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Algebraic structure of hypocoercive equations & application to large-time behavior of kinetic equ's

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Framework & goals

- Given an evolution eq: $\frac{\mathrm{d}}{\mathrm{d}t} f = -L\,f,\ t \geq 0;\quad L$ ... const-in-t operator
- Assume $-L$ is semi-dissipative, i.e. $\quad L_H := \frac{1}{2}(L + L^*) \geq 0$
- Assume L has a unique steady state: $L\,f_\infty = 0$

- L is called hypocoercive if: $\|f(t) - f_\infty\| \leq c\,e^{-\mu t}\|f_0 - f_\infty\|$, $t \geq 0$, $\forall f_0$;
  but typically without coercivity of L

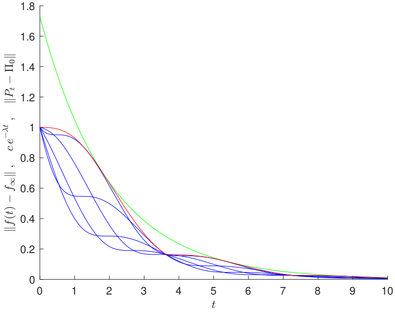
  for an ODE: hypocoercive = positive stable, i.e. $\quad \Re(\lambda_j(L)) \geq \mu > 0$

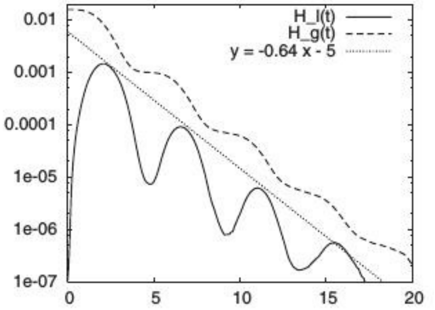
- Hypocoercivity is an interplay of 2 evolution mechanisms:
  a degenerate dissipation/relaxation & mixing for the non-dissipative "directions".

GOAL: Quantify the exponential decay behavior.

Framework & goals

- Wavy decay of $\|x(t)\|$ for ODE; ... of relative entropy w.r.t. the global Maxwellian (- - -$H_g(t)$)





sample trajectories $\|x(t)\|$,
envelope $\|e^{-Lt}\|$

simulation of 1+2 D Boltzmann eq. [Filbet-Mouhot-Pareschi 2006]

- Hypocoercivity index is a measure for the "degeneracy" of L.

GOAL: Connect algebraic structure (HC-index) to short+large time decay.

I. Algebraic structure, hypocoercivity index

Outline:

1. hypocoercive ODEs, Lyapunov's direct method
2. hypocoercivity index
3. hypocoercivity index for Hilbert-space evolution equations
4. hypocontractivity for discrete-t systems

Long-time decay for nonsymmetric ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $x^H \mathbf{C}_H x \geq \kappa \|x\|^2 \ \forall x$ (for a $\kappa > 0$); $\mathbf{C}_H := \frac{\mathbf{C}+\mathbf{C}^H}{2}$.

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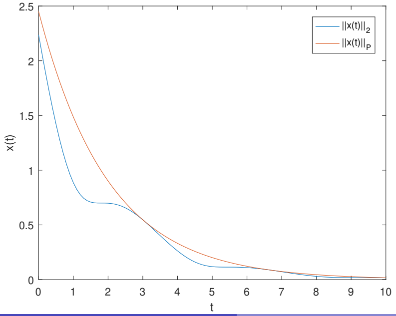
ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$ for (1).

- $\mathbf{C}$ not coercive $\Rightarrow$ no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of modified norm $\|x(t)\|_{\mathbf{P}} := \sqrt{x^H\,\mathbf{P}\,x}$; $\mathbf{P} := [2\ \ -1; -1\ 2]$



$\mathbf{P} > 0$ can be constructed from full spectral info of $\mathbf{C}$.

Constructing strictly decaying Lyapunov functionals is the main theme of large-t analysis of hypocoercive PDEs.

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov's direct method

Lemma 1

Let $\mathbf{C} \in \mathbb{C}^{n\times n}$ *be positive stable, i.e. spectral abscissa* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$.

