wavenumber explicit convergence analysis for the Helmholtz equation: hp-FEM and hp-BEM

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the Helmholtz model problem (FEM)

For a wavenumber $k > 0$ the goal is to find u such that

$$-\Delta u - k^2 u = f \quad \text{in } \Omega \tag{1a}$$
$$\partial_n u - \mathbf{i}ku = g \quad \text{on } \partial\Omega \tag{1b}$$

FACT: (1) uniquely solvable for all $f \in \left(H^1(\Omega)\right)'$, $g \in H^{-1/2}(\partial\Omega)$

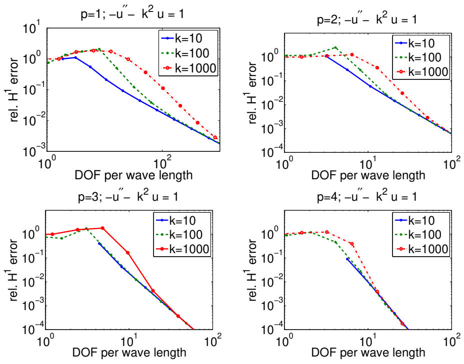
the problem of pollution

FACTS:

- FEM converges ASYMTOTICALLY quasi-optimally
- onset of asymptotic behavior is k-DEPENDENT

NUMERICAL EXAMPLE:

$$-u'' - k^2 u = 1 \quad \text{in } (0,1), \qquad u(0) = 0, \quad u'(1) - \mathbf{i}ku(1) = 0$$



OBSERVATION: high order methods are less prone to pollution

quasioptimality of hp-FEM

Theorem 1. *Let $\partial\Omega$ be analytic. Let $V_N = S^{p,1}(\mathscr{T}_h)$ be the H^1-conforming space of piecewise polynomials of degree p on a quasi-uniform mesh of mesh size h.*
Then there exist $c_1, c_2, C > 0$ independent of h, p, k s.t. for the model problem (1) the following is true:
Under the SCALE RESOLUTION CONDITION

$$\frac{kh}{p} \le c_1 \qquad \text{and} \qquad p \ge c_2 \log k \tag{2}$$

the hp-FEM approximation $u_N \in V_N$ exists and is QUASI-OPTIMAL*:*

$$\|u - u_N\|_{1,k} \le C \inf_{v \in V_N} \|u - v\|_{1,k}$$

$$\text{where} \qquad \|v\|_{1,k}^2 = \|\nabla v\|_{L^2(\Omega)}^2 + k^2 \|v\|_{L^2(\Omega)}^2.$$

UPSHOT: no pollution under the scale resolution condition (2)

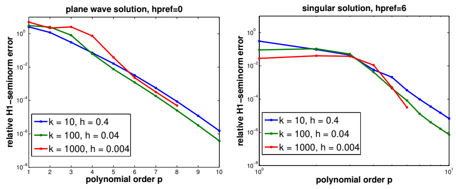
generalizations

Analogs of Theorem 1 are valid for

- POLYGONAL domains, if additionally geometric refinement is performed in an $O(1/k)$-neighborhood of vertices
- EXTERIOR Dirichlet problems, if the DtN-map is realized exactly

numerical examples on L-shaped domain

- **left:** plane wave solution, quasi-uniform meshes with kh =constant
- **right:** solution $J_{2/3}(kr)\cos\frac{2\pi}{3}\varphi$, quasi-uniform mesh with kh =const. augmented by geometric mesh refinement in $O(h)$-neighborhood of origin



OBSERVATION: left graph suggests stability already on quasi-uniform meshes with scale resolution condition (2) without mesh refinement near vertices

the Helmholtz model problem (BEM)

The SOUND SOFT SCATTERING problem is to find u such that for a bounded scatterer $\Omega \subset \mathbb{R}^d$

$$\begin{aligned} -\Delta u - k^2 u &= 0 && \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\ u &= g && \text{on } \partial\Omega \\ u & && \text{satisfies Sommerfeld radiation condition for } |x| \to \infty \end{aligned}$$

combined field integral equation formulation

The normal derivative $\varphi := \partial_n u$ satisfies

$$A\varphi = f \tag{3}$$
$$A := \frac{1}{2}\mathrm{Id} + K_k' - \mathbf{i}kV_k \tag{4}$$

where $f = \left(-D_k + \mathbf{i}kK + \frac{1}{2}\mathbf{i}k\,\mathrm{Id}\right) g$ and V_k, K_k, K_k', and D_k are the single, double, adjoint double layer, and hypersingular operators.

abstract BEM

The second kind integral equation (3) is viewed as posed on $L^2(\partial\Omega)$. For $V_N \subset L^2(\partial\Omega)$ the abstract BEM is: Find $\varphi_N \in V_N$ s.t.

$$\langle A\varphi, v\rangle_{L^2(\partial\Omega)} = \langle f, v\rangle_{L^2(\partial\Omega)} \quad \forall v \in V_N$$

FACTS:

