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Algebraic structure of hypocoercive equations & application to large-time behavior of kinetic equ's

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Framework & goals

- Given an evolution eq: $\frac{\mathrm{d}}{\mathrm{d}t} f = -L\, f, \; t \geq 0$; $\quad L$ ... const-in-t operator
- Assume $-L$ is semi-dissipative, i.e. $\quad L_H := \frac{1}{2}(L + L^*) \geq 0$
- Assume L has a unique steady state: $L\, f_\infty = 0$

- L is called hypocoercive if: $\|f(t) - f_\infty\| \leq c\, e^{-\mu t} \|f_0 - f_\infty\|$, $t \geq 0$, $\forall f_0$;
  but typically without coercivity of L

  for an ODE: hypocoercive = positive stable, i.e. $\quad \Re(\lambda_j(L)) \geq \mu > 0$

- Hypocoercivity is an interplay of 2 evolution mechanisms:
  a degenerate dissipation/relaxation & mixing for the non-dissipative "directions".

GOAL: Quantify the exponential decay behavior.

II. Kinetic equations with degenerate relaxations / modal decomposition

Outline:

1. Goldstein-Taylor equation: Lyapunov functional for sharp large-t behavior
2. Lorentz equation: uniform short-t behavior of modes
3. linearized/nonlinear BGK equations: approximate Lyapunov functionals

Goldstein-Taylor on torus: kinetic equation with discrete velocities ± 1

$$\begin{cases} \partial_t f_+(x,t) + \partial_x f_+(x,t) = \frac{\sigma(x)}{2}\big(f_-(x,t) - f_+(x,t)\big), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) = -\frac{\sigma(x)}{2}\big(f_-(x,t) - f_+(x,t)\big), \end{cases} \qquad t > 0,\ x \in \mathbb{T} \text{ (or } \mathbb{R})$$

- $f_\pm(\cdot)$ ... probability density in (x, v)-phase space
- $\sigma(x)$ ... local relaxation rate; $\ker \mathbf{C}_H = \{f_+(x) = f_-(x)\}$
- original applications: turbulent fluid motion, telegrapher's equation

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- original applications: turbulent fluid motion, telegrapher's equation
- mass density $u(x,t) := f_+ + f_-$, flux density $v(x,t) := f_+ - f_-$

$$\begin{cases} \partial_t u + \partial_x v &= 0, \\ \partial_t v + \partial_x u &= -\sigma(x) v, \end{cases}$$

$$f_\infty := \tfrac{1}{2}(f_{+,0} + f_{-,0})_{avg} := \tfrac{1}{4\pi}\int_0^{2\pi}[f_{+,0}(x) + f_{-,0}(x)]dx$$

- goal (on $\mathbb{T}$):
  sharp exp. decay estimates $u(t) \to u_\infty = 2f_\infty$, $v(t) \to v_\infty = 0$;
  ultimately for $\sigma(x)$

Modal decomposition of (u, v) on torus for $\sigma = const$:

$$\frac{d}{dt}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} = -\begin{pmatrix}0 & ik\\ ik & \sigma\end{pmatrix}\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix} =: -\mathbf{C}_k\begin{pmatrix}\widehat{u}(k)\\ \widehat{v}(k)\end{pmatrix}, \qquad k \in \mathbb{Z}.$$

$\widehat{u}(0,t) = \widehat{u}_0, \quad \widehat{v}(0,t) = \widehat{v}_0 e^{-\sigma t};$ $\qquad \sigma = const \Rightarrow$ all x-modes decoupled!

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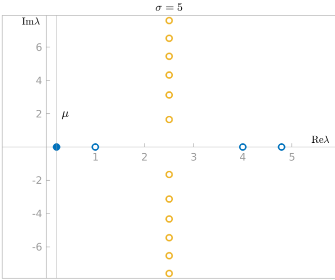
<u>Case I</u>: $0 < |k| < \frac{\sigma}{2}$: $\lambda_{\mp,k} = \frac{\sigma}{2} \mp \sqrt{\frac{\sigma^2}{4} - k^2} \in \mathbb{R}$,

$$\mathbf{P}_k^{(I)} := \begin{pmatrix}1 & -\frac{2ki}{\sigma}\\ \frac{2ki}{\sigma} & 1\end{pmatrix} \sim \begin{pmatrix}1 & -\frac{2}{\sigma}\partial_x\\ \frac{2}{\sigma}\partial_x & 1\end{pmatrix}$$

<u>Case II</u>: $|k| = \frac{\sigma}{2}$ ... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$



eigenvalues for $k \in \mathbb{N}$

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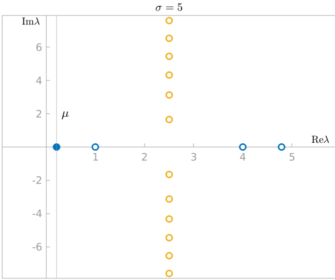

eigenvalues for $k \in \mathbb{N}$

<ins>Case II</ins>: $|k| = \frac{\sigma}{2}$ ... defective: $\mathcal{O}((1+t)e^{-\frac{\sigma}{2}t})$

