Entropy–entropy dissipation techniques for nonlinear higher-order PDEs

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1. Introduction

The goal of these lecture notes is to introduce in some aspects of entropy–entropy dissipation techniques. These techniques are used in order to understand the structure of nonlinear (higher-order) partial differential equations and the qualitative behavior of their solutions. In particular, we will detail a method recently developed in collaboration with Daniel Matthes (Pavia, Italy). Our approach is a more algebraic one; there exist also more geometric view points, see, for instance, the works of Carrillo, Otto, Savaré and many others.

We have in mind the following partial differential equations:

- Porous-medium equation:

$$u_t = \Delta(u^m), \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \Omega,$$

  where $\Omega = \mathbb{R}^d$ is the whole space or $\Omega = \mathbb{T}^d$ is the d-dimensional torus $\mathbb{T}^d \sim [0, L)^d$, $L > 0$, and $m > 1$. The solution $u(x, t)$ describes, for instance, the density of an unsaturated groundwater flow in a porous medium. If $0 < m < 1$, this equation is called the fast-diffusion equation. References: Vazquez, Bénilan/Crandell 1981.

- Thin-film equation:

$$u_t + \text{div}\, (u^\beta \nabla \Delta u) = 0, \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \Omega,$$

  where $\beta > 0$. The solution describes, for instance, the thickness of a thin film. The case $\beta = 1$ models the flow in a Hele-Shaw cell; $\beta = 3$ models the viscous flow on a solid surface without slip driven by surface tension. References: Bertozzi, Pugh, Grün, Garcke, dal Passo, Otto...

- Derrida-Lebowitz-Speer-Spohn (DLSS) equation:

$$u_t + \text{div}\, \left(u \nabla \left(\frac{\Delta \sqrt{u}}{\sqrt{u}}\right)\right) = 0, \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \Omega.$$

  The solution u is the limit of a random variable related to the deviation of the interface of a two-dimensional spin system from a straight line, derived by Derrida, Lebowitz, Speer, and Spohn in 1994. It also arises in quantum semiconductor modeling. Here, u is the electron density. References: Carrillo, Jüngel, Pinnau, Toscani, Unterreiter...

In order to explain what we mean by an entropy–entropy dissipation technique, we consider a simple example, the heat equation:

$$u_t = \Delta u, \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{T}^d.$$

It is well known that for integrable u_0, there exists a smooth nonnegative solution satisfying $\int u(x,t)dx = \int u_0(x)dx =: \bar{u}$ for all $t > 0$. Set $\bar{v} = \bar{u}/\text{vol}(\mathbb{T}^d)$. We introduce the following functionals which we call *entropies*:

$$E_2(t) = \int_{\mathbb{T}^d} (u - \bar{u})^2 dx, \quad E_1(t) = \int_{\mathbb{T}^d} u \log\left(\frac{u}{\bar{v}}\right) dx.$$

Observe that E_1 is nonnegative. Indeed, since $x \log(x/y) - x + y \geq 0$ for all x, $y \geq 0$, we obtain

$$0 \leq \int_{\mathbb{T}^d} \left(u \log\left(\frac{u}{\bar{v}}\right) - u + \bar{v}\right) dx = \int_{\mathbb{T}^d} u \log\left(\frac{u}{\bar{v}}\right) dx - \int_{\mathbb{T}^d} u\, dx + \int_{\mathbb{T}^d} \bar{v}\, dx = E_1.$$

We claim that both functionals E_1 and E_2 are nonincreasing in time. First we consider E_2. We obtain, using integration by parts,

$$\frac{dE_2}{dt} = 2 \int_{\mathbb{T}^d} (u - \bar{u}) u_t\, dx = 2 \int_{\mathbb{T}^d} (u - \bar{u}) \Delta u\, dx = -2 \int_{\mathbb{T}^d} |\nabla u|^2 dx \leq 0.$$

The expression on the right-hand side, $\int |\nabla u|^2 dx$, is called the *entropy dissipation* corresponding to E_2. This term allows to conclude more than the monotonicity of E_2. For this, we need the Poincaré inequality

$$\int_{\Omega} (u - \bar{u})^2 dx \leq C_P \int_{\Omega} |\nabla u|^2 dx \quad \text{for all } u \in H^1(\Omega).$$

For convex domains Ω, the constant satisfies $C_P \leq (\text{diam}(\Omega)/\pi)^2$ (Payne/Weinberger 1960); for $\Omega = \mathbb{T}$, the optimal constant is $C_P = (L/2\pi)^2$. The Poincaré inequality relates the entropy E_2 to the entropy dissipation. Then we conclude that

$$\frac{dE_2}{dt} \leq -2C_P^{-1} E_2.$$

By Gronwall's lemma (or just integrating the above differential inequality),

$$E_2(t) \leq E_2(0) e^{-2t/C_P}.$$

Hence, the solution u converges in the L^2 norm exponentially fast to the steady state $\bar{u}$ with rate $1/C_P$.

Next, we compute the derivative of E_1:

$$\frac{dE_1}{dt} = \int_{\mathbb{T}^d} \left(\log\left(\frac{u}{\bar{v}}\right) + 1\right) u_t\, dx = -\int_{\mathbb{T}^d} \nabla\left(\log\left(\frac{u}{\bar{v}}\right) + 1\right) \cdot \nabla u\, dx = -4 \int_{\mathbb{T}^d} |\nabla\sqrt{u}|^2 dx.$$

Again, we need an expression relating the entropy E_1 and the entropy dissipation. This is done by the logarithmic Sobolev inequality

$$\int_{\Omega} u \log \frac{u}{\bar{v}} dx \leq C_L \int_{\Omega} |\nabla\sqrt{u}|^2 dx \quad \text{for all } \sqrt{u} \in H^1(\Omega), \ u \geq 0.$$

If $\Omega = \mathbb{T}$, the constant C_L equals $L^2/2\pi^2$ (Rothaus 1980, Weissler 1980, or Dolbeault/Gentil/Jüngel 2006). Then we have

$$\frac{dE_1}{dt} \leq -4C_L^{-1}E_1 \quad \text{and} \quad E_1(t) \leq E_1(0)e^{-4t/C_L}.$$

The solution converges in the "entropy norm" exponentially fast to its steady state with rate $4/C_L$.

Sometimes, one is rather interested in the convergence of the solution in a Lebesgue norm, for instance in the L^1 norm. The above estimate allows to derive such a conclusion by means of the Csiszár-Kullback inequality:

$$\int_\Omega \left(u \log\left(\frac{u}{v}\right) - u + v \right) dx \geq C_K \left(\int_\Omega |u - v| dx \right)^2 \quad \text{for all } u, v \in L^1(\Omega), \ u, v \geq 0.$$

By Bartier/Dolbeault/Illner/Kowalczyk 2006, the constant $C_K \leq 1/4 \max\{\|u\|_{L^1}, \|v\|_{L^1}\}$. Therefore,

$$\|u - \bar{u}\|_{L^1(\mathbb{T}^d)} \leq \sqrt{\frac{E_1(0)}{C_K}} e^{-2t/C_L}.$$

The above example shows that the entropy–entropy dissipation method consists of the following ingredients:

- entropy functional,
- entropy–entropy dissipation inequality (depending on the PDE under consideration),
- study of the long-time behavior: relation between the entropy and entropy dissipation.

In the following section we will specify which kind of entropies are of interest. An important step is the derivation of the entropy–entropy dissipation inequality, which was easy in the case of the heat equation, but which is rather involved for the (fourth-order) equations mentioned above. An algorithmic construction method will be presented in section 3. Section 4 is concerned with additional results. We conclude in section 5 with some open problems.

