Between the Finite and the Infinite

J. P. Aguilera
Technische Universität Wien, 2019

Abstract

Various results in Proof Theory, Set Theory, and Recursion Theory are proved.

We introduce the *reflection order*, an ordering on patterns of iterated Σ^1_1- and Π^1_1-reflection for ordinals. Extending Aanderaa's theorem that $\pi^1_1 < \sigma^1_1$, we prove that σ^1_1 has rank ω in the reflection order and $\sigma^1_1(\sigma^1_1)$ has rank ω^ω. We prove that the reflection order is a prewellordering of length $\geq\varepsilon_0$. As part of the proof, we extend classical results of Aczel, Richter, and Gostanian.

We study the determinacy of games of transfinite length, mostly of length ω^2. We prove that clopen determinacy of length ω^2 is equivalent to σ-projective determinacy of length ω, to the existence of a certain uncountable sequence of canonical inner models with finitely many Woodin cardinals, and to the cut elimination theorem for Takeuti's Determinate Logic for languages with no atomic formulae of infinite arity. We obtain further characterizations of this in joint work with S. Müller and P. Schlicht.

We compute the consistency strength of determinacy for games of length ω^2 and payoff in other pointclasses, including the open sets, the F_σ sets, the Borel sets, the projective sets, and the σ-projective sets (the latter two jointly with S. Müller). We give new upper bounds for the consistency strength of clopen determinacy for games with length of the form $\omega^2 \cdot \beta$ and show that if there are ω^2 Woodin cardinals below a measurable cardinal, then there is a model of Kripke-Platek set theory satisfying $\mathsf{AD}_{\mathbb{R}}$.

We study proofs in finitary and infinitary systems with weakened eigenvariable conditions. Jointly with M. Baaz, we prove a speed-up theorem for cut-free proofs and characterize the sequents provable in intuitionistic ε-calculus. In the infinitary case, we characterize cut elimination theorems for these systems in terms of determinacy assertions, following Takeuti.

Jointly with D. Fernández-Duque, we prove that provability logic is strongly complete with respect to its topological semantics, and, in particular, with respect to the ordinal $\omega^\omega + 1$. More generally, we introduce a generalization of Icard spaces and show that provability logic is individually strongly complete with respect to every Icard space of large enough rank.

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Finally, we prove a completeness theorem for transfinite provability logic with respect to finite polytopologies based on Icard spaces.