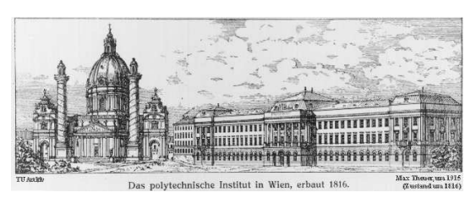

Optimal additive Schwarz methods for the hp-BEM: the hypersingular integral operator

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hypersingular integral operator D

$\Gamma = \partial\Omega =$ boundary of bounded, connected Lipschitz polyhedron

$$\mathrm{D}u := -\partial_{n_x}^{\text{int}} \int_\Gamma \partial_{n_y}^{\text{int}} G(x,y)u(y)\,ds(y), \qquad x \in \Gamma$$

$$G(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}$$

FACTS:

- $\mathrm{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$
- $\ker \mathrm{D} = \operatorname{span}\{1\}$
- D is semi-elliptic: $\langle \mathrm{D}u, u\rangle_\Gamma \geq c|u|^2_{H^{1/2}(\Gamma)}$

COMMON TECHNIQUE TO DEAL WITH SEMI-ELLIPTICITY:

- stabilized hypersingular operator $\widetilde{\mathrm{D}}$ defined by

$$\langle \widetilde{\mathrm{D}}u, v\rangle_\Gamma := \langle \mathrm{D}u, v\rangle_\Gamma + \alpha^2\langle u,1\rangle_\Gamma\langle v,1\rangle_\Gamma, \qquad \alpha > 0$$

where does D arise?

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_n u = g \quad \text{on } \Gamma$$

missing Cauchy datum $u|_\Gamma$ solves

$$\mathrm{D}u = \left(\frac{1}{2} - \mathrm{K}'\right)g, \qquad \mathrm{K}' = \text{adj. double layer operator}$$

in weak form:

$$\langle \mathrm{D}u, v\rangle_\Gamma = \left\langle (\frac{1}{2} - \mathrm{K}')g, v\right\rangle_\Gamma \qquad \forall v \in H^{1/2}(\Gamma).$$

hp-BEM for the hypersingular operator

- $\mathscr{T}_h =$ shape-regular affine meshes consisting of triangles
- $S^{p,1}(\mathscr{T}_h) := \{u \in C^0(\Gamma) \,|\, u|_K \in \mathscr{P}_p(K) \quad \forall K \in \mathscr{T}_h\}$
- Galerkin discretization leads to matrices

$$\mathbf{D}_{ij} = \langle \mathrm{D}\varphi_j, \varphi_i\rangle_\Gamma, \qquad \widetilde{\mathbf{D}}_{ij} = \langle \widetilde{\mathrm{D}}\varphi_j, \varphi_i\rangle_\Gamma,$$

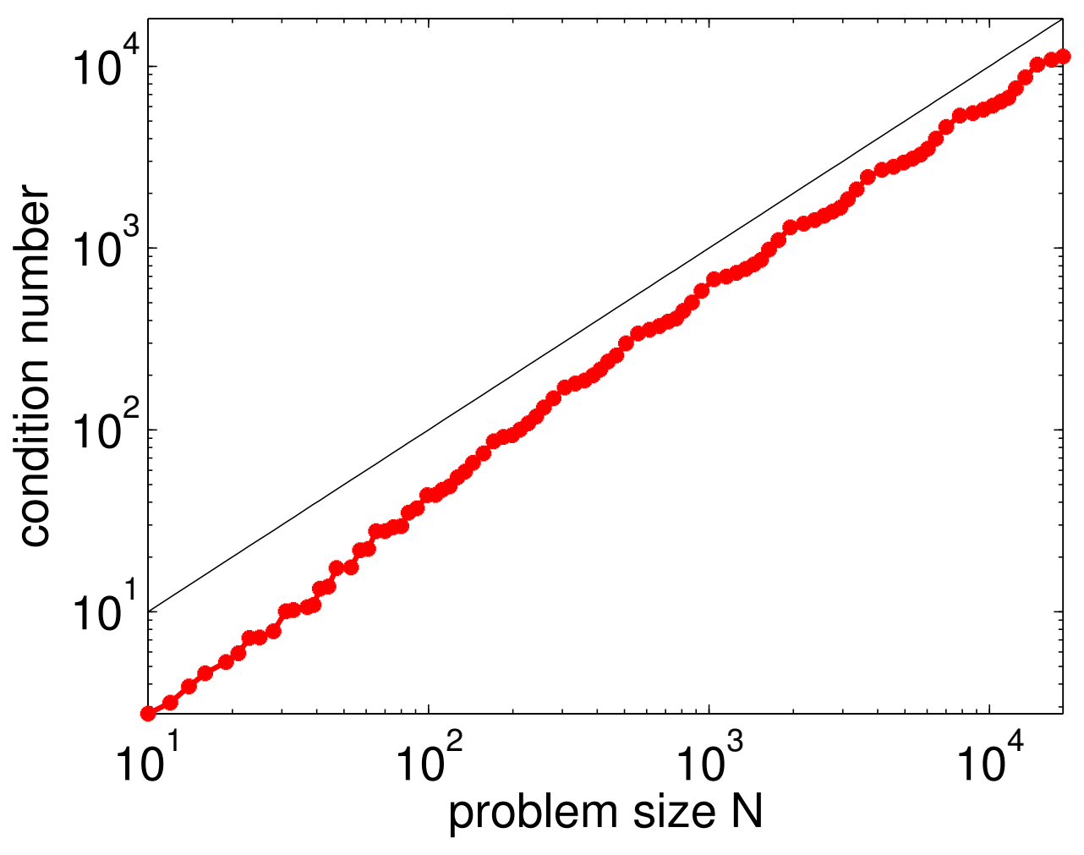
- condition number (quasi-uniform meshes, reasonable basis):

