$$\begin{aligned} \lambda(t_0 + s_0) + \epsilon^{-2}(|x_0 - y_0|^2 + |t_0 - s_0|^2) + \epsilon(|x_0|^2 + |y_0|^2) \\ \leq u(x_0, t_0) - w(y_0, s_0) - (u(0, 0) - w(0, 0)). \end{aligned}$$

The inequality shows (because of boundedness of u and w) that

$$|x_0 - y_0| + |t_0 - s_0| \leq C\epsilon, \quad \epsilon(|x_0|^2 + |y_0|^2) \leq C \tag{4.24}$$

with a constant $C > 0$ which is independent of ϵ. The last inequality implies with the Young inequality

$$\epsilon(|x_0| + |y_0|) = \epsilon^{1/4}\epsilon^{3/4}(|x_0| + |y_0|) \leq \epsilon^{1/2} + \epsilon^{3/2}(|x_0|^2 + |y_0|^2) \leq C\epsilon^{1/2}. \tag{4.25}$$

2^{nd} *estimate:* By the defintion of the maximum, we have $f(x_0, y_0, t_0, s_0) \geq f(x_0, x_0, t_0, t_0)$ which is equivalent to

$$\begin{aligned} u(x_0, t_0) - w(y_0, s_0) - \lambda(t_0 + s_0) - \epsilon^{-2}(|x_0 - y_0|^2 + |t_0 - s_0|^2) - \epsilon(|x_0|^2 + |y_0|^2) \\ \geq u(x_0, t_0) - w(x_0, t_0) - 2\lambda t_0 - 2\epsilon|x_0|^2. \end{aligned}$$

Equivalently, the inequality can be formulated as

$$\begin{aligned} &\epsilon^{-2}(|x_0 - y_0|^2 + |t_0 - s_0|^2) \\ &\qquad \leq w(x_0, t_0) - w(y_0, s_0) - \lambda(t_0 + s_0) + 2\lambda t_0 - \epsilon(|x_0|^2 + |y_0|^2 - 2|x_0|^2) \\ &\qquad = w(x_0, t_0) - w(y_0, s_0) + \lambda(t_0 - s_0) + \epsilon(x_0 + y_0) \cdot (x_0 - y_0). \end{aligned}$$

By the uniform continuity of w it follows

$$|w(x_0, t_0) - w(y_0, s_0)| \leq \omega_w(|x_0 - y_0| + |t_0 - s_0|),$$

where ω_w is a continuous function with $\omega_w(r) \to 0$ for $r \to 0$. Because of the bounds (4.24) and (4.25) we obtain

$$\epsilon^{-2}(|x_0 - y_0|^2 + |t_0 - s_0|^2) \leq \omega_w(|x_0 - y_0| + |t_0 - s_0|),$$

and hence

$$|x_0 - y_0| + |t_0 - s_0| \leq o(\epsilon).$$

This improves the first bound in (4.24).

3^{rd} *estimate:* Also u is uniformly continuous, i.e.,

$$|u(x_0, t_0) - u(y_0, s_0)| \leq \omega_u(|x_0 - y_0| + |t_0 - s_0|),$$

where ω_u is a continuous function with $\omega_u(r) \to 0$ for $r \to 0$. By (4.23) $f(x_0, y_0, t_0, s_0) \geq \sigma/2$, hence

$$\begin{aligned} \frac{\sigma}{2} &\leq f(x_0, y_0, t_0, s_0) \leq u(x_0, t_0) - w(y_0, s_0) \\ &= u(x_0, t_0) - u(x_0, 0) + \underbrace{u(x_0, 0) - w(x_0, 0)}_{=0} + w(x_0, 0) - w(x_0, t_0) + w(x_0, t_0) - w(y_0, s_0) \\ &\leq \omega_u(t_0) + \omega_w(t_0) + \underbrace{\omega_w(|x_0 - y_0| + |t_0 - s_0|)}_{=\omega_w(o(\epsilon))}. \end{aligned}$$

Let $\epsilon > 0$ be sufficiently small such that the above inequality implies $\omega_u(t_0) + \omega_w(t_0) \geq \sigma/4$. Then, we can show $t_0 \geq t^* > 0$ for some $t^* > 0$ and analogously $s_0 \geq t^* > 0$ for some $t^* > 0$.