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective*
   (i.e. geometric = algebraic multiplicity)
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{C}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{PC} + \mathbf{C}^H\mathbf{P} \geq 2\mu\mathbf{P}$ *(Lyapunov inequ.)*

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2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0\ : \quad \mathbf{PC} + \mathbf{C}^H\mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$.

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Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes z_j^H\,; \qquad z_j \ldots \text{ eigenvectors of } \mathbf{C}^H \qquad \square$$

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- If $\mathbf{C}$ is real $\Rightarrow$ $\mathbf{P}$ real.

Long-time decay of $\mathbf{P}$–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, $\mathbf{C}$ real):

  Let $\|x\|_P^2 := x^H \, \mathbf{P} \, x$.

$$\frac{d}{dt} \|x\|_{\mathbf{P}}^2 = -x^H \big(\underbrace{\mathbf{PC} + \mathbf{C}^H \mathbf{P}}_{\geq 2\mu \mathbf{P}} \big) x \leq -2\mu \|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \leq \|x(0)\|_{\mathbf{P}} \, e^{-\mu t}, \quad t \geq 0.$$

Long-time decay of P–norm

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$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \;\leq\; \|x(0)\|_{\mathbf{P}} \, e^{-\mu t}, \quad t \geq 0.$$

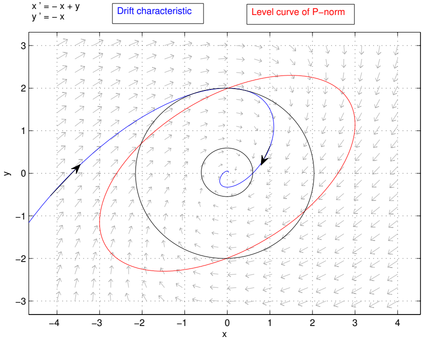
- motivation:
  Consider transformation $y := \mathbf{P}^{1/2}x \Rightarrow \dot{y} = -\hat{\mathbf{C}}y$, $\hat{\mathbf{C}} = \mathbf{P}^{1/2}\mathbf{C}\mathbf{P}^{-1/2}$.
  Then, spectral abscissa of $\hat{\mathbf{C}}_H$ is optimal $= \mu$.

- **P**–norm has been used for entropy/energy methods of kinetic equations (e.g. relaxation/BGK - also nonlinear, Fokker-Planck)

Decay of $\mathbf{P}$–norm

ex: $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2-axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x^* \mathbf{P}\, x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\rightarrow$ uniform decay of optimal Lyapunov funct. $\|x(t)\|_{\mathbf{P}}$ with sharp rate $\frac{1}{2}$.

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hypocoercive ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n$$

- Consider semi-dissipative systems $\mathbf{C}$:
  $\mathbf{C} = \mathbf{C}_H + \mathbf{C}_S \in \mathbb{C}^{n\times n}$; $\mathbf{C}_H^H = \mathbf{C}_H \geq 0$, $\mathbf{C}_S^H = -\mathbf{C}_S$
- Condition for hypocoercivity:
  No (non-trivial) subspace of $\ker \mathbf{C}_H$ is invariant under $\mathbf{C}_S$

Hypocoercivity index

Definition: Semi-dissipative system:

$$\dot{x} = -\mathbf{C}x, \qquad \text{with } \mathbf{C} = \mathbf{C}_H + \mathbf{C}_S \in \mathbb{C}^{n\times n} \tag{2}$$

if $\mathbf{C}_H \geq 0$ Hermitian; $\mathbf{C}_S$ is skew-Hermitian.

Definition 1 (Achleitner-AA-Carlen, KRM 2018)

The *hypocoercivity index* of $\mathbf{C}$ is the smallest integer $m_{HC} \in \mathbb{N}_0$, such that

$$\sum_{j=0}^{m_{HC}} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^H)^j > 0.$$

- $\mathbf{C}$ is coercive $\Leftrightarrow$ $\mathbf{C}_H > 0 \Leftrightarrow m_{HC} = 0$
- $\mathbf{C}$ is hypocoercive $\Leftrightarrow m_{HC} < \infty$ [not true for infinite dimensions]
- If $\mathbf{C}$ is hypocoercive: $\frac{n - \operatorname{rank} \mathbf{C}_H}{\operatorname{rank} \mathbf{C}_H} \leq m_{HC} \leq n - \operatorname{rank} \mathbf{C}_H$ $(\max = n-1)$
- m_{HC} describes the structural complexity of (2).