$$\operatorname{cond}\widetilde{\mathbf{D}} \sim \frac{p^2}{h}$$

$\Longrightarrow$ $\boxed{\text{preconditioning for iterative solution?}}$

numerical example: h-BEM

- 2D L-shaped domain
- adaptively refined mesh
- stabilized hypersingular operator



computation: T. Führer (TU Wien)

additive Schwarz preconditioning

- $\mathscr{V} =$ set of nodes
- $\omega_V :=$ patch associated with node $V \in \mathscr{V}$
- $V_h^1 := S^{1,1}(\mathscr{T}_h) =$ global space
- $V_V^p := \{u \in S^{p,1}(\mathscr{T}_h) \,|\, \operatorname{supp} u \subset \overline{\omega_V}\} =$ local spaces, $\qquad V \in \mathscr{V}$
- space decomposition

$$S^{p,1}(\mathscr{T}_h) = V_h^1 + \sum_{V\in\mathscr{V}} V_V^p$$

- induced preconditioner $\mathbf{B}^{-1}$ for $\widetilde{\mathbf{D}}$:
  - $R_h^T : V_h^1 \to S^{p,1}, \quad R_V^T : V_V^p \to S^{p,1}$ are natural embeddings;
  - $\widetilde{\mathbf{D}}_h =$ stabilized hypersingular matrix for V_h^1
  - $\widetilde{\mathbf{D}}_V =$ stabilized hypersingular matrix for V_V^p
  - action of preconditioner:

$$\mathbf{B}^{-1} := R_h^T \widetilde{\mathbf{D}}_h^{-1} R_h + \sum_{V\in\mathscr{V}} R_V^T \widetilde{\mathbf{D}}_V^{-1} R_V$$