<ins>Case III</ins>: $|k| > \frac{\sigma}{2}$: $\Re\lambda_{\pm,k} = \frac{\sigma}{2}$, $\mathbf{P}_k^{(III)} := \begin{pmatrix}1 & -\frac{i\sigma}{2k}\\ \frac{i\sigma}{2k} & 1\end{pmatrix} \sim \begin{pmatrix}1 & \frac{\sigma}{2}\partial_x^{-1}\\ -\frac{\sigma}{2}\partial_x^{-1} & 1\end{pmatrix}$

note: $\mathbf{P}_k^{(III)} \overset{|k|\to\infty}{\longrightarrow} \mathbf{I}$

Modal Lyapunov functional as pseudo-differential operator

<u>Situation A:</u> $0 < \sigma < 2$: all modes $k \neq 0$ in Case III

$$\mathcal{E}(\widehat{u},\widehat{v}) := \sum_{k\in\mathbb{Z}\setminus\{0\}} \left\| \begin{pmatrix} \widehat{u}(k) \\ \widehat{v}(k) \end{pmatrix} \right\|^2_{\mathbf{P}_k^{(III)}} + \left\| \begin{pmatrix} \widehat{u}(0) \\ \widehat{v}(0) \end{pmatrix} \right\|^2_2 \qquad \text{(modal form)}$$

$$= \underbrace{\|\tilde{u}\|^2_{L^2} + \|v\|^2_{L^2} - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx}_{=:E_\sigma(\tilde{u},v)\,, \qquad (\partial_x^{-1}f)_{avg}:=0}\,, \quad \tilde{u}(x) := u(x) - u_{avg} \quad \text{(non-modal)}$$

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Situation B: $\sigma = 2$: defective

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Situation B: $\sigma = 2$: defective

Situation C: $\sigma > 2$: Low modes in Case I determine global decay rate, high modes in Case III.

$\Rightarrow$ Mixture would yield a (complicated) ΨDO with non-smooth symbol.

Mixed modes in Situation C ($\sigma > 2$)

remedy:

Only modes $k = \pm 1$ determine global decay rate $\mu = \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}$.
$\Rightarrow$ Higher modes gives some leeway to adjust the decay estimate.
$\Rightarrow$ Suboptimal norms $\|.\|_{\mathbf{P}_k}$ can be used for $|k| \geq 2$:

Lemma 1

Let $\sigma > 2$*:* $\mathbf{P}_k := \begin{pmatrix} 1 & -\frac{2i}{k\sigma} \\ \frac{2i}{k\sigma} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{2}{\sigma}\partial_x^{-1} \\ -\frac{2}{\sigma}\partial_x^{-1} & 1 \end{pmatrix}$ *satisfies*

$$\mathbf{C}_k^* \mathbf{P}_k + \mathbf{P}_k \mathbf{C}_k \geq 2\mu \mathbf{P}_k \quad \forall k \neq 0\,. \tag{1}$$

$\Rightarrow \{\mathbf{P}_k\}_{k \neq 0}$ is "good enough" for optimal decay, since $\mathbf{P}_1 = \mathbf{P}_1^{(I)}$, all modes satisfy (1) with uniform rate μ.
also: $\{\mathbf{P}_k\} \sim$ simple ΨDO.

Decay of spatial Lyapunov functional – with symbol $\mathbf{P}_k$

$$E_\sigma(\tilde{u}, v) := \|\tilde{u}\|_{L^2}^2 + \|v\|_{L^2}^2 - \frac{\sigma}{2\pi}\int_0^{2\pi} \Re\left(\partial_x^{-1}\tilde{u}\overline{v}\right) dx\,, \quad \tilde{u}(x) := u(x) - u_{\mathrm{avg}}$$

Proposition 1

i) *If* $0 < \sigma < 2$*:* $E_\sigma\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \leq E_\sigma\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\sigma t}$.

ii) *If* $\sigma > 2$*:* $E_{\frac{4}{\sigma}}\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \leq E_{\frac{4}{\sigma}}\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\left(\sigma - \sqrt{\sigma^2 - 4}\right)t}$.

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Lemma 2

For $0 < \theta < 2$*:*

$$\left(1 - \frac{\theta}{2}\right)\left(\|f\|_{L^2}^2 + \|g\|_{L^2}^2\right) \le E_\theta\left(f, g\right) \le \left(1 + \frac{\theta}{2}\right)\left(\|f\|_{L^2}^2 + \|g\|_{L^2}^2\right)\ .$$

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Remark: The sharp rate from Proposition 1 can also be recovered from [Dolbeault-Mouhot-Schmeiser 2015] when optimizing a parameter $\delta(k)$.