2. Definitions and explanations

Let $u(x,t)$ be a nonnegative solution to a PDE for $x \in \Omega$, $t > 0$, and let E and P be functionals, defined by u and its derivatives. We call (E, P) an entropy–entropy production pair if and only if

$$(1) \qquad\qquad \frac{dE}{dt} + P \leq 0 \quad \text{and} \quad E, P \geq 0.$$

We are interested in the following entropies:

- Zeroth-order entropies:

$$E_\alpha = \frac{1}{\alpha(\alpha - 1)} \int_\Omega u^\alpha dx, \quad \alpha \notin \{0, 1\},$$

$$E_1 = \int_\Omega (u(\log u - 1) + 1)dx, \quad \alpha = 1,$$

$$E_0 = \int_\Omega (u - \log u)dx, \quad \alpha = 0.$$

- First-order entropies:

$$\tilde{E}_\alpha = \int_\Omega |\nabla(u^{\alpha/2})|^2 dx, \quad \alpha \neq 0.$$

Clearly, higher-order entropies can be also defined but we will study only entropies of zeroth or first order. The functional E_1 represents (up to a sign) the physical entropy. The functionals E_α give rise to estimates in the Lebesgue space L^α; $\tilde{E}_2$ provides a gradient estimate in L^2, whereas $\tilde{E}_1$ is called the Fisher information.

Examples for entropy productions are, in one space dimension,

$$P = \int_\Omega (u^{\alpha/2})_x^2 dx, \quad P = \int_\Omega (u^{\alpha/2})_{xx}^2 dx \quad \text{etc.}$$

In the following we explain which kind of informations can be drawn from inequalities like (1).

1. Long-time behavior. Let u be a solution of a PDE and (E, P) be an entropy–entropy dissipation pair satisfying $dE/dt + c_1 P \leq 0$, where $c_1 > 0$ is a constant. If there is a relation between the entropy and the entropy production of the form $E \leq c_2 P$ for some constant $c_2 > 0$ (like the Poincaré or logarithmic Sobolev inequality; see the previous section), we derive

$$\frac{dE}{dt} + \frac{c_1}{c_2} E \leq 0.$$

Thus, by Gronwall's lemma, $E(t) \leq E(0)e^{-c_1 t/c_2}$ and we have an exponential decay with rate c_1/c_2. In some situations (for instance, whole-space problems without confinement), we can only expect to obtain inequalities like $E^\gamma \leq c_2 P$ for some $\gamma > 1$ (see Carrillo/Dolbeault/Gentil/Jüngel 2006 for some examples) such that

$$\frac{dE}{dt} + \frac{c_1}{c_2} E^\gamma \leq 0.$$

Hence, after integration, $E(t)$ behaves like $t^{-1/(\gamma-1)}$ for $t \to \infty$. This gives an algebraic decay rate.

2. Existence of solutions. An existence proof of a nonlinear parabolic PDE of the type $u_t = F(u, \nabla u, \ldots)$ with some initial and boundary conditions may have the following steps. First, approximate the PDE appropriately. An example is to replace the derivative u_t by $(u(t) - u(t - \triangle t))/\triangle t$ and to solve the sequence of elliptic problems

$$u(t) - (\triangle t)F(u(t), \nabla u(t), \ldots) = (\triangle t)u(t - \triangle t),$$

where $z = u(t - \triangle t)$ is given. Second, define a fixed-point operator by "linearizing" the elliptic problem, $S : B \to B$, $w \mapsto u$, for instance,

$$u - \Delta u + \big((\triangle t)F(w, \nabla w, \ldots) - \Delta w\big) = (\triangle t)z.$$

If the existence of a fixed point will be proved by the Leray-Schauder fixed-point theorem, we need appropriate uniform estimates for all fixed points. We recall the fixed-point theorem.

> *Leray-Schauder's fixed-point theorem:* Let B be a Banach space, $S : B \times [0,1] \to B$ compact (i.e. $S(B_1(0))$ is a compact set, where $B_1(0) = \{x \in B : \|x\|_B < 1\}$) such that $S(x,0) = 0$ for all $x \in B$ and
>
> $$\exists c > 0 : \forall x \in B, \ \sigma \in [0,1], \ x = S(x, \sigma) : \|x\|_B \leq c.$$
>
> Then $x \mapsto S(x, 1)$ has a fixed point.

Third, let us suppose that for a given fixed point of S, i.e. a solution of the PDE under consideration (maybe including σ), we are able to prove an entropy–entropy dissipation inequality

$$(2) \qquad\qquad E(t) + \int_0^t P(s)ds \leq E(0).$$

Usually, the production term P contains derivatives (see above). The inequality should satisfy two requirements: (i) the estimate on $\int_0^t P ds$ should imply a bound for $\|u\|_X$ in a Banach space X and (ii) X is compactly embedded in B. Then the Leray-Schauder fixed-point theorem applies. In general, the crucial step of the existence proof is the estimate (2).

3. Regularity of solutions. For instance, in the case of the DLSS equation with initial data $\sqrt{u_0} \in H^1(\Omega)$, it is possible to prove for a solution u that (see Dolbeault/Gentil/Jüngel 2006)

$$\int_\Omega (\sqrt{u})_x^2(t)dx + c \int_0^t \int_\Omega \big((\sqrt{u})_{xx}^2 + (\sqrt{u})_{xxx}^2\big)dx \leq \int_\Omega (\sqrt{u_0})_x^2 dx.$$

Thus, since $u \in L^\infty(0, T; L^1(\Omega))$, $\sqrt{u} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega))$. This provides (slightly) more regularity than the definition of a weak solution to the DLSS equation.

4. Positivity of solutions. The proof of positivity is usually shown by means of the maximum principle. However, for higher-order equations, this principle generally does not apply, and other techniques have to be employed. The entropy–entropy dissipation inequalities can be helpful for some PDEs. The proof of positivity depends much on the studied PDE so that we present only an example, the 1D thin-film equation. Let the initial datum $u_0 \in H^1(0, 1)$ be positive, $\frac{7}{2} \leq \beta \leq 5$, and let u be a solution to

$$u_t + (u^\beta u_{xxx})_x = 0 \quad \text{in } (0, 1), \ t > 0, \quad u_x = u^\beta u_{xxx} = 0 \text{ for } x \in \{0, 1\}, \quad u(\cdot, 0) = u_0 > 0.$$

Beretta/Bertsch/dal Passo 1995 have shown that $u \in C^{1/8}([0, T]; C^{1/2}([0, 1]))$ and that for all $\frac{3}{2} \leq \alpha + \beta \leq 3$,

$$(3) \qquad \int_0^1 u^\alpha(x, t)dx \quad \text{is finite for all } t > 0.$$

(We will prove the second statement in section 3.) We claim that from this (entropy) estimate, the positivity of u follows. The following proof is taken from Beretta et al. 1995.

Suppose, by contradiction, that there is x_0 such that $u(x_0, t) = 0$. By the regularity on u, $0 \leq u(x, t) \leq C|x - x_0|^{1/2}$ uniformly in t. Let us take $\alpha = -2$ in (3). This is admissible since then $\frac{3}{2} \leq -2 + \beta \leq 3$ is equivalent to $\frac{7}{2} \leq \beta \leq 5$. We obtain

$$\infty > \int_0^1 u(x, t)^\alpha dx \geq C^\alpha \int_0^1 |x - x_0|^{\alpha/2} dx = C^{-2} \int_0^1 |x - x_0|^{-1} dx = \infty,$$

contradiction. Thus, $u(x, t) > 0$ for all x (and t).

5. *New functional inequalities.* Using the entropy–entropy dissipation method, we can also prove inequalities involving derivatives of functions like

$$\int_{\mathbb{T}} u^\alpha (\log u)_x^4 dx \leq C_\alpha \int_{\mathbb{T}} u^\alpha (\log u)_{xx}^2 dx, \quad C_\alpha = \frac{9}{\alpha^2}.$$

We refer to section 4 for details.