optimal additive Schwarz preconditioning

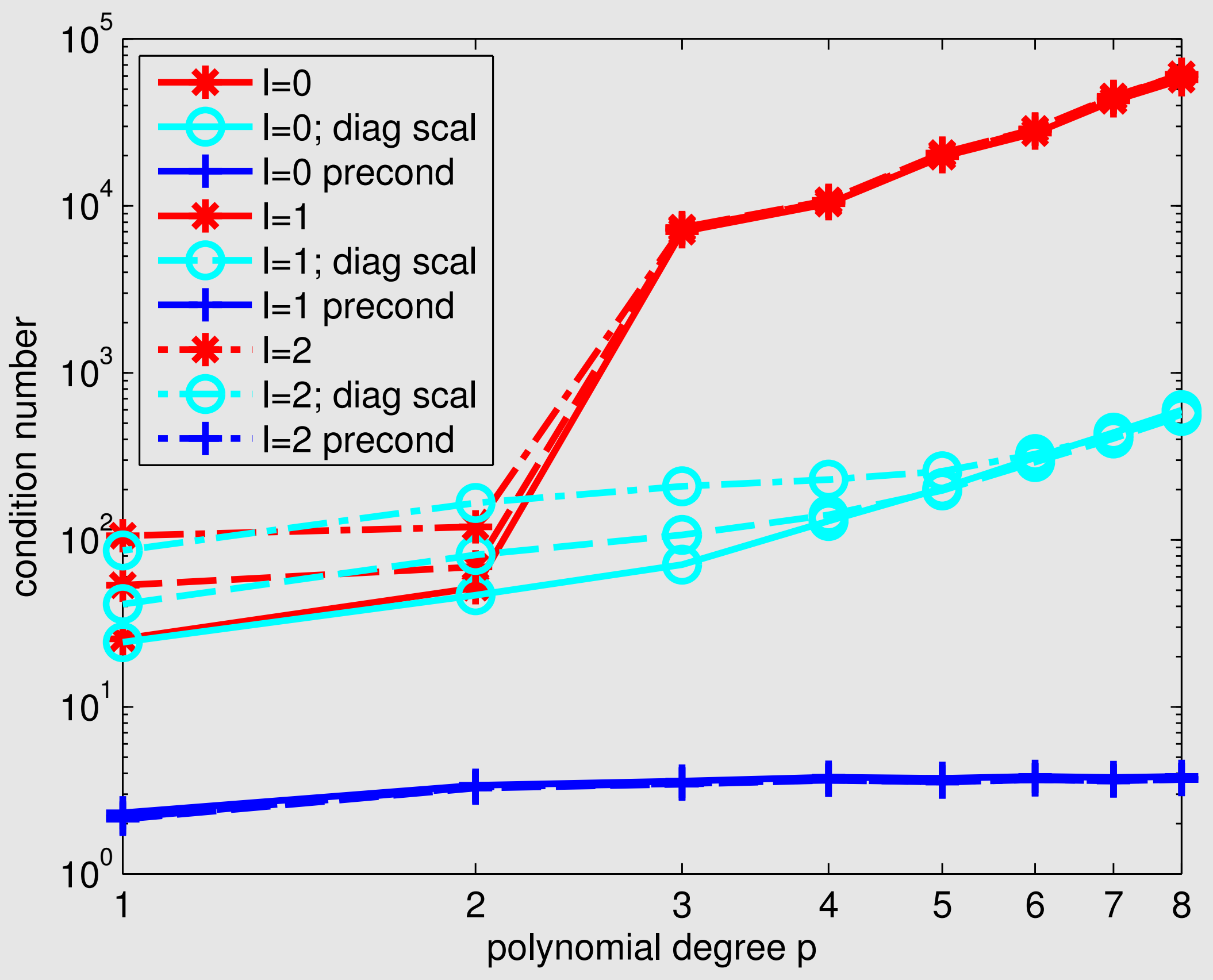
Theorem 1. $\operatorname{cond}(\mathbf{B}^{-1}\widetilde{\mathbf{D}}) \leq C$ *uniformly in h and p.*

REMARKS:

- quadrilateral elements → (Heuer 1999) exploiting *simultaneous* stability in L^2 and H^1 of Gauß-Lobatto interpolation of polynomials
- new technique needed for triangles, similar to (Schöberl, Melenk, Pechstein, and Zaglmayr 2008)
- Direct solver on V_h^1 could be replaced with multilevel preconditioners, e.g., (Feischl, Führer, Praetorius, and Stephan 2013)

numerical example: hp-BEM

- geometry: cube, side length 1
- stabilized hypersingular operator ($\alpha = 0.1$)
- meshes with 48 ($\ell = 0$), 192 ($\ell = 1$), 768 ($\ell = 2$) triangles
- hierarchical basis of (Karniadakis and Sherwin 1999)