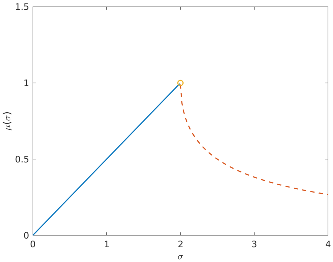
$\Rightarrow$ Decay in L^2 with sharp rate $\mu(\sigma)$:

Theorem 1 (AA-Einav-Signorello-Wöhrer, JSP 2021)

For $\sigma = const \neq 2$:

$$\left\| \begin{pmatrix} f_+(t) \\ f_-(t) \end{pmatrix} - \begin{pmatrix} f_\infty \\ f_\infty \end{pmatrix} \right\|_{L^2} \le C_\sigma \left\| \begin{pmatrix} f_{+,0} \\ f_{-,0} \end{pmatrix} - \begin{pmatrix} f_\infty \\ f_\infty \end{pmatrix} \right\|_{L^2} e^{-\mu(\sigma)t} \;, t \ge 0 \;;$$

$$C_\sigma := \begin{cases} \sqrt{\frac{2+\sigma}{2-\sigma}}, & 0 < \sigma < 2 \\ \sqrt{\frac{\sigma+2}{\sigma-2}}, & \sigma > 2 \end{cases}, \qquad \mu(\sigma) := \begin{cases} \frac{\sigma}{2}, & 0 < \sigma < 2 \\ \frac{\sigma}{2} - \sqrt{\frac{\sigma^2}{4} - 1}, & \sigma > 2 \end{cases}$$



Extension to $\sigma(x)$ on torus

- modal decomposition unfeasible: $\sigma \neq const \Rightarrow$ all x-modes coupled!
- But $E_\theta(\tilde{u}, v) := \|\tilde{u}\|_{L^2}^2 + \|v\|_{L^2}^2 - \frac{\theta}{2\pi} \int_0^{2\pi} \Re\left(\partial_x^{-1} \tilde{u} \overline{v}\right) dx$ is still a Lyapunov functional:

Theorem 2 (AA-Einav-Signorello-Wöhrer, JSP 2021)

For $0 < \sigma_{\min} \leq \sigma_{\max} < \infty$, *let* $\theta^* := \min\left(\sigma_{\min}, \frac{4}{\sigma_{\max}}\right)$. *Then*

$$E_{\theta^*}\left(u(t) - u_{\mathrm{avg}}, v(t)\right) \leq E_{\theta^*}\left(u_0 - u_{\mathrm{avg}}, v_0\right) e^{-\alpha^* t}, \quad t \geq 0$$

for an explicit $\alpha^*(\sigma_{\min}, \sigma_{\max}) > 0$.

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- pros: Strategy is applicable to BGK models with n velocities; $n = 3$ in [AESW '21].
- cons: Decay rate α^* is not optimal.
  Sharp rate for $\sigma(x)$ in [Bernard-Salvarani 2013] is based on equivalence to telegrapher's equation; but only for $n = 2$.

Goldstein-Taylor on $\mathbb{R}$ with $\sigma \equiv 1$: modal decomposition

$$\begin{cases} \partial_t f_+(x,t) + \partial_x f_+(x,t) = \dfrac{\sigma}{2}(f_- - f_+), \\ \partial_t f_-(x,t) - \partial_x f_-(x,t) = -\dfrac{\sigma}{2}(f_- - f_+), \end{cases} \qquad t > 0,\ x \in \mathbb{R}$$

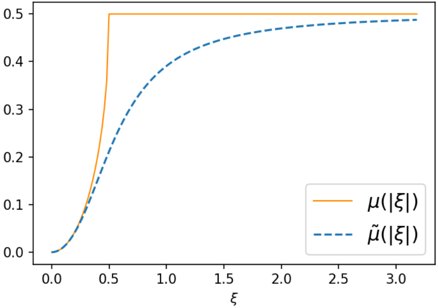
evolution of Fourier modes:

$$\frac{d}{dt}\begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} = -\begin{pmatrix} 0 & i\xi \\ i\xi & 1 \end{pmatrix}\begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} =: -\mathbf{C}_\xi \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix}, \qquad \xi \in \mathbb{R}.$$

Goldstein-Taylor on $\mathbb{R}$

3 problems:

- Decay rate $\mu(\xi)$ of small modes vanishes at $\xi = 0$.
  $\Rightarrow$ Global decay is only algebraic.



- Mixed mode–behavior, like Situation C on torus $\Rightarrow$ non-smooth ΨDO
- Defective case $|\xi| = \frac{\sigma}{2}$ is always present, cannot be skipped.
  $\Rightarrow$ Global approximation by $\tilde{\mu}(\xi)$ is one possible remedy.