3. Entropy construction method

3.1. **Idea of the method.** We introduce the technique by first studying a rather simple example, the 1D thin-film equation

$$(4) \qquad u_t + (u^\beta u_{xxx})_x = 0 \quad \text{in } \mathbb{T}, \quad u(\cdot, 0) = u_0.$$

Our goal is to derive estimates of the type

$$\frac{dE_\alpha}{dt} + P \leq 0,$$

where E_α is a zeroth-order entropy, $E_\alpha = \int u^\alpha dx / \alpha(\alpha - 1)$. For this, we assume that u is a positive smooth solution to (4), we take the time derivative of E_α and integrate by parts once:

$$\frac{dE_\alpha}{dt} = \frac{1}{\alpha - 1} \int_{\mathbb{T}} u^{\alpha-1} u_t dx = \int_{\mathbb{T}} u^{\alpha+\beta-2} u_x u_{xxx} dx =: -Q.$$

In order to show that $Q \geq 0$, we integrate by parts once more:

$$Q = (\alpha + \beta - 2) \int_{\mathbb{T}} u^{\alpha+\beta-3} \underbrace{u_x^2 u_{xx}}_{=\frac{1}{3}(u_x^3)_x} dx + \int_{\mathbb{T}} u^{\alpha+\beta-2} u_{xx}^2 dx$$

$$(5) \qquad = -\frac{1}{3}(\alpha + \beta - 2)(\alpha + \beta - 3) \int_{\mathbb{T}} u^{\alpha+\beta-4} u_x^4 dx + \int_{\mathbb{T}} u^{\alpha+\beta-2} u_{xx}^2 dx.$$

Now, $Q \geq 0$ if $(\alpha + \beta - 2)(\alpha + \beta - 3) \leq 0$ or if $2 \leq \alpha + \beta \leq 3$.

Although the above computation shows the claim, there are two disadvantages of this approach:

- One may wonder if the second term (5) can compensate the first term in case the factor $(\alpha + \beta - 2)(\alpha + \beta - 3)$ is negative. This is indeed the case. We will show below that $Q \geq 0$ if (and only if) $\frac{3}{2} \leq \alpha + \beta \leq 3$. Thus, the parameter range $\frac{3}{2} \leq \alpha + \beta < 2$ is not covered by the above computation.
- The integration by parts have been done in a non-systematic and ad-hoc way. Better results can be obtained by a different integration by parts (see below). Moreover, in several space dimensions, there are many possible integration by parts, and it is not clear how to apply them.

In view of these drawbacks, we suggest an approach which is based on a systematic use of integration by parts. For this, we describe integration by parts in a different way. The computation

$$(6) \quad Q = -\int_{\mathbb{T}} u^{\alpha+\beta-2} u_x u_{xxx} dx = (\alpha + \beta - 2) \int_{\mathbb{T}} u^{\alpha+\beta-3} u_x^2 u_{xx} dx + \int_{\mathbb{T}} u^{\alpha+\beta-2} u_{xx}^2 dx$$

can be written in the form

$$I_2 = \int_{\mathbb{T}} u^{\alpha+\beta} \left((\alpha + \beta - 2) \left(\frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} + \left(\frac{u_{xx}}{u} \right)^2 + \frac{u_x}{u} \frac{u_{xxx}}{u} \right) dx = \int_{\mathbb{T}} \left(u^{\alpha+\beta} \frac{u_x}{u} \frac{u_{xx}}{u} \right)_x dx = 0.$$

Then (6) corresponds to

$$Q = Q + c \cdot I_2 \quad \text{with } c = 1.$$

How many integration-by-parts rules do exist? There are three rules. The other two read as

$$I_1 = \int_{\mathbb{T}} u^{\alpha+\beta} \left((\alpha + \beta - 3) \left(\frac{u_x}{u} \right)^4 + 3 \left(\frac{u_x}{u} \right)^2 \frac{u_{xx}}{u} \right) dx = \int_{\mathbb{T}} \left(u^{\alpha+\beta} \left(\frac{u_x}{u} \right)^3 \right)_x dx = 0,$$

$$I_3 = \int_{\mathbb{T}} u^{\alpha+\beta} \left((\alpha + \beta - 1) \frac{u_x}{u} \frac{u_{xxx}}{x} + \frac{u_{xxxx}}{u} \right) dx = \int_{\mathbb{T}} \left(u^{\alpha+\beta} \frac{u_{xxx}}{u} \right)_x dx = 0.$$

The number of rules is determined by all integers p_1, p_2, and p_3 such that $1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 = 3$. Thus,

$$(p_1, p_2, p_3) = (3, 0, 0), (1, 1, 0), (0, 0, 1),$$

and there are exactly three rules. Then we can reformulate the problem of proving $Q \geq 0$ as

$$\exists c_1, c_2, c_3 \in \mathbb{R} : Q = Q + c_1 I_1 + c_2 I_2 + c_3 I_3 \geq 0.$$

Clearly, since $I_k = 0$ for $k = 1, 2, 3$, the above equality is trivial.

Now comes our main idea. We identify the integrands (up to the factor $u^{\alpha+\beta}$) as polynomials via the identification $\xi_1 \triangleq \frac{u_x}{u}$, $\xi_2 \triangleq \frac{u_{xx}}{u}$ etc. Therefore, with $\xi = (\xi_1, \xi_2, \xi_3)$,

$$Q \text{ corresponds to } S(\xi) = -\xi_1\xi_3,$$
$$I_1 \text{ corresponds to } T_1(\xi) = (\alpha + \beta - 3)\xi_1^4 + 3\xi_1^2\xi_2,$$
$$I_2 \text{ corresponds to } T_2(\xi) = (\alpha + \beta - 2)\xi_1^2\xi_2 + \xi_2^2 + \xi_1\xi_3,$$
$$I_3 \text{ corresponds to } T_3(\xi) = (\alpha + \beta - 1)\xi_1\xi_3 + \xi_4.$$

The polynomials T_k are termed shift polynomials. If we can solve the polynomial problem

(7) $$\exists c_1, c_2, c_3 \in \mathbb{R} : \forall \xi \in \mathbb{R}^3 : (S + c_1T_1 + c_2T_2 + c_3T_3)(\xi) \geq 0,$$

then we have a pointwise estimate for the integrands, and $Q \geq 0$ follows.

Remark 1. We show a much stronger estimate than just $Q = \int u^{\alpha+\beta} f(u, u_x, u_{xx}, \ldots)dx \geq 0$ since we try to prove that $f(u, u_x, u_{xx}, \ldots) \geq 0$ for all $x \in \mathbb{T}$. One may wonder if by this approach entropy estimates may get lost. It is possible to show (see Jüngel/Matthes 2006) that this is not the case for the 1D thin-film and DLSS equation. It is not known if this is true for the multi-dimensional situation or more general equations.

If the problem (7) is solved, we have only shown that the entropy is nonincreasing in time. In order to prove the stronger result $dE/dt + cP \geq 0$, where $c > 0$ is a constant and P a (nonnegative) entropy production, we can proceed in a similar way as above. Since $Q = -dE/dt$, we have to prove that there exists a constant $c > 0$ such that $-Q + cP \geq 0$ or, with the above integration-by-parts rules,

(8) $$\exists c_1, c_2, c_3 \in \mathbb{R}, \; c > 0 : -Q + cP + c_1I_1 + c_2I_2 + c_3I_3 \geq 0.$$

This problem is of the same type like (6) and therefore, it can be "translated" to a polynomial problem similar to (7). Possible entropy productions, for this fourth-order equation, are, for instance,

$$P = \int_{\mathbb{T}} (u^{(\alpha+\beta)/4})_x^4 dx, \quad P = \int_{\mathbb{T}} (u^{(\alpha+\beta)/2})_{xx}^2 dx, \quad P = \int_{\mathbb{T}} u^{\alpha+\beta-2}u_{xx}^2 dx.$$

We summarize our algorithm:
(1) Calculate the functional Q and "translate" it into a polynomial S. (This step depends on the equation at hand.)
(2) Determine the shift polynomials T_k which represent the integration-by-parts rules. (This step only depends on the order of the equation but not on the structure of the equation except for the parameter β.)
(3) Decide for which parameters α the problem (7) can be solved. This shows that E_α is nonincreasing in time.
(4) Decide for which parameters α the problem (8) can be solved. As a result, the entropy–entropy production inequality $dE_\alpha/dt + cP \leq 0$ holds.