4^{th} *estimate:* By definition f takes it a maximum at (x_0, y_0, t_0, s_0). Then $(x, t) \mapsto f(x, y_0, t, s_0)$ takes a maximum at (x_0, t_0) . By the definition of f this means that $u - v$ has a maximum at (x_0, t_0) where

$$v(x, t) = w(y_0, s_0) + \lambda(t + s_0) + \epsilon^{-2}(|x - y_0|^2 + |t - s_0|^2) + \epsilon(|x|^2 + |y_0|^2).$$

Since u is a viscosity solution of (4.18), by applying Lemma 4.22 it holds

$$0 \geq v_t(x_0,t_0) + H(x_0, \nabla v(x_0,t_0)) = \lambda + 2\epsilon^{-2}(t_0 - s_0) + H(x_0, 2\epsilon^{-2}(x_0 - y_0) + 2\epsilon x_0). \tag{4.26}$$

By a similar argument as above, $(y,s) \mapsto -f(x_0,y,t_0,s)$ has a minimum at (y_0,s_0). Hence, $w - v$ has a minimum at (y_0,s_0) where

$$v(y,s) = u(x_0,t_0) - \lambda(t_0+s) - \epsilon^{-2}(|x_0-y|^2 + |t_0-s|^2) - \epsilon(|x_0|^2 + |y|^2).$$

Also w is a viscosity solution of (4.18) which implies

$$0 \leq v_s(y_0,s_0) + H(y_0, \nabla v(y_0,s_0)) = -\lambda + 2\epsilon^{-2}(t_0 - s_0) + H(y_0, 2\epsilon^{-2}(x_0 - y_0) - 2\epsilon y_0).$$

Taking the difference of these inequalities and (4.26), we obtain

$$2\lambda \leq H(y_0, 2\epsilon^{-2}(x_0 - y_0) - 2\epsilon y_0) - H(x_0, 2\epsilon^{-2}(x_0 - y_0) + 2\epsilon x_0).$$

By assumption H is Lipschitz continuous (in the sense of the condition of the theorem). Then we obtain

$$\begin{aligned} 2\lambda &\leq H(y_0, 2\epsilon^{-2}(x_0 - y_0) - 2\epsilon y_0) - H(y_0, 2\epsilon^{-2}(x_0 - y_0) + 2\epsilon x_0) \\ &\quad + H(y_0, 2\epsilon^{-2}(x_0 - y_0) + 2\epsilon x_0) - H(x_0, 2\epsilon^{-2}(x_0 - y_0) + 2\epsilon x_0) \\ &\leq 2\epsilon L|x_0 + y_0| + L(1 + |2\epsilon^{-2}(x_0 - y_0 + 2\epsilon x_0|)|x_0 - y_0| \\ &\leq C\epsilon(|x_0| + |y_0|) + C|x_0 - y_0|(1 + \epsilon^{-2}|x_0 - y_0| + \epsilon|x_0|). \end{aligned}$$

Above we have proven $\epsilon(|x_0| + |y_0|) \leq C\epsilon^{1/2}$ and $|x_0 - y_0| + |t_0 - s_0| = o(\epsilon)$. Then it holds $\epsilon^{-2}|x_0 - y_0|^2 = o(1)$ and the limit $\epsilon \to 0$ in the above equation yields $|x_0 - y_0| \to 0$ (as $\epsilon \to 0$) and

$$\lambda \leq 0,$$

which is a contradiction to $\lambda > 0$. □

The question regarding the existence of viscosity solutions is more delicate and leads to control problems since the inequalities in the definition of the viscosity solution are related to the optimal conditions of the control theory. We cite here only a special existence result of Evans [10, Section 10.3.4, Theorem 3].

