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Global convergence of adaptive least-squares finite element methods for nonlinear PDEs

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joint work with Dirk Praetorius

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NumPDEs
Workgroup on Numerics of PDEs

Motivation

Least-squares minimization

Domain $\Omega \subset \mathbb{R}^d$ bounded, polyhedral, Lipschitz, $d \in \mathbb{N}$, Friedrichs const. C_{F}

Right-hand side $F = f - \operatorname{div} \boldsymbol{f} \in H^{-1}(\Omega)$ for $f \in L^2(\Omega)$ and $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$

Linear diffusion problem Find minimizer $(p^\star, u^\star) \in H(\operatorname{div},\Omega) \times H^1_0(\Omega)$ of

$$LS(f, \boldsymbol{f}; p, u) \coloneqq C_{\mathrm{F}}^2 \, \|f + \operatorname{div} p\|^2_{L^2(\Omega)} + \|\boldsymbol{f} + p - \sigma \nabla u\|^2_{L^2(\Omega)}$$

Fundamental equivalence For all $(p, u) \in H(\operatorname{div},\Omega) \times H^1_0(\Omega)$

$$LS(0, \mathbf{0}; p, u) \approx C_{\mathrm{F}}^2 \, \|\operatorname{div} p\|^2_{L^2(\Omega)} + \|p\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$$

Nonlinear least-squares minimization

Nonlinear diffusion problem Find minimizer $(p^\star, u^\star) \in H(\operatorname{div}, \Omega) \times H^1_0(\Omega)$ of

$$N(f, \boldsymbol{f}; p, u) \coloneqq C_{\mathrm{F}}^2 \, \|f + \operatorname{div} p\|^2_{L^2(\Omega)} + \|\boldsymbol{f} + p - \sigma(\nabla u)\|^2_{L^2(\Omega)}$$

Fundamental equivalence For all $(p, u), (q, v) \in H(\operatorname{div}, \Omega) \times H^1_0(\Omega)$

$$\begin{aligned} C_{\mathrm{F}}^2 \, \|\operatorname{div}(p - q)\|^2_{L^2(\Omega)} + \|p - q - (\sigma(\nabla u) - \sigma(\nabla v))\|^2_{L^2(\Omega)} \\ \approx C_{\mathrm{F}}^2 \, \|\operatorname{div}(p - q)\|^2_{L^2(\Omega)} + \|p - q\|^2_{L^2(\Omega)} + \|\nabla(u - v)\|^2_{L^2(\Omega)} \end{aligned}$$

Carstensen, B, Hellwig, Wriggers: *Numer. Math.* 139 (2018)

B, Carstensen, Tran: *Lect. Notes Appl. Comput. Mech.* (2022)

Selection of related literature

Newton-type methods

- [Bochev–Gunzburger, *Appl. Math. Lett.*, 1993] for Navier–Stokes equations
- [Chegini–Stevenson, *CMAM*, 2015] for semilinear PDEs

Gauss–Newton-type methods

- [Müller–Starke–Schwarz–Schröder, *SISC*, 2014] for hyperelasticity
- [Monnesland–Lee–Gunzburger–Yoon, *CAMWA*, 2016] for nonlinear Stokes equations
- [Westphal, *CMAM*, 2019] for Monge–Ampére equation
- [Brenner–Sung–Tan–Zhang, *CAMS*, 2024] for Monge–Ampére equation
- [Bertrand–Schneider, *CAMWA*, 2024] for sea-ice model

Local convergence analysis

- [Bertrand–Brodbeck–Ricken–Schneider, *preprint*, 2025]

Other minimum residual methods

- [Cantin–Heuer, *SINUM*, 2018] on discontinuous Petrov–Galerkin method

Zarantonello linearization

Operator equation Find $x^\star \in X$ with $\mathcal{B}(x^\star) = \mathcal{F}$ in X^*

Linear operator $\mathcal{A}\colon X \to X^*$ associated with scalar product

Zarantonello iteration Given damping $\delta > 0$ and $x^{k-1} \in X$, find $x_\star^k \in X$ with

$$\mathcal{A} x_\star^k = \mathcal{A} x^{k-1} + \delta \left[\mathcal{F} - \mathcal{B}(x^{k-1})\right]$$