Hypocoercivity index for $\dot{x} = -(\mathbf{C}_H + \mathbf{C}_S)x$

<u>ex:</u> $\mathbf{C}_H = \mathrm{diag}(0, 0, 1, 1)$

(a) $\mathbf{C}_S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, HC index $= 1$ (direct connection)

(b) $\widetilde{\mathbf{C}}_S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, HC index $= 2$ (indirect connection)

Define HC index via iterated commutators, Kalman cond.

1) *hypocoercivity index* of $\mathbf{C}$: smallest integer m_{HC} such that

$$\sum_{j=0}^{m_{HC}} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^H)^j > 0.$$

2) equivalent definition via iterated commutators: smallest integer m_{HC} such that

$$\sum_{j=0}^{m_{HC}} \mathbf{B}_j^H \mathbf{B}_j > 0 \quad \text{with } \mathbf{B}_0 := \sqrt{\mathbf{C}_H};\ \mathbf{B}_{j+1} := [\mathbf{B}_j, \mathbf{C}_S],\ j \in \mathbb{N}_0.$$

- related to *Hörmander's "rank r" bracket condition for hypoellipticity,* Villani's hypocoercivity method.

3) equivalent definition via *Kalman rank condition*: smallest integer m_{HC} such that

$$\text{rank}[\mathbf{C}_H, \mathbf{C}_S\mathbf{C}_H, ..., (\mathbf{C}_S)^{m_{HC}}\mathbf{C}_H] = n. \tag{3}$$

- (3) yields controllability of $\dot{x} = \mathbf{A}x + \mathbf{B}u$; $\quad \mathbf{A} \sim \mathbf{C}_S$, $\mathbf{B} \sim \mathbf{C}_H$.

Preservation of the hypocoercivity index

Lemma 2 (Achleitner-AA-Mehrmann, ELA 2022)

Let $-\mathbf{C}$ be semi-dissipative with HC-index $m_{HC} \in \mathbb{N}_0$. The following linear transformations preserve it:

- *inversion:* $\mathbf{C}^{-1}$ *[tricky proof]*
- *unitary congruence:* $\hat{\mathbf{C}} = \mathbf{U}\mathbf{C}\mathbf{U}^H$ *with* $\mathbf{U}$ *unitary [trivial by definition]*

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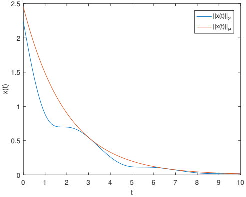
• Similarity transformations do *not* perserve it, in general:

ex: $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ has $m_{HC} = 1$.

But $\hat{\mathbf{C}} := \mathbf{P}^{1/2}\mathbf{C}\mathbf{P}^{-1/2}$ with $\mathbf{P} = [2\ -1; -1\ 2]$ has $m_{HC} = 0$:



$\|x(t)\|_{\mathbf{P}} = \|y(t)\|_2$
with $y := \mathbf{P}^{1/2}x$

Short-time decay / hypocoercivity index for $\dot{x} = -\mathbf{C}x$

characterization:

Theorem 2 (Achleitner-AA-Carlen, JDE 2023)

Let $-\mathbf{C}$ be semi-dissipative. Then its HC-index is $m_{HC} \in \mathbb{N}_0$ iff

$$\|e^{-\mathbf{C}t}\|_2 = 1 - c\, t^{2m_{HC}+1} + \mathcal{O}(t^{2m_{HC}+2}), \quad t \to 0+$$

- *Optimal $c > 0$ is explicit, in terms of $\mathbf{C}_H$, $\mathbf{C}_S$.*

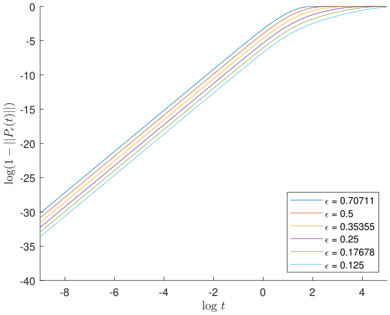
proof very technical:

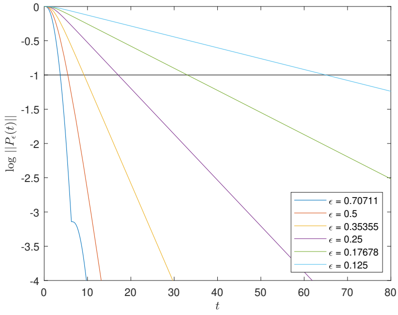
- lower bound: For $x_0 \in \bigcap_{j=0}^{m_{HC}-1} \ker\left(\sqrt{\mathbf{C}_H}\mathbf{C}_S^j\right)$ show $\|x(t)\|^2 \sim 1 - \tilde{c}\, t^{2m_{HC}+1}$.
- upper bound: Optimal c is not realized by a single (worst) trajectory, only as envelope of well-chosen family of initial conditions.
- use: $\|e^{-\mathbf{C}t}\|_2$ is real analytic on some $[0, t_0)$.

2 decay regimes: first polynomial, then exponential decay

ex: $\dot{x} = -\mathbf{C}x$, $\mathbf{C} = \mathbf{C}_H + \epsilon \mathbf{C}_S$

$$\mathbf{C}_H = \mathrm{diag}(0,\,0,\,1,\,1), \quad \mathbf{C}_S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad m_{HC} = 1$$



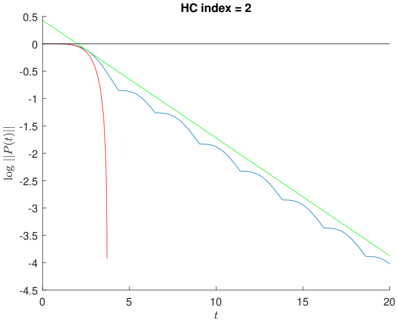


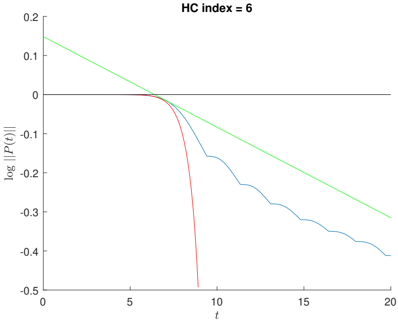
$\|e^{-\mathbf{C}t}\|_2 = 1 - \frac{\epsilon^2}{12} t^3 + \mathcal{O}(t^4)$

exponential decay for large time

Large HC-index delays exponential decay of $\|e^{-\mathbf{C}t}\|_2$

<u>ex:</u> $$\mathbf{C} = \begin{pmatrix} 0 & 1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad m_{HC} = n-1$$





$$\|e^{-\mathbf{C}t}\|_2, \qquad 1 - c\, t^{2m_{HC}+1}, \qquad c\, e^{-\mu t}$$

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Hypocoercivity

Definition 3 (Villani 2009)

Consider L on Hilbert space $\mathcal{H}$ with $\mathcal{K} := \ker L$; let $\tilde{\mathcal{H}} \hookrightarrow \mathcal{K}^\perp$ (densely) (e.g. $\mathcal{H}$ ... weighted L^2, $\tilde{\mathcal{H}}$ ... weighted H^1).
L is called hypocoercive on $\tilde{\mathcal{H}}$ if $\exists\, \lambda > 0,\ c \geq 1$:

$$\|\mathrm{e}^{-Lt} f_0\|_{\tilde{\mathcal{H}}} \leq c\, \mathrm{e}^{-\lambda t} \|f_0\|_{\tilde{\mathcal{H}}} \qquad \forall\, f_0 \in \tilde{\mathcal{H}}$$

- typically $c > 1$

Hypocoercivity for Hilbert space evolution equations (1)

$$\dot{x}(t) = -\mathbf{C}x(t), \quad x(0) = x_0 \in \mathcal{H} \text{ ... separable Hilbert space}$$

Let $\mathbf{C} \in \mathcal{B}(\mathcal{H})$, $-\mathbf{C}$ semi-dissipative (often in physical systems) with $\ker \mathbf{C} = \{0\}$ (for simplicity, otherwise project it out).

