Adaptive hp-Finite Element Methods for Two-Dimensional Elasticity with Tresca Friction

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Linear Elasticity with Tresca Friction

In linear elasticity of homogeneous isotropic materials with Tresca friction, the displacement field $\vec{u}(\vec{x})$ minimises the ENERGY FUNCTIONAL

$$\Pi(\vec{v}) := \frac{1}{2}a(\vec{v},\vec{v}) - L(\vec{v}) + j(\vec{v}),$$

where

$$\begin{aligned} a(\vec{v},\vec{w}) &:= \int_\Omega \sigma_{ik}(\vec{v})\varepsilon_{ik}(\vec{w})\mathrm{d}\vec{x}, && \text{(elastic energy)}\\ L(\vec{v}) &:= \int_\Omega F_i v_i \mathrm{d}\vec{x} + \int_{\Gamma_\mathrm{N}} \varphi_i v_i \mathrm{d}s_{\vec{x}}, && \text{(applied forces)}\\ j(\vec{v}) &:= \int_{\Gamma_\mathrm{C}} g|\vec{v}_t|\mathrm{d}s_{\vec{x}}. && \text{(friction)} \end{aligned}$$

Material properties

The SMALL STRAIN TENSOR $\varepsilon_{ij}(\vec{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is related to the STRESS TENSOR (σ_{ij}) by the HOOKE'S LAW

$$\sigma_{ij}(\vec{u}) := \lambda(\operatorname{trace}\varepsilon(\vec{u}))\delta_{ij} + 2\mu\varepsilon_{ij}(\vec{u}),$$

with the LAMÉ CONSTANTS λ and μ.
The friction model is determined by the FRICTION COEFFICIENT g.

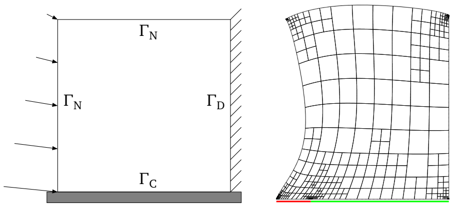
Variational inequality

Using the principle of virtual work, one obtains an equivalent formulation as a variational inequality,

$$a(\vec{u},\vec{v}-\vec{u}) + j(\vec{v}) - j(\vec{u}) \ge L(\vec{v}-\vec{u}),$$

where $\vec{v}$ varies over all admissible displacement fields (satisfying kinematic boundary conditions on Γ_D and contact conditions on Γ_C).

Model Problem: Geometry and Displacement



Red part: sliding on the contact boundary

Green part: sticking on the contact boundary

Primal-Dual Formulation

We have

$$j(\vec{v}) = \int_{\Gamma_\mathrm{C}} g\vec{v}_t \cdot \frac{\vec{v}_t}{|\vec{v}_t|}\mathrm{d}s_{\vec{x}} = \sup_{\vec{\mu}} \int_{\Gamma_\mathrm{C}} g\vec{v}_t \cdot \vec{\mu}\mathrm{d}s_{\vec{x}},$$

where the supremum varies over all functions $\vec{\mu}$ which are defined on the contact boundary and satisfy $|\vec{\mu}| \le 1$.
Defining

$$b(\vec{v},\vec{\mu}) := \int_{\Gamma_\mathrm{C}} g\vec{v}_t \cdot \vec{\mu}\mathrm{d}s_{\vec{x}},$$

the PRIMAL-DUAL FORMULATION reads:
Find an admissible displacement field $\vec{u}$, defined on Ω, and a function $\vec{\lambda}$, defined on the contact boundary Γ_C and bounded by 1, such that

$$\begin{aligned} a(\vec{u},\vec{v}-\vec{u}) + b(\vec{v}-\vec{u},\vec{\lambda}) &\ge L(\vec{v}-\vec{u}), && \text{(1a)}\\ b(\vec{u},\vec{\mu}-\vec{\lambda}) &\le 0, && \text{(1b)} \end{aligned}$$

where $\vec{v}$ varies over all admissible displacement fields and $\vec{\mu}$ over all functions on Γ_C which are bounded by 1.
Often, $\vec{\lambda}$ can be interpreted as the STRESS VECTOR of the displacement field $\vec{u}$.

Discretisation

We discretise Ω by a mesh with elements K, and associate an approximation order p_K with each element. The approximation $\vec{u}_N$ to $\vec{u}$ is elementwise a polynomial of degree p_K. Similarly, the approximation $\vec{\lambda}_N$ to $\vec{\lambda}$ is a piecewise polynomial of degree $p_K - 2$.

hp-Finite Element Method

The three versions of the finite element are

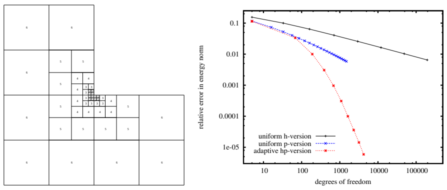
- the h-VERSION FEM, where convergence is obtained through mesh refinement,
- the p-VERSION FEM, where convergence is obtained by increasing the approximation order, and
- the hp-VERSION FEM, which is a combination of both methods.

In practice, the solutions are only piecewise smooth, as they have singularities at corners and transitions between different types of boundary conditions. In the frictional contact problem, singularities also arise at the a priori unknown transition points between sticking and sliding.

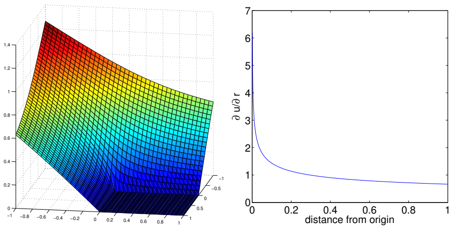
Provided the mesh resolves all singularities, the hp-FEM leads to EXPONENTIAL CONVERGENCE.