- $\partial\Omega$ smooth $\Longrightarrow$ abstract BEM is asymptotically quasi-optimal
- onset of quasi-optimal convergence may be k-dependent

quasi-optimality of hp-BEM

Theorem 2. *Let $\partial\Omega$ be analytic. Let $V_N = S^{p,0}(\mathscr{T}_h)$ be an $L^2(\partial\Omega)$-conforming space of piecewise polynomials of degree p on a mesh of mesh width h.*
Assume (3) is POLYNOMIALLY WELL-POSED*, i.e., for some C, α*

$$\|A^{-1}\|_{L^2 \leftarrow L^2} \le Ck^{\alpha}. \tag{5}$$

Then there exist constants $c_1, c_2 > 0$ such that under the
SCALE RESOLUTION CONDITION

$$\frac{kh}{p} \le c_1 \qquad \text{and} \qquad p \ge c_2 \log k \tag{6}$$

the hp-BEM is quasi-optimal:

$$\|\varphi - \varphi_N\|_{L^2(\partial\Omega)} \le 2 \inf_{v \in V_N} \|\varphi - v\|_{L^2(\partial\Omega)}$$

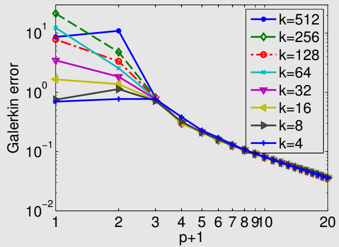
numerical example: hp-BEM

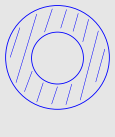
- geometry: annulus (radii of circles: $1/2$, $1/4$)
- $\mathscr{T}_h$ = quasi-uniform mesh with $N = 2k$ elements ($\rightarrow$ mesh size $h \sim 1/k$)
- approximation space: $V_N = S^{p,0}(\mathscr{T}_h)$
- $P_{h,p}$ = Galerkin projection $\varphi \mapsto \varphi_N$
- The Galerkin projection error is approximated as

$$\|\mathrm{Id} - P_{h,p}\| = \sup_{v \in S^{20,0}(\mathscr{T}_h)} \frac{\|(\mathrm{Id} - P_{h,p})v\|_{L^2(\partial\Omega)}}{\|v\|_{L^2(\partial\Omega)}}$$

- Galerkin Error = $\sqrt{\|\mathrm{Id} - P_{h,p}\|^2 - 1}$

circle in circle, r1=1/2, r2 =1/4, η = k





OBSERVATION: quasi-optimality already visible if kh/p is small

k-explicit regularity theory

FACTS:

- The Galerkin method is asymptotically quasi-optimal if applied to problems of the form “coercive + compact perturbation”
- The proof of this fact relies on APPROXIMATION properties of the ansatz space for the solutions of suitable ADJOINT problems
- Approximation properties of piecewise polynomial ansatz spaces are dictated by the REGULARITY of the function to be approximated

UPSHOT: need k-explicit regularity theory for Helmholtz problems

regularity by decomposition: FEM

Theorem 3. *Let $\partial\Omega$ be analytic. Then the solution u of (1) can be decomposed as*

$$u = u_{H^2} + u_A$$

where

$$\begin{aligned} \|u_{H^2}\|_{H^2(\Omega)} &\le C\left\{\|f\|_{L^2} + \|g\|_{H^{1/2}}\right\} \\ \|\nabla^n u_A\|_{L^2(\Omega)} &\le Ck^{3/2}\gamma^n \max\{k,n\}^n \left\{\|f\|_{L^2} + \|g\|_{H^{1/2}}\right\} \quad \forall n \in \mathbb{N}_0 \end{aligned}$$

GENERALIZATIONS:

- Analogous decomposition for polygonal Ω possible

regularity by decomposition: V_k

Theorem 4. *Let $\partial\Omega$ be analytic and choose $q \in (0,1)$. Then:*

$$V_k = V_0 + S_V + \mathscr{A}_V$$

where V_0 is the single layer operator for the Laplacian and:
(i) $S_V : L^2(\partial\Omega) \to H^3(\partial\Omega)$ and

$$\|S_V\|_{L^2 \leftarrow L^2} \lesssim qk^{-1}, \quad \|S_V\|_{H^1 \leftarrow L^2} \lesssim q, \quad \|S_V\|_{H^3 \leftarrow L^2} \lesssim k^2$$

(ii) $\mathscr{A}_V : L^2(\partial\Omega) \to$ space of analytic functions and

$$\|\nabla^n \mathscr{A}_V \varphi\|_{L^2(\partial\Omega)} \lesssim k^{3/2} \max\{k,n\}^n \gamma^n \|\varphi\|_{H^{-3/2}(\partial\Omega)} \quad \forall n \in \mathbb{N}_0$$

GENERALIZATIONS:

- analogous decompositions for K_k and K_k' available
- analogous decompositions in scales of Sobolev spaces available

regularity by decomposition: A^{-1}

Theorem 5. *Let $\partial\Omega$ be analytic. Assume* POLYNOMIAL WELL-POSEDNESS*, i.e., (5). Then: $A^{-1} = A_1 + \mathscr{A}_1$ where*
(i) $A_1 : L^2(\partial\Omega) \to L^2(\partial\Omega)$ with $\|A_1\|_{L^2 \leftarrow L^2} \le C$ independent of k
(ii) $\mathscr{A}_1 : L^2(\partial\Omega) \to$ space of analytic functions with

$$\|\nabla^n \mathscr{A}_1 \varphi\|_{L^2(\partial\Omega)} \le Ck^{\beta} \max\{k,n\}^n \|\varphi\|_{L^2(\partial\Omega)} \quad \forall n \in \mathbb{N}_0$$

for suitable $\beta \ge 0$.

Remark: Analogous decomposition for $(A')^{-1}$ available

essential ingredients for regularity results

- (piecewise) analyticity of the geometry
- polynomial well-posedness of the continuous problem: the norm of the solution operator grows at most algebraically in k.

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