ETH Swiss Federal
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Seminar for
Applied
Mathematics **SAM**

High Order Finite Element Methods: Theory & Implementation

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Robust Exponential Convergence for Singularly Perturbed Problems

Model Problem on curvilinear polygons

$$-\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon = f \quad \text{on } \Omega, \quad u_\varepsilon|_{\partial\Omega} = 0 \tag{1}$$

Theorem: On meshes with one layer of needle elements of width $O(p\varepsilon)$ in the Boundary Layer and geometric refinement in an $O(p\varepsilon)$ neighborhood of the vertices (cf. Fig. 1), the *hp*-FEM converges at a robust exponential rate, i.e., $\exists\ C$, $b > 0$ independent of ε s.t.

$$\|u_\varepsilon - u_{FE}\|_\varepsilon := \|u_\varepsilon - u_{FE}\|_{L^2(\Omega)} + \varepsilon\|\nabla(u_\varepsilon - u_{FE})\|_{L^2(\Omega)} \le Ce^{-bp}.$$

Numerical Example: Meshes

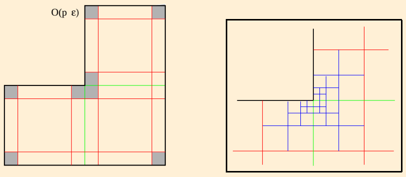
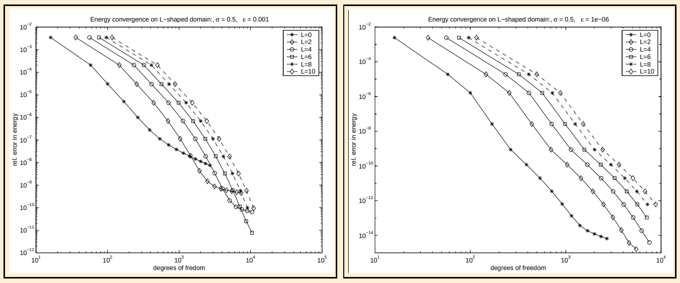
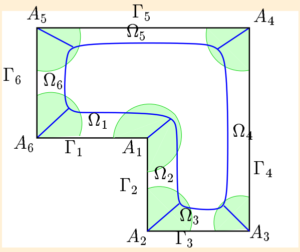


Fig. 1: *meshes with one layer of needle elements & geom. refinement near vertices*

Numerical Example: Energy Convergence for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$



Analytic Regularity



Theorem: Let Ω be a curvilinear polygon, f be analytic on $\overline{\Omega}$ and u_ε be the solution of (1). Then there are C, γ, $\alpha > 0$, $\beta \in [0,1)$ s.t. for each $\varepsilon \in (0,1]$ there is a decomposition of the form

$$u_\varepsilon = w_\varepsilon + \chi\, u_\varepsilon^{BL} + \widehat{\chi}\, u_\varepsilon^{CL} + r_\varepsilon$$

where χ is supported by a neighborhood of $\partial\Omega$, $\widehat{\chi}$ is supported by a neighborhood of the vertices, and w_ε, u_ε^{BL}, u_ε^{CL}, r_ε satisfy:

1. w_ε is analytic on $\overline{\Omega}$ and $\|\nabla^p w_\varepsilon\|_{L^\infty(\Omega)} \le C\gamma^p p!, \qquad \forall p \in N_0$.
2. remainder $r_\varepsilon \in H_0^1(\Omega)$ and $\|r_\varepsilon\|_\varepsilon \le C\exp(-\alpha/\varepsilon)$.
3. On Ω_j the boundary layer function u_ε^{BL} satisfies in boundary fitted coordinates
$$|\partial_{\rho_j}^t \partial_{\theta_j}^s u^{BL}(\rho_j, \theta_j)| \le C\max\{t!, \varepsilon^{-r}\}\gamma^{t+s} s!\, e^{-\alpha\rho_j/\varepsilon}, \quad \rho_j \ge 0.$$
4. The corner layer u_ε^{CL} satisfies on $(\Omega_j \cap B_j) \cup (\Omega_{j+1} \cap B_j)$
$$|u^{CL}(x)| \le C\exp(-\alpha r_j/\varepsilon), \qquad |\nabla^p u^{CL}(x)| \le C\varepsilon^{\beta-1}\, r_j^{1-p-\beta}\exp(-\alpha r_j/\varepsilon)$$
$(B_j = \text{neighborhood of } A_j,\ r_j = \operatorname{dist}(x, A_j))$

