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# Algebraic structure of hypocoercive equations & application to large-time behavior of kinetic equ's

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# Framework & goals

- Given an evolution eq: $\frac{\mathrm{d}}{\mathrm{d}t} f = -L\, f,\ t \geq 0;\quad L$ ... const-in-$t$ operator
- Assume $-L$ is semi-dissipative, i.e. $\quad L_H := \frac{1}{2}(L + L^*) \geq 0$
- Assume $L$ has a unique steady state: $L\, f_\infty = 0$

- $L$ is called hypocoercive if: $\|f(t) - f_\infty\| \leq c\, e^{-\mu t} \|f_0 - f_\infty\|$, $t \geq 0$, $\forall f_0$;
  but typically without coercivity of $L$

  for an ODE: hypocoercive $=$ positive stable, i.e. $\quad \Re(\lambda_j(L)) \geq \mu > 0$

- Hypocoercivity is an interplay of 2 evolution mechanisms:
  a degenerate dissipation/relaxation & mixing for the non-dissipative "directions".

GOAL: Quantify the exponential decay behavior.

# III. Degenerate Fokker-Planck equations with (non)linear drift

## Outline:

1. long-time decay of Fokker-Planck equations
2. short-time decay of Fokker-Planck equations
3. non-symmetric Fokker-Planck equations: optimal decay
4. non-$L^2$ Lyapunov functionals; non-quadratic confinement potential

# degenerate Fokker-Planck equations

$$f_t \;=\; \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: -Lf\,, \quad x \in \mathbb{R}^d \tag{1}$$

with degenerate $0 \le \mathbf{D} \in \mathbb{R}^{d\times d}$ is degenerate parabolic;
(symmetric part of) $L$ is not coercive.

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(symmetric part of) $L$ is not coercive.

**Definition 1 (Villani 2009)**

Consider $L$ on Hilbert space $\mathcal{H}$ with $\mathcal{K} = \ker L$; let $\tilde{\mathcal{H}} \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. $\mathcal{H}$ ... weighted $L^2$, $\tilde{\mathcal{H}}$ ... weighted $H^1$).
$L$ is called hypocoercive on $\tilde{\mathcal{H}}$ if $\exists\, \lambda > 0,\ c \geq 1$:

$$\|\mathrm{e}^{-Lt} f_0\|_{\tilde{\mathcal{H}}} \leq c\, \mathrm{e}^{-\lambda t} \|f_0\|_{\tilde{\mathcal{H}}} \qquad \forall\, f_0 \in \tilde{\mathcal{H}}$$

- typically $c > 1$

# hypocoercive Fokker-Planck equation

$$f_t = \mathrm{div}\left(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\right)$$

can be normalized such that $\mathbf{D} = \mathbf{C}_H$ (from now assumed).
Then $f_\infty(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$; $\mathcal{H} := L^2(f_\infty^{-1})$.

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Condition A for hypocoercivity:

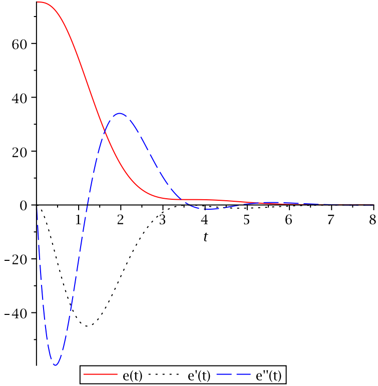
1. No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
   (equivalent: $L$ is hypoelliptic.)
2. Let $\mathbf{C}_H \in \mathbb{R}^{d\times d} \geq 0$.

   $\Rightarrow$ $\mathbf{C}$ is positive stable (i.e. $\Re\lambda_C > 0$).
   $\exists$ confinement potential; drift towards $x = 0$.

- hypoelliptic + confinement = hypocoercive (for FP eq.)