It remains to solve the problem (7). This is a decision problem which is well-know in real algebraic geometry. It was shown by Tarski in 1951 that such problems can be always treated in the following sense:

> A quantified statement about polynomials can be reduced to a quantifier-free statement about polynomials in an algorithmic way.

Solution algorithms for the above quantifier elimination problem have been implemented, for instance, in `Mathematica`. There are also tools specialized on quantifier elimination, like `QEPCAD` (Quantifier Elimination using Partial Cylindrical Algebraic Decomposition), see Collins/Hong 1991. The advantage of these algorithms is that the solution is complete and exact. The disadvantage is that the complexity of the algorithms is doubly exponential in the number of variables ξ_i and c_i. An alternative approach is given by the sum-of-squares (SOS) method. This method tries to write the polynomial as a sum of squares. Therefore, the answer may be not complete since there are polynomials which are nonnegative but which cannot be written as a sum of squares. Fortunately, for some decision problems arising from 1D fourth-order equations, we can solve the problems directly without going into real algebraic geometry. This will be explained below.

We notice that our method is formal in the sense that positive smooth solutions have to be assumed in order to justify the calculations. The proofs can be made rigorous if an appropriate approximation of the original equation is available which (i) allows for positive smooth solutions and (ii) does not destroy the entropic structure. Clearly, such an approximation depends much on the specific equation.

3.2. **The general scheme.** In the following we only present the general scheme for spatial one-dimensional equations. A systematic treatment of multi-dimensional problems is in progress (but see section 4). We are concerned with equations of the type

$$u_t = \left(u^{\beta+1} q\left(\frac{u_x}{u}, \frac{u_{xx}^2}{u}, \ldots, \frac{u_{x\ldots x}}{x} \right) \right)_x \quad \text{in } \mathbb{T}, \ t > 0,$$

where the derivatives $u_{x\ldots x}$ are up to order $k-1$ (with even k) and $q(\xi_1, \ldots, \xi_{k-1})$ is a real polynomial,

$$q(\xi_1, \ldots, \xi_{k-1}) = \sum_{p_1, \ldots, p_{k-1}} c_{p_1, \ldots, p_{k-1}} \xi_1^{p_1} \cdots \xi_k^{p_{k-1}}$$

such that at most those coefficients $c_{p_1, \ldots, p_{k-1}} \in \mathbb{R}$ with $1 \cdot p_1 + \cdots + (k-1) \cdot p_{k-1} = k-1$ are nonzero. We denote the set of those polynomials as Σ_{k-1}. In this notation, $q \in \Sigma_{k-1}$. An example is the thin-film equation with $k = 4$ and $q(\xi) = \xi_3$. In order to distinguish between the polynomial q and its differential operator, we set

$$D_q(u) = q\left(\frac{u_x}{u}, \frac{u_{xx}^2}{u}, \ldots, \frac{u_{x\ldots x}}{x} \right).$$

Let u be a positive smooth solution to

$$u_t = (u^{\beta+1} D_q(u))_x, \quad u(\cdot, 0) = u_0,$$

where $q \in \Sigma_{k-1}$, with periodic boundary conditions. We need some definitions. For this, let $s(u)$ be one of the following functions:

$$s(u) = \frac{u^\alpha}{\alpha(\alpha-1)}, \quad s(u) = u(\log u - 1) + 1, \quad s(u) = u - \log u,$$

where $\alpha \in \mathbb{R}$, $\alpha \notin \{0, 1\}$. Notice that $s''(u) = u^{\alpha-1}$.

- The function $E(t) = \int s(u(x,t))dx$ is called an *entropy* if $E(t)$ is nonincreasing.
- The function $P(t) = \int u^{\alpha+\beta} D_p(u)dx$ with $p \in \Sigma_k$ is called an *entropy production* for the entropy E if there exists $c > 0$ such that for all $t > 0$,

$$\frac{dE}{dt} + cP \leq 0.$$

- The entropy E is called *generic* if P is an entropy production for all $p \in \Sigma_k$.

Step 1: characteristic polynomials. Taking the derivative of a function E and integrating by parts gives

$$\frac{dE}{dt} = \int_{\mathbb{T}} s'(u)u_t dx = -\int_{\mathbb{T}} s''(u)u_x(u^{\beta+1}D_q(u))dx = -\int_{\mathbb{T}} u^{\alpha+\beta}\frac{u_x}{u}D_q(u)dx.$$

For the entropy–entropy dissipation method, we need to modify the polynomial corresponding to $(u_x/u)D_q(u)$. Therefore, we call $s_0 \in \Sigma_k$ a *characteristic polynomial* if

$$\frac{dE}{dt} = -\int_{\mathbb{T}} u^{\alpha+\beta}D_{s_0}(u)dx.$$

Clearly, there is at least one characteristic polynomial, namely $s_0(\xi) = \xi_1 q(\xi_1, \ldots, \xi_{k-1})$, which is called a canonical (characteristic) polynomial.

Step 2: shift polynomials. The integration-by-parts rules can be formalized as follows. We introduce for $\gamma \in \mathbb{R}$ the operator $\delta_\gamma : \Sigma_{k-1} \to \Sigma_k$ by

$$(u^\gamma D_p(u))_x = u^\gamma D_{\delta_\gamma p}(u), \quad p \in \Sigma_{k-1}.$$

An explicit calculation shows that the image of the monomial $p(\xi) = \xi_1^{p_1} \cdots \xi_{k-1}^{p_{k-1}}$ is

$$(9) \qquad \delta_\gamma p(\xi) = (\gamma - p_1 - \cdots - p_{k-1})\xi_1 p(\xi) + p_1 \frac{\xi_2}{\xi_1}p(\xi) + \cdots + p_{k-1}\frac{\xi_k}{\xi_{k-1}}p(\xi).$$

Example 2. The three monomials $r_1(\xi) = \xi_1^3$, $r_2(\xi) = \xi_1\xi_2$, $r_3(\xi) = \xi_3$ form a basis of Σ_3. Then formula (9) gives

$$T_1(\xi) := \delta_{\alpha+\beta}r_1(\xi) = (\alpha + \beta - 3)\xi_1^4 + 3\xi_1^2\xi_2,$$
$$T_2(\xi) := \delta_{\alpha+\beta}r_2(\xi) = (\alpha + \beta - 2)\xi_1^2\xi_2 + \xi_2^2 + \xi_1\xi_3,$$
$$T_3(\xi) := \delta_{\alpha+\beta}r_3(\xi) = (\alpha + \beta - 1)\xi_1\xi_3 + \xi_4.$$

These polynomials are a basis of the linear space $\delta_{\alpha+\beta}\Sigma_3$. They express our integration-by-parts rules.

In view of the above example, we choose a basis of monomials $r_i \in \Sigma_{k-1}$, $i = 1, \ldots, d$. Then the functions $T_i = \delta_{\alpha+\beta}r_i$ are also linearly independent. We will call them *shift polynomials*. We have to solve the following problem:

$$\exists c_1, \ldots, c_d \in \mathbb{R} : \forall \xi \in \mathbb{R}^k : s(\xi) = (s_0 + c_1T_1 + \cdots + c_dT_d)(\xi) \geq 0.$$

If this is true then E is an entropy. This decision problem can be simplified by eliminating some integration-by-parts rules which are not useful. In terms of polynomial manipulations, this leads to the notion of a normal form.

Step 3: decision problems. We call a characteristic polynomial $p \in \Sigma_k$ to be in *normal form* if for each j, the highest exponent with which ξ_j occurs in p is even. Otherwise, $\xi_j^a \to -\infty$ if $\xi_j \to -\infty$ and the polynomial cannot be nonnegative.