Definition 4

1. $\mathbf{C} = \mathbf{C}_H + \mathbf{C}_S$ ($\mathbf{C}_H^* = \mathbf{C}_H \geq 0$, $\mathbf{C}_S^* = -\mathbf{C}_S$) is *hypocoercive* if

$$\|e^{-\mathbf{C}t} x_0\|_{\mathcal{H}} \leq c\, e^{-\lambda t} \|x_0\|_{\mathcal{H}} \quad \forall x_0 \in \mathcal{H}\,.$$

[= generator of a uniformly exponentially stable semigroup]

2. The *hypocoercivity index* of $\mathbf{C}$ is the smallest $m_{HC} \in \mathbb{N}_0$, such that

$$\sum_{j=0}^{m_{HC}} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^*)^j \geq \kappa \mathbf{I} \quad \text{for some } \kappa > 0\,.$$

[Uniform lower bound $\kappa\mathbf{I}$ in $\mathcal{H}$ is important.]

Hypocoercivity for Hilbert space evolution equations (2)

Lemma 3 (Achleitner-AA-Mehrmann-Nigsch, JFA 2025)

Let $\mathbf{C} = \mathbf{C}_H + \mathbf{C}_S \in \mathcal{B}(\mathcal{H})$. If $\mathbf{C}$ is hypocoercive, its index is finite.

Proof:

- Hypocoercivity condition $\sum_{j=0}^{m} \mathbf{C}_S^j \mathbf{C}_H (\mathbf{C}_S^*)^j \geq \kappa \mathbf{I}$ is equivalent to

$$\operatorname{span}\Big(\bigcup_{j=0}^{m} \operatorname{range}\big(\mathbf{C}_S^j \sqrt{\mathbf{C}_H}\big)\Big) = \mathcal{H}\,. \quad \text{[cp. Kalman rank cond.]} \qquad (4)$$

- indirect: If (4) holds with $m = \infty$, it also holds for some finite m: By Baire's category theorem, $\mathcal{H}$ cannot be the union of meager sets $\big\{\operatorname{range}\big(\mathbf{C}_S^j \sqrt{\mathbf{C}_H}\big)\big\}_j$, hence one of them is non-meager. So, by the open mapping theorem, a finite combination is already surjective. □

Rem: $m_{HC} = \infty$ is possible for unbounded generators.

Hypocoercivity for Hilbert space evolution equations (3)

Theorem 5 (reverse direction)

Let $-\mathbf{C} \in \mathcal{B}(\mathcal{H})$ *be semi-dissipative (for* $\dot{x} = -\mathbf{C}x$*). Then*

$$\mathbf{C} \textit{ is hypocoercive iff } \; m_{HC} \in \mathbb{N}_0 \,.$$

Rem: LHS is a global-in-t decay result; RHS a (local) generator property.

Hypocoercivity for Hilbert space evolution equations (3)

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Rem: LHS is a global-in-t decay result; RHS a (local) generator property.

Proof ($\Leftarrow$):

- equivalent condition for HC-index: $\sum_{j=0}^{m} (\mathbf{C}^*)^j \mathbf{C}_H \mathbf{C}^j \geq \kappa \mathbf{I}$
- Lyapunov functional: $\|x\|_{\mathbf{P}}^2 := \langle x, \mathbf{P}x \rangle$, $\mathcal{B}(\mathcal{H}) \ni \mathbf{P} := \displaystyle\sum_{j=0}^{m} (\mathbf{C}^*)^j \mathbf{C}^j \geq \mathbf{I}$,

$$\frac{d}{dt}\|x(t)\|_{\mathbf{P}}^2 = -2\sum_{j=0}^{m} \langle \mathbf{C}^j x, \mathbf{C}_H \mathbf{C}^j x \rangle \leq -2\kappa \|x\|_{\mathcal{H}}^2 \leq -\frac{2\kappa}{\|\mathbf{P}\|}\|x(t)\|_{\mathbf{P}}^2 \qquad \square$$

This gives exp. decay of $\|x(t)\|_{\mathcal{H}}$, $\|x(t)\|_{\mathbf{P}}$.

short-time behavior: characterization by hypocoercivity index

Theorem 6 (JFA 2025)

Let $-\mathbf{C} \in \mathcal{B}(\mathcal{H})$ be semi-dissipative. Then its HC-index is $m_{HC} \in \mathbb{N}_0$ iff

$$\|e^{-\mathbf{C}t}\|_{\mathcal{B}(\mathcal{H})} = 1 - c\, t^{2m_{HC}+1} + o(t^{2m_{HC}+1}), \quad t \to 0+$$

Rem: Remainder is only $o(t^{2m_{HC}+1})$, since $\|e^{-\mathbf{C}t}\|_{\mathcal{B}(\mathcal{H})}$ is in general not analytic close to $t = 0$ [but OK for matrices $\mathbf{C}$].