# typical decay of degenerate Fokker-Planck equation

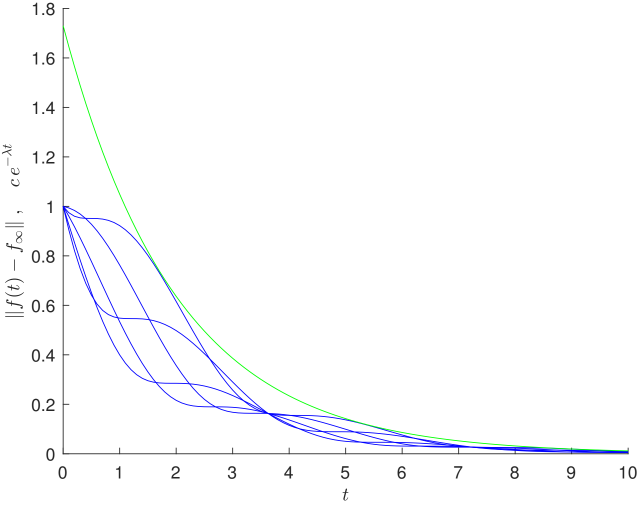
decay of $e(t) := \|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1})}$ :




degenerate FP eq. with $\mathbf{D} \geq 0$: $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$

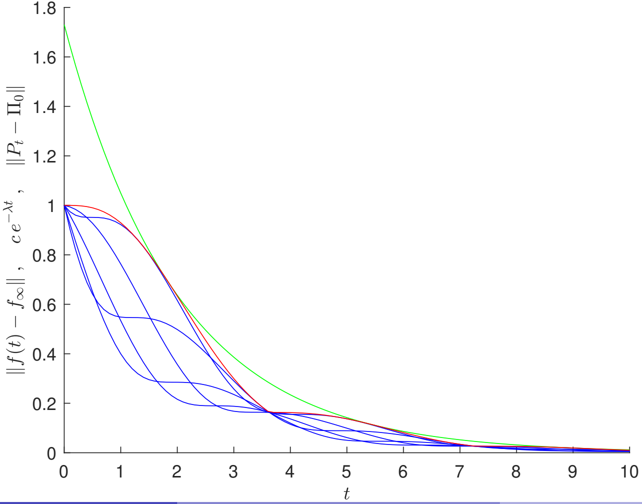
# decay estimates for Fokker-Planck equations

Goal 1: best exponential decay $\|f(t) - f_\infty\|_{\mathcal{H}} \leq c\, e^{-\lambda t} \|f(0) - f_\infty\|_{\mathcal{H}}$



# decay estimates for Fokker-Planck equations

Goal 2: find exact PDE-propagator norm $\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})}$ $\Rightarrow$ Goal 1



# propagator norm of (normalized) Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf\,, \quad \mathbf{D} = \mathbf{C}_H$$

**key Theorem 1 (AA-Signorello-Schmeiser 2022)**

*Let $L$ satisfy Condition A (i.e. $L$ is hypocoercive). Then*

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2\,, \quad t \geq 0$$

$\Pi_0$ ... *projection on* $\operatorname{span}[f_\infty]$

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<u>ex:</u> [Gadat-Miclo '13] $f_t = -vf_x + axf_v + (vf)_v + f_{vv}$; $f_\infty(x,v) = c\,e^{-\frac{a}{2}x^2 - \frac{v^2}{2}}$

normalized Fokker-Planck equ: $\mathbf{C}_a = \begin{pmatrix} 0 & -\sqrt{a} \\ \sqrt{a} & 1 \end{pmatrix}, \quad a > 0$

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = C_a(t)\,\exp\Big(-\frac{1-\sqrt{(1-4a)_+}}{2}\,t\Big),$$

$C_a(t) = \mathcal{O}(1)$ for $a \neq \frac{1}{4}$, $\quad C_{1/4}(t) = \mathcal{O}(t),\ t \to \infty$