Approximated modal decay function

- $\tilde{\mathbf{P}}(\xi) := \begin{pmatrix} 1 & -\frac{2\,i\,\xi}{1+4\,\xi^2} \\ \frac{2\,i\,\xi}{1+4\,\xi^2} & 1 \end{pmatrix}$ is a good approximation of the optimal scaling matrices $\mathbf{P}_\xi^{(I)}$ as $\xi \to 0$ and of $\mathbf{P}_\xi^{(III)}$ as $|\xi| \to \infty$.
- $\Rightarrow$ sharp modal spectral gap is replaced by $\tilde{\mu}(\xi) = \frac{1}{2} - \frac{1}{2\sqrt{1+4\,\xi^2\,(1+4\,\xi^2)}}$, avoids the defective case $|\xi| = \frac{\sigma}{2}$ at the kink of $\mu(\xi)$.

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- global Lyapunov functional on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(\widehat{u},\widehat{v}) := \int_{\mathbb{R}} \left\| \begin{pmatrix} \widehat{u}(\xi) \\ \widehat{v}(\xi) \end{pmatrix} \right\|^2_{\tilde{\mathbf{P}}(\xi)} d\xi = \|u\|^2 + \|v\|^2 - 4 \int_{\mathbb{R}} u\,\partial_x \underbrace{\left(1 - 4\,\partial_x^2\right)^{-1}}_{\geq \mathbf{I}} v\,dx$$

- Decay proof uses modal decay estimates for each fixed $\xi \neq 0$:

$$\left\| \begin{pmatrix} \widehat{u}(\xi,t) \\ \widehat{v}(\xi,t) \end{pmatrix} \right\|_2^2 \leq \operatorname{cond}(\tilde{\mathbf{P}}(\xi))\, e^{-2\,\tilde{\mu}(\xi)\,t} \left\| \begin{pmatrix} \widehat{u}(\xi,0) \\ \widehat{v}(\xi,0) \end{pmatrix} \right\|_2^2, \quad t \geq 0\,. \quad (2)$$

algebraic decay

Theorem 3 (AA-Dolbeault-Schmeiser-Wöhrer 2021)

Let $y := \binom{u}{v}$, *let* $u_0,\ v_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. *Then:*

$$\|y(\cdot, t)\|^2_{L^2(\mathbb{R})}$$
$$\leq \inf_{0<R\leq\frac{\sqrt{5}-1}{4}} \left(\frac{1+2\,R+4\,R^2}{1-2\,R+4\,R^2} \min\left\{2R, \sqrt{\frac{\pi}{2t}}\right\} \|y_0\|^2_{L^1(\mathbb{R})} + 3e^{-2\tilde{\mu}(R)t} \|y_0\|^2_{L^2(\mathbb{R})} \right)$$

- High modes (with $|\xi| > R$) decay exponentially in $L^2(\mathbb{R})$.
- Low modes (with $|\xi| < R$) decay algebraically, using $\|\widehat{y}_0\|^2_{\mathrm{L}^\infty(\mathbb{R}_\xi)}$:

$$\int_{|\xi|\leq R} |\widehat{y}(\xi, t)|^2 \, d\xi \overset{(2)}{\leq} \underbrace{\int_{|\xi|\leq R} \mathrm{cond}(\tilde{\mathbf{P}}(\xi))\, e^{-2\,\tilde{\mu}(\xi)\,t}\, d\xi}_{=\mathcal{O}(t^{-1/2})} \|\widehat{y}_0\|^2_{\mathrm{L}^\infty(\mathbb{R}_\xi)}$$

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1. Goldstein-Taylor equation: Lyapunov functional for sharp large-t behavior
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3. linearized/nonlinear BGK equations: approximate Lyapunov functionals

PDE-Example: Lorentz kinetic equation for $f(\mathbf{x}, \mathbf{v}, t)$

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = -\mathbf{C}_H f := \left(\frac{1}{\mathrm{meas}(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} f \,\mathrm{d}\mathbf{v} - f \right), \quad \mathbb{T}^d \times \mathbb{S}^{d-1} \times \mathbb{R}^+$$

- original model for electron transport in metals

Proposition 2 (Golse 2008, Ukai 1986)

- $-\mathbf{v}\cdot\nabla_{\mathbf{x}} - \mathbf{C}_H$ *generates a* C_0*-semigroup on* $\mathcal{H} := L^2(\mathbb{T}^d \times \mathbb{S}^{d-1})$.
- *exponential convergence to (const)* f_∞*; but rate was not quantified.*

decay in Lorentz equation for $f(\mathbf{x}, \mathbf{v}, t)$

Def. (HC-index for unbounded operator in Lorentz eq.): $m_{HC}(\mathbf{C}) = 1$ if

$$\mathbf{C}_H + \mathbf{C}_S \mathbf{C}_H \mathbf{C}_S^* \geq \kappa \mathbf{I} \quad \text{on } \left\{ f \in \mathcal{H} \,\middle|\, \frac{\partial^2 f}{\partial \mathbf{x}^2} \in \mathcal{H}, \int f \,\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} = 0 \right\} \tag{3}$$

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Theorem 4 (Achleitner-AA-Mehrmann-Nigsch, JFA 2025)

(a) *For $d = 2$: Hypocoercivity cond.* (3) *holds with* $\kappa = \frac{3-\sqrt{5}}{2}$, $m_{HC} = 1$.