Example 3. We are looking for the normal forms for $s_0 + c_1 T_1 + \cdots + c_d T_d$ in the case of the thin-film equation. Recall that $s_0(\xi) = -\xi_1 \xi_3$. Since T_3 contains $\xi_4 = \xi_4^1$ and 1 is odd, $c_3 = 0$. The constant c_2 must be chosen to eliminate $\xi_3 = \xi_3^1$. Hence, $c_2 = 1$. There is no restriction on c_1. Thus, the normal forms are given by

$$s := s_0 + c_1 \cdot T_1 + 1 \cdot T_2 + 0 \cdot T_3 = (\alpha + \beta - 3)c_1 \xi_1^4 + (\alpha + \beta - 2 + 3c_1)\xi_1^2 \xi_2 + \xi_2^2.$$

Step 4: entropy production. Finally, we turn to an algebraic formulation of the entropy production $P = \int u^{\alpha+\beta} D_p(u) dx$. To prove $dE/dt + cP \le 0$ for some $c > 0$, it is sufficient to show that

$$s(\xi) - c \cdot p(\xi) \ge 0.$$

Again, a decision problem has to be solved. Recall that an entropy E is generic if this inequality is true for all $p \in \Sigma_k$, with constants c depending on p. The idea behind this notion is that, for small $c > 0$, $s - cp$ is a polynomial with an c-small perturbation in the coefficients.

3.3. **Solution of some decision problems.** The notion of normal forms allows to reduce the number of variables c_i in the decision problem. In this subsection, we give some lemmas by which decision problems for polynomials up to sixth order can be solved.

Lemma 4. *Let the real polynomial* $p(\xi_1, \xi_2) = a_1 \xi_1^4 + a_2 \xi_1^2 \xi_2 + a_3 \xi_2^2$ *be given. Then the quantified expression*

$$\forall \xi_1, \xi_2 \in \mathbb{R} : \ p(\xi_1, \xi_2) \ge 0$$

is equivalent to the quantifier-free statement that

$$\text{either } a_3 > 0 \quad \text{and} \quad 4a_1 a_3 - a_2^2 \ge 0$$
$$\text{or } a_3 = a_2 = 0 \quad \text{and} \quad a_1 \ge 0.$$

Lemma 5. *Let the real polynomial*

$$p(\xi_1, \xi_2, \xi_3) = a_1 \xi_1^6 + a_2 \xi_1^4 \xi_2 + a_3 \xi_1^3 \xi_3 + a_4 \xi_1^2 \xi_2^2 + a_5 \xi_1 \xi_2 \xi_3 + \xi_3^2$$

be given. Then the quantified formula

$$\forall \xi_1, \xi_2, \xi_3 \in \mathbb{R} : \ p(\xi_1, \xi_2, \xi_3) \ge 0$$

is equivalent to the quantifier free formula

$$\text{either } 4a_4 - a_5^2 > 0 \quad \text{and} \quad 4a_1 a_4 - a_1 a_5^2 - a_2^2 - a_3^2 a_4 + a_2 a_3 a_5 \ge 0$$
$$\text{or } 4a_4 - a_5^2 = 2a_2 - a_3 a_5 = 0 \quad \text{and} \quad 4a_1 - a_3^2 \ge 0.$$

The (elementary) proofs of these lemmas can be found in Jüngel/Matthes 2006. We can employ Lemma 4 to solve the decision problem for the 1D thin-film equation,

$$\exists c_1 \in \mathbb{R} : \forall \xi \in \mathbb{R}^2 : (s_0 + c_1 T_1 + T_2)(\xi) = (\alpha + \beta - 3)c_1 \xi_1^4 + (\alpha + \beta - 2 + 3c_1)\xi_1^2 \xi_2 + \xi_2^2 \geq 0.$$

Since the coefficient for ξ_2^2 is positive, the nonnegativity is guaranteed if and only if

$$0 \leq 4a_1 a_3 - a_2^2 = 4c_1(\alpha + \beta - 3) - (\alpha + \beta - 2 + 3c_1)^2$$
$$= -9\Big(c_1 + \frac{1}{9}(\alpha + \beta)\Big)^2 - \frac{8}{9}(\alpha + \beta)^2 + 4(\alpha + \beta) - 4.$$

Choosing the maximizing value $c_1 = -(\alpha + \beta)/9$, this inequality is satisfied if and only if

$$0 \leq -8(\alpha + \beta)^2 + 36(\alpha + \beta) - 36.$$

The roots of the polynomial $x \mapsto -8x^2 + 36x - 36$ are $x_1 = 3/2$ and $x_2 = 3$. Therefore, the inequality is valid if and only if $\frac{3}{2} \leq \alpha + \beta \leq 3$.

Entropy productions are derived by showing $(s_0 + c_1 T_1 - cp)(\xi) \geq 0$ for some (small) $c > 0$. If, for instance, the entropy production is given by $P = \int u^{\alpha + \beta - 2} u_{xx}^2 dx$, we have $p(\xi) = \xi_2^2$, and we need to solve

$$\exists c_1 \in \mathbb{R}, \; c > 0 : \forall \xi \in \mathbb{R}^2 : (\alpha + \beta - 3)c_1 \xi_1^4 + (\alpha + \beta - 2 + 3c_1)\xi_1^2 \xi_2 + (1 - c)\xi_2^2 \geq 0.$$

It can be easily seen that this problem is solvable if $\frac{3}{2} < \alpha + \beta < 3$.

In the nongeneric cases $\alpha + \beta = 3/2$ and $\alpha + \beta = 3$, there may be still specific entropy productions. For instance, let $\alpha + \beta = 3/2$. We choose $c_1 = -1/6$ in the polynomial $s_0 + c_1 T_1 + T_2$ which gives

$$(s_0 + c_1 T_1 + T_2)(\xi) = \frac{1}{4}\xi_1^4 + \xi_1^2 \xi_2 + \xi_2^2 = \Big(\xi_2 - \frac{1}{2}\xi_1^2\Big)^2.$$

The corresponding functional reads as

$$(10) \qquad \frac{dE_\alpha}{dt} = -\int_{\mathbb{T}} u^{3/2}\Big(\frac{u_{xx}}{u} - \frac{u_x}{2u}\Big)^2 dx = -\int_{\mathbb{T}} u^{1/2}(u^{1/2})_{xx}^2 dx.$$

Similarly, the case $\alpha + \beta = 3$ can be treated: Choosing $c_1 = -1/3$ leads to

$$(11) \qquad \frac{dE_\alpha}{dt} = -\int_{\mathbb{T}} u u_{xx}^2 dx.$$

We have shown:

Theorem 6. *The functionals $E_\alpha(t)$ are nonincreasing in time (along solutions of the 1D thin-film equation) if $\frac{3}{2} \leq \alpha + \beta \leq 3$. Moreover, if $\frac{3}{2} < \alpha + \beta < 3$, there exists $c > 0$ such that*

$$\frac{dE_\alpha}{dt} + c\int_{\mathbb{T}} \big(u^{\alpha + \beta - 2} u_{xx}^2 + (u^{(\alpha+\beta)/2})_{xx}^2 + (u^{(\alpha+\beta)/4})_x^4\big) dx \leq 0.$$

Furthermore, if $\alpha + \beta = 3/2$ or $\alpha + \beta = 3$, the inequalities (10), (11), respectively, hold.

More precisely, we have shown the last inequality only for the first entropy production term but the proof for the remaining terms is similar.