$H^{1/2}$-stable space decompositions

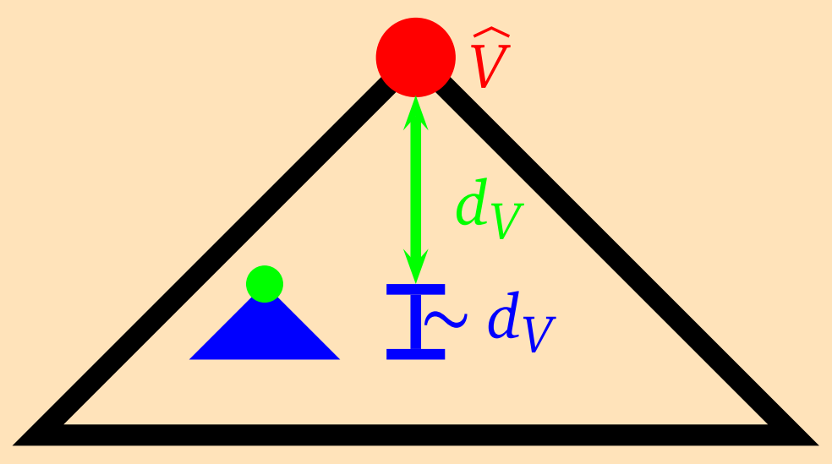
Theorem 2. *Every $u \in S^{p,1}(\mathscr{T}_h)$ can be written as $u = u_0 + \sum_{V\in\mathscr{V}} u_V$, where $u_0 \in S^{1,1}(\mathscr{T}_h)$ and $u_V \in V_V^p$. Furthermore,*

$$\begin{aligned} \|u_0\|_{H^{1/2}(\Gamma)} &\leq C\|u\|_{H^{1/2}(\Gamma)} \\ \sum_{V\in\mathscr{V}} |u_V|^2_{H^{1/2}(\omega_V)} + h_V^{-1}\|u_V\|^2_{L^2(\omega_V)} &\leq C\|u\|^2_{H^{1/2}(\Gamma)} \end{aligned}$$

REMARKS:

- analogous result in $H^s(\Gamma)$, $s \in (0,1)$.

vertex part: averaging operators $\mathscr{A}^V$ and $\mathscr{A}_0^V$



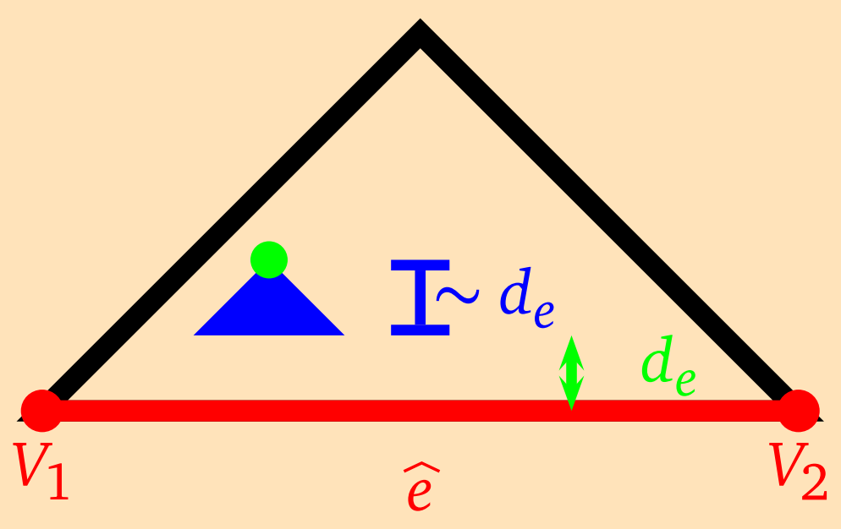
FACTS:

- $\mathscr{A}^V u(V) = u(V)$ if u is sufficiently smooth
- $\mathscr{A}^V$ stable in L^2 and H^1
- $\mathscr{A}_0^V$ stable in L^2 and H^1
- $\|{d_V}^{-1}(u - \mathscr{A}^V u)\|_{L^2} \lesssim \|\nabla u\|_{L^2}$

modified vertex part operator $\mathscr{A}_0^V$

$$\mathscr{A}_0^V u := \mathscr{A}^V u - \text{ correction to enforce 'zero' opposite vertex } \widehat{V}$$

edge part: averaging operators $\mathscr{A}^e$ and $\mathscr{A}_0^e$



FACTS:

- $\mathscr{A}^e u|_{\widehat{e}} = u|_{\widehat{e}}$ if u is sufficiently smooth
- $\mathscr{A}^e$ stable in L^2 and H^1

modified edge part operator $\mathscr{A}_0^e$

$$\begin{aligned} \mathscr{A}_0^e u &:= \mathscr{A}^e u - \text{ correction to enforce 'zero' opposite edge } \widehat{e} \\ \|\mathscr{A}_0^e u\|_{L^2} &\lesssim \|u\|_{L^2}, \\ \|\mathscr{A}_0^e u\|_{H^1} &\lesssim \|u\|_{H^1} + \|{d_{V_1}}^{-1} u\|_{L^2} + \|d_{V_2}^{-1} u\|_{L^2} \end{aligned}$$

interpolation norms of polynomial spaces

Theorem 3. *Set*
$X_0 = (S^{p,1}(\mathscr{T}_h), \|\cdot\|_{L^2(\Gamma)})$,
$X_1 = (S^{p,1}(\mathscr{T}_h), \|\cdot\|_{H^1(\Gamma)})$.