- Zarantonello: *Technical Report* 160 (1960)
- Heid, Wihler: *Calcolo* 57 (2020)
- Heid, Wihler: *Math. Comp.* 89 (2020)
- Heid, Praetorius, Wihler: *CMAM* 21 (2021)
- Brunner et al. *IMA JNA* 44 (2024)

Convergence of Zarantonello iteration

Nonlinear operator Assume that $\mathcal{B}\colon X \to X^*$ satisfies, for all $x, y, z \in X$

(MON) strong monotonicity $\alpha\, |||x - y|||_{\mathcal{A}}^2 \leq \langle \mathcal{B}(x) - \mathcal{B}(y), x - y \rangle$

(LIP) Lipschitz continuity $\langle \mathcal{B}(x) - \mathcal{B}(y), z \rangle \leq L\, |||x - y|||_{\mathcal{A}}\, |||z|||_{\mathcal{A}}$

Contraction For $0 < \delta < \delta^\star := 2\alpha/L^2$, there exists $0 < \rho_{\mathrm{Z}} < 1$ such that

$$|||x^\star - x_\star^k|||_{\mathcal{A}} \leq \rho_{\mathrm{Z}}\, |||x^\star - x^{k-1}|||_{\mathcal{A}}$$

Zarantonello: *Technical Report* 160 (1960)

Zarantonello LSFEM

Nonlinear diffusion problem

First-order system of PDE Flux-like variable $p^\star := \sigma(\nabla u^\star) - \boldsymbol{f}$ leads to

$$f + \operatorname{div} p^\star = 0 \quad \& \quad \boldsymbol{f} + p^\star - \sigma(\nabla u^\star) = 0 \iff -\operatorname{div} \sigma(\nabla u^\star) = f - \operatorname{div} \boldsymbol{f}$$

Nonlinear mapping $\sigma \colon \mathbb{R}^d \to \mathbb{R}^d$ Fréchet differentiable

Derivative $\mathrm{D}\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ with, for all $\xi, a, b \in \mathbb{R}^d$

(N1) ellipticity $\quad \Lambda_1 \, |a|^2 \leq (\mathrm{D}\sigma(\xi)a) \cdot a$

(N2) boundedness $\quad |(\mathrm{D}\sigma(\xi)a) \cdot b| \leq \Lambda_2 \, |a| \, |b|$

Zarantonello least-squares formulation

Employ scalar weights $\omega_1,\ \omega_2 > 0$

Linearized first-order system of PDEs

$$
\begin{aligned}
-\omega_1 \operatorname{div} p_\star^k &= -\omega_1 \operatorname{div} p^{k-1} + \delta\omega_1 \left[f + \operatorname{div} p^{k-1}\right]\\
p_\star^k - \omega_2^2 \nabla u_\star^k &= p^{k-1} - \omega_2^2 \nabla u^{k-1} - \delta \left[\boldsymbol{f} + p^{k-1} - \sigma(\nabla u^{k-1})\right]
\end{aligned}
$$

Given (p^{k-1}, u^{k-1}), find minimizer $(p_\star^k, u_\star^k) \in H(\operatorname{div}, \Omega) \times H_0^1(\Omega)$ of

$$
\begin{aligned}
Z_k(f, \boldsymbol{f}; p, u) := C_{\mathrm{F}}^2 \omega_1^2 \, &\| \operatorname{div}(p - p^{k-1}) + \delta \, [f + \operatorname{div} p^{k-1}] \|_{L^2(\Omega)}^2\\
&+ \| p - p^{k-1} - \omega_2^2 \, (\nabla u - \nabla u^{k-1}) + \delta \, [\boldsymbol{f} + p^{k-1} - \sigma(\nabla u^{k-1})] \|_{L^2(\Omega)}^2
\end{aligned}
$$