Theorem: Let Ω be a curvilinear polygon, f analytic on $\overline{\Omega}$. Then there are C, $K > 0$, $\beta \in [0,1)$ depending only on f, $\partial\Omega$ such that for all $0 < \varepsilon \le 1$

$$\|u_\varepsilon\|_\varepsilon \le C, \qquad \|\Phi_{p,\beta,\varepsilon}\nabla^{p+2}u_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^{-2}K^p \max\{p!, \varepsilon^{-p}\} \qquad \forall p \in N_0,$$
$$\Phi_{p,\beta,\varepsilon}(x) = \min\{1, (r/z)^{p+\beta}\}, \quad r = \min_j \operatorname{dist}(x, A_j), \quad z = \min\{1, \varepsilon(p+1)\}.$$

Efficient *hp*-FEM Implementation

joint with Prof. C. Schwab & K. Gerdes (Chalmers University of Technology)

Point of departure: Main computational cost of *hp*-FEM is the quadrature in the generation of the local stiffness matrices.

Requirements for quadrature

1. Robustness: allow for overintegration to handle distorted elements, parameter-dependent problems (e.g., shells), vector-valued problems
2. Efficiency: allow variable/anisotropic polynomial degree distribution to minimize number of unknows

Key Ingredients of Spectral Galerkin Algorithm

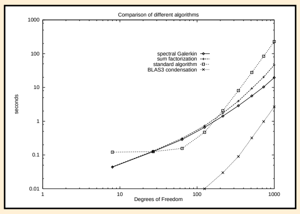
1. efficient forms of sum factorization (as in spectral methods)
2. adapt shape functions to the quadrature rule
3. can be formulated for all typical element types: quadrilaterals, triangles, hexahedra, tetrahedra, prisms, wedges.

Fig. 2: *Work estimates in dependence on spectral order p and amount of overintegration $q \ge 0$:*

work	2D	3D
standard	$p^4(p+q)^2$	$p^6(p+q)^3$
sum fact.	$p^4(p+q)$	$p^6(p+q)$
Spec. Gal.	$p^4(1+q)$	$qp^6 + p^5$

Numerical Example: CPU time for 3D Poisson Problem

Fig. 3: *CPU times to generate stiffness matrix on hexahedron for 3-D Poisson problem: Standard Algorithm, Sum Factorization, Spectral Galerkin, $p = 1, \dots, 9$.*



Operator Adapted Methods: The Partition-of-Unity Method

joint with I. Babuška, University of Texas at Austin

Key Features of the Partition of Unity Method

1. provides a framework for constructing ansatz spaces that are adapted to the differential operator under consideration
2. the resulting ansatz spaces are conforming
3. generalizes classical h-, p-, and *hp*-FEM

Example: Helmholtz equation at large wave numbers

$$-\Delta u - k^2 u = 0, \quad \text{on } \Omega = (0,1)^2, \quad \partial_n u + iku = g \quad \text{on } \partial\Omega$$

local approximation spaces: plane waves of order p:

$$V^p = \left\{\exp\left(\frac{2\pi i}{p} j\right), j = 0, \dots, p-1\right\}$$

partition of unity: standard bilinear hat-functions

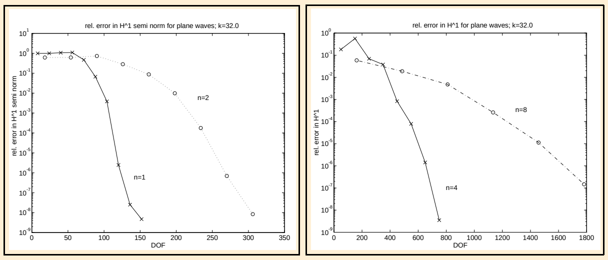


Fig. 4: *H^1 convergence for $k = 32$ and different number of patches*