# sharp long-time decay of (normal.) Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf\,, \quad \mathbf{D} = \mathbf{C}_H \tag{2}$$

## Corollary 1 (of key Theorem)

*Let $\mathbf{C} \in \mathbb{R}^{d\times d}$ be non-defective and satisfy Condition A (i.e. $\mathbf{C}$ is hypocoercive). Let $(c_1, \mu)$ be the optimal constants for $\dot{x} = -\mathbf{C}x$ in estimate*

$$\|x(t)\|_2 \le c_1\, e^{-\mu t}\|x_0\|, \quad t \ge 0.$$

*Then, they are optimal for (2):*

$$\|f(t) - f_\infty\|_{\mathcal{H}} \le c_1\, e^{-\mu t}\|f_0 - f_\infty\|_{\mathcal{H}}, \quad \int_{\mathbb{R}^d} f_0(x)\,dx = 1$$

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<u>ex:</u> For $d = 2$: $\Re\lambda_1^{\mathbf{C}} = \Re\lambda_2^{\mathbf{C}}$ ; $\quad c_1 = \sqrt{\operatorname{cond}(\mathbf{P})}$ (from $\mathbf{PC} + \mathbf{C}^\top\mathbf{P} \ge 2\mu\mathbf{P}$)

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# short-time decay of Fokker-Planck equation

ex: [Gadat-Miclo '13] $f_t = -vf_x + axf_v + (vf)_v + f_{vv} := -L_a f$

normal. Fokker-Planck: $\mathbf{C}_a = \begin{pmatrix} 0 & -\sqrt{a} \\ \sqrt{a} & 1 \end{pmatrix}$, hypocoercivity index $= 1$

$$\text{for } a \geq \frac{1}{4}: \quad \|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - \frac{a}{6}t^3 + o(t^3),\ t \to 0+$$

Conjecture: Decay "power 3 should be seen as an order of the hypocoercivity of the operator $L_a$."

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GOAL: Make this connection concrete, in analogy to ODEs.

# short-time decay of Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big), \quad \mathbf{D} = \mathbf{C}_H \tag{3}$$

**Definition 2**

The *hypocoercivity index of* (3) *is the smallest integer* $m_{HC} \in \mathbb{N}_0$*, such that* $\displaystyle\sum_{j=0}^{m_{HC}} \mathbf{C}_S^j \mathbf{D} (\mathbf{C}_S^\top)^j > 0.$

*(Also valid for* (3) *not normalized, i.e.* $\mathbf{D} \neq \mathbf{C}_H$*.)*

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**Corollary 3 (of key Theorem: $\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2$)**

*The HC-index of* (3) *is* $m_{HC}$ *iff (with some* $c > 0$*)*

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - ct^{2m_{HC}+1} + \mathcal{O}(t^{2m_{HC}+2}), \quad t \to 0+ \ .$$

$m_{HC}$ *determines the short-t decay; high index is "bad" for quick decay.*

proof: HC-index of (3) = HC-index of ODE ($\dot{x} = -\mathbf{C}x$).

# short-time decay of Fokker-Planck: second interpretation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf\,, \quad \text{with HC-index } m_{HC} \in \mathbb{N}_0$$

- Then: short-time regularization:

**Theorem 4 ([Villani '09] for kinetic FP; [AA-Erb '14] for HC-index)**

$$\left\| \nabla \frac{f(t)}{f_\infty} \right\|_{L^2(f_\infty)} \le c\, t^{-(m_{HC}+\frac{1}{2})} \left\| \frac{f_0}{f_\infty} - 1 \right\|_{L^2(f_\infty)}, \quad 0 < t \le \delta \qquad (4)$$

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- For Fokker-Planck eq. this is equivalent to the short time decay:

**Proposition 1 (AA-Schmeiser-Signorello 2022)**

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - ct^a + o(t^a), \quad t \to 0+$$

*iff regularization* (4) *holds with rate* $t^{-a/2}$.