(b) *short-time decay:*

$$1 - c_1 t^3 \leq \| e^{-\mathbf{C}t} - \Pi_\infty \|_{\mathcal{B}(\mathcal{H})} \leq 1 - c_2 t^3, \quad 0 \leq t \leq \tau \,,$$

Π_∞ ... *orthogonal projection on* $\text{span}[f_\infty]$.

(c) *long-time decay:*

$$\| f(t) - f_\infty \|_{\mathcal{H}} \leq \sqrt{3} e^{-\lambda t} \| f_0 - f_\infty \|_{\mathcal{H}}, \quad 0 \leq t \,, \quad \forall f_0 \in \mathcal{H} \,,$$

$\lambda \approx 0.0857$

decay in Lorentz equation

Proof:

- Fourier decomposition on $\mathbf{x}$-torus $\mathbb{T}^2$:

$$\partial_t f_{\mathbf{n}} = -(\mathbf{C}_H + i\mathbf{v}\cdot\mathbf{n}) f_{\mathbf{n}} =: -(\mathbf{C}_H - \mathbf{J}_{\mathbf{n}}) f_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}^2, \quad t > 0\,,$$

  $\mathbf{C}_H \sim \mathrm{diag}(..., 1, 1, 0, 1, 1, ...)$, $\mathbf{J}_{\mathbf{n}} = -\mathbf{J}_{\mathbf{n}}^* \in \mathcal{B}(\mathcal{H})$.

- Check hypocoercivity condition for each spatial mode $\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}$.

Part (a) – hypocoercivity index:

- $\mathbf{C}_H$ is invariant under rotations of $\mathbb{S}^1_{\mathbf{v}}$.
  $\Rightarrow$ Just analyze $\mathbf{J}_{(n_1,0)} = n_1 \mathbf{J}_{(1,0)}$ as "infinite matrices" corresponding to the angular modes $e^{ij\varphi}$ for $\mathbf{v} \in \mathbb{S}^1$.
- $m_{HC}(\mathbf{C}_H - \mathbf{J}_{\mathbf{n}}) = 1, \quad \forall\, \mathbf{n} \neq 0.$

decay in Lorentz equation

Proof-Part (c) – long-time decay:

- goal: uniform exponential decay of all modes:

$$\|f_{\mathbf{n}}(t)\|_{L^2(\mathbb{S}^1)} \le \min\left[1,\ \underbrace{\sqrt{\tfrac{2|\mathbf{n}|+1}{2|\mathbf{n}|-1}}}_{=\sqrt{\operatorname{cond}(\mathbf{P}_{|\mathbf{n}|})}}\ e^{-\lambda t}\right]\|f_{\mathbf{n}}(0)\|_{L^2(\mathbb{S}^1)} \le \sqrt{3}e^{-\lambda t}\|f_{\mathbf{n}}(0)\|_{L^2(\mathbb{S}^1)}$$

- strategy: show a Lyapunov inequality for $\mathbf{C}_{|\mathbf{n}|} := \mathbf{C}_H - |\mathbf{n}|\mathbf{J}_{(1,0)}$, $|\mathbf{n}| \ge 1$:

$$\mathbf{C}^*_{|\mathbf{n}|}\mathbf{P}_{|\mathbf{n}|} + \mathbf{P}_{|\mathbf{n}|}\mathbf{C}_{|\mathbf{n}|} \ge 2\lambda\mathbf{P}_{|\mathbf{n}|}\,,$$

with $\quad \mathbf{P}_{|\mathbf{n}|} := \left[\begin{array}{c|c|c|c} \ddots & & & \\ \quad 1 & & & \\ \hline & 1 & -i\frac{\alpha}{|\mathbf{n}|} & \\ \hline & i\frac{\alpha}{|\mathbf{n}|} & 1 & \\ \hline & & & 1 \quad \\ & & & \ddots \end{array}\right] > 0, \quad$ for some $\alpha \in \mathbb{R}$.