Theorem 4.24 (Hopf-Lax formula as viscosity solution)**.** *Let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex function with $\lim_{|p|\to\infty} H(p)/|p| = \infty$ and let the initial value $u_0 : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous and bounded. Then, there exists a unique viscosity solution to*

$$u_t + H(\nabla u) = 0 \quad \text{in } \mathbb{R}^n,\ t > 0, \quad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^n.$$

We note that the viscosity solution is explicitly given by

$$u(x,t) = \min_{y\in\mathbb{R}^n} \Big(tL\Big(\frac{x-y}{t}\Big) + u_0(y)\Big),$$

where L is the so-called Legendre-Transformation of H:

$$L(q) = \sup_{p\in\mathbb{R}^n} (p\cdot q - H(p)), \quad q \in \mathbb{R}^n.$$

4.4 Logarithmic equation of forth order

In this section we present an example for a forth order equation, namely, the *Derrida-Lebowitz-Speer-Spohn equation* (DLSS equation)

$$u_t + (u(\ln u)_{xx})_{xx} = 0 \quad \text{in } \Omega,\ t > 0, \quad u(0) = u_0, \tag{4.27}$$

on an interval $\Omega = (0, 1)$, with the boundary conditions

$$u(0, t) = u(1, t) = 1, \quad u_x(0, t) = u_x(1, t) = 0, \quad t > 0. \tag{4.28}$$

In a certain sense the equation can be interpreted as a generalization of the heat equation applied to higher derivatives since the last equation can be written as $u_t = u_{xx} = (u(\ln u)_x)_x$. The equation (4.27) is parabolic since it can be formulated as

$$u_t + u_{xxxx} - \left(\frac{u_x^2}{u}\right)_{xx} = 0,$$

and the main part $u_t + u_{xxxx}$ is parabolic. The equation (4.27) describes the limit of a certain probability density in the context of a particle system with spin or the density of electrons in a specific material of a semi conductor.

The equation (4.27) is of particular interest since solutions of non-negative initial values should also be non-negative due to physical reasons. For differential equations of second order this property can be shown via the maximum principle. This is for equations of higher order not possible in general. One can construct a counter example for the equation $u_{xxxx} = f$ with appropriate boundary conditions (see exercises). Via a special technique this is possible to prove non-negativity of solutions to (4.27). In general, there exists no comparison principle.

To prove the existence of non-negative solutions to (4.27)-(4.28), we present two techniques: exponential variable transformation and semi-discretization in time. For the frist technique we write $u = e^w$ with a new function w and reformulate (4.27) to

$$(e^w)_t + (e^w w_{xx})_{xx} = 0.$$

In this formulation $u = e^w > 0$ if $w \in L^\infty$. The function w satisfies the boundary condition $w = 0$ and $w_x = 0$ on $\partial\Omega = \{0, 1\}$. We can look for a weak solution $w \in H_0^2(\Omega)$. In the second technique we solve the semi-continuous equation:

$$\frac{1}{\tau}(e^{w_k} - e^{w_{k-1}}) = -(e^{w_k}(w_k)_{xx})_{xx}, \quad k = 1, \dots, N,$$

where $\tau N = T$ for a given $T > 0$. This is a recursive equation: For given w_{k-1} we look for a solution $w_k \in H_0^2(\Omega)$. The advantage of this equation is its ellipticity and and that we do not have to consider the additional time dependence. The weak formulation is given by

$$\frac{1}{\tau}\int_0^1 (e^{w_k} - e^{w_{k-1}})v\,\mathrm{d}x = -\int_0^1 e^{w_k}(w_k)_{xx}v_{xx}\,\mathrm{d}x \quad \text{for all } v \in H_0^2(\Omega). \tag{4.29}$$

The right hand side defines for given e^{w_k} a bilinear form and allows the application of the Lax-Milgram lemma. We continue as in Jüngel and Pinnau[15].

Proposition 4.25. *Let $u_0 \geq 0$ be measurable and $\int_\Omega (u_0 - \ln u_0)\mathrm{d}x$ be finite. Then, there exists a weak solution $w_k \in H_0^2(\Omega)$ of (4.29).*

Proof. In the following we assume that we can write $u_0 = e^{w_0}$ for a function $w_0 \in L^\infty(\Omega)$. This is possible if $u_0 \in L^\infty(\Omega)$ is strictly positive. More general initial values u_0 can be treated using an approximation argument, which we do not discuss further here.