# Proof of main result (step 1)

**key Theorem 2 (AA-Schmeiser-Signorello 2022)**

*Let* $L = -\operatorname{div}\Big(\mathbf{D}\,\nabla\cdot + \mathbf{C}\,x\,\cdot\Big)$ *satisfy Condition A (i.e. L is hypocoercive). Then*

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2, \quad t \geq 0$$

$\Pi_0$ ... *projection on* $\operatorname{span}[f_\infty]$, $f_\infty = c\, e^{-|x|^2/2}$

- $L$ ... nonsymmetric. Still, $\exists$ a partially orthogonal decomposition:

$$\mathcal{H} := L^2(f_\infty^{-1}) = \bigoplus_{m\in\mathbb{N}_0}^{\perp} V^{(m)}; \quad V^{(m)} = \operatorname{span}[g_\alpha(x) := (-1)^{|\alpha|}\nabla^\alpha f_\infty,\ |\alpha| = m]$$

$$\sigma(L) = \Big\{\sum_{j=1}^d \alpha_j\lambda_j,\ \alpha \in \mathbb{N}_0^d\Big\}; \quad \lambda_j \ldots \text{eigenvalues of } \mathbf{C} \in \mathbb{R}^{d\times d}$$

# main proof (step 2): evolution in subspaces $V^{(m)}$ decouple!

$d_\alpha(t)$ ... coefficient of $g_\alpha(x)$, $\alpha \in \mathbb{N}_0^d$, $x \in \mathbb{R}^d$

<u>ex. $d = 2$:</u> • $m = 0$: $d_{(0,0)}(t) = const$ ... steady state.

- $m = 1$: $\frac{\mathrm{d}}{\mathrm{d}t}\binom{d_{(1,0)}}{d_{(0,1)}} = -\mathbf{C}\binom{d_{(1,0)}}{d_{(0,1)}}$

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- $m=2$: $\begin{pmatrix} d_{(2,0)} \\ d_{(1,1)} \\ d_{(0,2)} \end{pmatrix}$ ... impractical !

better: $D^{(2)}(t) := \begin{pmatrix} d_{(2,0)} & d_{(1,1)}/2 \\ d_{(1,1)}/2 & d_{(0,2)} \end{pmatrix}(t) \in \mathbb{R}^{2\times 2}$

$$\frac{\mathrm{d}}{\mathrm{d}t} D^{(2)} = -\left(\mathbf{C}D^{(2)} + D^{(2)}\mathbf{C}^\top\right)$$

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- $m \geq 3$: $D^{(m)}(t)$ ... symmetric $m$-order tensor

$$\frac{\mathrm{d}}{\mathrm{d}t} D^{(m)}(t) = -m \,\mathrm{Sym}\big( \underbrace{\mathbf{C} \odot D^{(m)}(t)}_{\text{mult. on 1st index}} \big) \quad \text{... tensored drift ODE}$$

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$\Rightarrow$ FP = 2nd quantization of drift ODE in Bosonic Fock space of $\mathbb{R}^2$

# main proof (step 3): evolution in subspaces $V^{(m)}$

• ingredient for evolution equation in $V^{(m)}$:
rank-1 decomposition of order-$m$ tensors:

$$D^{(m)} = \sum_{k=1}^{s} \mu_k v_k^{\otimes m}, \quad \mu_k \in \mathbb{R},\ v_k \in \mathbb{R}^d \tag{5}$$

**Lemma 1**

*Let (5) be the decomposition of $D^{(m)}(0)$. Then, the evolution in $V^{(m)}$ is given by*

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- subspaces $V^{(m)}$ are orthogonal
- decay behavior of $\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})}$ determined only by 1st subspace $\square$

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# non-symmetric Fokker-Planck equations: optimal decay

- given: $f_\infty(x) = c_{\mathbf{K}} \exp\Big(-\frac{x^T \mathbf{K}^{-1} x}{2}\Big),\ \ x \in \mathbb{R}^d$ (non-isotropic)
- Q: Which $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C} x f)$ gives fastest decay in the sense:

$$\|f(t) - f_\infty\|_{L^2(f_\infty^{-1})} \le c\, e^{-\lambda t} \|f_0 - f_\infty\|_{L^2(f_\infty^{-1})}, \quad \forall\, f_0 \in \mathcal{H};\ t \ge 0,$$

with (1) maximal $\lambda > 0$ and (2) minimal $c \ge 1$ ?