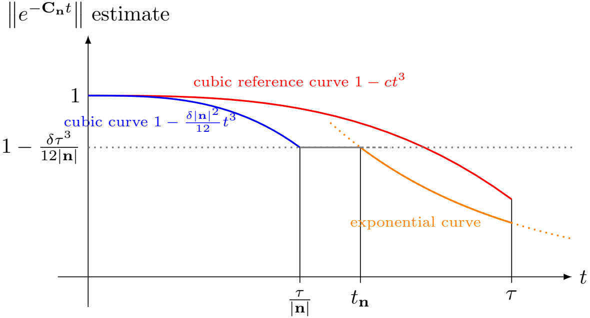
decay in Lorentz equation

Proof-Part (b) – short-time decay:

• goal: uniform polynomial decay of all modes:

$$\|e^{-\mathbf{C}_\mathbf{n} t}\|_{\mathcal{B}(L^2(\mathbb{S}^1))} \le 1 - ct^3, \quad 0 \le t \le \tau,\ \mathbf{n} \ne 0.$$

• strategy: combine the individual short-time decay (Part a) and long-time decay (Part c); with $\mathrm{cond}(\mathbf{P}_{|\mathbf{n}|}) = \frac{2|\mathbf{n}|+1}{2|\mathbf{n}|-1} \overset{|\mathbf{n}|\to\infty}{\longrightarrow} 1$:



Outline:

1. Goldstein-Taylor equation: Lyapunov functional for sharp large-t behavior
2. Lorentz equation: uniform short-t behavior of modes
3. linearized/nonlinear BGK equations: approximate Lyapunov functionals

Space-inhomogen. Bhatnagar-Gross-Krook model (1954)

- application: gas dynamics, computational fluid dynamics
- (vs. Boltzmann eq.) simplified kinetic model; relaxation operator:

$$f_t + v \cdot \nabla_x f = M_f(x,v,t) - f(x,v,t)\ , \quad t \geq 0\ ,\ x \in \mathbb{T}^d\ ,\ v \in \mathbb{R}^d$$

- relaxation towards x-local Maxwellian M_f with same hydrodynamic moments as f $\rightarrow$ local-in-x dissipation:

$$M_f(x,v) = \frac{\rho(x)}{(2\pi T(x))^{\frac{d}{2}}}\, e^{-\frac{|v-u(x)|^2}{2T(x)}}\ ,$$

$$\text{density} \quad \rho(x) := \int_{\mathbb{R}^d} f(x,v)\, dv\ ,$$

$$\text{mean velocity} \quad u(x) := \frac{1}{\rho(x)} \int_{\mathbb{R}^d} v f(x,v)\, dv\ ,$$

$$\text{temperature} \quad T(x) := \frac{1}{d\rho(x)} \int_{\mathbb{R}^d} |v-u(x)|^2 f(x,v)\, dv\ .$$

- transport operator $-v \cdot \nabla_x$ leads to uniformity in x $\quad\rightarrow\quad$ hypocoercive

1) Linear continuous velocity BGK model: Fourier modes

$$f_t + v f_x = M_T(v) \int_{\mathbb{R}} f(x,v,t)\, dv - f(x,v,t)\ ; \quad x \in \mathbb{T}\ ,\ v \in \mathbb{R},\ t > 0$$

- x-Fourier modes $k \in \mathbb{Z}$ decouple; Hermite function basis in v

$\Rightarrow$ consider "infinite vector" $\hat{\mathbf{f}}_k(t) \in \ell^2(\mathbb{N}_0)$ for each fixed mode k:

$$\partial_t \hat{\mathbf{f}}_k + ik\sqrt{T}\,\mathbf{L}_1 \hat{\mathbf{f}}_k = \mathbf{L}_2 \hat{\mathbf{f}}_k\ , \quad t \ge 0;\ k \in \mathbb{Z}\ ,$$

$$\mathbf{L}_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}\ , \quad \mathbf{L}_2 = \mathrm{diag}(0,\ -1,\ -1,\ \cdots)$$

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• need: spectral gap of the "infinite matrix" $\mathbf{C}_k := ik\sqrt{T}\,\mathbf{L}_1 - \mathbf{L}_2$, uniform in $k \in \mathbb{Z}$.

• sharp estimate is difficult for ∞-dimensional case

Continuous velocity BGK model: decay of Fourier modes

$\rightarrow$ approximate transform. matrices $\mathbf{P}_k$: ansatz for upper left 2×2 block:

$$\mathbf{P}_k^{2\times 2} = \begin{pmatrix} 1 & -i\alpha/k \\ i\alpha/k & 1 \end{pmatrix}, \quad \text{remaining diagonal} = 1.$$

Motivation: mix 0^{th}, 1^{st} mode of $\mathbf{L}_2 = \mathrm{diag}(0, -1, \cdots)$

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Lyapunov fct: $\quad e(f) := \sum_{k\in\mathbb{Z}} \left\langle (f_k(v) - M_T(v)), \tilde{\mathbf{P}}_k(f_k(v) - M_T(v))\right\rangle_{L^2(M_T^{-1})}$

$\tilde{\mathbf{P}}_k$ ... bounded operator on $L^2(M_T^{-1}(v))$, represented by matrix $\mathbf{P}_k$.