We use the fixed point theorem of Leray-Schauder. Let $e^{w_{k-1}} \in L^1(\Omega)$. Further let $\bar{w} \in H^1(\Omega)$ and $\sigma \in [0,1]$. Let $w \in H_0^2(\Omega)$ be the unique solution of the linear problem

$$\int_\Omega e^{\bar{w}} w_{xx} v_{xx} \mathrm{d}x = \frac{\sigma}{\tau} \int_\Omega (e^{w_{k-1}} - e^{\bar{w}}) v \mathrm{d}x, \quad v \in H_0^2(\Omega).$$

The left hand side defines a continuous coercive bilinear form $a(w,v)$. This form is coercive since by the continuous embedding of $H^1(\Omega)$ to $L^\infty(\Omega)$ (only in one space dimension!) $\bar{w}$ is pointwise essentially bounded and hence

$$a(w,w) = \int_\Omega e^{\bar{w}} w_{xx}^2 \mathrm{d}x \geq \exp(-\|\bar{w}\|_{L^\infty(\Omega)}) \int_\Omega w_{xx}^2 \mathrm{d}x.$$

Applying twice the Poincaré inequality we obtain

$$\|w\|_{H^2(\Omega)}^2 \leq C_1 \|w_{xx}\|_{L^2(\Omega)}^2 \leq C_2(\bar{w}) a(w,w),$$

i.e., coercivity of a in $H_0^2(\Omega)$. This defines the fixed point operator $S : H^1(\Omega) \times [0,1] \to H^1(\Omega)$, $S(\bar{w}, \sigma) = w$. It holds $S(\bar{w}, 0) = 0$, since for $\sigma = 0$ one has to solve the problem

$$(e^{\bar{w}} w_{xx})_{xx} = 0 \quad \text{in } \Omega, \quad w = w_x = 0 \quad \text{on } \partial\Omega$$

and the unique solution is given by $w = 0$. The continuity of S can be shown using the methods of Chapter 2 and Chapter 3. By the compact embedding $H_0^2(\Omega) \hookrightarrow H^1(\Omega)$ compactness of S follows. It remains to show uniform estimates for all fixed points of S.

Let $w \in H_0^2(\Omega)$ be a fixed point of S. Then, w solves the equation

$$\int_\Omega e^{w} w_{xx} v_{xx} \mathrm{d}x = -\frac{\sigma}{\tau} \int_\Omega (e^{w_{k-1}} - e^{w}) v \mathrm{d}x, \quad v \in H_0^2(\Omega).$$

We use the wisely chosen test function $v = 1 - e^{-w} \in H_0^2(\Omega)$ and obtain by $(1 - e^{-w})_{xx} = e^{-w}(w_{xx} - w_x^2)$

$$\frac{\sigma}{\tau} \int_\Omega (e^{w} - e^{w_{k-1}})(1 - e^{-w}) \mathrm{d}x + \int_\Omega w_{xx}(w_{xx} - w_x^2) \mathrm{d}x = 0. \tag{4.30}$$

The inequality $e^x \geq x + 1$ for all $x \in \mathbb{R}$ yields

$$\begin{aligned} \int_\Omega (e^{w} - e^{w_{k-1}})(1 - e^{-w}) \mathrm{d}x &= \int_\Omega (e^{w} - e^{w_{k-1}} - 1 + \underbrace{e^{w_{k-1} - w}}_{\geq w_{k_1} - w + 1}) \mathrm{d}x \\ &\geq \int_\Omega (e^{w} - w) \mathrm{d}x - \int_\Omega (e^{w_{k-1}} - w_{k_1}) \mathrm{d}x. \end{aligned}$$

For the other integral in (4.30) we use the boundary conditions

$$\begin{aligned} \int_\Omega w_{xx}(w_{xx} - w_x^2) \mathrm{d}x &= \int_\Omega w_{xx}^2 \mathrm{d}x - \frac{1}{3} \int_\Omega (w_x^3)_x \mathrm{d}x = \int_\Omega w_{xx}^2 \mathrm{d}x - \frac{1}{3} \Big(w_x(1)^3 - w_x(0)^3 \Big) \\ &= \int_\Omega w_{xx}^2 \mathrm{d}x, \end{aligned}$$

since $w_x(1) = w_x(0) = 0$. Hence by (4.30)

$$\frac{\sigma}{\tau}\int_\Omega (e^w - w)\mathrm{d}x + \int_\Omega w_{xx}^2 \mathrm{d}x \le \frac{\sigma}{\tau}\int_\Omega (e^{w_{k-1}} - w_{k-1})\mathrm{d}x. \tag{4.31}$$

We use the inquality $e^x - x \ge 1$ to deduce that $\|w_{xx}\|_{L^2(\Omega)} \le C(w_{k-1})$. Via the Poincaré inequality we obtain a uniform estimate for w in $H^2(\Omega)$. By the theorem of Leray-Schauder we obtain the existence of a solution to (4.29).