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with (1) maximal $\lambda > 0$ and (2) minimal $c \ge 1$ ?

- Fix $\mathrm{Tr}\,(\mathbf{D}) = d$, for a well-posed problem.

- Well known: In FP equations, non-symmetric drift can speed up convergence to equilibrium (compared to symmetric FP evolution).

# non-symmetric Fokker-Planck: optimal decay results

**Theorem 5 (Guillin-Monmarché 2016)**

1. $\lambda_{opt} = \lambda_{max}(\mathbf{K}^{-1})$
2. $\mathbf{D}_{opt}$ *constructive, singular:* $\mathrm{range}(\mathbf{D}_{opt}) \subset$ *eigenspace of* $\lambda_{max}(\mathbf{K}^{-1})$

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**Theorem 6 (AA-Signorello, KRM 2022)**

1. $\forall\, c > 1$: $\exists\, \mathbf{C}_{opt} = \mathbf{C}_{opt}(c)$ *(constructive) such that*

$$\|f(t) - f_\infty\|_{L^2(f_\infty^{-1})} \le c\, e^{-\lambda_{opt} t} \|f_0 - f_\infty\|_{L^2(f_\infty^{-1})}, \quad t \ge 0.$$

2. *If* $\lambda_{max}$ *is a simple eigenvalue of* $\mathbf{K}^{-1}$*, then:*

$$\mathrm{rank}(\mathbf{D}_{opt}) = 1, \quad \textit{HC-index of } L_{\mathbf{C},\mathbf{D}} = d - 1 \quad \textit{(i.e. maximal)}$$

For $\lambda_{max}(\mathbf{K}^{-1})$ simple: high index is "good", gives fastest decay for large $t$.
$\Rightarrow$ opposing influence of HC-index for short / large $t$ !

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# extensions: non-spectral tools, entropy method

so far:

- spectral decomposition only for FP equation with linear drift:
  $f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big)$ ,
  or *kinetic FP equation* with quadratic confinement potential $V(x)$:
  $f_t = -vf_x + V_x\, f_v + (vf)_v + f_{vv}$ .
- only $L^2$-type norms as Lyapunov functional: $\|\cdot\|_{L^2(f_\infty^{-1})}$ .

new:

- non-quadratic potentials $V(x) \sim$ nonlinear drift fields $\mathbf{C}x$ .
- relative entropies as Lyapunov functionals.

# admissible relative entropies (for entropy method)

for probability densities $f_{1,2}$:

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^d} \psi\left(\frac{f_1}{f_2}\right) f_2 \,\mathrm{d}x \geq 0 \qquad \text{... relative entropy}$$

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$

$e_\psi(f_1|f_2) = 0 \iff f_1 = f_2$

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$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^d} \psi\left(\frac{f_1}{f_2}\right) f_2 \,\mathrm{d}x \geq 0 \qquad \dots \text{ relative entropy}$$

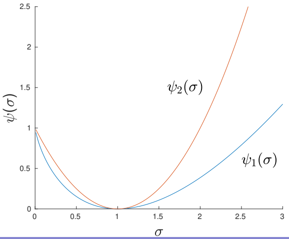
$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \qquad \dots$ entropy generators

$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$

$e_\psi(f_1|f_2) = 0 \iff f_1 = f_2$

examples: 1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$ (Boltzmann entropy)