3.4. Second example: DLSS equation.

We consider the DLSS equation

$$u_t + (u(\log u)_{xx})_{xx} = 0 \quad \text{in } \mathbb{T}, \ t > 0, \quad u(\cdot, 0) = u_0.$$

Since

$$-(u(\log u)_{xx})_x = u\left(-\left(\frac{u_x}{u}\right)^3 + 2\frac{u_x}{u}\frac{u_{xx}}{u} - \frac{u_{xxx}}{u}\right),$$

we have $u_t = (uD_p(u))_x$ with $p(\xi) = -\xi_1^3 + 2\xi_1\xi_2 - \xi_3$. Hence, the (canonical) characteristic polynomial is $s_0(\xi) = \xi_1 p(\xi) = -\xi_1^4 + 2\xi_1^2\xi_2 - \xi_1\xi_3$. The shift polynomials are the same as for the thin-film equation choosing $\beta = 0$:

$$T_1(\xi) = (\alpha - 3)\xi_1^4 + 3\xi_1^2\xi_2,$$
$$T_2(\xi) = (\alpha - 2)\xi_1^2\xi_2 + \xi_2^2 + \xi_1\xi_3,$$
$$T_3(\xi) = (\alpha - 1)\xi_1\xi_3 + \xi_4.$$

Similarly, the most general normal form is given by

$$s_0 + c \cdot T_1 + 1 \cdot T_2 + 0 \cdot T_3 = (c(\alpha - 3) - 1)\xi_1^4 + (3c + \alpha)\xi_1^2\xi_2 + \xi_2^2.$$

This polynomial is nonnegative for all $\xi \in \mathbb{R}^3$ if, by Lemma 4,

$$0 \le 4a_1a_3 - a_2^2 = 4(c(\alpha - 3) - 1) - (3c + \alpha)^2 = -9c^2 - 2(\alpha + 6)c - (4 + \alpha^2)$$
$$= -9\left(c + \frac{1}{9}(\alpha + 6)\right)^2 - \frac{8}{9}\alpha\left(\alpha - \frac{3}{2}\right).$$

Choosing the maximizing value $c = -(\alpha + 6)/9$, we obtain

$$0 \le -\frac{8}{9}\alpha\left(\alpha - \frac{3}{2}\right),$$

This inequality is satisfied if and only if $0 \le \alpha \le 3/2$. As before, the generic entropies are those for which $4a_1a_3 - a_2^2 > 0$, corresponding to $0 < \alpha < 3/2$.

Next, we turn to the nongeneric entropies $\alpha = 0$ and $\alpha = 3/2$. For $\alpha = 0$, we have

$$s_0(\xi) + cT_1(\xi) + T_2(\xi) = -(3c + 1)\xi_4^1 + 3c\xi_1^2\xi_2 + \xi_2^2.$$

Taking $c = -2/3$ gives $(s_0 + cT_1 + T_2)(\xi) = \xi_1^4 - 2\xi_1^2\xi_2 + \xi_2^2 = (\xi_1^2 - \xi_2)^2$ which translates to the entropy production

$$\int_{\mathbb{T}} \left(\left(\frac{u_x}{u}\right)^2 - \frac{u_{xx}}{u}\right)^2 dx = \int_{\mathbb{T}} (\log u)_{xx}^2 dx.$$

Similarly, if $\alpha = 3/2$, we take $c = -5/6$, leading to the entropy production

$$4 \int_{\mathbb{T}} u^{1/2}(u^{1/2})_{xx}^2 dx.$$

We summarize these results in the following theorem.

Theorem 7. *The functionals $E_\alpha(t)$ are nonincreasing in time (along solutions of the 1D DLSS equation) if $0 \le \alpha \le 3/2$. Moreover, if $0 < \alpha < 3/2$, there exists $c > 0$ such that*

$$\frac{dE_\alpha}{dt} + c \int_{\mathbb{T}} \left(u^{\alpha-2}u_{xx}^2 + (u^{\alpha/2})_{xx}^2 + (u^{\alpha/4})_x^4\right) dx \le 0.$$

Furthermore,

$$\frac{dE_\alpha}{dt} + \int_{\mathbb{T}} (\log u)^2_{xx} dx \le 0 \quad \text{if } \alpha = 0,$$

$$\frac{dE_\alpha}{dt} + 4 \int_{\mathbb{T}} u^{1/2} (u^{1/2})^2_{xx} dx \le 0 \quad \text{if } \alpha = \frac{3}{2}.$$

4. ADDITIONAL RESULTS

4.1. **Higher-order entropies.** Clearly, the approach of the previous section also applies to higher-order entropies, like the first-order entropies

$$E = \int_\Omega (u^{\alpha/2})^2_x dx, \quad \alpha > 0.$$

Consider again the 1D thin-film equation. Then, taking the derivative,

$$\frac{dE}{dt} = 2 \int_\Omega (u^{\alpha/2})_x (u^{\alpha/2})_{tx} dx = -2 \int_\Omega (u^{\alpha/2})_{xx} \frac{\alpha}{2} u^{\alpha/2-1} u_t dx$$

$$= -\alpha \int_\Omega (u^{\alpha/2})_{xx} u^{\alpha/2-1} (u^\beta u_{xxx})_x dx.$$

Thus, we have to find all integration-by-parts rules involving a total of six derivatives. It can be seen that there are seven integration-by-parts rules giving seven shift polynomials. Some of the shift polynomials do not need to be taken into account (i.e., the normal form does not contain them), like

$$T(\xi) = (\alpha + \beta - 1)\xi_1 \xi_5 + \xi_6,$$

since ξ_6 appears in odd order. Writing down the normal form leads to the following decision problem:

$$\exists c_1, c_2 \in \mathbb{R} : \forall \xi \in \mathbb{R}^3 : \quad (\alpha + \beta - 5)c_1 \xi_1^6 + (5c_1 + (\alpha + \beta - 4)c_2)\xi_1^4 \xi_2 + 3c_2 \xi_1^2 \xi_2^2$$

$$+ (\tfrac{1}{2}(\alpha^2 - 5\alpha + 6)\xi_1^3 \xi_3 + (2\alpha - 4)\xi_1 \xi_2 \xi_3 + \xi_3^2 \ge 0.$$

The quantifier elimination can be performed using Lemma 5 for polynomials in Σ_6 in three variables up to order six. The result is displayed in Figure 1. (This result has been first found by Laugesen 2005.) Notice that there is always a trivial first-order entropy corresponding to $\alpha = 2$, reading $E = \int u_x^2 dx$. In fact, the corresponding entropy–entropy inequality can be easily obtained by differentiation:

$$\frac{d}{dt} \int_{\mathbb{T}} u_x^2 dx = 2 \int_{\mathbb{T}} u_x u_{tx} dx = -2 \int_{\mathbb{T}} u_{xx} u_t dx = 2 \int_{\mathbb{T}} u_{xx} (u^\beta u_{xxx})_x dx = -2 \int_{\mathbb{T}} u^\beta u_{xxx}^2 dx.$$

A similar result can be derived for the DLSS equation. The functional $E = \int (u^{\alpha/2})^2_x dx$ is an entropy if α lies in between the two reals roots of $20 - 100\alpha + 53\alpha^2$, i.e. $\alpha \in (0, 2274\dots, 1.6593\dots)$.

The situation is more complicated concerning second-order entropies since even for fourth-order equations, polynomials of order 8 need to be solved. We have not found second-order entropies for the thin-film or DLSS equation using quantifier elimination or
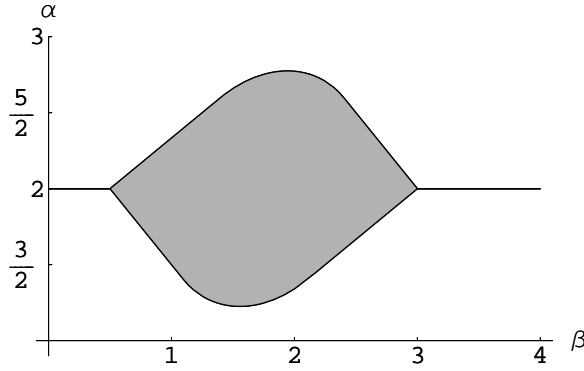
FIGURE 1. Values of α and β providing an entropy.

the sum-of-squares method. This does not mean that there are no second-order entropies for these equations since our proof is based on pointwise estimates of the integrand.