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discrete-time systems (ODEs) — asymptotic stability

- consider t-discretization of $\dot{x} = -\mathbf{C}x$: $\quad x_{k+1} = \mathbf{A}x_k$
- goal: discrete analog of hypocoercivity

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spectral norm: $\|\mathbf{A}\|_2 = \sqrt{\lambda_{max}(\mathbf{A}^H\mathbf{A})} = \sigma_{max}(\mathbf{A})$

Definition: $\mathbf{A} \in \mathbb{C}^{n\times n}$ is ...

- *contractive* if $\sigma_{max}(\mathbf{A}) < 1$;
- *semi-contractive* if $\sigma_{max}(\mathbf{A}) \leq 1$;
- *hypocontractive* (or asymptotically stable) if all $|\lambda(\mathbf{A})| < 1$.

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Definition 7

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be semi-contractive. Its *hypocontractivity index* $m_{dHC} \in \mathbb{N}_0$ is the smallest integer m, such that

$$\sum_{j=0}^{m} (\mathbf{A}^H)^j(\mathbf{I} - \mathbf{A}^H\mathbf{A})\mathbf{A}^j > 0 \ .$$

- $\mathbf{A}$ is contractive iff $m_{dHC} = 0$ (i.e. $\mathbf{A}^H\mathbf{A} < \mathbf{I}$).

discrete short-time behavior: characterization by hypocontractivity index

Theorem 8 (Achleitner-AA-Mehrmann, ELA 2022)

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be semi-contractive (i.e. $\sigma_{max}(\mathbf{A}) \leq 1$). Its hypocontractivity index is $m_{dHC} \in \mathbb{N}_0$ iff

$$\|\mathbf{A}^j\|_2 = 1, \text{ for all } j = 1, ..., m_{dHC}, \quad \text{and } \|\mathbf{A}^{m_{dHC}+1}\|_2 < 1 \ .$$

Mid-point rule: connection of hypocoercivity/-contractivity

- Consider mid-point-discretization of $\dot{x} = -\mathbf{C}x$:
  $\frac{x_{k+1}-x_k}{h} = -\mathbf{C}\frac{x_{k+1}+x_k}{2}$
- e.g. with $h = 1$: $x_{k+1} = \mathbf{A}x_k$ with $\mathbf{A} = (\mathbf{I} - \frac{\mathbf{C}}{2})(\mathbf{I} + \frac{\mathbf{C}}{2})^{-1}$

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Theorem 9 (Achleitner-AA-Mehrmann, ELA 2022)

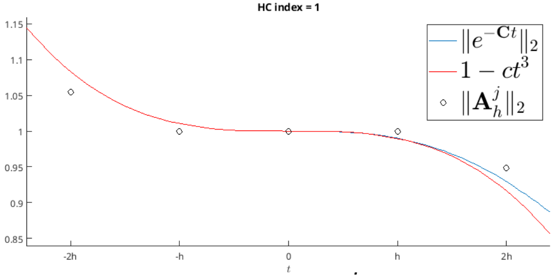
Let -1 not be an eigenvalue of $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then:

- $\mathbf{C}$ *is hypocoercive* $\Leftrightarrow$ $\mathbf{A}$ *is hypocontractive.*
- $-\mathbf{C}$ *is semi-dissipative* $\Leftrightarrow$ $\mathbf{A}$ *is semi-contractive.*
- *The hypocoercivity and hypocontractivity indices coincide:*
  $m_{HC}(\mathbf{C}) = m_{dHC}(\mathbf{A})$
- *For the scaled Cayley transform* $\mathbf{A}_h := (\mathbf{I} - \frac{h}{2}\mathbf{C})(\mathbf{I} + \frac{h}{2}\mathbf{C})^{-1}$, $h > 0$, $m_{dHC}(\mathbf{A}_h)$ *is constant in* h.

