Towards a better understanding of $\mathcal{C}^p$ functions and definable $\mathcal{C}^p$ functions of several variables

Rafał Pierzchała

The first aim of my talk is to present the following result concerning definable $\mathcal{C}^p$ functions.

Theorem 1. *Assume that $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is an open set, is a function definable in some polynomially bounded o-minimal structure (e.g., a semialgebraic function), and let $p \in \mathbb{N}$. Then:*

(i) If, for each $\nu \in \{1, \dots, N\}$, the partial derivative $\dfrac{\partial^p f}{\partial x_\nu^p}$ exists and is continuous in Ω, then f is of class $\mathcal{C}^p$.

(ii) If, for each $\nu \in \{1, \dots, N\}$, the partial derivative $\dfrac{\partial^p f}{\partial x_\nu^p}$ exists and is locally bounded in Ω, then $f \in \mathcal{C}^{p-1,1}(\Omega)$, i.e., f is of class $\mathcal{C}^{p-1}$ and, for each $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = p-1$, the partial derivative $D^\alpha f$ is locally Lipschitz in Ω.

The above result is connected with the following problem. Assume that $\omega : [0, +\infty) \to [0, +\infty)$ is a modulus of continuity. Assume moreover that $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is an open set, and let $p \in \mathbb{N}_0$. Consider the following two conditions:

(i) f is of class $\mathcal{C}^p$ and, for each $a \in \Omega$, there exist an open neighbourhood $W_a \subset \Omega$ and $C_a > 0$ such that, for each $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = p$ and each $y, z \in W_a$,

$$|D^\alpha f(y) - D^\alpha f(z)| \le C_a \omega(|y-z|)\,.$$

(ii) For each $\nu \in \{1, \dots, N\}$, the partial derivative $\dfrac{\partial^p f}{\partial x_\nu^p}$ exists. Moreover, for each $a \in \Omega$, there exist an open neighbourhood $U_a \subset \Omega$ and $M_a > 0$ such that, for each $\nu \in \{1, \dots, N\}$ and each $y, z \in U_a$ with $y - z \in \mathbb{R}e_\nu$,[1]

$$\left| \frac{\partial^p f}{\partial x_\nu^p}(y) - \frac{\partial^p f}{\partial x_\nu^p}(z) \right| \le M_a \omega(|y-z|)\,.$$

(If $|\alpha| = 0$, then $D^\alpha f := f$. If $p = 0$, then $\dfrac{\partial^p f}{\partial x_\nu^p} := f$.)

Problem 2. *What assumptions on ω should be made to guarantee that conditions (i) and (ii) are equivalent?*

The second aim of my talk is to provide a complete answer to this problem.

[1] By $e_1, \dots, e_N$ we denote the canonical basis in the vector space $\mathbb{R}^N$.