4.2. **Multi-dimensional equations.** In principle, the strategy for the one-dimensional PDEs can be generalized in a straight forward way to multi-dimensional equations. In this situation, we have to deal with polynomial variables for all the partial derivatives. Integration-by-parts rules are obtained by differentiating products in all variables. Practically, this strategy is useless since it leads to polynomial expression in many variables ξ_k and a huge number of shift polynomials T_i. A better approach is not to incorporate all products of differential expressions. As this is still current research, we will only sketch some ideas.

Our main idea is to exploit the symmetry of the problem. For instance, consider the problem of deriving first-order entropies for a fourth-order equation in d space dimensions. Then the polynomials are a linear combination of 14 $O(d)$-invariant scalar expressions like

$$\|\nabla^3 u\|^3, \quad \nabla \Delta u \cdot \nabla^2 u \cdot \nabla u, \quad |\nabla u|^6, \ldots,$$

where $\nabla^2 u$ denotes the Hessian of u and $\nabla^3 u$ the corresponding 3-tensor. Then we perform the following steps:

(1) Restrict to those integration-by-parts rules which can be written in the above 14 expressions alone. This leads to 7 decision variables.
(2) Perform a rotation and/or dilation to eliminate all dependencies on first derivatives. The remaining polynomials are at most of order three.
(3) Eliminating the third-order terms gives a quadratic problem in $\mathbb{R}^N$, where the variables represent derivatives of second and third order.
(4) Decomposing $\mathbb{R}^N$ into certain subspaces, it is enough to consider the quadratic problem on the subspaces. This yields a semi-definite programming problem in 9 variables with 7 parameters.

Example 8. In this example we detail the problem of finding zeroth-order entropies for the thin-film equation,

$$u_t + \operatorname{div}(u^\beta \nabla \Delta u) = \quad \text{in } \mathbb{T}^d, \ t > 0, \quad u(\cdot, 0) = u_0,$$

we proceed slightly different than above (see Jüngel/Matthes 2006). There are 7 scalar expressions containing exactly four derivatives:

$$\eta_G^4 = u^{-4}|\nabla u|^4, \quad \eta_L^2 = u^{-2}(\Delta u)^2, \quad \operatorname{trace}(\eta_H^2) = u^{-2}\operatorname{trace}(\nabla^2 u)^2,$$
$$\eta_G^2 \eta_L = u^{-3}|\nabla u|^2 \Delta u, \quad \eta_G^2 \eta_H = u^{-1}(\nabla u)^\top \nabla^2 u(\nabla u), \quad \eta_G \eta_T = u^{-2}\nabla \Delta u \cdot \nabla \Delta u,$$
$$\eta_D = u^{-1}\Delta^2 u.$$

The expressions on the left-hand sides have to be read as formal symbols and not as products. There are four shift polynomials of interest:

$$T_1(\eta) = (\alpha + \beta - 3)\eta_G^4 + \eta_G^2 \eta_L + 2\eta_G^2 \eta_H,$$
$$T_2(\eta) = (\alpha + \beta - 2)\eta_G^2 \eta_L + \eta_L^2 + \eta_G \eta_T,$$
$$T_3(\eta) = (\alpha + \beta - 2)\eta_G^2 \eta_H + \operatorname{trace}(\eta_H^2) + \eta_G \eta_T,$$
$$T_4(\eta) = (\alpha + \beta - 1)\eta_G \eta_T + \eta_D.$$

The (canonical) characteristic polynomial is $s_0(\eta) = -\eta_G \eta_H$. It can be shown that the general normal form reads as follows:

$$(s_0 + c_1 T_1 + c_2 T_2 + c_3 T_3)(\eta) = c_1(\alpha + \beta - 3)\eta_G^4 + c_2 \eta_L^2 + c_3 \operatorname{trace}(\eta_H^2)$$
$$+ ((\alpha + \beta - 2)c_2 + c_1)\eta_G^2 \eta_L + ((\alpha + \beta - 2)c_3 + 2c_1)\eta_G^2 \eta_H,$$

and c_2 and c_3 satisfy the relation $c_2 + c_3 = 1$. Actually, it is even possible to reduce the quantifier elimination problem to three scalar variables $\|\eta_G\|$, $(\operatorname{trace}(\eta_H^2))^{1/2}$, and η_L and one decision variable. Then, employing the algebra tool QEPCAD, one can show that the zeroth-order functionals are entropies if $3/2 \leq \alpha + \beta \leq 3$. This is the same condition as in the one-dimensional case.

4.3. **New functional inequalities.** The entropy–entropy dissipation approach can be also used to prove some functional inequalities. As an example, we will show that for all positive smooth functions u, it holds

(12) $$\int_\mathbb{T} u^\alpha (\log u)_x^4 dx \leq \frac{9}{\alpha^2} \int_\mathbb{T} u^\alpha (\log u)_{xx}^2 dx, \quad \alpha > 0.$$

This inequality resembles the logarithmic Sobolev inequality, mentioned in the introduction,

$$\int_\mathbb{T} u^2 \log \frac{u^2}{M^2} dx \leq C \int_\mathbb{T} u^2 (\log u)_x^2 dx,$$

where $M^2 = \int u^2 dx / L$ and $\mathbb{T} \sim [0, L]$.

Inequality (12) can be written as

$$0 \leq \int_\mathbb{T} u^\alpha \big((\log u)_{xx}^2 - c(\log u)_x^4\big) dx = \int_\mathbb{T} \left((1 - c)\left(\frac{u_x}{u}\right)^4 - 2\left(\frac{u_x}{u}\right)^2 \frac{u_{xx}}{u} + \left(\frac{u_{xx}}{u}\right)^2\right).$$

Writing the integrand as a polynomial, (12) is shown if

$$\exists c > 0 : \forall \xi \in \mathbb{R}^2 : q(\xi) = (1 - c)\xi_1^4 - 2\xi_1^2\xi_2 + \xi_2^2 \geq 0.$$

There is only one integration-by-parts rule which is relevant here. Thus we have to show:

$$\exists c_1 \in \mathbb{R}, \ c > 0 : \forall \xi \in \mathbb{R}^2 : (q + c_1 T_1)(\xi) = (1 - c - c_1(\alpha - 3))\xi_1^4 + (3c - 2)\xi_1^2\xi_2 + \xi_2^2 \geq 0.$$

We apply Lemma 4 and obtain eventually the relation

$$9c_1^2 + 4\alpha c_1 + 4c \leq 0,$$

which is true if and only if $9c \leq \alpha^2$. It can be shown (see Jüngel/Matthes 2006 for details) that this choice is optimal.

It is also possible, by the same technique, to prove multi-dimensional inequalities. For instance, we have shown in Jüngel/Matthes 2007:

Theorem 9. *Let $u \in H^2(\mathbb{T}^d) \cap W^{1,4}(\mathbb{T}^d) \cap L^\infty(\mathbb{T}^2)$, $d \geq 2$, and assume that $\inf_\mathbb{T} u > 0$. Then, for any $0 < \gamma < 2(d+1)/(d+2)$,*

$$(13) \qquad \frac{1}{2(\gamma - 1)} \int_{\mathbb{T}^d} \sum_{i,j=1}^d u^2 \partial_{ij}^2(\log u) \partial_{ij}^2(u^{2(\gamma-1)}) dx \geq \kappa_\gamma \int_{\mathbb{T}^d} (\Delta u^\gamma)^2 dx,$$

if $\gamma \neq 1$, or

$$\int_{\mathbb{T}^d} \sum_{i,j=1}^d u^2 |\partial_{ij}^2(\log u)|^2 dx \geq \kappa_1 \int_{\mathbb{T}^d} (\Delta u)^2 dx,$$

if $\gamma = 1$, respectively, where

$$(14) \qquad \kappa_\gamma = \frac{p(\gamma)}{\gamma^2(p(\gamma) - p(0))} \quad and \quad p(\gamma) = -\gamma^2 + \frac{2(d+1)}{d+2}\gamma - \left(\frac{d-1}{d+2}\right)^2.$$