- Same connection for $\mathbf{C}$, $\mathbf{A} \in \mathcal{B}(\mathcal{H})$.

explain exponent "mismatch": hypocoercivity/-contractivity

- hypocoercivity: $\phi(t) := \|e^{-\mathbf{C}t}\|_2 \sim 1 - c\, t^{2m_{HC}+1}$
- hypocontractivity: $\|\mathbf{A}_h^{m_{HC}+1}\|_2 < 1, \quad \mathbf{A}_h := (\mathbf{I} - \frac{h}{2}\mathbf{C})(\mathbf{I} + \frac{h}{2}\mathbf{C})^{-1}$



Let $m_{HC} = 1$. Idea: odd extension of $\|\mathbf{A}_h^j\|_2$ to $j < 0$, like $1 - ct^3$.

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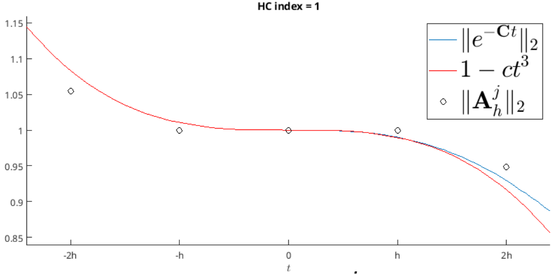

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- structure-preserving numerical method w.r.t. hypocoercivity $\forall m_{HC}$:
- consistent finite difference analog of $\phi(t)$:

$$\frac{\Delta^{2m+1}[\|\mathbf{A}_h^j\|_2, j = -m-1, ..., m+1]}{h^{2m+1}} \xrightarrow{h\to 0} \phi^{(2m+1)}(0) = -(2m+1)!\, c$$

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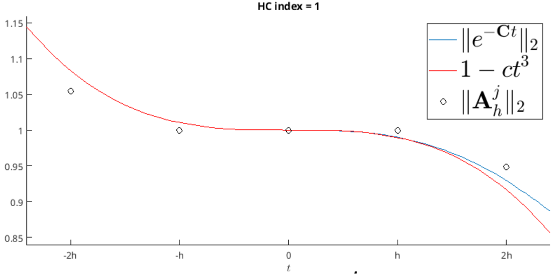

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θ-schemes, $\theta \neq \frac{1}{2}$: no match hypocoercivity/-contractivity

- Consider θ-schemes for $\dot{x} = -\mathbf{C}x$:
  $\frac{x_{k+1}-x_k}{h} = -\mathbf{C}\big(\theta x_{k+1} + (1-\theta)x_k\big), \quad \theta \in [0,1]$
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Theorem 10 (AA-Egger, 2026)

(a) *Let $\frac{1}{2} < \theta \leq 1$, let $\mathbf{C}$ be hypocoercive and $-\mathbf{C}$ be semi-dissipative. Then, for any $h > 0$, $\mathbf{A}_h$ is contractive, i.e. $m_{dHC}(\mathbf{A}_h) = 0$ (for any $m_{HC}(\mathbf{C}) \in \mathbb{N}_0$).*

(b) *Let $0 \leq \theta < \frac{1}{2}$, let $\mathbf{C}$ be hypocoercive.*
  - *If $m_{HC}(\mathbf{C}) = 0$, then $\mathbf{A}_h$ is contractive for $h \in (0, h_0)$ with some $h_0 > 0$, i.e. $m_{dHC}(\mathbf{A}_h) = 0$.*
  - *If $m_{HC}(\mathbf{C}) \geq 1$, then $\mathbf{A}_h$ is <u>not</u> semi-contractive for any $h > 0$.*

Hence: The more implicit schemes are too dissipative; the more explicit schemes lose semi-contractivity for (only) hypocoercive evolutions.

Conclusion

- Hypocoercivity index characterizes the short-time decay of ODEs and Hilbert space evolution equations $\dot{x} = -\mathbf{C}x$, $\mathbf{C} \in \mathcal{B}(\mathcal{H})$: first polynomial, then exponential decay of $\|e^{-\mathbf{C}t}\|$.
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References

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