Zarantonello least-squares formulation

Euler–Lagrange equation For all $(q, v) \in H(\operatorname{div}, \Omega) \times H^1_0(\Omega)$

$$\mathcal{A}(p^k_\star, u^k_\star; q, v) = \mathcal{A}(p^{k-1}, u^{k-1}; q, v) + \delta \left[\mathcal{F}(q, v) - \mathcal{B}(p^{k-1}, u^{k-1}; q, v)\right]$$

$$\begin{aligned}
\mathcal{A}(p, u; q, v) &\coloneqq C_{\mathrm{F}}^2 \omega_1^2 \, (\operatorname{div} p,\ \operatorname{div} q)_{L^2(\Omega)} + (p - \omega_2^2 \, \nabla u,\ q - \omega_2^2 \, \nabla v)_{L^2(\Omega)} \\
\mathcal{B}(p, u; q, v) &\coloneqq C_{\mathrm{F}}^2 \omega_1^2 \, (\operatorname{div} p,\ \operatorname{div} q)_{L^2(\Omega)} + (p - \sigma(\nabla u),\ q - \omega_2^2 \, \nabla v)_{L^2(\Omega)} \\
\mathcal{F}(q, v) &\coloneqq -C_{\mathrm{F}}^2 \omega_1^2 \, (f,\ \operatorname{div} q)_{L^2(\Omega)} - (\boldsymbol{f},\ q - \omega_2^2 \, \nabla v)_{L^2(\Omega)}
\end{aligned}$$

$$|||(p, u)|||^2_{\mathcal{A}} = C_{\mathrm{F}}^2 \omega_1^2 \, \| \operatorname{div} p\|^2_{L^2(\Omega)} + \|p - \omega_2^2 \, \nabla u\|^2_{L^2(\Omega)}$$

Properties of $\mathcal{B}$

Miracle 1 For the choice $\omega_2^2 := \Lambda_2^2/\Lambda_1$ and $\omega_1^2 := 2\omega_2^2/\Lambda_1 = 2\Lambda_2^2/\Lambda_1^2$, the operator $\mathcal{B}$ satisfies (MON) and (LIP) with respect to $|||\cdot|||_{\mathcal{A}}$ and

$$\alpha_{\mathrm{LS}} = \frac{\Lambda_1^2}{8\Lambda_2^2} \qquad L_{\mathrm{LS}} \leq 12 \qquad \Longrightarrow \qquad \delta^\star_{\mathrm{LS}} \geq \frac{\Lambda_1^2}{576\Lambda_2^2}$$

Miracle 2 Every solution $(p^\star, u^\star)$ to the operator equation

$$C_{\mathrm{F}}^2\omega_1^2\,(f + \operatorname{div} p^\star,\ \operatorname{div} q)_{L^2(\Omega)} + (\boldsymbol{f} + p^\star - \sigma(\nabla u^\star),\ q - \omega_2^2\,\nabla v)_{L^2(\Omega)} = 0$$

minimizes the nonlinear LS functional N

🎓 Riveros Neira: Master Thesis. Supervisor: Prof. M. Karkulik (2023)

📄 B, Praetorius: *preprint* (2025)

Discretization

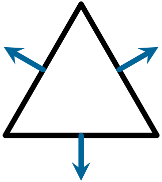
Given (p_h^{k-1}, u_h^{k-1}), the **discrete minimizers** (p_h^k, u_h^k) satisfy, for all (q_h, v_h)

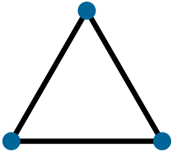
$$\mathcal{A}(p_h^k, u_h^k; q_h, v_h) = \mathcal{A}(p_h^{k-1}, u_h^{k-1}; q_h, v_h) + \delta\,[\mathcal{F}(q_h, v_h) - \mathcal{B}(p_h^{k-1}, u_h^{k-1}; q_h, v_h)]$$

$$p_h^k, q_h \in \mathrm{RT}^m(\mathcal{T}) \coloneqq \left\{ r_h \in H(\mathrm{div}, \Omega) \;:\; \forall T \in \mathcal{T},\ r_h|_T \in \mathrm{P}^m(T; \mathbb{R}^d) + \mathrm{P}^m(T)\,\mathrm{id} \right\}$$

$$u_h^k, v_h \in \mathrm{S}_0^{m+1}(\mathcal{T}) \coloneqq \left\{ w_h \in H_0^1(\Omega) \;:\; \forall T \in \mathcal{T},\ w_h|_T \in \mathrm{P}^{m+1}(T) \right\}$$