□

It is indeed possible to show that the solution to (4.29) is unique (see [15]). But, we will not use this result in the following. By solving the recursive estimate (4.31), we obtain directly an estimate which is uniformly in k:

Lemma 4.26. *Let $w_0, \dots, w_N$ be solutions of the recursively given equation (4.31). Then for all $k = 1, \dots, N$ it holds:*

$$\int_\Omega (e^{w_k} - w_k)\mathrm{d}x + \tau\int_\Omega (w_k)_{xx}^2 \mathrm{d}x \le \int_\Omega (e^{w_0} - w_0)\mathrm{d}x.$$

To obtain the solution to the original equation (4.27), we have to take the limit $\tau \to 0$. Since the proof is quite long, we only sketch the most important ideas. Define the piecewise constant in time function

$$w^{(N)}(x,t) = w_k(x), \quad x \in \Omega,\ t \in ((k-1)\tau, k\tau],\ k = 1, \dots, N.$$

Lemma 4.26 shows that $w^{(N)}(t)$ is bounded in $H^2(\Omega)$. Integrating in time $t \in (0,T)$ (which since the function is piecwise constant yields a sum over $k = 1, \dots, N$), we obtain

$$\|w^{(N)}\|_{L^2(0,T;H^2(\Omega))} = \tau\sum_{k=1}^N \|w_k\|_{H^2(\Omega)}^2 \le C_1, \tag{4.32}$$

and $C_1 > 0$ is independent of N and τ. The main difficulty is to obtain an estimate for the time derivative of $w^{(N)}$, to apply the Lemma of Aubin. Unfortunately, the time-derivative of $w^{(N)}$ does not exists (as an element of a Sobolev space), since the function is piecewise constant in t. Further, we only have accesss to the time-discrete derivative of $e^{w^{(N)}}$, but not of $w^{(N)}$. The first problem can be solved by using a refinement of the Lemma of Aubin.

Theorem 4.27. *Let X, B and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Futher, let $1 \le p < \infty$ and $\sigma_\tau : L^p(\tau, T; X) \to L^p(0, T-\tau, X)$, $(\sigma_\tau u)(t) = u(t-\tau)$ be the shift operator. Let (u_τ) be a sequence which is piecewise constant in t and bounded in $L^p(0,T;X)$ and for which there exists a constant $C > 0$ such that*

$$\|u_\tau - \sigma_\tau u_\tau\|_{L^1(\tau,T;Y)} \le C\tau \quad \text{for all } \tau > 0.$$

Then (u_τ) is relatively compact in $L^p(0,T;B)$.

The second problem can be solved by showing in addition to (4.32) an estimate for $e^{w^{(N)}}$. To do this we need the following version of the Gagliardo-Nirenberg inequality.

Lemma 4.28 (Gagliardo-Nirenberg inequality)**.** *Let $\Omega \in \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^1$ ($n \geq 1$), $1 \leq p, q \leq \infty$ and $u \in H^2(\Omega) \cap L^q(\Omega)$. Then for a constant $C > 0$ (independent of u):*

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|u\|^{\theta}_{H^2(\Omega)}\|u\|^{1-\theta}_{L^q(\Omega)},$$

where $\theta \in [0,1]$ is given by

$$1 - \frac{n}{p} = \theta\Big(2 - \frac{n}{2}\Big) - (1-\theta)\frac{n}{q}.$$

We show the following estimate for $e^{w^{(N)}}$.