Exponential convergence of hp-FEM

We consider a two-dimensional model problem with a singularity at the reentrant corner.



Graded mesh and convergence plot



Solution and directional derivative $\partial u/\partial r$ of solution

hp-FEM for Variational Inequalities

Difficulty

The standard approach in finite element methods is to use CONFORMING METHODS, i. e., the discrete solution is required to be admissible for the continuous problem. For LOWEST ORDER h-VERSION FEM, the non-penetration condition and the condition $|\vec{\mu}_N| \le 1$ can be ensured by requiring them at the nodes of the mesh. For HIGHER ORDER METHODS, however, the pointwise constraint $|\vec{\mu}_N| \le 1$ cannot be enforced by evaluation in a few a priori chosen points.

Solution

Higher order methods thus have to be NON-CONFORMING. Following (Maischak and Stephan 2005), we relax the pointwise constraint by requiring it only at GAUSS QUADRATURE POINTS.

Theorem 1 (Dörsek 2008)**.** *The resulting non-conforming approximation scheme converges.*

hp-FEM for Variational Inequalities

A priori Error Estimate

Theorem 2 (Dörsek 2008)**.** *There exists a constant $C > 0$ such that for the solutions $(\vec{u}_N, \vec{\lambda}_N)$ of the discrete problems and the solution $(\vec{u}, \vec{\lambda})$ of the continuous problem, we have the error estimates*

$$\begin{aligned} \|\vec{u}-\vec{u}_N\|^2_{\mathrm{H}^1(\Omega)} \le C\Big[& b(\vec{u},\vec{\lambda}_N-\vec{\mu}) + b(\vec{u},\vec{\lambda}-\vec{\mu}_N)\\ & + \beta_N^{-1}\left(\|\vec{u}-\vec{v}_N\|^2_{\mathrm{H}^1(\Omega)} + \|\vec{\lambda}-\vec{\mu}\|^2_{\tilde{\mathrm{H}}^{-1/2}(\Gamma_\mathrm{C})}\right)\Big] \end{aligned}$$

and

$$\begin{aligned} \|\vec{\lambda}-\vec{\lambda}_N\|_{\mathrm{H}^{-1\tilde{/}2}(\Gamma_\mathrm{C}} \le {} & C\beta_N^{-1}\times\\ & \times\left(\|\vec{u}-\vec{u}_N\|_{\mathrm{H}^1(\Omega)} + \|\vec{\lambda}-\vec{\mu}_N\|_{\tilde{\mathrm{H}}^{-1/2}(\Gamma_\mathrm{C}}\right), \end{aligned}$$

where

$$\beta_N := \left(p_{N,\max}\log^{3/2} p_{N,\max}\right)^{-1} \qquad (2)$$

and $\vec{v}_N$ and $\vec{\mu}_N$ vary over all discretely admissible and $\vec{\mu}$ over all continuously admissible functions.

The KEY INGREDIENT OF THE PROOF is a novel hp-mortar projection operator.

A posteriori Error Estimation

In adaptivity, error estimators serve two purposes:

- First, they provide a stopping criterion in adaptive algorithms by estimating the error $\|\vec{u}-\vec{u}_N\|$, and
- second, they serve to steer adaptive mesh refinements by identifying elements with large errors.

Error Indicators

We generalise the h-FEM residual error indicator for friction problems developed in (Han 2005) to the hp-FEM using techniques from (Melenk and Wohlmuth 2001).
The LOCAL ERROR INDICATOR for an element K of the mesh with element width h_K and approximation degree p_K is given by

$$\eta_K^2 := h_K^2 p_K^{-2}\|\vec{r}_K\|^2_{\mathrm{L}^2(K)} + h_K p_K^{-1}\sum_{E\subseteq\partial K}\|\vec{R}_E\|^2_{\mathrm{L}^2(E)}.$$

- $\vec{r}_K$ describes the deviation from the equilibrium equations.
- For E AN INNER BOUNDARY, $\vec{R}_E$ measures the jump of the stress vector across E.
- For $E \subseteq \Gamma_\mathrm{C}$, $\vec{R}_E$ measures by how much $\vec{\lambda}$ fails to be the stress vector corresponding to the displacement field $\vec{u}_N$.

Reliability and Efficiency

Theorem 3 (Dörsek 2008)**.** *The error indicator*

- *always provides an upper bound for the true error (*RELIABILITY*), and*
- *provides a lower bound for the true error given that the mesh resolves the transition point on the contact boundary (*EFFICIENCY*).*

hp-Adaptivity

The fundamental question in hp-adaptivity is to decide whether to do an h- or a p-refinement of an element where the error indicator is high.

Marking Strategy

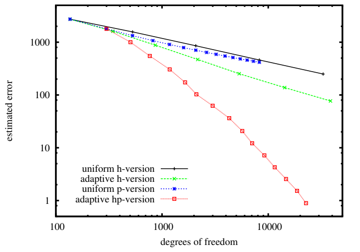
For elements with a large error,

- we p-REFINE those where the solution is smooth, and
- h-REFINE the others.

Smoothness indicator

- We EXPAND the numerical solution into a LEGENDRE SERIES and
- DETERMINE THE DECAY RATE of the coefficients.
- FAST DECAY indicates smoothness.

hp- vs. h-Adaptivity for a Friction Problem



- Adaptive h-FEM is significantly better than uniform h- or p-FEM.
- Only adaptive hp-FEM delivers highly accurate results.

Conclusions

- The presented hp-adaptivity yields exponential convergence.
- It locates the a priori unknown transition points from sticking to sliding on the contact boundary.
- All singularities are resolved.
- A generalisation to three space dimensions is possible.

References

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