$m = 0$



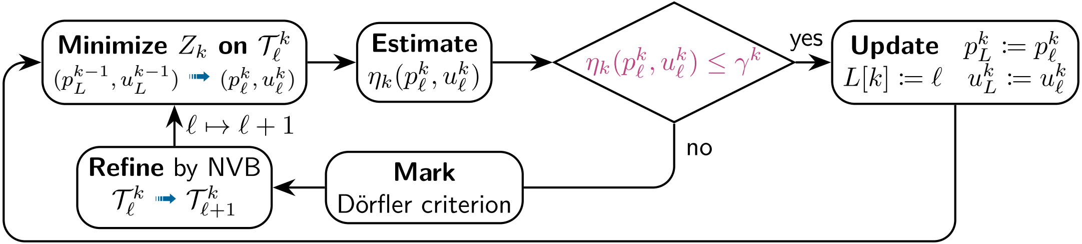


Adaptive algorithm

Adaptive Uzawa-type algorithm

Built-in discretization error estimator $\quad \eta_k(q,v)^2 := Z_k(f,\boldsymbol{f};q,v)$

Input: Initial mesh $\mathcal{T}_0^1 := \mathcal{T}_0$, initial iterates $p_0^0 := p_L^0 \in \mathrm{RT}^m(\mathcal{T}_0^1)$, $u_0^0 := u_L^0 \in \mathrm{S}_0^{m+1}(\mathcal{T}_0^1)$, marking parameter $0 < \theta \leq 1$, reduction parameter $0 < \gamma < 1$



Minimize Z_k **on** $\mathcal{T}_\ell^k$

$(p_L^{k-1}, u_L^{k-1}) \Rightarrow (p_\ell^k, u_\ell^k)$

Estimate

$\eta_k(p_\ell^k, u_\ell^k)$

$\eta_k(p_\ell^k, u_\ell^k) \leq \gamma^k$

yes

Update $\quad p_L^k := p_\ell^k$

$L[k] := \ell \quad u_L^k := u_\ell^k$

no

Mark

Dörfler criterion

Refine by NVB

$\mathcal{T}_\ell^k \Rightarrow \mathcal{T}_{\ell+1}^k$

$\ell \mapsto \ell + 1$

$\mathcal{T}_0^{k+1} := \mathcal{T}_L^k$ (nested iteration), $\quad \ell := 0, \quad k \mapsto k+1$

Output: Sequentially ordered meshes $\mathcal{T}_\ell^k$ with solutions $(p_\ell^k, u_\ell^k) \in \mathrm{RT}^m(\mathcal{T}_\ell^k) \times \mathrm{S}_0^{m+1}(\mathcal{T}_\ell^k)$

Führer, Praetorius: *CMAM* 75 (2018)

Properties of adaptive algorithm

For *any* initial mesh $\mathcal{T}_0$, guess (p_0^0, u_0^0), and parameters $0 < \theta \leq 1$, $0 < \gamma < 1$

Adaptive refinement The inner loop terminates after finitely many steps $L[k]$

R-linear convergence There exist $C_{\mathrm{lin}} > 0$ and $0 < \rho < 1$ such that

$$|||(p^\star - p_L^k, u^\star - u_L^k)||| \approx |||(p^\star - p_L^k, u^\star - u_L^k)|||_{\mathcal{A}} \leq C_{\mathrm{lin}}\, \rho^k$$

Führer, Praetorius: *CMAM* 80 (2020)

Gantner, Stevenson: *M2AN* 55 (2021)

Führer, Praetorius: *CMAM* 75 (2018)

B, Praetorius: *preprint* (2025)

Numerical experiment

Porous media flow

Material parameters $k_1 = 0.2,\ k_2 = 20$

Forchheimer's law

$$-p = \sigma(\nabla u) = \frac{2\,\nabla u}{k_1 + \sqrt{k_1^2 + k_2\,|\nabla u|}} =: \phi(|\nabla u|)\nabla u$$