2) $\psi_p(\sigma) = [\sigma^p - 1 - p(\sigma - 1)]/(p-1), \quad 1 < p \leq 2$



# entropy method for linear symmetric FP equation

$$f_t = \operatorname{div}\left(\mathbf{D}\cdot[\nabla f + f\nabla A(x)]\right), \quad t>0; \quad \mathbf{D}\geq 0 \tag{6}$$

If $f_0$ is probability density $\Rightarrow f(\cdot,t)$ is probability density.
$f_\infty(x) = e^{-A(x)}$ ... (unique) normalized steady state

**Lemma 2 (entropy dissipation)**

*Let $f(t)$ solve* (6).

$$\begin{aligned}\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|f_\infty) &= -\int_{\mathbb{R}^d} \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla^\top \frac{f(t)}{f_\infty}\cdot\mathbf{D}\cdot\nabla\frac{f(t)}{f_\infty}\, f_\infty\,\mathrm{d}x \\ &=: -I_\psi(f(t)|f_\infty) \leq 0 \text{ ... \textit{(negative) Fisher information}}\end{aligned}$$

- $e_\psi$ ... Lyapunov functional

# Step 1: exponential decay of entropy dissipation

Let $0 < \mathbf{D} \in \mathbb{R}^{d\times d}$; $f_\infty(x) = e^{-A(x)}$.

**Theorem 7 (Bakry-Emery '84; AA-Markowich-Toscani-Unterreiter '01)**

*Let $I_\psi(f_0|f_\infty) < \infty$. Let $\mathbf{D}, A$ satisfy a*

$$\boxed{\textit{Bakry - Emery condition} \qquad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1}} \quad \forall x \in \mathbb{R}^d$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty)\,, \qquad t \geq 0$$

BEC: $A$ . . . uniformly convex if $\mathbf{D} = \mathbf{I}$

- robust w.r.t. many nonlinear perturbations

# Step 2: exponential decay of relative entropy for $\mathbf{D} > 0$

**Theorem 8**

*Let* $\mathbf{D}$, $A$ *satisfy BEC:* $\frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t}\, e_\psi(f_0|f_\infty), \qquad t \geq 0$$

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Proof: from proof of Theorem 7 :

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \qquad \Big| \int_t^\infty \dots \,\mathrm{d}t$$

Since $I(t)$, $e(t) \xrightarrow{t\to\infty} 0$:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} e(t)}_{=-I(t)} \leq -2\lambda_1 e(t) \qquad (= \text{Poincaré/Log. Sobolev-type inequality}) \qquad \square$$

# new entropy method for degen. FP: $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f)$

- $e'(t) = 0$ for some $f \neq f_\infty \quad \Rightarrow \quad$ entropy dissipation/Fisher info:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^d} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \, \mathrm{d}x =: -I_\psi(f) \leq 0$$

is "useless" as Lyapunov functional. BEC $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1}$ meaningless.

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$\Rightarrow$ define modified "entropy dissipation" as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^d} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{>0} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \,\mathrm{d}x \geq 0$$

goal: estimate between $S_\psi(f(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} S_\psi(f(t))$ for "good" choice of $\mathbf{P} = \mathbf{P}^* > 0$ with $\mathbf{PC} + \mathbf{C}^*\mathbf{P} \geq 2\mu \mathbf{P}$ (if $\mathbf{C}$ is non-defective).

Then: $\mathbf{P} \geq c_P\, \mathbf{D}$ $\Rightarrow$ $S_\psi(f(t)) \geq c_P\, I_\psi(f(t)) \searrow 0$ as $t \to \infty$

# Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty} \, f_\infty \, \mathrm{d}x \geq 0$$

## Proposition 2

$\mu := \min\{\Re\, \lambda_C\}$ *for drift matrix* $\mathbf{C}$. *Let* $f_0$ *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \; \textit{weighted } H^1\textit{–seminorm})$$