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Theorem 5

Let $T = 1$. $\Rightarrow \|f(t) - f_\infty\|_{L^2((0,2\pi)\times\mathbb{R};M_1(v)^{-1})}$ decays with rate $= 0.273...$

This estimated rate is off by factor 2.5 (compared to numerical result).

approximate transformation: justification of ansatz for $\mathbf{P}$

Consider $\quad \dot{x} = -\mathbf{C}x \ , \quad \mathbf{C} \in \mathbb{C}^{n\times n} \ , \quad \mathbf{C} = \mathbf{C}_H + \mathbf{C}_S \ .$

Theorem 6

Let $\mathbf{C}_H \geq 0$, $\dim(\ker(\mathbf{C}_H)) = 1$, $\mathbf{C}_S^H = -\mathbf{C}_S$.
W.l.o.g. $\mathbf{C}_H = \operatorname{diag}(0, c_2, ..., c_n)$, $c_j > 0$.
Let $\mathbf{C}_S = (\gamma_{jk})$, $\gamma_{12} \neq 0$ *(coupling of mode 1 & 2).*
Then: $\exists\, \mathbf{P} > 0$ *with*

$$\mathbf{P}^{2\times 2} = \begin{pmatrix} 1 & \lambda \\ \bar{\lambda} & 1 \end{pmatrix} ,$$

rest is $\mathbf{I}$, *such that* $\mathbf{PC} + \mathbf{C}^H\mathbf{P} > 0$.

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Proof: Choose $\lambda = re^{i\varphi}$ (with "good" φ).

Lowest eigenvalue $\tilde{\mu}(r)$ of $\mathbf{A}(r) := \mathbf{P}(r)\mathbf{C} + \mathbf{C}^H\mathbf{P}(r)$ is continuous with $\tilde{\mu}(0) = 0$ and differentiable at 0: $\frac{d\tilde{\mu}}{dr}(0) > 0$. $\square$

- Simplified ansatz for $\mathbf{P}$ always works, if only 1 conserved quantity.

2) 1D nonlinear (toy) BGK model

$$f_t(x,v,t) + vf_x = M_f(x,v,t) - f \; ; \quad x \in \mathbb{T} \, , \; v \in \mathbb{R}$$

M_f ... local Maxwellian with same mass & "temperature" (=2^{nd} moment) as f; but mean velocity := zero:

$$M_f(x,v) = \frac{\rho^{3/2}}{\sqrt{2\pi P}} e^{-\frac{v^2 \rho}{2P}} \; ; \quad \rho(x) = \int f \, dv \, , \quad P(x) = \int v^2 f \, dv$$

1D linearized BGK model

- Linearized model around temperature $= T := \frac{1}{2\pi}\int_0^{2\pi} P^I(x)\,dx$.
  perturbation/deviation from steady state $h := M_T - f$;
  in x Fourier / in v Hermite expansion:

$$\partial_t \hat{\mathbf{h}}_k(t) + ik\sqrt{T}\,\mathbf{L}_1 \hat{\mathbf{h}}_k(t) = \mathbf{L}_3 \hat{\mathbf{h}}_k(t)\,, \quad k \in \mathbb{Z};$$

$$\mathbf{L}_3 = \mathrm{diag}(0, -1, 0, -1, -1, \cdots)\,, \quad \mathbf{L}_1 = \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & \ddots \end{pmatrix}.$$

- 2 conserved quantities
- $\mathbf{L}_1$ couples mode 0 to mode 1; mode 2 to mode 3 (and 1)

1D linearized BGK model - simplified Lyapunov functional

$\Rightarrow$ simplified ansatz for $\mathbf{P}_k$:

$$\begin{pmatrix} 1 & -i\alpha/k & 0 & 0 \\ i\alpha/k & 1 & 0 & 0 \\ 0 & 0 & 1 & -i\beta/2k \\ 0 & 0 & i\beta/2k & 1 \end{pmatrix} \xrightarrow{k\to\infty} \mathbf{I}$$

as its upper-left 4×4 block; rest is $\mathbf{I}$.

- $\alpha = \beta = \frac{1}{3}$ yields for some $\mu_0 > 0$, with $\mathbf{C}_k := ik\mathbf{L}_1 - \mathbf{L}_3$:

$$\mathbf{P}_k\mathbf{C}_k + \mathbf{C}_k^*\mathbf{P}_k \geq 2\mu_0\mathbf{P}_k \ , \quad k\text{–uniformly.}$$

This is the essence for the following decay result:

1D linearized BGK model - exponential decay

Lyapunov functional (with $h = f - M_T$):

$$e_\gamma(f) := \sum_{k\in\mathbb{Z}} (1+k^2)^\gamma \langle h_k(v), \mathbf{P}_k h_k(v)\rangle_{L^2(M_T^{-1})} \sim \|f - M_T\|^2_{\mathcal{H}^\gamma}$$

with $\mathcal{H}^\gamma := H^\gamma(0,2\pi) \otimes L^2(\mathbb{R}_v; M_T^{-1})$, $\forall\, \gamma \geq 0$.