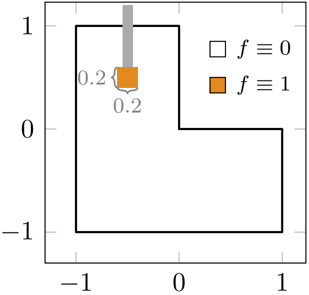
Assumptions For $0 \leq t \leq T := 10^{-2}$

$$0 < \Lambda_1 := \phi(T) + T\phi'(T) \leq \phi(t) + t\phi'(t) \leq k_1^{-1} =: \Lambda_2$$

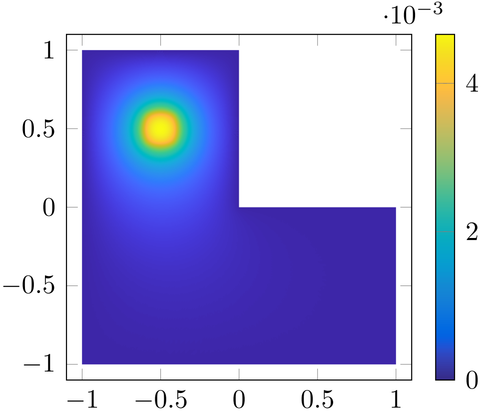
Weights

$\Lambda_1 \approx 1.1835$ $\quad \omega_1^2 = 2\Lambda_2^2/\Lambda_1^2 \approx 35.6971$

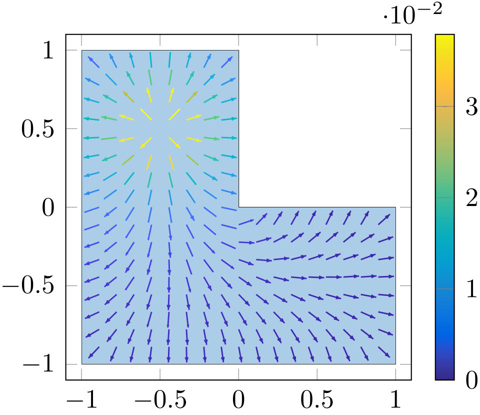
$\Lambda_2 \approx 5$ $\quad \omega_2^2 = \Lambda_2^2/\Lambda_1 \approx 21.1238$



Discrete solution

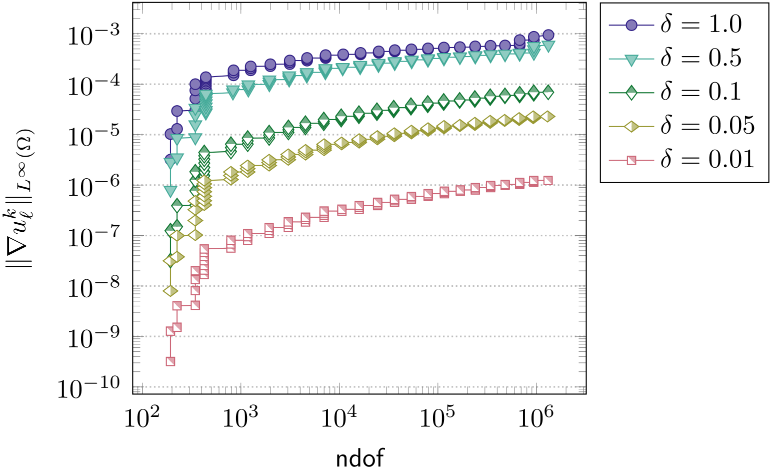


discrete pressure $u_1^{46} \in \mathrm{S}_0^1(\mathcal{T}_1^{46})$



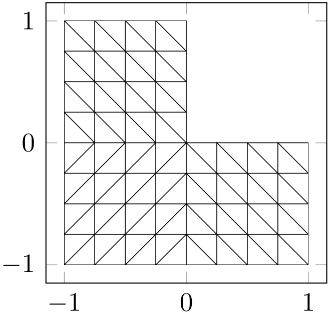
discrete flux $p_1^{46} \in \mathrm{RT}^0(\mathcal{T}_1^{46})$