Lemma 4.29. *It holds for N which is independent of the constant $C > 0$:*

$$\|e^{w^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(\Omega))} + \frac{1}{\tau}\|e^{w^{(N)}} - \sigma_\tau e^{w^{(N)}}\|_{L^{10/9}(\tau,T;H^{-2}(\Omega))} \leq C_2. \tag{4.33}$$

Proof. 1. First we note that for $x \geq 0$ it holds

$$e^x - x \geq 1 + x + \frac{x^2}{2} - x = 1 + \frac{x^2}{2} \geq x$$

(the last inequality is equivalent to $(x-1)^2 + 1 \geq 0$); further for $x < 0$

$$e^x - x \geq -x = |x|.$$

Hence by the estimate of Lemma 4.26

$$C(w_0) \geq \int_\Omega (e^{w_k} - w_k)\mathrm{d}x \geq \int_{\{w_k \geq 0\}} w_k \mathrm{d}x + \int_{\{w_k < 0\}} |w_k|\mathrm{d}x = \int_\Omega |w_k|\mathrm{d}x$$

for all $k \geq 0$. This implies

$$\|w^{(N)}\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{4.34}$$

where $C > 0$ is a generic constant which is independent of N. The same estimate of Lemma 4.26 shows that

$$\|e^{w^{(N)}}\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{4.35}$$

2. We apply the Gagliardo-Nirenberg inequality for $p = \infty$ and $q = 1$:

$$\|w_x^{(N)}\|^{5/2}_{L^{5/2}(0,T;L^\infty(\Omega))} = \int_0^T \|w_x^{(N)}\|^{5/2}_{L^\infty(\Omega)}\mathrm{d}t \leq C\int_0^T \|w^{(N)}\|^{5\theta/2}_{H^2(\Omega)}\|w^{(N)}\|^{5(1-\theta)/2}_{L^1(\Omega)}\mathrm{d}t,$$

where $1 = \theta(2 - 1/2) - (1-\theta) = 5\theta/2 - 1$, hence $\theta = 4/5$. This yields

$$\|w_x^{(N)}\|^{5/2}_{L^{5/2}(0,T;L^\infty(\Omega))} \leq C\|w^{(N)}\|^{1/2}_{L^\infty(0,T;L^1(\Omega))}\int_0^T \|w^{(N)}\|^2_{H^2(\Omega)}\mathrm{d}t \leq C$$

by (4.34) and (4.32). Further, we obtain

$$\begin{aligned}\|(e^{w^{(N)}})_x\|^{5/2}_{L^{5/2}(0,T;L^\infty(\Omega))} &= \int_0^T \Big(\int_\Omega |e^{w^{(N)}} w_x^{(N)}|\mathrm{d}x\Big)^{5/2}\mathrm{d}t \\ &\le \int_0^T \Big(\int_\Omega e^{w^{(N)}}\mathrm{d}x\Big)^{5/2}\|w_x^{(N)}\|^{5/2}_{L^\infty(\Omega)}\mathrm{d}t \\ &\le \sup_{t\in(0,T)}\Big(\int_\Omega e^{w^{(N)}}\mathrm{d}x\Big)^{5/2}\int_0^T \|w_x^{(N)}\|^{5/2}_{L^\infty(\Omega)}\mathrm{d}t \\ &\le \|e^{w^{(N)}}\|^{5/2}_{L^\infty(0,T;L^1(\Omega)}|w_x^{(N)}\|^{5/2}_{L^{5/2}(0,T;L^\infty(\Omega))} \le C\end{aligned}$$

by (4.35) and the above estimate for $w_x^{(N)}$. This proves the first claim.

3. We bound the discrete time derivative of $e^{w^{(N)}}$:

$$\begin{aligned}\tau^{-1}\|e^{w^{(N)}} - \sigma_\tau e^{w^{(N)}}\|^{10/9}_{L^{10/9}(\tau,T;H^{-2}(\Omega))} &= \|(e^{w^{(N)}} w_{xx}^{(N)})_{xx}\|^{10/9}_{L^{10/9}(\tau,T;H^{-2}(\Omega))} \\ &\le \|e^{w^{(N)}} w_{xx}^{(N)}\|^{10/9}_{L^{10/9}(\tau,T;L^2(\Omega))} \\ &\le \int_0^T \|e^{w^{(N)}}\|^{10/9}_{L^\infty(\Omega)}\|w^{(N)}\|^{10/9}_{H^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Applying the Hölder inequality with $p = 9/4$ and $p' = 9/5$, it follows

