Abstract

In this work, we study stochastic Volterra integral equations (SVIEs) of convolution type, frameworks for Hilbert space-valued Markovian lifts of the latter, small-time limit theorems for both classes of processes and three independent proof concepts for verifying that solutions to sufficiently non-degenerate SVIEs cannot possess the Markov property. While the latter appears as a unanimously agreed fact in the literature, so far a rigorous mathematical proof thereof has been absent.

In the first part of the thesis, we prove a small-time central limit theorem (CLT) for the finite-dimensional distributions of sufficiently smooth transformations of solutions to stochastic Volterra integral equations as well as its extension to a functional CLT. Here we focus on kernels with small-time power law asymptotics and coefficients satisfying linear growth and Hölder conditions. Moreover, as an application of the obtained functional CLT in the context of rough volatility models, we derive asymptotic pricing formulae for digital calls on the realized variance in three different regimes showcasing high robustness with regard to the parameters of the underlying volatility process. The results hold in particular for the rough Heston model.

Furthermore, we introduce a framework of Hilbert space-valued Markovian lifts for SVIEs which is particularly suited for scalar completely monotone kernels whose Bernstein measures fulfill a mild moment condition. We proceed by deriving a corresponding small-time CLT and a functional CLT also for transformations of the Markovian lift under suitable continuous linear functionals. As a special case, this contains in particular the above CLT for the Volterra process itself, provided that a certain projection operator is applicable.

As an application of the CLT framework, the first approach presented for proving the failure of the time-homogeneous Markov property for stochastic Volterra integral equations unveils a connection with the non-Markovianity of the Gaussian limit process obtained in the above CLT. In particular, we show that once this limit process is not a Markov process, also the original Volterra process cannot possess the time-homogeneous Markov property. Except for the generally Markovian SDE case, our result contains SVIEs with Riemann-Liouville and gamma kernels for the full range of the Hurst parameter. Furthermore, our second method is specifically designed for SVIEs with affine drifts. As for these processes there exist semi-explicit formulae for expectations and conditional expectations, we can show by solving certain integro-functional equations that an SVIE solution with an exponential or constant initial curve which fulfills the time-homogeneous Markov property necessarily has to possess a kernel of the same type. When the drift vanishes, analogous results are obtained under structural assumptions on the diffusion coefficient. Moreover, based on the semi-explicit formula for conditional expectations, we show that by the non-measurability of a certain path functional, the Volterra square-root process cannot even possess the time-inhomogeneous Markov property.

Finally, as our most universal result on the failure of the time-inhomogeneous Markov

property, we consider general d-dimensional SVIEs with matrix-valued Volterra kernels and a potentially random initial curve which is measurable with respect to the information provided at time 0. In an abstract framework for Markovian lifts of these SVIEs onto a suitable separable Hilbert space, which includes the previously introduced lift as a special case, we prove the failure of the Markov property once there exists an admissible, suitably non-degenerate second Volterra kernel that can be obtained by perturbations of the original Volterra kernel, while similar approximations have to converge also for the initial curve, if it is random. Such perturbations are usually realized via applications of appropriate bounded linear operators to the corresponding lift. In particular, this framework covers the classes of completely monotone and absolutely continuous kernels which fulfill certain technical integrability conditions required for constructing the corresponding lift. For example, our results include d-dimensional SVIEs with (diagonal) rough, regular and shift-regularized Riemann-Liouville and gamma kernels and Hölder continuous drift and diffusion coefficient.