

Abstract

This thesis is devoted to rate-optimal adaptive finite element methods (AFEMs) for semilinear elliptic partial differential equations (PDEs). It considers a model problem with a nonlinear reaction term, where the operator associated to the PDE is *locally* Lipschitz continuous. By equilibrating various error sources, this thesis proves that all presented algorithms are rate-optimal, i.e., a suitable error quantity converges with optimal decay rate with respect to the number of degrees of freedom of the discretization. The main contributions are the following:

First, we investigate a goal-oriented AFEM (GOAFEM) for the semilinear model problem where the principal aim is to approximate a linear functional (*quantity of interest*) evaluated at the exact, but unknown solution with optimal convergence rates. By means of established duality techniques, the approximation error can be estimated by a product of two approximation errors. This product structure allows that convergence rates add up, contributing substantially to the attractiveness of GOAFEMs in practice. For nonlinear problems, the approximation error in the goal first leads to a noncomputable theoretical dual problem that depends on the unavailable exact solution. To make the goal error accessible, we replace this by a computable practical dual problem. A suitable marking strategy for the refinement allows for the proof of R-linear convergence: contraction of the error product regardless of which error component determines the marked elements. Moreover, we show optimal convergence rates with respect to the number of degrees of freedom of the discretization. This, for the first time, extends the literature on rate-optimal GOAFEM to a model problem with underlying nonlinear PDE.

Second, to efficiently solve the nonlinear model problem, we consider an AFEM, where the number of linearization steps is also steered adaptively. Under the assumption that all arising linear systems can be solved at linear cost, the proposed algorithm, coined as adaptive iteratively linearized finite element method (AILFEM), is cost-optimal. This means that the suitable error quantity decays with optimal convergence rates with respect to the (theoretical) overall computational cost that is needed to obtain the numerical approximation. The main challenge in the numerical analysis of locally Lipschitz continuous problems is to ensure that all iterates are uniformly bounded. Having achieved this, we prove full R-linear convergence, i.e., contraction of an error quantity independently of the adaptivity parameters and regardless of whether we refine the mesh or perform a linearization step. For sufficiently small adaptivity parameters, we eventually establish optimal convergence rates with respect to the theoretical computational cost of the proposed AILFEM.

Third, we analyze the preceding AILFEM, where the linearized problem is additionally solved with an iterative algebraic solver. This perturbation of the exact linearization procedure significantly increases the technicalities to verify uniform boundedness of all iterates. As before, we prove full R-linear convergence for an error quantity that now consists of error components stemming from discretization, linearization, and the algebraic solver. We conclude optimal rates with respect to computational complexity. Importantly, all steps in the AILFEM strategy can now rigorously be realized in linear complexity. Hence, optimal convergence rates with respect to overall computational cost can indeed be understood as optimal convergence rates with respect to computation time. This is also observed in numerical experiments.