$$\begin{aligned}\frac{1}{\tau}\|e^{w^{(N)}} - \sigma_\tau e^{w^{(N)}}\|^{10/9}_{L^{10/9}(\tau,T;H^{-2}(\Omega))} &\le \Big(\int_0^T \|e^{w^{(N)}}\|^{5/2}_{L^\infty(\Omega)}\Big)^{4/9}\Big(\int_0^T \|w^{(N)}\|^2_{H^2(\Omega)}\Big)^{5/9} \\ &= \|e^{w^{(N)}}\|^{4/9\cdot 5/2}_{L^{5/2}(0,T;L^\infty(\Omega))}\|w^{(N)}\|^{5/9\cdot 1/2}_{L^2(0,T;H^2(\Omega))} \le C\end{aligned}$$

by (4.34) and (4.35). This proves the lemma. □

We can also apply the lemma of Aubin to $e^{w^{(N)}}$ with $X = W^{1,1}(\Omega)$, $B = L^2(\Omega)$ and $Y = H^{-2}(\Omega)$. Note that the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ is compact in one space dimension. We obtain the existence of a subsequence with

$$e^{w^{(N')}} \to u \quad \text{in } L^{5/2}(0,T;L^2(\Omega)) \quad \text{for } N' \to \infty.$$

Since $N' \to \infty$ it holds $\tau' = T/N' \to 0$. By (4.32) it holds further

$$w^{(N')} \rightharpoonup w \quad \text{in } L^2(0,T;H^2(\Omega)).$$

By (4.33) it follows

$$\frac{1}{\tau}(e^{w^{(N)}} - \sigma_\tau e^{w^{(N)}}) \rightharpoonup y \quad \text{in } L^{10/9}(0,T;H^{-2}(\Omega)).$$

Then $y = u_t$ due to the uniqueness of the limit.

We claim that $u = e^w$ holds. Since $e^{w^{(N')}} \to u$ pointwise almost everywhere it holds $w^{(N')} = \ln e^{w^{(N')}} \to \ln u$ pointwise almost everywhere. This does not say anything about the integrability of $\ln u$. But $w^{(N')} \rightharpoonup w$ in $L^2(0,T;H^2(\Omega))$, i.e., it holds $\ln u = 0 \in L^2(0,T;H^2(\Omega))$. Hence, it holds $u = e^w$ almost everywhere in $\Omega \times (0,T)$ and we have $e^w \in L^{5/2}(0,T;L^2(\Omega))$. Note that we can not deduce $u > 0$, since w is not necessary an element in $L^\infty(\Omega \times (0,T))$ (we only know that $w \in L^2(0,T;L^\infty(\Omega))$).

We can take now the limit $\tau' \to 0$ in the weak formulation (4.29) and obtain a solution to (4.27)-(4.28). The solution is non-negative since by $e^{w^{(N)}} > 0$ we obtain $u = \lim_{N'\to\infty} e^{w^{(N')}} \ge 0$. We summarise:

Theorem 4.30 (Global existence for the DLSS equation)**.** *Let $u_0 \geq 0$ is measurable function with $\int_\Omega (u_0 - \ln u_0) \mathrm{d}x < \infty$ and $T > 0$. Then, there exists a weak solution $u \geq 0$ of (4.27)-(4.28) with*

$$u \in L^{5/2}(0,T;W^{1,1}(\Omega)), \quad u_t \in L^{10/9}(0,T;H^{-2}(\Omega)), \quad \ln u \in L^2(0,T;H_0^2(\Omega)).$$

If $u_0 \geq C > 0$ is strictly positive, one could guess that the weak solution is also positive and not only non-negative. For sufficiently small and sufficiently large $t > 0$ this can be proven. The general case is still open. The difficulty is given by the fact that the maximum principle is not available. We have avoided the problem in the proof of non-negativity by the tranformation $u^{(N)} = e^{w^{(N)}} > 0$. Unfortunately, we loose the strict inequality in the limit $N' \to 0$ and obtain only $u = \lim_{N' \to \infty} \geq 0$. For further results we refer to the work by Jüngel and Matthes [14].

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