Uniform bound of pressure gradient



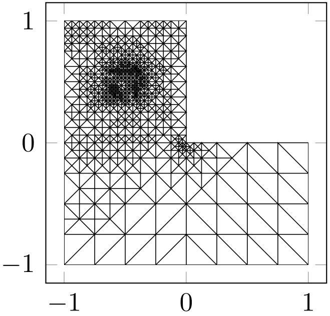
$$\theta = 0.3 \qquad \gamma = 0.9$$

Adaptive meshes



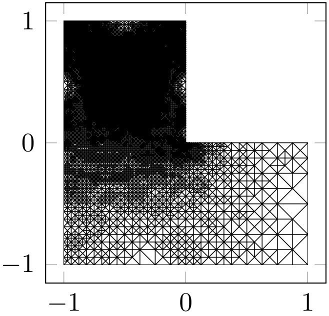
$\mathcal{T}_0^1$

96 triangles



$\mathcal{T}_1^{17}$

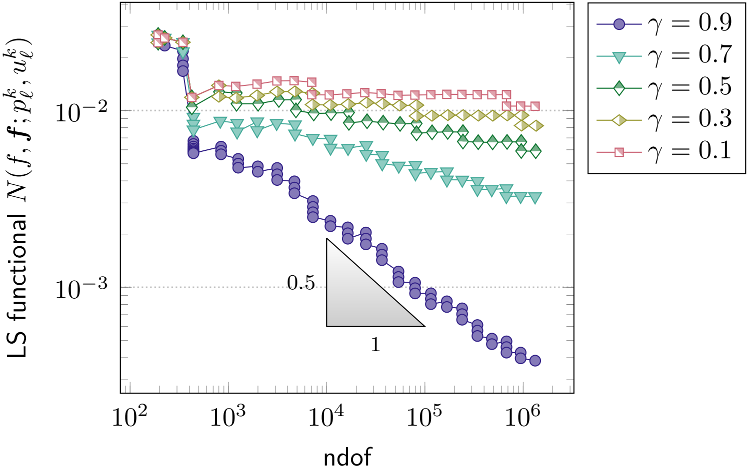
1 568 triangles



$\mathcal{T}_0^{34}$

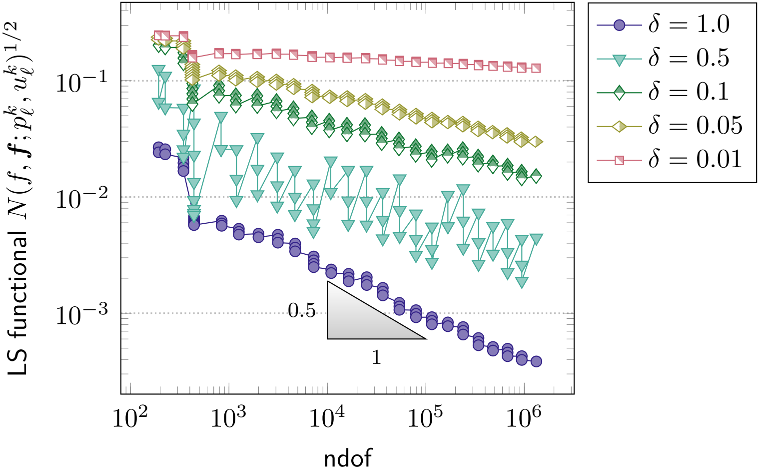
39 717 triangles

Reduction parameter γ



$$\delta = 1 \qquad \theta = 0.3$$

Damping parameter δ



$\theta = 0.3 \qquad \gamma = 0.9$

Conclusion

- ✓ application to Zarantonello-linearized PDEs
- ✓ stabilization and symmetrization
- ✓ flexibility in ansatz spaces
- ✓ independence of scaling of Ω
- ✓ general right-hand side $f - \operatorname{div} \boldsymbol{f} \in H^{-1}(\Omega)$
- ✓ built-in adaptivity with global convergence

Thank you for your attention!

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P. Bringmann and D. Praetorius (2025). *Global convergence of adaptive least-squares finite element methods for nonlinear PDEs*. Preprint. arXiv: 2509.01531

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