1. *If all* $\lambda_C^{\min}$ *are non-defective* $\Rightarrow \quad S_\psi(f(t)) \leq e^{-2\mu t} S_\psi(f_0)\,, \quad t \geq 0;$

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2. *If one* $\lambda_C^{\min}$ *is defective*
   $\Rightarrow \quad S_\psi(f(t), \varepsilon) \leq e^{-2(\mu-\varepsilon)t} S_\psi(f_0, \varepsilon)\,, \quad t \geq 0.$

- spectral gap of $\mathbf{C}$ $\rightarrow$ sharp decay rate of Fokker-Planck equation

# Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t} S_\psi(f(t)) = -\int \psi''\Big(\frac{f(t)}{f_\infty}\Big) u(t)^\top \quad \underbrace{\Big[\mathbf{PC} + \mathbf{C}^\top \mathbf{P}\Big]}_{\geq 2\mu \mathbf{P} \text{ ... replaces BEC}} \quad u(t) f_\infty \,\mathrm{d}x$$

$$-2\int \underbrace{\mathrm{Tr}\,(\mathbf{XY})}_{\geq 0} f_\infty \,\mathrm{d}x \leq -2\mu\, S_\psi(f(t))\,; \quad u(t) := \nabla \frac{f(t)}{f_\infty}$$

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use $\quad \mathbf{X}(x) \; := \; \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{\mathrm{IV}} \end{pmatrix} \Big(\frac{f}{f_\infty}\Big) \geq 0 \quad ,$

since $\;\det \mathbf{X} \; = \; \frac{1}{2}\psi''\psi^{\mathrm{IV}} - (\psi''')^2 \geq 0$ (for admissible rel. entropies) ;

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$$\mathbf{Y}(x) := \begin{pmatrix} \mathrm{Tr}\,(\mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}\frac{\partial u}{\partial x}) & u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u \\ u^\top \mathbf{D}\frac{\partial u}{\partial x}\mathbf{P}u & (u^\top \mathbf{P}u)\,(u^\top \mathbf{D}u) \end{pmatrix} \geq 0\,,$$

with Cauchy-Schwarz

# Step 2: exponential decay of relative entropy

**Theorem 9**

*Let $f_0$ satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right)\left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \,\mathrm{dx} < \infty \,.$$

$$\Rightarrow \quad \boxed{e_\psi(f(t)|f_\infty) \overset{\text{Log. Sobolev ineq.}}{\leq} c\, S_\psi(f(t)) \leq c\, e^{-2\mu t} S_\psi(f_0)\,, \quad t \geq 0}$$

*( reduced rate for a defective $\lambda_C^{\min}$: $2(\mu - \varepsilon)$ )*

# kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \, \mathrm{div}_v(vf)\, ; \quad x, v \in \mathbb{R}^d$$

*steady state* factorizes in $x$, $v$: $\quad f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2} + V(x)\right]}$

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**Theorem 10 (Arnold-Erb, Achleitner-Arnold 2015)**

*Let* $d = 1$; $V(x) = \frac{\omega_0^2}{2}|x|^2 + \tilde{V}(x)$ *... given confinement potential with*
$\sqrt{\max V''(x)} - \sqrt{\min V''(x)} \le \nu$

$$\Rightarrow \qquad e_\psi(f(t)|f_\infty) \le c\, S_\psi(f_0) e^{-2\kappa t}, \qquad t \ge 0$$

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- $\tilde V$ ... $\mathcal{O}(1)$ perturbation
- Greens function not explicit (in contrast to linear drift fields)

# Conclusion

- Hypocoercivity index characterizes the regularization rate and short-time decay of Fokker-Planck equations $f_t = \mathrm{div}(\mathbf{C}[\nabla f + xf])$: first polynomial, then exponential decay of $\|e^{-Lt}\|$.
- non-symmetric FP equation: fastest decay when large HC-index.
- entropy method for degenerate FP equations: modified "Fisher information" as (sharp) Lyapunov functional

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