Then $\qquad \|u\|_{(X_0,X_1)_{\theta,2}} \sim \|u\|_{H^\theta(\Gamma)} \qquad \forall u \in S^{p,1}(\mathscr{T}_h)$

REMARKS:

- h-version: use stable projections, (Arioli and Loghin 2009)
- p-version single element: (Bernardi, Dauge, and Maday 2007)
- proof relies on Theorem 4, a lifting in weighted spaces, and the characterization of interpolation spaces as trace spaces

refined $H^{1/2}$-stable decomposition

- $\mathscr{E} =$ set of edges of triangulation
- edge patches $\omega_e =$ elements that have e as an edge
- vertex patch space V_V: $\{u \in S^{p,1} \,|\, \operatorname{supp} u \subset \overline{\omega}_V,\ u$ is "rotationally symmetric" on $\omega_V\}$
- edge patch space V_e: $\{u \in S^{p,1} \,|\, \operatorname{supp} u \subset \overline{\omega}_e,\ u$ is symmetric w.r.t. $e\}$
- element space V_K: $\{u \in S^{p,1} \,|\, \operatorname{supp} u \subset \overline{K}\}$

Theorem 4. *Every $u \in S^{p,1}(\mathscr{T}_h)$ can be written as $u = u_0 + \sum_{V\in\mathscr{V}} u_V + \sum_{e\in\mathscr{E}} u_e + \sum_{K\in\mathscr{T}_h} u_K$ where*

$$\begin{aligned} \|u_0\|_{H^{1/2}(\Gamma)} &\leq C\|u\|_{H^{1/2}(\Gamma)} \\ \sum_{V\in\mathscr{V}} |u_V|^2_{H^{1/2}(\omega_V)} + h_V^{-1}\|u_V\|^2_{L^2(\omega_V)} &\leq C\|u\|^2_{H^{1/2}(\Gamma)} \\ \sum_{e\in\mathscr{E}} |u_e|^2_{H^{1/2}(\omega_e)} + h_e^{-1}\|u_e\|^2_{L^2(\omega_e)} &\leq C\|u\|^2_{H^{1/2}(\Gamma)} \\ \sum_{K\in\mathscr{T}_h} |u_K|^2_{H^{1/2}(K)} + h_K^{-1}\|u_K\|^2_{L^2(K)} &\leq C\|u\|^2_{H^{1/2}(\Gamma)} \end{aligned}$$

REMARKS:

- Decomposition is even well-defined on L^2 (not just $S^{p,1}(\mathscr{T}_h)$)
- Analogous results in $H^s(\Gamma)$, $s \in (0,1)$

space decomposition

- Decomposition is defined on $L^2(\Omega)$ and stable in Sobolev spaces
- If u is in $S^{p,1}(\mathscr{T}_h)$, then the terms of the decompositions are in V_V, V_e, and V_K

Decomposition is obtained in several steps:

1. h-depencence:
   - $u_0 =$ (piecewise linear) Clément/Scott-Zhang interpolant
   - set $u_1 := u - u_0$
2. vertex part:
   - for each ω_V pick an element $K_V \subset \omega_V$
   - define u_V on K_V by $\mathscr{A}_0^V u_1$
   - define u_V on ω_V "by rotation" (and by zero outside ω_V)
   - set $u_2 := u_1 - \sum_{V\in\mathscr{V}} u_V$
3. edge part:
   - for each ω_e pick an element $K_e \subset \omega_e$
   - define u_e on K_e by $\mathscr{A}_0^e u_2$
   - define u_e on ω_e "by symmetry " (and by zero outside ω_e)
   - set $u_3 := u_2 - \sum_{e\in\mathscr{E}} u_e$
4. element part: $u_K = u_3|_K$ for all $K \in \mathscr{T}_h$

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