Theorem 7 (Achleitner-AA-Carlen 2016)

Let $T = 1$ and $\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} \binom{1}{v^2} f^I \, dvdx = \binom{1}{1}$. Then

$$e_\gamma(f(t)) \leq e^{-t/25} e_\gamma(f^I) \qquad t \geq 0;\ \forall\, \gamma \geq 0.$$

Remark: Estimated rate off by factor 18 (compared to numerical result).

1D nonlinear BGK model - local exponential stability

Theorem 8 (Achleitner-AA-Carlen 2016)

Additional assumptions (over Theorem 7): Let $\gamma > \frac{1}{2}$, $\|f^I - M_1\|_{\mathcal{H}^\gamma} < \delta_\gamma$ (=explicit constant). Then

$$e_\gamma(f(t)) \le e^{-t/25} e_\gamma(f^I)\ , \quad t \ge 0.$$

Proof-idea:

- Rewrite nonlinear BGK like linearized BGK + higher order remainder.
- Quantitative spectral gap compensates "linearization error".

1D nonlinear BGK model - local exponential stability

Proof: Rewrite nonlinear BGK like linearized BGK (use $h = f - M_1$):

$$h_t + v h_x = Q_2 h + R_f$$

- linearized BGK-operator (use $T = 1$): $Q_2 h(x, v) =$

$$= \Big(\frac{3}{2} - \frac{v^2}{2T}\Big) M_T(v) \int h\, dv + \Big(\frac{v^2}{2T^2} - \frac{1}{2T}\Big) M_T(v) \int v^2 h\, dv - h(x, v)$$

- nonlinear remainder: $\|R_f\|_{\mathcal{H}^\gamma} \le c\|f - M_1\|^2_{\mathcal{H}^\gamma}$,
  due to bounds on the perturbation of the moments:

$$\Big\| \int_{\mathbb{R}} h\, dv \Big\|_{L^\infty(0,2\pi)} + \Big\| \int_{\mathbb{R}} v^2 h\, dv \Big\|_{L^\infty(0,2\pi)} \overset{\text{Sobolev}}{\le} c\|f - M_1\|_{\mathcal{H}^\gamma} , \gamma > \frac{1}{2} .$$

$$\Rightarrow \quad \frac{d}{dt} e_\gamma(f) \le -\underbrace{0.0412}_{>1/25} \underbrace{e_\gamma(f)}_{=\mathcal{O}(h^2)} + c\|h\|^3_{\mathcal{H}^\gamma} . \qquad \square$$

3) True BGK model, linearized around f_∞

Let $T = 1$, $d = 1$:

$$\partial_t \hat{\mathbf{h}}_k + ik\,\mathbf{L}_1 \hat{\mathbf{h}}_k = \mathbf{L}_4 \hat{\mathbf{h}}_k\,, \quad k \in \mathbb{Z}; \tag{4}$$

$$\mathbf{L}_4 = \mathrm{diag}(0, 0, 0, -1, -1, \cdots)$$

- 3 conserved quantities (mass, mean velocity, temperature)
- iterated hypocoercive structure: $\circ\!\longrightarrow\!\circ\!\longrightarrow\!\circ\!\longrightarrow\!\bullet$ (index $= 3$): Tridiagonal matrix $\mathbf{L}_1$ ($\sim$ transport operator) couples mass-mode to velocity-mode to energy-mode to the dissipative mode 3.

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$\Rightarrow$ simplified ansatz for $\mathbf{P}_k$ (with some $\alpha,\, \beta,\, \gamma \in \mathbb{R}$) - in analogy to $\mathbf{L}_1$:

$$\begin{pmatrix} 1 & -i\alpha/k & 0 & 0 \\ i\alpha/k & 1 & -i\beta/k & 0 \\ 0 & i\beta/k & 1 & -i\gamma/k \\ 0 & 0 & i\gamma/k & 1 \end{pmatrix}$$

as its upper-left 4×4 block; rest is $\mathbf{I}$.

Theorem: exponential decay for (4) [also for $d = 2, 3$: index $= 2$]

Conclusion

- Modal decomposition (on $\mathbb{T}^d$) to obtain uniform/sharp exponential decay estimates for modal ODEs $\rightarrow$ exponential large-t decay.
- Hypocoercivity on $\mathbb{R}$: splitting of low/high modes $\rightarrow$ algebraic decay.
- nonlinear BGK: spectral gap compensates linearization error.

Conclusion

- Modal decomposition (on $\mathbb{T}^d$) to obtain uniform/sharp exponential decay estimates for modal ODEs $\rightarrow$ exponential large-t decay.
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