The constant κ_γ is positive if

$$\frac{(\sqrt{d} - 1)^2}{d+2} < \gamma < \frac{(\sqrt{d} + 1)^2}{d+2}.$$

We sketch the proof of the above theorem. *First*, we define the variables in which will wish to work. Let θ, λ, and μ be defined by

$$\theta = \frac{|\nabla u|}{u}, \quad \lambda = \frac{1}{d}\frac{\Delta u}{u}, \quad (\lambda + \mu)\theta^2 = \frac{\nabla u^\top}{u}\frac{\nabla^2 u}{u}\frac{\nabla u}{u},$$

where $\nabla^2 u$ denotes the Hessian of u. Furthermore, it is possible to show (see [8]) that the following expression defines the variable $\rho \geq 0$:

$$\frac{\|\nabla^2 u\|^2}{u^2} = \left(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2\right).$$

With these definitions, we can write the left-hand side J and the right-hand side K of (13) as

$$J = \frac{1}{2(\gamma - 1)} \int_{\mathbb{T}^d} \sum_{i,j=1}^{d} u^2 \partial_{ij}^2 (\log u) \partial_{ij}^2 (u^{2(\gamma-1)}) dx$$

$$= \int_{\mathbb{T}^d} u^{2\gamma} \left(d\lambda^2 + \frac{d}{d-1}\mu^2 + \rho^2 - 2(2-\gamma)(\lambda+\mu)\theta^2 + (3-2\gamma)\theta^4 \right) dx,$$

$$K = \frac{1}{\gamma^2} \int_{\mathbb{T}^d} (\Delta u^\gamma)^2 dx = \int_{\mathbb{T}^d} u^{2\gamma} (d\lambda + (\gamma-1)\theta^2)^2 dx.$$

Second, we identify those integration by parts which are useful for the analysis:

$$I_1 = \int_{\mathbb{T}^d} \operatorname{div}(u^{2\gamma-2}(\nabla^2 u - \Delta u I) \cdot \nabla u) dx = 0,$$

$$I_2 = \int_{\mathbb{T}^d} \operatorname{div}(u^{2\gamma-3}|\nabla u|^2 \nabla u) dx = 0.$$

Both integrals can be written in terms of θ, λ, μ, and ρ. (The precise expressions can be found in Jüngel/Matthes 2007.) The problem of finding a constant $c_0 > 0$ such that $J - c_0 K \geq 0$ can now be formulated as:

Find c_0, c_1, c_2 such that $J - c_0 K = J - c_0 K + c_1 I_1 + c_2 I_2 \geq 0$.

A tedious computation shows that $J - c_0 K + c_1 I_1 + c_2 I_2$ equals

$$J - c_0 K = \int_{\mathbb{T}^d} u^{2\gamma} (d\lambda^2 a_1 + \lambda\theta^2 a_2 + Q(\theta, \mu, \rho)) dx,$$

where

$$a_1 = 1 - dc_0 - (d-1)c_1,$$
$$a_2 = 2(\gamma - 1)(1 - dc_0 - (d-1)c_1) + (d+2)c_2 - 2,$$

and $Q(\theta, \mu, \rho)$ depends on θ, μ, and ρ but not on λ.

Third, we simplify the problem in the following way. We choose to eliminate λ by fixing c_1 and c_2 such that $a_1 = a_2 = 0$. Then $Q(\theta, \mu, \rho)$ becomes

$$Q(\theta, \mu, \rho) = \frac{1}{(d-1)^2(d+1)} (\underbrace{b_1\mu^2 + 2b_2\mu\theta^2 + b_3\theta^4}_{\geq 0 \text{ if } c_0 \leq p(\gamma)/(p(\gamma)-p(0))} + \underbrace{b_4\rho^2}_{\geq 0 \text{ if } c_0 < 1}),$$

where b_i depend on d, c_0, and γ. Thus, $Q \geq 0$ if $0 \leq c_0 \leq p(\gamma)/(p(\gamma) - p(0))$, and choosing $\kappa_\gamma = c_0/\gamma^2$ gives the conclusion.

Elimination of λ from the above integrand is certainly not the only strategy to simplify the problem in such a way that it can be analytically solved. However, there is strong evidence from numerical studies of the multivariate polynomial that this strategy leads to the optimal values for c_0, at least for γ close to one.

We notice that the inequality (14) for $\gamma = 1$ has been employed in the existence proof of the multi-dimensional DLSS equation. Indeed, one of the main steps of the existence

proof is the derivation of a priori estimates. For instance, ony may employ $\log u$ as a test function in the weak formulation of

$$u_t + \sum_{i,j} \partial_{ij}^2 (u \partial_{ij}^2 \log u) = 0.$$

Then

$$\partial_t \int_{\mathbb{T}^d} u(\log u - 1)dx = \int_{\mathbb{T}^d} u_t \log u \, dx = -\sum_{i,j} \int_{\mathbb{T}^d} u(\partial_{ij}^2 \log u)^2 dx,$$

and (14) shows that

$$\partial_t \int_{\mathbb{T}^d} u(\log u - 1)dx + 4\kappa_1 \int_{\mathbb{T}^d} (\Delta u)^2 dx \le 0.$$

This provides essentially an H^2 estimate for the solution $u(\cdot, t)$ and it is the key of the existence proof. Again, we refer to Jüngel/Matthes 2007 for details.

5. Open problems

By the presented algorithmic entropy construction method, many properties for nonlinear PDEs can be derived. However, the method is still under development, and there are many open problems. We mention only a few:

- The results are all valid for positive smooth solutions to the corresponding PDEs. In order to make the computations rigorous, it is necessary to find an appropriate approximation of the PDE which satisfies two constraints: It should allow for smooth positive solutions, and the approximation should not destroy the entropy structure of the equation. For the thin-film equation, the term u^β has been regularized (see Bernis/Friedman 1990). Concerning the DLSS equation, we have employed the exponential variable $u = e^y$ (see Jüngel/Pinnau 2000, Jüngel/Matthes 2007). It would be nice to have a general strategy for the approximation of a nonlinear PDE.
- We can allow for compound equations which are homogeneous in u, like the "destabilized" thin-film equation

$$u_t + (u^\beta u_{xxx} + qu^\beta u_x)_x = 0 \quad \text{in } \mathbb{T}.$$

If $q < 0$, both terms in the brackets have the right sign (in the sense of well-posedness) and entropy estimates are straight forward (just add the admissible entropies for the thin-film and the porous-medium equation). However, if $q > 0$, the sign of the second-order term has a destabilizing effect. It is still possible to apply our method, and we found entropy estimates as long as $q < (2\pi/L)^2$ (L being the length of the 1D torus). We do not know how to handle an equation of the type

$$u_t + (u^\beta u_{xxx} + qu^\gamma u_x)_x = 0 \quad \text{in } \mathbb{T},$$

if $q > 0$ and $\beta \ne \gamma$ since our method is based on homogenity in u.

- Our method applies so far only to equations but not to systems. For instance, it is well-known that there are entropies for cross-diffusion systems like

$$(b(u))_t - \operatorname{div}(A(u)\nabla u) = f(u),$$

where $u(x,t) \in \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone function (in the sense of operator theory), and $A(u)$ is a symmetric positive definite $\mathbb{R}^n \times \mathbb{R}^n$ matrix. There are many applications in which such systems appears (see, for instance, Degond/Génieys/Jüngel 1997). Another example are the Euler equations of gas dynamics (although they are hyperbolic conservation laws and not parabolic equations). Is it possible to find new entropies for these systems of PDEs by applying the entropy construction method?

- The quantifier elimination algorithms are valid for all polynomials. Actually, we are only interested in a subclass of polynomials,

$$P(\xi) = \sum_{p_1,\ldots,p_k} c_{p_1,\ldots,p_k} \xi_1^{p_1} \cdots \xi_k^{p_k}$$

satisfying $1 \cdot p_1 + \cdots + k \cdot p_k = k$. Is it possible to speed up the existing algorithms for such polynomials?

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